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Adjunctions and Galois connections in complete co-residuated lattices

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Abstract. In this paper, using distance function based on complete co-residuated lattices, we investigate adjunctions, Galois connections and join (meet) preserving maps between various operations as extensions of Zadeh powerset operations. We give their examples.

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1. Introduction

Ward et al. [1] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2, 3] investigated the properties of fuzzy Galois connections and fuzzy closure operators on a residuated lattice which supports part of foundation of theoretic computer science. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

As a dual sense of complete residuated lattice, Zheng et al. [11] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng et al. [12] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \odot, \&)$ where $(L, \&)$ is a complete residuated lattice and $(L, \odot)$ is a complete co-residuated lattice. Kim and Ko [13] introduced the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces in complete co-residuated lattices. Moreover, Oh and Kim [14, 15] investigated the properties of Alexandrov fuzzy topologies, distance functions, join preserving maps, join approximation maps fuzzy complete lattices using distance functions instead of fuzzy partially orders in complete co-residuated lattices.

Bělohlávek [2, 3] introduced the notion of formal concepts with $R \in L^{X\times Y}$ on a complete residuated lattice $(L, \odot, \rightarrow)$. A formal fuzzy concept is a pair $(A, B) \in$

$L^X \times L^Y$ such that $F(A) = B, G(B) = A$ where $F : L^X \to L^Y, G : L^Y \to L^X$ are defined as

$$F(A)(y) = \bigwedge_{x\in X}(A(x) \to R(x,y)),$$
$$G(B)(x) = \bigwedge_{y\in Y}(B(y) \to R(x,y)).$$

Moreover, (F,G) is a Galois connection, i.e., $e_{L^Y}(B, F(A)) = e_{L^X}(A, G(B))$, where e_{L^Y} is a partially order defined as $e_{L^Y}(B, F(A)) = \bigwedge_{y\in Y}(B(y) \to F(A)(y))$.

Georgescu and Popescu [16] proposed attribute-oriented fuzzy concept lattices. A attribute-oriented fuzzy concept is a pair $(A,B) \in L^X \times L^Y$ such that $F(A) = B, G(B) = A$, where $F : L^X \to L^Y, G : L^Y \to L^X$ are defined as

$$F(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y)),$$
$$G(B)(x) = \bigwedge_{y\in Y}(R(x,y) \to B(y)).$$

Moreover, (F,G) is a adjunction, i.e., $e_{L^Y}(F(A), B) = e_{L^X}(A, G(B))$.

Our aim of this paper, using the distance functions d_{L^X} instead of fuzzy partially ordered sets e_{L^X} based on complete co-residuated lattices, we investigate adjunctions, Galois connections and join (meet) preserving maps on Alexandrov topologies. As applications of this paper, using adjunctions and Galois connections, we define a formal fuzzy concept and an attribute-oriented fuzzy concept in Remark 3.5.

Rodabough [5] introduced the adjoint function theorem using the adjunctions. He showed that $(f^\to, f^\leftarrow)$ is an adjunction, where Zadeh's powersets operators $f^\to : L^X \to L^Y, f^\leftarrow : L^Y \to L^X$ are defined as

$$f^\to(A)(y) = \bigvee_{f(x)=y} A(x),\ f^\to(B)(x) = B(f(x)).$$

As extensions of Zadeh's powersets operators from fuzzy sets to fuzzy sets, four types of operations [17, 18] are investigated. Using adjunctions, Galois connections and distance functions, we study various operators from Alexandrov topologies to Alexandrov topologies in co-residuated lattices.

2. Preliminaries

Definition 2.1 ([11, 12, 13, 14, 15])**.** An algebra $(L, \wedge, \vee, \oplus, \bot, \top)$ is called a *complete co-residuated lattice,* if it satisfies the following conditions:

(C1) $L = (L, \vee, \wedge, \bot, \top)$ is a complete lattice, where $\bot$ is the bottom element and $\top$ is the top element,

(C2) $a = a \oplus \bot$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$,

(C3) $(\bigwedge_{i\in\Gamma} a_i) \oplus b = \bigwedge_{i\in\Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge\{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x)$, $(\alpha \oplus A)(x) = \alpha \oplus A(x)$, $\alpha_X(x) = \alpha$.

Put $n(x) = \top \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law.*

Lemma 2.2 ([13, 14, 15]). *Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \le z$, $x \oplus y \le x \oplus z$, $y \ominus x \le z \ominus x$ and $x \ominus z \le x \ominus y$.*

(2) $(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ *and* $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.

(3) $(\bigwedge_{i\in\Gamma} x_i) \ominus y \le \bigwedge_{i\in\Gamma}(x_i \ominus y)$

(4) $x \ominus (\bigvee_{i\in\Gamma} y_i) \le \bigwedge_{i\in\Gamma}(x \ominus y_i)$.

(5) $x \ominus x = \bot$, $x \ominus \bot = x$ *and* $\bot \ominus x = \bot$. *Moreover,* $x \ominus y = \bot$ *iff* $x \le y$.

(6) $y \oplus (x \ominus y) \ge x$, $y \ge x \ominus (x \ominus y)$ *and* $(x \ominus y) \oplus (y \ominus z) \ge x \ominus z$.

(7) $x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.

(8) $x \ominus y \ge (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \ge (x \ominus z) \ominus (y \ominus z)$, $y \ominus x \ge (z \ominus x) \ominus (z \ominus y)$ *and* $(x \oplus y) \ominus (z \oplus w) \le (x \ominus z) \oplus (y \ominus w)$.

(9) $x \oplus y = \bot$ *iff* $x = \bot$ *and* $y = \bot$.

(10) $(x \oplus y) \ominus z \le x \oplus (y \ominus z)$ *and* $(x \ominus y) \oplus z \ge x \ominus (y \ominus z)$.

(11) $(\bigvee_{i\in\Gamma} x_i) \ominus (\bigvee_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(12) $(\bigwedge_{i\in\Gamma} x_i) \ominus (\bigwedge_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(13) *If L satisfies a double negative law and $n(x) = \top \ominus x$, then $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ and $x \ominus y = n(y) \ominus n(x)$.*

Definition 2.3 ([13, 14, 15]). Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x, x) = \bot$ for all $x \in X$,

(M2) $d_X(x, y) \oplus d_X(y, z) \ge d_X(x, z)$ for all $x, y, z \in X$,

(M3) if $d_X(x, y) = d_X(y, x) = \bot$, then $x = y$.

The pair (X, d_X) is called a *distance space.*

Remark 2.4 ([13, 14, 15]). Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.2 (5) and (6), (L, d_L) is a distance space. For $\tau \subset L^X$, we define a function $d_\tau : \tau \times \tau \to L$ as $d_\tau(A, B) = \bigvee_{x\in X}(A(x) \ominus B(x))$. Then (τ, d_τ) is a distance space.

In this paper, we assume $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ is a complete co-residuated lattice.

Definition 2.5 ([15]). Let (X, d_X) be a distance space and $A \in L^X$.

(1) A point x_0 is called a *fuzzy join* of A, denoted by $x_0 = \sqcup_X A$, if it satisfies

(J1) $A(x) \ge d_X(x, x_0)$,

(J2) $\bigvee_{x\in X}(d_X(x, y) \ominus A(x)) \ge d_X(x_0, y)$.

The pair (X, d_X) is called *fuzzy join complete,* if $\sqcup_X A$ exists for each $A \in L^X$.

A point x_1 is called a *fuzzy meet* of A, denoted by $x_1 = \sqcap_X A$, if it satisfies

(M1) $A(x) \ge d_X(x_1, x)$,

(M2) $\bigvee_{x\in X}(d_X(y, x) \ominus A(x)) \ge d_X(y, x_1)$.

The pair (X, d_X) is called *fuzzy meet complete*, if $\sqcap_X A$ exists for each $A \in L^X$.

The pair (X, d_X) is called *fuzzy complete*, if $\sqcap_X A$ and $\sqcup_X A$ exists for each $A \in L^X$.

Theorem 2.6 ([15]). *Let (X, d_X) be a distance space and $\Phi \in L^X$.*

(1) *A point x_0 is a fuzzy join of Φ iff* $\bigvee_{x\in X}(d_X(x, y) \ominus \Phi(x)) = d_X(x_0, y)$.

(2) *A point x_1 is a fuzzy meet of Φ iff* $\bigvee_{x\in X}(d_X(y, x) \ominus \Phi(x)) = d_X(y, x_1)$.

(3) *If $\sqcup_X \Phi$ is a fuzzy join of $\Phi \in L^X$, then it is unique. Moreover, if $\sqcap_X \Phi$ is a fuzzy meet of $\Phi \in L^X$, then it is unique.*

Definition 2.7 ([15]). (1) A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X, provided that it satisfies the following conditions:

(A1) if $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$,

(A2) if $A \in \tau$ and $\alpha \in L$, then $\alpha_X, A \ominus \alpha, A \oplus \alpha \in \tau$.

The pair (X, τ) is called an *Alexandrov topological space* on X.

Theorem 2.8 ([15]). *Let (X, d_X) be a distance space. We define*

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x,y) \geq A(y)\}$$
$$\tau_{d_X^{-1}} = \{A \in L^X \mid A(x) \oplus d_X(y,x) \geq A(y)\}.$$

Then the properties hold.

(1) *τ_{d_X} and $\tau_{d_X^{-1}}$ are Alexandrov topologies.*

(2) *$(\tau_{d_X}, d_{\tau_{d_X}})$ and $(\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ are complete lattices.*

(3) *$\tau_{d_X} = \{\bigvee_{x\in X} A(x) \oplus d_X(x,-) \mid A \in L^X\}$ and $\tau_{d_X^{-1}} = \{\bigvee_{x\in X} A(x) \oplus d_X(-,x) \mid A \in L^X\}$.*

Definition 2.9 ([15]). Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \to Y$ be a map. Define $f^* : L^X \to L^Y$ as

$$f^*(A)(y) = \begin{cases} \top, & \text{if } f^{-1}(\{y\}) = \varnothing, \\ \bigwedge A(x), & \text{if } x \in f^{-1}(\{y\}). \end{cases}$$

(1) f is called a *join (resp. meet) preserving map* if $f(\sqcup_X A) = \sqcup_{L^X} f^*(A)$ (resp. $f(\sqcap_X A) = \sqcap_{L^X} f^*(A)$) for each $A \in L^X$ with $\sqcup_X A$ (resp.$\sqcap_X A$) exists.

(2) f is called a *join-meet (resp. meet-join) preserving map* if $f(\sqcup_X A) = \sqcap_{L^X} f^*(A)$ (resp. $f(\sqcap_X A) = \sqcup_{L^X} f^*(A)$) for each $A \in L^X$ with $\sqcup_X A$ (resp.$\sqcap_X A$) exists.

(3) f is called an *(resp. dual) embedding map* if f is injective an $d_X(x,y) = d_X(f(x), f(y))$ (resp. $d_X(x,y) = d_X(f(y), f(x))$) for each $x, y \in X$.

Theorem 2.10 ([15]). *Let (X, d_X) be a distance space.*

(1) *Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ as $f(x) = (d_X)_x$. Then f is an embedding map. Moreover, if $\sqcup_X A$ exists, then*

$$\sqcup_{\tau_{d_X}} f^*(A) = \bigvee_{x\in X}(d_X(x,-)) \ominus A(x)) = f(\sqcup_X A),$$
$$\sqcap_{\tau_{d_X}} f^*(A) = \bigwedge_{z\in X}(A(z) \oplus d_X(z,-)).$$

If $A \in \tau_{d_X}$, then $\sqcap_{\tau_{d_X}} f^*(A) = A$.

(2) *Define $g : (X, d_X) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$. Then g is a dual embedding map. Moreover, if $\sqcap_X A$ exists, then*

$$\sqcup_{\tau_{d_X^{-1}}} g^*(A) = \bigvee_{x\in X}(d_X(-,x)) \ominus A(x)) = g(\sqcap_X A),$$
$$\sqcap_{\tau_{d_X^{-1}}} g^*(A) = \bigwedge_{z\in X}(A(z) \oplus d_X(-,z)).$$

If $A \in \tau_{d_X^{-1}}$, then $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = A$.

Theorem 2.11 ([15]). *Let (X, d_X) and (Y, d_Y) be distance spaces. Define $f^\oplus, f^{s\oplus} : L^X \to L^Y$ and $f^{\leftarrow}_\oplus, f^{s\leftarrow}_\oplus : L^X \to L^Y$ as*

$$f^\oplus(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_Y(f(x), y)),$$
$$f^{s\oplus}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_Y(y, f(x))),$$

$$f_{\oplus}^{\leftarrow}(B)(x) = \bigwedge_{z\in X}(B(f(z)) \oplus d_X(z,x)),$$
$$f_{\oplus}^{s\leftarrow}(B)(x) = \bigwedge_{z\in X}(B(f(z)) \oplus d_X(x,z)).$$

Then the following properties hold.

(1) *If* $f : (X, d_X) \to (Y, d_Y)$ *is a map with* $d_X(x, y) \geq d_Y(f(x), f(y))$ *for each* $x, y \in X$, *then* $d_Y(\sqcup_Y f^{s\oplus}(A), f(\sqcup_X A)) = \bot$ *and* $d_Y(f(\sqcap_X A), \sqcap_Y f^{\oplus}(A)) = \bot$, *for each* $A \in L^X$.

(2) $d_{L^X}(B, A) \geq d_{L^Y}(f^{\oplus}(B), f^{\oplus}(A))$ *and* $d_{L^X}(B, A) \geq d_{L^Y}(f^{s\oplus}(B), f^{s\oplus}(A))$.

(3) $d_{L^Y}(C, D) \geq d_{L^X}(f_{\oplus}^{\leftarrow}(C), f_{\oplus}^{\leftarrow}(E))$ *and* $d_{L^Y}(C, D) \geq d_{L^X}(f_{\oplus}^{s\leftarrow}(C), f_{\oplus}^{s\leftarrow}(E))$.

(4) $f^{\oplus}(A) \in \tau_{d_Y}$ *and* $f^{s\oplus}(A) \in \tau_{d_Y^{-1}}$.

(5) $f_{\oplus}^{\leftarrow}(A) \in \tau_{d_X}$ *and* $f_{\oplus}^{s\leftarrow}(A) \in \tau_{d_X^{-1}}$.

3. Adjunctions, Galois connections and various operations

Definition 3.1. Let (X, d_X) and (Y, d_Y) be distance spaces. Let $f : X \to Y$ and $g : Y \to X$ be maps.

(i) The pair (f, g) is called an *adjunction*, if for $x, y \in X$, $d_Y(y, f(x)) = d_X(g(y), x)$ for each $x \in X, y \in Y$.

(ii) The pair (f, g) is called a *Galois connection*, if for $x, y \in X$, $d_Y(f(x), y) = d_X(g(y), y)$ for each $x \in X, y \in Y$.

Theorem 3.2. *Let* (X, d_X) *and* (Y, d_Y) *be distance spaces and* $f : (X, d_X) \to (Y, d_Y)$ *be a map with* $d_X(x, z) \geq d_Y(f(x), f(z))$ *for each* $x, z \in X$. *Let* $f^{\oplus}, f^{s\oplus} : L^X \to L^Y$ *and* $f_{\oplus}^{\leftarrow}, f_{\oplus}^{s\leftarrow} : L^X \to L^Y$ *be defined as Theorem 2.11. Then the following properties hold.*

(1) $f^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_Y^{-1}}$ *and* $f_{\oplus}^{s\leftarrow} : \tau_{d_Y^{-1}} \to \tau_{d_X^{-1}}$ *are well-defined,* $d_{\tau_{d_X^{-1}}}(A, A_1) \geq d_{\tau_{d_Y^{-1}}}(f^{s\oplus}(A), f^{s\oplus}(A_1))$ *and* $d_{\tau_{d_Y^{-1}}}(B, B_1) \geq d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(B), f_{\oplus}^{s\leftarrow}(B_1))$.

(2) *The pair* $(f^{s\oplus}, f_{\oplus}^{s\leftarrow})$ *is an adjunction, i.e.,* $d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) = d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(B), A)$ *for each* $A \in \tau_{d_X^{-1}}, B \in \tau_{d_Y^{-1}}$.

(3) $f^{\oplus} : \tau_{d_X} \to \tau_{d_Y}$ *and* $f_{\oplus}^{\leftarrow} : \tau_{d_Y} \to \tau_{d_X}$ *are well-defined,* $d_{\tau_{d_X}}(A, A_1) \geq d_{\tau_{d_Y}}(f^{\oplus}(A), f^{\oplus}(A_1))$ *and* $d_{\tau_{d_Y}}(B, B_1) \geq d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(B), f_{\oplus}^{\leftarrow}(B_1))$.

(4) *The pair* $(f^{\oplus}, f_{\oplus}^{\leftarrow})$ *is an adjunction, i.e.,* $d_{\tau_{d_Y}}(B, f^{\oplus}(A)) = d_{\tau_{d_Y}}(f_{\oplus}^{\leftarrow}(B), A)$ *for each* $A \in \tau_{d_X}, B \in \tau_{d_Y}$.

(5) *Let* $f^{\oplus} : \tau_{d_X} \to \tau_{d_Y}$ *be a map in (4). Define* $g : \tau_{d_Y} \to \tau_{d_X}$ *as* $g(B) = \bigwedge\{A \in \tau_{d_X} \mid f^{\oplus}(A) \geq B\}$. *Then* $g = f_{\oplus}^{\leftarrow}$.

(6) *Let* $f_{\oplus}^{\leftarrow} : \tau_{d_Y} \to \tau_{d_X}$ *be a map in (4). Define* $h : \tau_{d_X} \to \tau_{d_Y}$ *as* $h(A) = \bigvee\{B \in \tau_{d_Y} \mid f_{\oplus}^{\leftarrow}(B) \leq A\}$. *Then* $h = f^{\oplus}$.

(7) *Let* $f^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_Y^{-1}}$ *be a map in (1). Define* $g : \tau_{d_Y^{-1}} \to \tau_{d_X^{-1}}$ *as* $g(B) = \bigwedge\{A \in \tau_{d_X^{-1}} \mid f^{s\oplus}(A) \geq B\}$. *Then* $g = f_{\oplus}^{s\leftarrow}$.

(8) *Let* $f_{\oplus}^{s\leftarrow} : \tau_{d_Y^{-1}} \to \tau_{d_X^{-1}}$ *be a map in (1). Define* $h : \tau_{d_X^{-1}} \to \tau_{d_Y^{-1}}$ *as* $h(A) = \bigvee\{B \in \tau_{d_Y^{-1}} \mid f_{s\oplus}^{\leftarrow}(B) \leq A\}$. *Then* $h = f^{s\oplus}$.

Proof. (1) For each $A, A_1 \in \tau_{d_X^{-1}}, B, B_1 \in \tau_{d_Y^{-1}}$, by Theorem 2.11, we have

$d_{\tau_{d_X^{-1}}}(A, A_1) \leq d_{\tau_{d_Y^{-1}}}(f^{s\oplus}(A), f^{s\oplus}(A_1))$ and $d_{\tau_{d_Y^{-1}}}(B, B_1) \leq d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(B), f_{\oplus}^{s\leftarrow}(B_1))$.

(2) For each $A \in \tau_{d_X^{-1}}, B \in \tau_{d_Y^{-1}}$, we get

$$\begin{aligned}
d_{\tau_{d_Y^{-1}}}(B, f^{s\oplus}(A)) &= \bigvee_{y\in X}(B(y) \ominus f^{s\oplus}(A)(y)) \\
&= \bigvee_{y\in X}(B(y) \ominus \bigwedge_{x\in X}(A(x) \oplus d_Y(y, f(x)))) \\
&\quad \text{[By Lemma 2.2 (2,7)]} \\
&= \bigvee_{x\in X}\bigvee_{y\in X}((B(y) \ominus d_Y(y, f(x))) \ominus A(x)) \\
&\geq \bigvee_{x\in X}(B(f(x)) \ominus A(x)) \\
&\geq \bigvee_{y\in X}(\bigwedge_{z\in X}(B(f(z)) \oplus d_X(x,z)) \ominus A(x)) \\
&= d_{\tau_{d_Y^{-1}}}(f_{\oplus}^{s\leftarrow}(B), A).
\end{aligned}$$

Let $a \geq d_{\tau_{d_Y^{-1}}}(f_{\oplus}^{s\leftarrow}(B), A)$ be given. Then we get

$$a \oplus A(x) \geq f_{\oplus}^{s\leftarrow}(B)(x) = \bigwedge_{z\in X}(B(f(z)) \oplus d_X(x,z)).$$

Thus we have

$$\begin{aligned}
a \oplus f^{s\oplus}(A)(y) &= \bigwedge_{x\in X}(a \oplus A(x) \oplus d_Y(y, f(x))) \\
&\geq \bigwedge_{x\in X}(\bigwedge_{z\in X}(B(f(z)) \oplus d_X(x,z)) \oplus d_Y(y, f(x))) \\
&\geq \bigwedge_{z\in X}(B(f(z)) \oplus \bigwedge_{x\in X}(d_Y(f(x), f(z)) \oplus d_Y(y, f(x)))) \\
&\geq \bigwedge_{z\in X}(B(f(z)) \oplus d_Y(y, f(z))) \\
&\geq B(y). \ \text{[Because } B \in \tau_{d_Y^{-1}}\text{]}
\end{aligned}$$

So $a \geq B(y) \ominus f^{s\oplus}(A)(y)$. It implies $d_{\tau_{d_Y^{-1}}}(f_{\oplus}^{s\leftarrow}(B), A) \geq d_{\tau_{d_X^{-1}}}(B, f^{s\oplus}(A))$.

(3) and (4) are similarly proved as (1) and (2) respectively.

(5) Since $d_{\tau_{d_Y}}(B, f^{\oplus}(f_{\oplus}^{\leftarrow}(B))) = d_{\tau_{d_Y}}(f_{\oplus}^{\leftarrow}(B), f_{\oplus}^{\leftarrow}(B)) = \perp$, by (4), we get

$$f^{\oplus}(f_{\oplus}^{\leftarrow}(B)) \geq B \text{ and } f_{\oplus}^{\leftarrow}(B) \in \tau_{d_X}.$$

Then $g(B) \leq f_{\oplus}^{\leftarrow}(B)$. Since $d_{\tau_{d_Y}}(B, f^{\oplus}(\bigwedge_{i\in I} A_i)) = d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(B), \bigwedge_{i\in I} A_i) = \bigvee_{i\in I} d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(B), A_i) = \bigvee_{i\in I} d_{\tau_{d_Y}}(B, f^{\oplus}(A_i)) = d_{\tau_{d_Y}}(B, \bigwedge_{i\in I} f^{\oplus}(A_i))$, we have

$$f^{\oplus}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} f^{\oplus}(A_i).$$

Thus $f^{\oplus}(g(B)) \geq B$. So we get

$$\top = d_{\tau_{d_Y}}(B, f^{\oplus}(g(B))) = d_{\tau_{d_Y}}(f_{\oplus}^{\leftarrow}(B), g(B)), \ f_{\oplus}^{\leftarrow}(B) \leq g(B).$$

Hence the result holds.

(6) Since $d_{\tau_{d_Y}}(f_{\oplus}^{\leftarrow}(f^{\oplus}(A)), A) = d_{\tau_{d_X}}(f^{\oplus}(A), f^{\oplus}(A)) = \perp$, by (4), we have

$$f_{\oplus}^{\leftarrow}(f^{\oplus}(A)) \leq A \text{ and } f^{\oplus}(A) \in \tau_{d_Y}.$$

Then $h(A) \geq f^{\oplus}(B)$. Since $d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(\bigvee_{i\in I} B_i), A) = d_{\tau_{d_Y}}(\bigvee_{i\in I} B_i, f^{\oplus}(A))$
$= \bigvee_{i\in I} d_{\tau_{d_Y}}(B_i, f^{\oplus}(A)) = \bigvee_{i\in I} d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(B_i), A) = d_{\tau_{d_X}}(\bigvee_{i\in I} f_{\oplus}^{\leftarrow}(B_i), A)$, we get

$$f_{\oplus}^{\leftarrow}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} f_{\oplus}^{\leftarrow}(B_i).$$

Thus $f_{\oplus}^{\leftarrow}(h(A)) \leq A$. So

$$\top = d_{\tau_{d_X}}(f_{\oplus}^{\leftarrow}(h(A)), A) = d_{\tau_{d_Y}}(h(A), f_{\oplus}(A)), \ h(A) \leq f_{\oplus}^{\leftarrow}(B).$$

Hence the result holds.

(7) and (8) are similarly proved as (5) and (6) respectively. □

Theorem 3.3. *Let (X, d_X) and (Y, d_Y) be distance spaces and $f : (X, d_X) \to (Y, d_Y)$ be a map with $d_X(x, z) \geq d_Y(f(x), f(z))$ for each $x, z \in X$. Then the following properties hold.*

(1) *If $f^{s\oplus} : (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}}) \to (\tau_{d_Y^{-1}}, d_{\tau_{d_Y^{-1}}})$ and $f_{\oplus}^{s\leftarrow} : (\tau_{d_Y^{-1}}, d_{\tau_{d_Y^{-1}}}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$, then for all $\mathcal{U} \in L^{\tau_{d_X^{-1}}}$ and $\mathcal{W} \in L^{\tau_{d_Y^{-1}}}$,*

$$f^{s\oplus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U}) = \sqcap_{\tau_{d_Y^{-1}}}(f^{s\oplus})^*(\mathcal{U}) \text{ and } \sqcup_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow})^*(\mathcal{W}) = f_{\oplus}^{s\leftarrow}(\sqcup_{\tau_{d_Y^{-1}}}\mathcal{W}).$$

(2) *If $f^{\oplus} : (\tau_{d_X}, d_{\tau_{d_X}}) \to (\tau_{d_Y}, d_{\tau_{d_Y}})$ and $f_{\oplus}^{\leftarrow} : (\tau_{d_Y}, d_{\tau_{d_Y}}) \to (\tau_{d_X}, d_{\tau_{d_X}})$, then for all $\mathcal{U} \in L^{\tau_{d_X}}$ and $\mathcal{W} \in L^{\tau_{d_Y}}$,*

$$f^{\oplus}(\sqcap_{\tau_{d_X}}\mathcal{U}) = \sqcap_{\tau_{d_Y}}(f^{\oplus})^*(\mathcal{U}) \text{ and } \sqcup_{\tau_{d_X}}(f_{\oplus}^{\leftarrow})^*(\mathcal{W}) = f_{\oplus}^{\leftarrow}(\sqcup_{\tau_{d_Y}}\mathcal{W}).$$

Proof. (1) Let $\mathcal{U} \in L^{\tau_{d_X^{-1}}}$. Then We have

$$\begin{aligned}
d_{\tau_{d_Y^{-1}}}(C, \sqcap_{\tau_{d_Y^{-1}}}(f^{s\oplus})^*(\mathcal{U})) &= \bigvee_{B\in\tau_{d_Y^{-1}}}(d_{\tau_{d_Y^{-1}}}(C, B) \ominus (f^{s\oplus})^*(\mathcal{U})(B)) \\
&= \bigvee_{B\in\tau_{d_Y^{-1}}}(d_{\tau_{d_Y^{-1}}}(C, B) \ominus \bigwedge_{f^{s\oplus}(D)=B}\mathcal{U}(D)) \\
&= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_Y^{-1}}}(C, f^{s\oplus}(D)) \ominus \mathcal{U}(D)) \\
&= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(C), D) \ominus \mathcal{U}(D)) \\
&\quad \text{[By Theorem 3.2 (2)]} \\
&= d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(C), \sqcap_{\tau_{d_X^{-1}}}\mathcal{U})) \\
&= d_{\tau_{d_Y^{-1}}}(C, f^{s\oplus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U})).
\end{aligned}$$

Thus we get $f^{s\oplus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U}) = \sqcap_{\tau_{d_Y^{-1}}}(f^{s\oplus})^*(\mathcal{U})$.

Now let $\mathcal{W} \in L^{\tau_{d_Y^{-1}}}$. Then we get

$$\begin{aligned}
d_{\tau_{d_X^{-1}}}(\sqcup_{\tau_{d_X^{-1}}}(f_{s\oplus}^{\leftarrow})^*(\mathcal{W}), C) &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D, C) \ominus (f_{\oplus}^{s\leftarrow})^*(\mathcal{W})(D)) \\
&= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D, C) \ominus \bigwedge_{f_{\oplus}^{s\leftarrow}(E)=D}(\mathcal{W}(E)) \\
&= \bigvee_{E\in\tau_{d_Y^{-1}}}(d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(E), C) \ominus \mathcal{W}(E)) \\
&= \bigvee_{E\in\tau_{d_Y^{-1}}}(d_{\tau_{d_Y^{-1}}}(E, f^{s\oplus}(C)) \ominus \mathcal{W}(E)) \\
&\quad \text{[By Theorem 3.2 (2)]} \\
&= d_{\tau_{d_Y^{-1}}}(\sqcup_{\tau_{d_Y^{-1}}}\mathcal{W}, f^{s\oplus}(C)) \\
&= d_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow}(\sqcup_{\tau_{d_Y^{-1}}}\mathcal{W}), C).
\end{aligned}$$

Thus we have $\sqcup_{\tau_{d_X^{-1}}}(f_{\oplus}^{s\leftarrow})^*(\mathcal{W}) = f_{\oplus}^{s\leftarrow}(\sqcup_{\tau_{d_Y^{-1}}}\mathcal{W})$.

(2) It is similarly proved as (1). □

Remark 3.4. Let $([0, 1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined as $n(x) = 1 - x$,

$$x \oplus y = (x + y) \wedge 1, \quad x \ominus y = (x - y) \vee 0.$$

Let X, Y be sets and $f : X \to Y$ a function. Define $d_X \in L^{X\times X}, d_Y \in L^{Y\times Y}$ as

$$d_X(x, z) = \begin{cases} 0, & \text{if } z = x, \\ 1, & \text{if } z \neq x, \end{cases} \quad d_Y(y, w) = \begin{cases} 0, & \text{if } y = w, \\ 1, & \text{if } y \neq w. \end{cases}$$

Then we easily show that d_X and d_Y are distance functions. Since f is a function, $d_X(x,z) \geq d_Y(f(x), f(z))$. Thus we have

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x,y) \geq A(y)\} = L^X = \tau_{d_X^{-1}}.$$

Moreover, $\tau_{d_Y} = L^Y = \tau_{d_Y^{-1}}$. For f^* in Definition 2.9, we obtain

$$f^{\oplus}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_Y(f(x), y)) = f^{s\oplus}(A)(y) = f^*(A)(y),$$
$$f^{\leftarrow}_{\oplus}(B)(x) = \bigwedge_{z\in X}(B(f(z)) \oplus d_X(z,x)) = B(f(x)) = f^{s\leftarrow}_{\oplus}(B)(x).$$

For each $A \in \tau_{d_X} = L^X, B \in \tau_{d_Y} = L^Y$, we get

$$d_{L^Y}(B, f^{\oplus}(A)) = d_{L^Y}(B, f^*(A)) = d_{L^X}(f^{\leftarrow}_{\oplus}(B), A) = d_{L^X}(f^{\leftarrow}(B), A).$$

So $(f^*, f^{\leftarrow})$ is an adjunction. It is the concept of Zadeh's powerset operations (See [5]).

From Theorem 3.3, it is clear that for all $\mathcal{U} \in L^{L^X}$ and $\mathcal{W} \in L^{L^Y}$,

$$f^{\oplus}(\sqcap_{L^X}\mathcal{U}) = \sqcap_{L^Y}(f^{\oplus})^*(\mathcal{U}) \text{ and } \sqcup_{L^X}(f^{\leftarrow}_{\oplus})^*(\mathcal{W}) = f^{\leftarrow}_{\oplus}(\sqcup_{L^Y}\mathcal{W}).$$

Remark 3.5. Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Using adjunctions and Galois connections, we will define a formal fuzzy concept and an attribute-oriented fuzzy concept as follows:

Let $F : L^X \to L^Y, G : L^Y \to L^X$ be maps where X is a set of objects and Y is a set of attributes. If (F, G) is a Galois connection, i.e., $d_{L^Y}(F(A), B) = d_{L^X}(G(B), A)$, then a formal fuzzy concept is a pair $(A, B) \in L^X \times L^Y$ such that $F(A) = B, G(B) = A$ as a Bělohlávek's sense (See [2, 3]).

If (F, G) is an adjunction, i.e., $d_{L^Y}(B, F(A)) = d_{L^X}(G(B), A)$, then an attribute-oriented fuzzy concept is a pair $(A, B) \in L^X \times L^Y$ such that $F(A) = B, G(B) = A$ as a Georgescu and Popescu's sense (See [16]).

Theorem 3.6. *Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \to Y$ be a map. Define $f^{\ominus}, f^{s\ominus} : L^X \to L^Y$ and $f^{\leftarrow}_{\ominus}, f^{s\leftarrow}_{\ominus} : L^Y \to L^X$ as*

$$f^{\ominus}(A)(y) = \bigvee_{x\in X}(d_Y(f(x), y) \ominus A(x)),$$
$$f^{s\ominus}(A)(y) = \bigvee_{x\in X}(d_Y(y, f(x)) \ominus A(x)),$$

$$f^{\leftarrow}_{\ominus}(B)(x) = \bigvee_{z\in X}(d_X(z,x) \ominus B(f(z))),$$
$$f^{s\leftarrow}_{\ominus}(B)(x) = \bigvee_{z\in X}(d_X(x,z) \ominus B(f(z))).$$

For each $A, C \in L^X$ and $B, D \in L^Y$, the followings hold.

(1) $d_{L^X}(A, C) \geq d_{L^Y}(f^{\ominus}(C), f^{\ominus}(A))$ *and* $d_{L^X}(A, C) \geq d_{L^Y}(f^{s\ominus}(C), f^{s\ominus}(A))$.

(2) $d_{L^Y}(B, D) \geq d_{L^X}(f^{\leftarrow}_{\ominus}(D), f^{\leftarrow}_{\ominus}(B))$ *and* $d_{L^Y}(B, D) \geq d_{L^X}(f^{s\leftarrow}_{\ominus}(D), f^{s\leftarrow}_{\ominus}(B))$.

(3) $f^{\ominus}(A) \in \tau_{d_Y}$ *and* $f^{s\ominus}(A) \in \tau_{d_Y^{-1}}$.

(4) $f^{\leftarrow}_{\ominus}(B) \in \tau_{d_Y}$ *and* $f^{s\leftarrow}_{\ominus}(B) \in \tau_{d_Y^{-1}}$.

Proof. (1) For $A, C \in L^X$,

$$\begin{aligned} & d_{L^Y}(f^{\ominus}(C), f^{\ominus}(A)) \\ &= \bigvee_{x\in X}(d_Y(f(x), y) \ominus C(x)) \ominus \bigvee_{x\in X}(d_Y(f(x), y) \ominus A(x)) \\ &\leq \bigvee_{x\in X}((d_Y(f(x), y) \ominus C(x)) \ominus (d_Y(f(x), y) \ominus A(x))) \text{ [By Lemma 2.2 (8,11)]} \\ &\leq \bigvee_{x\in X}(A(x) \ominus C(x)). \end{aligned}$$

Similarly, $d_{L^X}(A, C) \geq d_{L^Y}(f^{s\ominus}(C), f^{s\ominus}(A))$.

(2) For $B, C \in L^Y$,

$$\begin{aligned}
&d_{L^X}(f^{\leftarrow}_{\ominus}(C), f^{\leftarrow}_{\ominus}(B))\\
&= \bigvee_{x\in X}(d_X(z,x)\ominus C(f(x)))\ominus \bigvee_{x\in X}(d_X(x,z)\ominus B(f(x)))\\
&\leq \bigvee_{x\in X}((d_X(x,z)\ominus C(f(x)))\ominus (d_X(x,z)\ominus B(f(x))))\\
&\quad \text{[By Lemma 2.2 (8,11)]}\\
&\leq \bigvee_{x\in X}(B(f(x))\ominus C(f(x)))\leq d_{L^Y}(B,C).
\end{aligned}$$

Similarly, $d_{L^Y}(B,C) \geq d_{L^X}(f^{s\leftarrow}_{\ominus}(C), f^{s\leftarrow}_{\ominus}(A))$.

(3) For $A \in L^X$,

$$\begin{aligned}
&A(x)\oplus f^{\ominus}(A)(y)\oplus d_Y(y,w)\\
&= A(x)\oplus (\bigvee_{x\in X}(d_Y(f(x),y)\ominus A(x)))\oplus d_Y(y,w)\\
&\geq d_Y(f(x),y))\oplus d_Y(y,w)\geq d_Y(f(x),w).
\end{aligned}$$

Then $f^{\ominus}(A)(y)\oplus d_Y(y,w)\geq f^{\ominus}(A)(w)$ and $f^{\ominus}(A)\in \tau_{d_X}$.

Other case is similarly proved.

(4) For $B \in L^Y$,

$$\begin{aligned}
&B(f(z))\oplus f^{s\leftarrow}_{\ominus}(B)(x)\oplus d_X(w,x)\\
&= B(f(z))\oplus (\bigvee_{z\in X}((d_X(x,z)\ominus B(f(z)))\oplus d_X(w,x)\\
&\geq d_X(x,z))\oplus d_X(w,x)\geq d_X(w,z).
\end{aligned}$$

Then $f^{s\leftarrow}_{\ominus}(B)(x)\oplus d_X(w,x)\geq f^{s\leftarrow}_{\ominus}(B)(w)$ and $f^{s\leftarrow}_{\ominus}(B)\in \tau_{d_X^{-1}}$. □

Theorem 3.7. *Let (X, d_X) and (Y, d_Y) be distance spaces and $f : (X, d_X) \to (Y, d_Y)$ be a map with $d_X(x, y) \leq d_Y(f(x), f(y))$ for all $x, y \in X$. Let $f^{\ominus}, f^{s\ominus} : L^X \to L^Y$ and $f^{\leftarrow}_{\ominus}, f^{s\leftarrow}_{\ominus} : L^Y \to L^X$ be defined as Theorem 3.6. Then the following properties hold.*

(1) Two operations $f^{\ominus} : \tau_{d_X^{-1}} \to \tau_{d_Y}$, $f^{s\leftarrow}_{\ominus} : \tau_{d_Y} \to \tau_{d_X^{-1}}$ satisfy $d_{\tau_{d_Y}}(f^{\ominus}(A), B) \leq d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(B), A)$ and $f^{\ominus}(f^{s\leftarrow}_{\ominus}(B)) \leq B$. Moreover, if f is surjective and $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then the pair $(f^{\ominus}, f^{s\leftarrow}_{\ominus})$ is a Galois connection, i.e., $d_{\tau_{d_Y}}(f^{\ominus}(A), B) = d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(B), A)$.

(2) Two operations $f^{s\ominus} : \tau_{d_X} \to \tau_{d_Y^{-1}}$, $f^{\leftarrow}_{\ominus} : \tau_{d_Y^{-1}} \to \tau_{d_X}$ satisfy $d_{\tau_{d_Y^{-1}}}(f^{s\ominus}(A), B) \leq d_{\tau_{d_X}}(f^{\leftarrow}_{\ominus}(B), A)$. Moreover, if f is surjective and $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then the pair $(f^{s\ominus}, f^{\leftarrow}_{\ominus})$ is a Galois connection, i.e., $d_{\tau_{d_Y^{-1}}}(f^{s\ominus}(A), B) = d_{\tau_{d_X}}(f^{\leftarrow}_{\ominus}(B), A)$.

Proof. (1) For $A \in \tau_{d_X^{-1}}, B \in \tau_{d_Y}$,

$$\begin{aligned}
d_{\tau_{d_Y}}(f^{\ominus}(A), B) &= \bigvee_{y\in Y}(f^{\ominus}(A)(y)\ominus B(y))\\
&= \bigvee_{y\in Y}(\bigvee_{x\in X}(d_Y(f(x),y)\ominus A(x))\ominus B(y))\\
&= \bigvee_{x\in X}(\bigvee_{y\in Y}(d_Y(f(x),y)\ominus B(y))\ominus A(x))\\
&\quad \text{[By Lemma 2.2 (2,7)]}\\
&\geq \bigvee_{x\in X}(\bigvee_{z\in X}(d_Y(f(x),f(z))\ominus B(f(z)))\ominus A(x))\\
&\geq \bigvee_{x\in X}(\bigvee_{z\in X}(d_X(x,z)\ominus B(f(z)))\ominus A(x))\\
&= \bigvee_{x\in X}(f^{s\leftarrow}_{\ominus}(B)(x)\ominus A(x)) = d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(B), A).
\end{aligned}$$

Moreover, we get

$$\bot = d_{\tau_{d_Y}}(f^{\ominus}(A), f^{\ominus}(A)) \geq d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(f^{\ominus}(A)), A) = \bot.$$

Then $f^{s\leftarrow}_{\ominus}(f^{\ominus}(A))) \leq A$.

If f is surjective and $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$, then we have

$$\begin{aligned} d_{\tau_{d_Y}}(f^{\ominus}(A),B) &= \bigvee_{y\in Y}(f^{\ominus}(A)(y)\ominus B(y)) \\ &= \bigvee_{y\in Y}(\bigvee_{x\in X}(d_Y(f(x),y)\ominus A(x))\ominus B(y)) \\ &= \bigvee_{x\in X}(\bigvee_{y\in Y}(d_Y(f(x),y)\ominus B(y))\ominus A(x)) \\ &= \bigvee_{x\in X}(\bigvee_{z\in X}(d_Y(f(x),f(z))\ominus B(f(z)))\ominus A(x)) \\ &= \bigvee_{x\in X}(\bigvee_{z\in X}(d_X(x,z)\ominus B(f(z)))\ominus A(x)) \\ &= \bigvee_{x\in X}(f^{s\leftarrow}_{\ominus}(B)(x)\ominus A(x)) = d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(B),A). \end{aligned}$$

Thus $d_{\tau_{d_Y}}(f^{\ominus}(A),B) = d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(B),A)$.

(2) It is similarly proved as (1). □

Theorem 3.8. *Let (X,d_X) and (Y,d_Y) be distance spaces and $f:(X,d_X)\to(Y,d_Y)$ be a surjective map with $d_X(x,z)=d_Y(f(x),f(z))$ for each $x,z\in X$. Let $f^{\ominus},f^{s\ominus}: L^X\to L^Y$ and $f^{\leftarrow}_{\ominus},f^{s\leftarrow}_{\ominus}:L^Y\to L^X$ be defined as Theorem 3.6. Then the following properties hold.*

(1) *If $f^{\ominus}:(\tau_{d_X^{-1}},d_{\tau_{d_X^{-1}}})\to(\tau_{d_Y},d_{\tau_{d_Y}})$ and $f^{s\leftarrow}_{\ominus}:(\tau_{d_Y},d_{\tau_{d_Y}})\to(\tau_{d_X^{-1}},d_{\tau_{d_X^{-1}}})$, then for all $\mathcal{U}\in L^{\tau_{d_X^{-1}}}$ and $\mathcal{W}\in L^{\tau_{d_Y}}$,*

$$f^{\ominus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U}) = \sqcup_{\tau_{d_Y}}(f^{\ominus})^*(\mathcal{U}) \text{ and } \sqcup_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus})^*(\mathcal{W}) = f^{s\leftarrow}_{\ominus}(\sqcap_{\tau_{d_Y}}\mathcal{W}).$$

(2) *If $f^{s\ominus}:(\tau_{d_X},d_{\tau_{d_X}})\to(\tau_{d_Y^{-1}},d_{\tau_{d_Y^{-1}}})$ and $f^{\leftarrow}_{\ominus}:(\tau_{d_Y^{-1}},d_{\tau_{d_Y^{-1}}})\to(\tau_{d_X},d_{\tau_{d_X}})$, then for all $\mathcal{U}\in L^{\tau_{d_X}}$ and $\mathcal{W}\in L^{\tau_{d_Y^{-1}}}$,*

$$f^{s\ominus}(\sqcap_{\tau_{d_X}}\mathcal{U}) = \sqcup_{\tau_{d_Y^{-1}}}(f^{s\ominus})^*(\mathcal{U}) \text{ and } \sqcup_{\tau_{d_X}}(f^{\leftarrow}_{\ominus})^*(\mathcal{W}) = f^{\leftarrow}_{\ominus}(\sqcap_{\tau_{d_Y^{-1}}}\mathcal{W}).$$

(3) *Let $f^{\ominus}:(\tau_{d_X^{-1}},d_{\tau_{d_X^{-1}}})\to(\tau_{d_Y},d_{\tau_{d_Y}})$ be a map. Define $g:(\tau_{d_Y},d_{\tau_{d_Y}})\to(\tau_{d_X^{-1}},d_{\tau_{d_X^{-1}}})$ as $g(B)=\bigwedge\{A\in\tau_{d_X^{-1}}\mid f^{\ominus}(A)\le B\}$. Then $g=f^{s\leftarrow}_{\ominus}$.*

(4) *Let $f^{s\ominus}:(\tau_{d_X},d_{\tau_{d_X}})\to(\tau_{d_Y^{-1}},d_{\tau_{d_Y^{-1}}})$ be a map. Define $h:(\tau_{d_Y^{-1}},d_{\tau_{d_Y^{-1}}})\to(\tau_{d_X},d_{\tau_{d_X}})$ as $h(B)=\bigwedge\{A\in\tau_{d_X}\mid f^{s\ominus}(A)\le B\}$. Then $h=f^{\leftarrow}_{\ominus}$.*

Proof. (1) Let $\mathcal{U}\in L^{\tau_{d_X^{-1}}}$. Then we have

$$\begin{aligned} d_{\tau_{d_Y}}(\sqcup_{\tau_{d_Y}}(f^{\ominus})^*(\mathcal{U}),C) &= \bigvee_{B\in\tau_{d_Y}}(d_{\tau_{d_Y}}(B,C)\ominus(f^{\ominus})^*(\mathcal{U})(B)) \\ &= \bigvee_{B\in\tau_{d_Y}}(d_{\tau_{d_Y}}(B,C)\ominus\bigwedge_{f^{\ominus}(D)=B}\mathcal{U}(D)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_Y}}(f^{\ominus}(D),C)\ominus\mathcal{U}(D)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(C),D)\ominus\mathcal{U}(D)) \\ &\qquad \text{[By Theorem 3.7 (1)]} \\ &= d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(C),\sqcap_{\tau_{d_X^{-1}}}\mathcal{U})) \\ &= d_{\tau_{d_Y}}(f^{\ominus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U}),C). \end{aligned}$$

Thus we get $f^{\ominus}(\sqcap_{\tau_{d_X^{-1}}}\mathcal{U}) = \sqcup_{\tau_{d_Y}}(f^{\ominus})^*(\mathcal{U})$

Now let $\mathcal{W}\in L^{\tau_{d_Y^{-1}}}$. The we have

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(\sqcup_{\tau_{d_X^{-1}}}(f^{\leftarrow}_{s\ominus})^*(\mathcal{W}),C) &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D,C)\ominus(f^{s\leftarrow}_{\ominus})^*(\mathcal{W})(D)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D,C)\ominus\bigwedge_{f^{s\leftarrow}_{\ominus}(E)=D}(\mathcal{W}(E)) \\ &= \bigvee_{E\in\tau_{d_Y}}(d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(E),C)\ominus\mathcal{W}(E)) \end{aligned}$$

$$= \bigvee_{E\in\tau_{d_Y}}(d_{\tau_{d_Y}}(f^{\ominus}(C),E)\ominus\mathcal{W}(E))$$
[By Theorem 3.7 (1)]
$$= d_{\tau_{d_Y}}(f^{\ominus}(C),\sqcap_{\tau_{d_Y}}\mathcal{W})$$
$$= d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(\sqcap_{\tau_{d_Y}}\mathcal{W}),C).$$

Thus we get $\sqcup_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus})^{*}(\mathcal{W}) = f^{s\leftarrow}_{\ominus}(\sqcap_{\tau_{d_Y}}\mathcal{W})$.

(3) Since $d_{\tau_{d_Y}}(f^{\ominus}(f^{s\leftarrow}_{\ominus}(B)),B) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\ominus}(B),f^{s\leftarrow}_{\ominus}(B)) = \bot$, by (1), we have

$$f^{\ominus}(f^{s\leftarrow}_{\ominus}(B)) \le B \text{ and } f^{s\leftarrow}_{\ominus}(B)\in\tau_{d_X^{-1}}.$$

Then $g(B)\le f^{s\leftarrow}_{\ominus}(B)$. Since $d_{\tau_{d_Y}}(f^{\ominus}(\bigwedge_{i\in I}A_i),B) = d_{\tau_{d_X}}(f^{s\leftarrow}_{\ominus}(B),\bigwedge_{i\in I}A_i) = \bigvee_{i\in I}d_{\tau_{d_X}}(f^{s\leftarrow}_{\ominus}(B),A_i) = \bigvee_{i\in I}d_{\tau_{d_Y}}(f^{\ominus}(A_i),B) = d_{\tau_{d_Y}}(\bigvee_{i\in I}f^{\ominus}(A_i),B)$, we have $f^{\ominus}(\bigwedge_{i\in I}A_i) = \bigvee_{i\in I}f^{\ominus}(A_i)$. Thus $f^{\ominus}(g(B))\le B$. So $\top = d_{\tau_{d_Y}}(f^{\ominus}(g(B)),B) = d_{\tau_{d_Y}}(f^{s\leftarrow}_{\ominus}(B),g(B))$, $f^{s\leftarrow}_{\ominus}(B)\le g(B)$. Hence the result holds.

(2) and (4) are similarly proved as (1) and (3) respectively. □

Example 3.9. Let $([0,1],\le,\vee,\wedge,\oplus,\ominus,0,1)$ be a complete co-residuated lattice defined as $n(x)=1-x$,

$$x\oplus y=(x+y)\wedge 1,\quad x\ominus y=(x-y)\vee 0.$$

Let $X=\{a,b,c\}$ be a set and $A,B\in[0,1]^X$ with

$$A(x)=0.3, A(y)=0.2, A(z)=0.5, B(x)=0.6, B(y)=0.3, B(z)=0.5.$$

Define $d_X\in L^{X\times X}$ as

$$d_X=\begin{pmatrix} 0 & 0.5 & 0.8\\ 0.7 & 0 & 0.6\\ 0.4 & 0.6 & 0\end{pmatrix}$$

Then we easily show that d_X is a distance function. Moreover,

$$A=\bigwedge_{x\in X}(A(x)\oplus d_X(x,-)) = \bigwedge_{x\in X}(A(x)\oplus d_X(-,x))$$
$$B=\bigwedge_{x\in X}(B(x)\oplus d_X(x,-)) = \bigwedge_{x\in X}(B(x)\oplus d_X(-,x)).$$

Thus by Theorem 2.8 (3), $A,B\in\tau_{d_X}, A,B\in\tau_{d_X^{-1}}$.

(1) Define $f:(X,d_X)\to(\tau_{d_X},d_{\tau_{d_X}})$ as $f(x)=(d_X)_x$, where $(d_X)_x(z)=d_X(x,z)$ in Theorem 2.10 (1). Moreover, $d_X(x,y)=d_{\tau_{d_X}}(f(x),f(y))$. By Theorem 3.2, we obtain $f^{\oplus}:\tau_{d_X}\to\tau_{d_{\tau_{d_X}}}$, $f^{\leftarrow}_{\oplus}:\tau_{d_{\tau_{d_X}}}\to\tau_{d_X}$,where $\tau_{d_{\tau_{d_X}}}=\{\alpha\in L^{L^X}\mid \alpha(A)\oplus d_{\tau_{d_X}}(A,B)\ge\alpha(B)\}$. For $A,B\in\tau_{d_X}$,

$$f^{\oplus}(A)(B) = \bigwedge_{x\in X}(d_{\tau_{d_X}}((d_X)_x,B)\oplus A(x)$$
$$= (0.3\oplus 0.3)\wedge(0.1\oplus 0.2)\wedge(0.3\oplus 0.5)$$
$$= 0.3,$$

$$f^{\leftarrow}_{\oplus}(\Psi)(x) = \bigwedge_{z\in X}(d_X(z,x)\oplus\Psi(f(z))),$$
$$f^{\oplus}(A)(f(-)) = \bigwedge_{z\in X}(d_{\tau_{d_X}}((d_X)_z,(d_X)_-)\oplus A(z))$$
$$= \bigwedge_{z\in X}(d_X(z,-)\oplus A(z))$$
$$= (0.3,0.2,0.5),$$

$$f^{\leftarrow}_{\oplus}(f^{\oplus}(A))(-) = \bigwedge_{z\in X}(d_X(z,-)\oplus f^{\oplus}(A)(f(z)))$$
$$= \bigwedge_{z\in X}(d_X(z,-)\oplus A(z))$$

$= (0.3, 0.2, 0.5) = A.$

Then $d_{\tau_{d_{\tau_{d_X}}}}(\Psi, f^{\oplus}(A)) = d_{\tau_{d_X}}(f^{\leftarrow}_{\oplus}(\Psi), A)$, i.e., $(f^{\oplus}, f^{\leftarrow}_{\oplus})$ is an adjunction.

(2) Define $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$, where $(d_X)^x(z) = d_X(z, x)$ in Theorem 2.10 (2). Moreover, $d_X^{-1}(x, y) = d_{\tau_{d_X^{-1}}}(g(x), g(y))$. By Theorem 3.2, we obtain $g^{s\oplus} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}}$, $g^{s\leftarrow}_{\oplus} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}}$. Then

$$\begin{aligned} g^{s\oplus}(A)(B) &= \textstyle\bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}(B, (d_X)^x) \oplus A(x)) \\ &= (0.6 \oplus 0.3) \wedge (0.3 \oplus 0.2) \wedge (0.5 \oplus 0.5) \\ &= 0.5, \end{aligned}$$

$$\begin{aligned} g^{s\oplus}(A)(g(-)) &= \textstyle\bigwedge_{x\in X}(d_{\tau_{d_X^{-1}}}(g(-), (d_X)^x) \oplus A(x) \\ &= \textstyle\bigwedge_{x\in X}(d_X(x, -) \oplus A(x) = A, \end{aligned}$$

$$\begin{aligned} &g^{s\leftarrow}_{\oplus}(\Psi)(x) = \textstyle\bigwedge_{z\in X}(d_X(x, z) \oplus \Psi(f(z))), \\ &g^{s\leftarrow}_{\oplus}(g^{s\oplus}(A))(-) = \textstyle\bigwedge_{z\in X}(d_X(-, z) \oplus A(z)) = A. \end{aligned}$$

Thus $d_{\tau_{d_{\tau_{d_X^{-1}}}}}(\Psi, g^{s\oplus}(A)) = d_{\tau_{d_X^{-1}}}(g^{s\leftarrow}_{\oplus}(\Psi), A)$, i.e., $(g^{s\oplus}, g^{s\leftarrow}_{\oplus})$ is an adjunction.

(3) Since $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ as $f(x) = (d_X)_x$, where $d_X(x, y) = d_{\tau_{d_X}}(f(x), f(y))$ for each $x, y \in X$, by Theorem 3.7 and (1), we obtain

$$f^{\ominus} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X}}},\ f^{s\leftarrow}_{\ominus} : \tau_{d_{\tau_{d_X}}} \to \tau_{d_X^{-1}}.$$

But f is not surjective and $d_{\tau_{d_{\tau_{d_X}}}}(f^{\ominus}(C), D) \leq d_{\tau_{d_X^{-1}}}(f^{s\leftarrow}_{\ominus}(D), C)$ and $f^{\ominus}(f^{s\leftarrow}_{\ominus}(D)) \leq D$. For $A, B \in \tau_{d_X^{-1}}$,

$$\begin{aligned} f^{\ominus}(A)(B) &= \textstyle\bigvee_{x\in X}(d_{\tau_{d_X}}((d_X)_x, B) \ominus A(x)) \\ &= (0.3 \ominus 0.3) \vee (0.1 \ominus 0.2) \vee (0.3 \ominus 0.5) \\ &= 0, \end{aligned}$$

$$\begin{aligned} f^{\ominus}(A)(f(-)) &= \textstyle\bigvee_{x\in X}(d_{\tau_{d_X}}((d_X)_x, (d_X)_-) \ominus A(x)) \\ &= \textstyle\bigvee_{x\in X}(d_X)(x, -) \ominus A(x)) = (0.5, 0.2, 0.5), \end{aligned}$$

$$f^{s\leftarrow}_{\ominus}(\Psi)(x) = \textstyle\bigvee_{z\in X}(d_X(x, z) \ominus \Psi(f(z))),$$

$$\begin{aligned} f^{s\leftarrow}_{\ominus}(f^{\ominus}(A))(-) &= \textstyle\bigvee_{z\in X}(d_X(-, z) \ominus f^{\ominus}(A)(f(z))) \\ &= (0.3, 0.2, 0.4) \\ &< A. \end{aligned}$$

(4) Since $g : (X, d_X^{-1}) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$, where $d_X^{-1}(x, y) = d_{\tau_{d_X^{-1}}}(g(x), g(y))$ for each $x, y \in X$, by Theorem 3.7, we obtain

$$g^{s\ominus} : \tau_{d_X^{-1}} \to \tau_{d_{\tau_{d_X^{-1}}}},\ g^{\leftarrow}_{\ominus} : \tau_{d_{\tau_{d_X^{-1}}}} \to \tau_{d_X^{-1}}.$$

But g is not surjective and $d_{\tau_{d_{\tau_{d_X^{-1}}}}}(g^{s\ominus}(C), D) \leq d_{\tau_{d_X^{-1}}}(g^{\leftarrow}_{\ominus}(D), C)$ and $g^{s\ominus}(g^{\leftarrow}_{\ominus}(D)) \leq D$. For $A, B \in \tau_{d_X^{-1}}$,

$$\begin{aligned} g^{s\ominus}(A)(B) &= \bigvee_{x\in X}(d_{\tau^{-1}_{d_X}}(B,(d_X)^x)\ominus A(x)) \\ &= (0.6\ominus 0.3)\vee(0.3\ominus 0.2)\vee(0.5\ominus 0.5) \\ &= 0.3, \end{aligned}$$

$$\begin{aligned} g^{s\ominus}(A)(g(-)) &= \bigvee_{x\in X}(d_{\tau^{-1}_{d_X}}((d_X)^-,(d_X)^x)\ominus A(x)) \\ &= \bigvee_{x\in X}(d_X)(x,-)\ominus A(x)) \\ &= (0.5,0.2,0.5), \end{aligned}$$

$$g^{\leftarrow}_{\ominus}(\Psi)(x) = \bigvee_{z\in X}(d_X(z,x)\ominus \Psi(g(z))),$$

$$\begin{aligned} g^{\leftarrow}_{\ominus}(g^{s\ominus}(A))(-) &= \bigvee_{z\in X}(d_X(z,-)\ominus g^{s\ominus}(A)(g(z))) \\ &= (0.5,0.1,0.4). \end{aligned}$$

Then $0 = d_{\tau_{d^{-1}_{\tau_{d^{-1}_X}}}}(g^{s\ominus}(A), g^{s\ominus}(A)) < d_{\tau_{d_X}}(g^{\leftarrow}_{\ominus}(g^{s\ominus}(A)), A) = 0.2$.

Let $Y = \{x,y,z\}$ and $f : X \to Y$ be a function as $f(a) = f(b) = x, f(c) = y$. Define $d_Y \in L^{Y\times Y}$ as

$$d_Y = \begin{pmatrix} 0 & 0.4 & 0.9 \\ 0.3 & 0 & 0.5 \\ 0.7 & 0.4 & 0 \end{pmatrix}$$

Then $d_X(a,b) \geq d_Y(f(a), f(b))$ for all $a, b \in X$. The properties of Theorems 3.2 and 3.3 hold.

4. Conclusion

Using distance functions, we have investigated adjunctions, Galois connections and join (meet) preserving maps between various operations based on co-residuated lattices. As applications for adjunctions and Galois connections, we can define a formal fuzzy concept and an attribute-oriented fuzzy concept in Remark 3.5. As extensions of Rodabough's the adjoint function theorem using the adjunctions, we have studied various operators from Alexandrov topologies to Alexandrov topologies in co-residuated lattices.

In the future, by using the concepts of adjunctions, Galois connections and join (meet) preserving maps between various operations, information systems and decision rules are investigated in co-residuated lattices.

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