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On a convexity in fuzzy normed linear spaces

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Abstract. In this paper, we introduce the definitions of uniform nonsquareness, property (B_k) and B-convexity in fuzzy setting and study their implications.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh [1] in 1965. Matloka [2] introduced in 1986 the class of bounded and convergent sequences of fuzzy numbers with respect to the Hausdorff metric. Recently, Zararsiz [3] introduced and studied the sequence spaces of fuzzy numbers.

Katsaras [4] first introduced in 1984 the idea of fuzzy norm on a linear space. After that, Felbin [5], Cheng and Moderson [6] introduced the definition of fuzzy norm on a linear space in different approach. Following Cheng and Moderson [6], Bag and Samanta [7] introduced the definition of fuzzy norm on a linear space. The different types of convergent sequences are investigated by Cho and Lee [8, 9] and some geometric properties are studied in fuzzy setting [10]. Fuzzy radius, fuzzy normal structure and fuzzy uniform convexity were studied in [10]. Mukherjee and Bag [11] introduced the idea of strictly convex and strong strictly convex fuzzy normed linear spaces and studied some properties of such spaces.

In this paper, we introduce the notion of uniform nonsquareness, property (B_k) and B-convexity in fuzzy normed linear spaces and their implications.

This paper is organized, as follows. Section 2 provides some preliminary results. In Section 3, we introduce the definition of uniform nonsquareness in fuzzy setting. We also show that fuzzy uniform convexity implies fuzzy uniform nonsquareness and the converse does not hold, in general. In Section 4, we give the definitions of property (B_k) and B-convexity in fuzzy setting and study their implications.

2. Preliminaries

Let us recall [12] that a continuous t-norms is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian topological monoid with unit 1. The continuous t-norms frequently used are the minimum, $a * b = \min\{a, b\}$, the usual product, $a * b = ab$ and the bounded difference, $a * b = \max\{0, a + b - 1\}$.

Various definitions of fuzzy normed spaces have been investigated by several authors. In this paper, we take the definition of fuzzy normed spaces introduced by Bag and Samanta [7].

Definition 2.1. Let X be a linear space over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Let N be a fuzzy subset of $X \times \mathbb{R}$ and $*$ be a continuous t-norm. Then N is called a *fuzzy norm* on X, if for all $x, y \in X$ and $c \in \mathbb{F}$,

(N1) for all $t \leq 0$, $N(x, t) = 0$,
(N2) for all $t > 0$, $N(x, t) = 1$ if and only if $x = 0$,
(N3) for all $t > 0$ and $c \neq 0$, $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$,
(N4) for all $s, t \in \mathbb{R}$, $N(x + y, s + t) \geq N(x, s) * N(y, t)$,
(N5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ such that $\lim_{t\to\infty} N(x, t) = 1$.

The pair $(X, N, *)$ will be referred to as a *fuzzy normed linear space.*

In Definition 2.1, it is stated that $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$. But the following inequality from (N4) shows that $N(x, \cdot)$ is non-decreasing: For $t > s$,

$$N(x, t) \geq N(x, s) * N(0, t - s) = N(x, s) * 1 = N(x, s).$$

In this paper, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and positive integers, respectively.

3. Fuzzy uniform nonsquareness

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X and S_X, we denote the closed unit ball and the unit sphere of X, respectively. $(X, \|\cdot\|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\|x - y\| > \epsilon$, $\left\|\frac{1}{2}(x + y)\right\| \leq 1 - \delta$. $(X, \|\cdot\|)$ is said to be uniformly nonsquare (UNS) if there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\left\|\frac{1}{2}(x - y)\right\| > 1 - \delta$, $\left\|\frac{1}{2}(x + y)\right\| \leq 1 - \delta$. An uniformly convex Banach space is uniformly nonsquare and the converse does not hold, in general [13, 14, 15]. The modulus of convexity a normed space X is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \epsilon\right\}.$$

The coefficient of convexity of a Banach space X is the number

$$\epsilon_0(X) = \sup\{\epsilon \geq 0 : \delta(\epsilon) = 0\}.$$

It is well known fact that a Banach space X is uniformly nonsquare if and only if $\epsilon_0(X) < 2$. The following example is found in [15].

Example 3.1. Let Q_i, $i = 1, 2, 3, 4$, denote the ith quadrant in $\mathbb{R}^2$, and for $x = (x_1, x_2) \in \mathbb{R}^2$, set

$$\|x\| = \begin{cases} (x_1^2+x_2^2)^{\frac{1}{2}} & \text{if } x \in Q_1 \cup Q_3 \\ |x_1|+|x_2| & \text{if } x \in Q_2 \cup Q_4 \end{cases}$$
$$= \begin{cases} \|x\|_2 & \text{if } x_1 \cdot x_2 \geq 0 \\ \|x\|_1 & \text{if } x_1 \cdot x_2 \leq 0. \end{cases}$$

Let $(\mathbb{R}^2, \|\cdot\|)$ be denoted by X_G. Then $\epsilon_0(X_G) \leq 2^{\frac{1}{2}}$ [15]. This implies that X_G is uniformly nonsquare. For $x = e_1 = (1,0)$ and $y = -e_2 = (0,-1)$ in X_G,

$$\|x-y\| = 2^{\frac{1}{2}} \text{ and } \|x+y\| = 2.$$

This means that X_G is not uniformly convex.

The definition of uniform convexity in fuzzy setting is found in [10].

Definition 3.2 ([10]). A fuzzy normed space $(X, N, *)$ is said to be *uniformly convex*, if for $0 < \epsilon \leq 2$, there exists $0 < \delta < 1$ such that for $x, y \in X$,

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t) \geq \alpha\} \leq 1, \sup_{\alpha\in(0,1)} \inf\{t>0 : N(y,t) \geq \alpha\} \leq 1$$

and

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x-y,t) \geq \alpha\} > \epsilon$$

imply that

$$\sup_{\alpha\in(0,1)} \inf\left\{t>0 : N\left(\frac{x+y}{2}, t\right) \geq \alpha\right\} \leq 1-\delta.$$

We now define uniform nonsquareness in fuzzy normed spaces.

Definition 3.3. A fuzzy normed space $(X, N, *)$ is said to be *uniformly nonsquare* if there exists $0 < \delta < 1$ such that for $x, y \in X$,

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t) \geq \alpha\} \leq 1, \sup_{\alpha\in(0,1)} \inf\{t>0 : N(y,t) \geq \alpha\} \leq 1$$

and

$$\sup_{\alpha\in(0,1)} \inf\left\{t>0 : N\left(\frac{x-y}{2}, t\right) \geq \alpha\right\} > 1-\delta$$

imply that

$$\sup_{\alpha\in(0,1)} \inf\left\{t>0 : N\left(\frac{x+y}{2}, t\right) \geq \alpha\right\} \leq 1-\delta.$$

We need the following lemmas which will be used in this paper.

Lemma 3.4. *Let $(X, N, *)$ be a fuzzy normed space. Then for $\lambda \in \mathbb{R}$ and $x, y \in X$,* $\sup_{\alpha\in(0,1)} \inf\{t>0 : N(\lambda x,t) \geq \alpha\} = |\lambda| \sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t) \geq \alpha\}$

Proof. If $\lambda = 0$, then

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(\lambda x,t) \geq \alpha\} = 0 = |\lambda| \sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t) \geq \alpha\}.$$

For $\lambda \neq 0$,

$$\begin{aligned} |\lambda| \sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t)\geq\alpha\} &= \sup_{\alpha\in(0,1)} \inf\{|\lambda|\, t>0 : N(x,t)\geq\alpha\} \\ &= \sup_{\alpha\in(0,1)} \inf\left\{s>0 : N\left(x, \frac{s}{|\lambda|}\right)\geq\alpha\right\} \\ &= \sup_{\alpha\in(0,1)} \inf\left\{s>0 : N\left(\lambda x, s\right)\geq\alpha\right\}. \end{aligned}$$

□

Lemma 3.5. *Let $(X,N,*)$ be a fuzzy normed space and $\alpha*\alpha=\alpha$ for all $\alpha\in\mathbb{R}$. Then* $sup_{\alpha\in(0,1)} \inf\{t>0 : N(x+y,t)\geq\alpha\}$
$\leq \sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t)\geq\alpha\} + \sup_{\alpha\in(0,1)} \inf\{t>0 : N(y,t)\geq\alpha\}$.

Proof. Since $N(x+y,s+t)\geq N(x,t)*N(y,s)\geq\alpha*\alpha=\alpha$,

$$\begin{aligned} &\sup_{\alpha\in(0,1)} \inf \{t>0 : N(x,t)\geq\alpha\} + \sup_{\alpha\in(0,1)} \inf\{s>0 : N(y,s)\geq\alpha\} \\ &\quad\geq \sup_{\alpha\in(0,1)} (\inf\{t>0 : N(x,t)\geq\alpha\} + \inf\{s>0 : N(y,s)\geq\alpha\}) \\ &\quad\geq \sup_{\alpha\in(0,1)} \inf\{s+t>0 : N(x,t)\geq\alpha,\ N(y,s)\geq\alpha\} \\ &\quad\geq \sup_{\alpha\in(0,1)} \inf\{s+t>0 : N(x+y,s+t)\geq\alpha\} \\ &\quad= \sup_{\alpha\in(0,1)} \inf\{t>0 : N(x+y,t)\geq\alpha\}. \end{aligned}$$

□

We now investigate the relationship between uniform convexity and unform nonsquareness in fuzzy normed spaces.

Theorem 3.6. *Uniform convexity implies unform nonsquareness in fuzzy normed spaces.*

Proof. Suppose that a fuzzy normed space $(X,N,*)$ is uniformly convex. Then for $\epsilon_0=1$, there exists $0<\delta_0<1$ such that for $x,y\in X$,

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t)\geq\alpha\}\leq 1,\ \sup_{\alpha\in(0,1)} \inf\{t>0 : N(y,t)\geq\alpha\}\leq 1$$

and

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x-y,t)\geq\alpha\}>\epsilon_0$$

imply that

$$\sup_{\alpha\in(0,1)} \inf\left\{t>0 : N\left(\frac{x+y}{2}, t\right)\geq\alpha\right\}\leq 1-\delta_0.$$

Let $\delta=\inf\left\{\frac{1}{2},\delta_0\right\}$. If for $x,y\in X$,

$$\sup_{\alpha\in(0,1)} \inf\{t>0 : N(x,t)\geq\alpha\}\leq 1,\ \sup_{\alpha\in(0,1)} \inf\{t>0 : N(y,t)\geq\alpha\}\leq 1$$

and

$$\sup_{\alpha\in(0,1)} \inf \left\{ t>0 : N\left(\frac{x-y}{2}, t\right) \geq \alpha \right\} > 1-\delta,$$

then

$$\sup_{\alpha\in(0,1)} \inf \left\{ t>0 : N\left(x-y, t\right) \geq \alpha \right\} > 2(1-\delta) \geq \epsilon_0,$$

by Lemma 3.4. This implies that

$$\sup_{\alpha\in(0,1)} \inf \left\{ t>0 : N\left(\frac{x+y}{2}, t\right) \geq \alpha \right\} \leq 1-\delta_0 \leq 1-\delta.$$

This completes the proof. □

The converse of Theorem 3.6 does not hold, in general.

Example 3.7. Let X_G be the uniformly nonsquare normed space which is not uniformly convex (Example 3.1). Define a function $N : X_G \times \mathbb{R} \to [0,1]$ by

$$N(x,t) = \begin{cases} 1 & \text{if } t > \|x\| \\ \frac{1}{2} & \text{if } 0 < t \leq \|x\| \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $(X_G, N, \min)$ is a fuzzy normed space. We note that

$$\sup_{\alpha\in(0,1)} \inf \{ t>0 : N(x,t) \geq \alpha \} = \|x\|.$$

Since X_G is the uniformly nonsquare normed space which is not uniformly convex, the fuzzy normed space $(X_G, N, \min)$ is not uniformly convex but uniformly nonsquare.

By Theorem 3.6 and Example 3.7, we can get that Uniform convexity implies unform nonsquareness in fuzzy normed spaces and the converse does not hold, in general.

4. The property (B_k)

For $k \geq 2$ and $\delta > 0$, a normed space X is said to be (k,δ)-convex if for all $x_1, x_2, \cdots, x_k$ in B_X, there exist $\epsilon_i \in \{1,-1\}$, $i = 1,2,\cdots,k$ such that $\|\epsilon_1 x_1 + \cdots + \epsilon_k x_k\| \leq k(1-\delta)$. X is B-convex if there exists $k \geq 2$ and $\delta > 0$ for which X is (k,δ)-convex. We now define the following.

Definition 4.1. A Banach space X *has the property* (B_k) for $k \geq 2$, if there exists $\delta > 0$ such that for $x_1, x_2, \cdots, x_k \in B_X$, there exist $\epsilon_i \in \{1,-1\}$, $i = 1,\cdots,k$ with

$$\|\epsilon_1 x_1 + \cdots + \epsilon_k x_k\| \leq k(1-\delta).$$

We note that X is B-convex if it has property (B_k) for some k and uniformly nonsquare if it has property (B_2).

We now define the property (B_k) in fuzzy normed spaces.

Definition 4.2. A fuzzy normed space $(X, N, *)$ *has the property* (B_k), $k \geq 2$, if there exists $\delta > 0$ such that for $x_1, x_2, \cdots, x_k \in X$,

$$\sup_{\alpha \in (0,1)} \inf \{t > 0 : N(x_i, t) \geq \alpha\} \leq 1, \text{ for } i = 1, 2, \cdots, k$$

there exist $\epsilon_i \in \{ 1, -1\}$, $i = 1, \cdots, k$ with

$$\sup_{\alpha \in (0,1)} \inf \left\{ t > 0 : N\left(\frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_i, t \right) \geq \alpha \right\} \leq 1 - \delta.$$

A fuzzy normed space $(X, N, *)$ is said to be B-convex if it has property (B_k) for some k. We note that A fuzzy normed space $(X, N, *)$ is uniformly nonsquare if it has property (B_2).

Theorem 4.3. *Let* $(X, N, *)$ *be a fuzzy normed space and* $\alpha * \alpha = \alpha$ *for all* $\alpha \in \mathbb{R}$. *Then the property* (B_k) *in fuzzy normed spaces implies* (B_{k+1}) *for* $k \geq 2$.

Proof. The proof is by contradiction. Assume the assertion were false, i.e., suppose $(X, N, *)$ has no the property (B_{k+1}). Then for all $n \in \mathbb{N}$, there exist $x_1^{(n)}, x_2^{(n)}, \cdots, x_{k+1}^{(n)} \in X$ with $\sup_{\alpha \in (0,1)} \inf \left\{ t > 0 : N(x_i^{(n)}, t) \geq \alpha \right\} \leq 1$ for $i = 1, 2, \cdots, k+1$ such that

$$\sup_{\alpha \in (0,1)} \inf \left\{ t > 0 : N\left(\frac{1}{k+1} \sum_{i=1}^{k+1} \epsilon_i x_i, t \right) \geq \alpha \right\} \geq 1 - \frac{1}{n}$$

for all $\epsilon_i \in \{ 1, -1\}$, $i = 1, \cdots, k+1$. By Lemma 3.4 and Lemma 3.5,

$$\begin{aligned} & \sup_{\alpha \in (0,1)} \inf \left\{ t > 0 : N\left(\sum_{i=1}^{k} \epsilon_i x_i^{(n)}, t \right) \geq \alpha \right\} \\ \geq \; & \sup_{\alpha \in (0,1)} \inf \left\{ t > 0 : N\left(\sum_{i=1}^{k+1} \epsilon_i x_i^{(n)}, t \right) \geq \alpha \right\} - \sup_{\alpha \in (0,1)} \left\{ t > 0 : N\left(\epsilon_{k+1} x_{k+1}^{(n)}, t \right) \geq \alpha \right\} \\ > \; & (k+1)\left(1 - \frac{1}{n}\right) - 1 \geq k \left(1 - \frac{2}{n}\right). \end{aligned}$$

This implies that $(X, N, *)$ has no the property (B_k). We get the contradiction. □

Corollary 4.4. *The property* (B_k) *in normed spaces implies* (B_{k+1}) *for* $k \geq 2$.

Proof. Let X be a normed space. Suppose that X has the property (B_k). Define a function $N : X \times \mathbb{R} \to [0, 1]$ by

$$N(x, t) = \begin{cases} 1 & \text{if } t > \|x\| \\ \frac{1}{2} & \text{if } 0 < t \leq \|x\| \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $(X, N, \min)$ is a fuzzy normed space. We note that

$$\sup_{\alpha \in (0,1)} \inf \{t > 0 : N(x, t) \geq \alpha\} = \|x\|. \tag{4.1}$$

Thus $(X, N, \min)$ has the property (B_k). By Theorem 4.3, it has the property (B_{k+1}). Therefore a normed space X has the property (B_{k+1}) by (4.1). □

We now consider the converse of Corollary 4.4.

Example 4.5. Consider $l_1^k = \big(\mathbb{R}^k, \|\cdot\|_1\big)$. Then l_1^k does not have the property (B_k), since

$$\|\epsilon_1 e_1 + \cdots + \epsilon_k e_k\|_1 = 1$$

for all $\epsilon_i = \pm 1$, where e_i is usual unit vector of l_1^k.

We show that l_1^k has the property (B_{k+1}). Let $x_1, \cdots, x_{k+1} \in B_{l_1^k}$. We may assume $x_i \neq 0$ for $i = 1, \cdots, k+1$. Since dim $l_1^k = k$, there exist not all zero scalars $a_1, \cdots, a_{k+1}$ such that $\sum_{i=1}^{k+1} a_i x_i = 0$. Without loss of generality, we may assume $\max_{1\leq i\leq k+1} |a_i| = a_{k+1}$. Let $b_i = \frac{a_i}{|a_{k+1}|}$ then $|b_i| \leq 1$ and $\sum_{i=1}^{k+1} b_i x_i = 0$.

Let $\epsilon_i = \operatorname{sgn}(b_i)$, where $\operatorname{sgn}(\cdot)$ is the sign function defined by

$$\operatorname{sgn}(a) = \begin{cases} \frac{a}{|a|} & \text{if } a \neq 0 \\ 1 & \text{if } a = 0. \end{cases}$$

Then we have

$$\begin{aligned} \left\|\sum_{i=1}^{k+1} \epsilon_i x_i\right\|_1 &\leq \left\|\sum_{i=1}^{k+1} b_i x_i + \sum_{i=1}^{k+1} (\epsilon_i - b_i) x_i\right\|_1 \\ &= \left\|\sum_{i=1}^{k+1} (\epsilon_i - b_i) x_i\right\|_1 \\ &= \left\|\sum_{i=1}^{k} (\epsilon_i - b_i) x_i\right\|_1 \leq k, \end{aligned}$$

since $\epsilon_{k+1} = b_{k+1}$ and $0 \leq \epsilon_i - b_i \leq 1$. Thus we get

$$\frac{1}{k+1}\left\|\sum_{i=1}^{k+1} \epsilon_i x_i\right\|_1 \leq 1 - \frac{1}{k+1}.$$

This implies that l_1^k has the property (B_{k+1}).

The converse of Theorem 4.3 does not hold.

Example 4.6. We note that $l_1^k = \big(\mathbb{R}^k, \|\cdot\|_1\big)$ has the property (B_{k+1}) but not (B_k), for $k \geq 2$. Define a function $N : l_1^k \times \mathbb{R} \to [0,1]$ by

$$N(x,t) = \begin{cases} 1 & \text{if } t > \|x\| \\ \frac{1}{2} & \text{if } 0 < t \leq \|x\| \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $(l_1^k, N, \min)$ is a fuzzy normed space. We note that

$$\sup_{\alpha\in(0,1)} \inf\{t > 0 : N(x,t) \geq \alpha\} = \|x\|.$$

Since $l_1^k = \big(\mathbb{R}^k, \|\cdot\|_1\big)$ has the property (B_{k+1}) but not (B_k) for $k \geq 2$, the fuzzy normed space $(l_1^k, N, \min)$ has the property (B_{k+1}) but not (B_k) for $k \geq 2$.

By Theorem 4.3 and Example 4.6, if $(X, N, *)$ be a fuzzy normed space and $\alpha * \alpha = \alpha$, for all $\alpha \in \mathbb{R}$, then the property (B_k) implies the property (B_{k+1}) and the converse does not hold, in general.

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