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Alexandrov L-preuniform filter spaces

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Abstract. In this paper, we introduce the notion of Alexandrov L-(neighborhood) filters and Alexandrov L-preuniform filters as a topological viewpoint of fuzzy rough sets. We investigate the relations among Alexandrov L-neighborhood filters, L-fuzzy preorders and Alexandrov L-preuniform filter structures. Moreover, we investigate their topological properties and give their examples. As an application for a fuzzy information system, Alexandrov L-neighborhood filters, L-fuzzy preorders and Alexandrov L-preuniform filters are studied.

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1. Introduction

Eklund and Gähler [1] introduced the notion of fuzzy filters as a point-based approach to fuzzy topology on completely distributive complete lattice. Gähler [2, 3] investigated the categorical relations among L-neighborhood filters, L-fuzzy topologies and L-fuzzy topological structures. Höhle [4, 5] introduced L-filters and L-topological structures on algebraic structures (cqm lattices, quantales, MV-algebras) for many valued logics [4, 5, 6, 7, 8, 9]. Kim [10] studied L-filter bases on commutative quantales.

Jäger [11] developed stratified L-convergence structures based on the concepts of L-filters where L is a complete Heyting algebra. Yao [12] extended stratified L-convergence structures to complete residuated lattices and investigated between stratified L-convergence structures and L-fuzzy topological spaces.

Zhang [13, 14, 15] defined a strong L-topology on the concepts of fuzzy complete lattices. As an extension of Yao [12], Fang [16, 17] introduced L-ordered convergence structures on L-ordered filters and investigated between L-ordered convergence structures and strong L-topological spaces. Many researchers developed topological structures using L-filters [18, 19, 20, 21, 22].

Pawlak [23, 24] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of classical rough sets, many researchers [25, 26, 27, 28, 29, 30] developed L-lower and L-upper approximation operators in complete residuated lattices. By using this concepts, information systems and decision rules were investigated in complete residuated lattices [6, 31].

An interesting and natural research topic in rough set theory is the study of rough set theory and topological structures. Lai [32] and Ma [33] investigated the Alexandrov L-topology and lattice structures of L-fuzzy rough sets determined by lower and upper sets.

Kim [2, 12, 13, 14, 15] introduced the notion of Alexandrov L-(neighborhood) filters as a topological viewpoint of fuzzy rough sets and studied the relations among fuzzy preorders, Alexandrov L-(neighborhood) filters, Alexandrov topologies and Alexandrov L-convergence structures in complete residuated lattices.

The aim of this paper is to study Alexandrov L-neighborhood filters, L-fuzzy preorders and Alexandrov L-preuniform filters in fuzzy information systems.

In this paper, we introduce the notion of Alexandrov L-(neighborhood) filters and Alexandrov L-preuniform filters as a topological viewpoint of fuzzy rough sets in a complete residuated lattice. We investigate the relations among Alexandrov L-neighborhood filters, reflexive L-fuzzy relations, Alexandrov L-toplogies and Alexandrov L-preuniform filters. Moreover, we investigate their topological properties and give their examples. As an application for a fuzzy information system, Alexandrov L-neighborhood filters, L-fuzzy preorders and Alexandrov L-preuniform filters are studied in Example 3.7.

2. Preliminaries

Definition 2.1 ([4, 5, 6, 7, 8, 9])**.** An algebra $(L, \leq, \wedge, \vee, \odot, \to, \bot, \top)$ is called a *complete residuated lattice*, if it satisfies the following conditions:

(L1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$,

(L2) $(L, \odot, \top)$ is a commutative monoid,

(L3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we always assume that $(L, \leq, \wedge, \vee, \odot, \to, ^*, \bot, \top)$ is complete residuated lattice with a negation $x^* = x \to \bot$ and $(x^*)^* = x$.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \to A), (\alpha \odot A), \alpha_X \in L^X$ as $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$, $\alpha_X(x) = \alpha$.

Lemma 2.2 ([4, 5, 6, 7, 8, 9])**.** *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $\top \to x = x$, $\bot \odot x = \bot$.

(2) *If* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$.

(3) $x \leq y$ *iff* $x \to y = \top$.

(4) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$.

(5) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$.
(6) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$.
(7) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(8) $(x \to y) \odot (z \to w) \le (x \odot z) \to (y \odot w)$ *and* $x \to y \le (x \odot z) \to (y \odot z)$.
(9) $(x \to y) \odot (y \to z) \le x \to z$.
(10) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.
(11) $x \to y \le (y \to z) \to (x \to z)$ *and* $x \to y \le (z \to x) \to (z \to y)$.
(12 $(x \odot y^*)^* = x \to y$ *and* $x \to y = y^* \to x^*$.

Definition 2.3 ([6, 32])**.** Let X be a set. A function $e_X : X \times X \to L$ is said to be:
(E1) *reflexive*, if $e_X(x, x) = \top$ for all $x \in X$,
(E2) *transitive*, if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$,
(E3) if $e_X(x, y) = e_X(y, x) = \top$, then $x = y$.
If e satisfies (E1) and (E2), (X, e_X) is an *L-fuzzy preordered set.* If e_X satisfies (E1), (E2) and (E3), (X, e_X) is an *L-fuzzy partially ordered set.*

Example 2.4. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then (L, e_L) is an L-fuzzy partially ordered set.
(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A, B) = \bigwedge_{x\in X} (A(x) \to B(x)).$$

Then (L^X, e_{L^X}) is an L-fuzzy partially ordered set from Lemma 2.2 (9).

Definition 2.5 ([13, 14, 15])**.** Let (X, e_X) be an L-fuzzy partially ordered set and $A \in L^X$.
(i) A point x_0 is called a *join* of A, denoted by $x_0 = \sqcup A$, if it satisfies
(J1) $A(x) \le e_X(x, x_0)$,
(J2) $\bigwedge_{x\in X}(A(x) \to e_X(x, y)) \le e_X(x_0, y)$.
(ii) A point x_1 is called a *meet* of A, denoted by $x_1 = \sqcap A$, if it satisfies
(M1) $A(x) \le e_X(x_1, x)$,
(M2) $\bigwedge_{x\in X}(A(x) \to e_X(y, x)) \le e_X(y, x_1)$.

Remark 2.6. Let (X, e_X) be an L-fuzzy partially ordered set and $\Phi \in L^X$.
(1) If x_0 is a join of Φ, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y) = e_X(y_0, y) = \top$ implies $x_0 = y_0$. Similarly, if a meet of Φ exist, then it is unique.
(2) A point x_0 is a join of Φ iff $\bigwedge_{x\in X}(\Phi(x) \to e_X(x, y)) = e_X(x_0, y)$.
(3) A point x_1 is a meet of Φ iff $\bigwedge_{x\in X}(\Phi(x) \to e_X(y, x)) = e_X(y, x_1)$.

Remark 2.7. Let (L, e_L) be an L-fuzzy partially ordered set and $A \in L^L$.
(1) Since x_0 is a join of A iff

$$\begin{aligned}&\bigwedge_{x\in L}(A(x) \to e_L(x, y))\\ &= \bigwedge_{x\in L}(A(x) \to (x \to y))\\ &= \bigvee_{x\in L}(x \odot A(x)) \to y\\ &= e_L(x_0, y)\\ &= x_0 \to y,\end{aligned}$$

we have $x_0 = \sqcup A = \bigvee_{x\in L}(x \odot A(x))$.
(2) Since x_0 is a join of A iff $\bigwedge_{x\in L}(A(x) \to e_L(x, y)$

$$\begin{aligned} &= \bigwedge_{x\in L}(A(x)\to(y\to x)) \\ &= \bigwedge_{x\in L}(y\to(A(x)\to x)) \\ &= y\to\bigwedge_{x\in L}(A(x)\to x) \\ &= y\to\sqcap A, \end{aligned}$$

we get $\sqcap A=\bigwedge_{x\in L}(A(x)\to x)$.

Remark 2.8. Let (L^X, e_{L^X}) be an L-fuzzy partially ordered set and $\Phi\in L^{L^X}$.

(1) $\sqcup\Phi=\bigvee_{A\in L^X}(\Phi(A)\odot A)$, from the following

$$e_{L^X}(\sqcup\Phi, B)=\bigwedge_{A\in L^X}(\Phi(A)\to e_{L^X}(A,B))=e_{L^X}(\bigvee_{A\in L^X}(\Phi(A)\odot A),B).$$

(2) $\sqcap\Phi=\bigwedge_{A\in L^X}(\Phi(A)\to A)$, from the following

$$\begin{aligned} e_{L^X}(B,\sqcup\Phi) &= \bigwedge_{A\in L^X}(\Phi(A)\to e_{L^X}(B,A)) \\ &= \bigwedge_{A\in L^X} e_{L^X}(B,(\Phi(A)\to A)) \\ &= e_{L^X}(B,\bigwedge_{A\in L^X}(\Phi(A)\to A)). \end{aligned}$$

Definition 2.9 ([13, 14, 15])**.** Let (X, e_X) be an L-fuzzy partially ordered set. The pair (X, e_X) is called a *fuzzy join (resp. meet) complete lattice*, if $\sqcup\Phi$ (resp. $\sqcap\Phi$) exists for each $\Phi\in L^X$.

The pair (X, e_X) is called a *fuzzy complete lattice*, if $\sqcup\Phi$ and $\sqcap\Phi$ exist for each $\Phi\in L^X$.

Definition 2.10 ([13, 14, 15])**.** Let (X, e_X) and (Y, e_Y) be fuzzy complete lattices and $\psi: X\to Y$ a map.

(i) ψ is a *join preserving map*, if $\psi(\sqcup\Phi)=\sqcup\psi^{\to}(\Phi)$ for all $\Phi\in L^X$, where $\psi^{\to}(\Phi)(y)=\bigvee_{\psi(x)=y}\Phi(x)$.

(ii) ψ is a *meet preserving map*, if $\psi(\sqcap\Phi)=\sqcap\psi^{\to}(\Phi)$ for all $\Phi\in L^X$.

(iii) ψ is an *order preserving map*, if $e_X(x,y)\le e_Y(\psi(x),\psi(y))$ for all $x,y\in X$.

Definition 2.11 ([18, 22])**.** Let (L^X, e_{L^X}) and (L, e_L) be L-fuzzy partially ordered sets. A map $\mathcal{F}:(L^X, e_{L^X})\to(L,e_L)$ is called an *Alexandrov L-filter* on X, if $\mathcal{F}(\sqcap\Phi)=\sqcap\mathcal{F}^{\to}(\Phi)$ for all $\Phi\in L^{L^X}$. Let $AF(X)$ denote the set of all Alexandrov L-filters on X.

Theorem 2.12 ([18, 22])**.** *A map $\mathcal{F}: L^X\to L$ is an Alexandrov L-filter on X iff it satisfies the following conditions:*

(F1) $\mathcal{F}(\bigwedge_{i\in\Gamma}A_i)=\bigwedge_{i\in\Gamma}\mathcal{F}(A_i)$ *for all* $A_i\in L^X$,

(F2) $\mathcal{F}(\alpha\to A)=\alpha\to\mathcal{F}(A)$ *for all* $A\in L^X$ *and* $\alpha\in L$.

Definition 2.13 ([18, 22])**.** A family $\mathcal{N}_X=\{\mathcal{N}^x\mid x\in X\}$ is called an *Alexandrov L-neighborhood system* on X, if for $x\in X$, a map $\mathcal{N}^x: L^X\to L$ satisfies:

(N1) $\mathcal{N}^x$ is an Alexandrov L-filter on X,

(N2) $\mathcal{N}^x(A)\le A(x)$ for all $A\in L^X$.

The pair $(X,\mathcal{N}_X)$ is called an *Alexandrov L-neighborhood space.*

An Alexandrov L-neighborhood system on X is *topological*, if (TN) $\mathcal{N}^x(A)\le\mathcal{N}^x(\mathcal{N}^-(A))$ for all $\mathcal{N}^-(A)\in L^X$ such that $\mathcal{N}^-(A)(y)=\mathcal{N}^y(A)$ for all $y\in X$.

Definition 2.14 ([18, 22, 26, 27])**.** A subset $\tau\subset L^X$ is called an *Alexandrov L-topology* on X, if it satisfies the following conditions:

(AT1) $\alpha_X\in\tau$,

(AT2) if $A_i \in \tau$ for all $i \in \Gamma$, then $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$,
(AT3) if $A \in \tau$ and $\alpha \in L$, then $\alpha \odot A, \alpha \to A \in \tau$.
The pair (X, τ) is called an *Alexandrov L-topological space*.

3. Alexandrov preuniform L-filter spaces

Definition 3.1. (i) An Alexandrov L-filter $\mathcal{W} : L^{X\times X} \to L$ is called an *Alexandrov L-preuniform filter* on $X \times X$, if $\mathcal{W} \leq \bigwedge_{x\in X}[(x, x)]$, where $[(x, x)](u) = u(x, x)$ for each $u \in L^{X\times X}$.

(ii) An Alexandrov L-preuniform filter $\mathcal{W}$ is called an *Alexandrov L-quasiuniform filter* on $X\times X$, if $\bigvee_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)})\odot\mathcal{W}^*(\top^*_{(y,z)})) \leq \mathcal{W}^*(\top^*_{(x,z)})$, where $\top^*_{(x,y)}(z, w) = \bot$ if $(z, w) = (x, y)$ and $\top$, otherwise.

The pair $(X, \mathcal{W})$ is called an *Alexandrov L-preuniform (resp. L-quasiuniform) filter space.*

Theorem 3.2. (1) *A map $\mathcal{W} : L^{X\times X} \to L$ is an Alexandrov L-preuniform (resp. L-quasiuniform) filter on $X \times X$ iff there exists a reflexive L-fuzzy relation (resp. L-fuzzy preorder) $e_\mathcal{W} \in L^{X\times X}$ with $e_\mathcal{W}(x, y) = \mathcal{W}^*(\top^*_{(x,y)})$ such that $\mathcal{W}(u) = \bigwedge_{x,z\in X}(e_\mathcal{W}(x, z) \to u(x, z))$ for all $u \in L^{X\times X}$.*

(2) *For each $x \in X$, a map $\mathcal{N}^x : L^X \to L$ is an Alexandrov (resp. topological) L-neighborhood filter on X iff there exists a reflexive L-fuzzy relation (resp. L-fuzzy preorder) $e_\mathcal{N} \in L^{X\times X}$ such that $\mathcal{N}^x(A) = \bigwedge_{z\in X}(e_\mathcal{N}(x, z) \to A(z))$ for all $A \in L^X$.*

(3) *Let $\mathcal{N}^x : L^X \to L$ be an Alexandrov L-neighborhood filter on X for each $x \in X$. Define $\mathcal{W}_\mathcal{N} : L^{X\times X} \to L$ as*

$$\mathcal{W}_\mathcal{N}(u) = \bigwedge_{x\in X} \mathcal{N}^x(u(x, -)).$$

Then $\mathcal{W}_\mathcal{N}$ is an Alexandrov L-preuniform filter on $X \times X$ such that

$$\mathcal{W}_\mathcal{N}(u) = \bigwedge_{x,y\in X} ((\mathcal{N}^x(\top^*_y))^* \to u(x, y)).$$

If $\{\mathcal{N}^x \mid x \in X\}$ is topological, then $\mathcal{W}_\mathcal{N}$ is an Alexandrov L-quasiuniform filter on $X \times X$.

(4) *Let $\mathcal{W} : L^{X\times X} \to L$ be an Alexandrov L-preuniform (resp. L-quasiuniform) filter on $X \times X$. Define $\mathcal{N}^x_\mathcal{W} : L^X \to L$ as*

$$\mathcal{N}^x_\mathcal{W}(A) = \bigwedge_{y\in X} (\mathcal{W}^*(\top^*_{(x,y)})) \to A(y)).$$

Then $\mathcal{N}^x_\mathcal{W}$ is an Alexandrov (resp. topological) L-neighborhood filter on X such that

$$\mathcal{W}_{\mathcal{N}_\mathcal{W}} = \mathcal{W}.$$

(5) *If $\mathcal{N}^x : L^X \to L$ be an Alexandrov L-neighborhood filter on X for each $x \in X$, then $\mathcal{N}^x_{\mathcal{W}_\mathcal{N}} = \mathcal{N}^x$.*

Proof. (1) ($\Rightarrow$) Let $\mathcal{W}$ be an Alexandrov L-quasiuniform filter on $X \times X$. For all $u \in L^{X\times X}$, since $u = \bigwedge_{x,z\in X}(u^*(x, z) \to \top^*_{(x,z)})$, by Theorem 2.12 (F1) and (F2), we have

$$\mathcal{W}(u) = \mathcal{W}(\bigwedge_{x,z\in X}(u^*(x, z) \to \top^*_{(x,z)})$$

$$= \bigwedge_{x,z\in X}(u^*(x,z) \to \mathcal{W}(\top^*_{(x,z)}))$$
$$= \bigwedge_{x,z\in X}((\mathcal{W}^*(\top^*_{(x,z)})) \to u(x,z)).$$

Put $e_{\mathcal{W}}(x,y) = \mathcal{W}^*(\top^*_{(x,y)})$. Then we get

$$\begin{aligned} e_{\mathcal{W}}(x,x) &= \mathcal{W}^*(\top^*_{(x,x)}) \\ &\geq \bigvee_{x\in X}[(x,x)]^*(\top^*_{(x,x)}) \\ &= \bigvee_{x\in X}[(x,x)]^*(\top^*_{(x,x)}) \\ &= \top_{(x,x)}(x,x) \\ &= \top, \end{aligned}$$

$$\begin{aligned} \bigvee_{y\in X}(e_{\mathcal{W}}(x,y)\odot e_{\mathcal{W}}(y,z)) &= \bigvee_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)})\odot \mathcal{W}^*(\top^*_{(y,z)})) \\ &\leq \mathcal{W}^*(\top^*_{(x,z)}) \\ &= e_{\mathcal{W}}(x,z). \end{aligned}$$

Thus $e_{\mathcal{W}}$ is an L-fuzzy preorder. Moreover, $\mathcal{W}(u) = \bigwedge_{x,z\in X}(e_{\mathcal{W}}(x,z) \to u(x,z))$.

($\Leftarrow$) From Theorem 2.12, the result holds $e_{\mathcal{W}}(x,y) = \mathcal{W}^*(\top^*_{(x,y)})$ and from:

(F1) For all $u_i \in L^{X\times X}$,

$$\begin{aligned} \mathcal{W}(\bigwedge_{i\in\Gamma} u_i) &= \bigwedge_{x,z\in X}(e_{\mathcal{W}}(x,z) \to \bigwedge_{i\in\Gamma} u_i(x,z)) \\ &= \bigwedge_{i\in\Gamma}\bigwedge_{z\in X}(e_{\mathcal{W}}(x,z) \to u_i(x,z)) \\ &= \bigwedge_{i\in\Gamma}\mathcal{W}(u_i). \end{aligned}$$

(F2) For all $u \in L^{X\times X}$ and $\alpha \in L$, by Lemma 2.2 (7),

$$\begin{aligned} \mathcal{W}(\alpha \to u) &= \bigwedge_{x,z\in X}(e_{\mathcal{W}}(x,z) \to (\alpha \to u(x,z))) \\ &= \alpha \to \bigwedge_{x,z\in X}(e_{\mathcal{W}}(x,z) \to u(x,z))) \\ &= \alpha \to \mathcal{W}(u). \end{aligned}$$

(2) Let $\mathcal{N}^x$ be an Alexandrov topological L-neighborhood filter. Then we have: for $A = \bigwedge_{y\in X}(A^*(y) \to \top^*_y)$,

$$\begin{aligned} \mathcal{N}^x(A) &= \mathcal{N}^x(\bigwedge_{y\in X}(A^*(y) \to \top^*_y)) \\ &= \bigwedge_{y\in X}(A^*(y) \to \mathcal{N}^x(\top^*_y)) \quad (\mathcal{N}^x(\top^*_y) \\ &= e^*_{\mathcal{N}}(x,y)) \\ &= \bigwedge_{y\in X}(e_{\mathcal{N}}(x,y) \to A(y)). \end{aligned}$$

Thus $e_{\mathcal{N}}(x,x) \geq (\mathcal{N}^x(\top^*_x))^* \geq \top_x(x) = \top$, i.e., $e_{\mathcal{N}}$ is reflexive. Since $\mathcal{N}^x(\mathcal{N}^-(\top^*_z)) = \mathcal{N}^x(\top^*_z)$ and $\mathcal{N}^-(\top^*_z) = \bigwedge_{y\in X}((\mathcal{N}^y(\top^*_z))^* \to \top^*_y)$,

$$\begin{aligned} \mathcal{N}^x(\mathcal{N}^-(\top^*_z)) &= \mathcal{N}^x(\bigwedge_{y\in X}((\mathcal{N}^y(\top^*_z))^* \to \top^*_y)) \\ &= \bigwedge_{y\in X}((\mathcal{N}^y(\top^*_z))^* \to \mathcal{N}^x(\top^*_y)) \\ &= \mathcal{N}^x(\top^*_z) \end{aligned}$$
$$\text{iff } \bigvee_{y\in X}((\mathcal{N}^y(\top^*_z))^* \odot (\mathcal{N}^x(\top^*_y))^*) = (\mathcal{N}^x(\top^*_z))^*.$$

So $e_{\mathcal{N}}$ is an L-fuzzy preorder, from the following:

$$\begin{aligned} \bigvee_{y\in X}(e_{\mathcal{N}}(y,z)\odot e_{\mathcal{N}}(x,y)) &= \bigvee_{y\in X}((\mathcal{N}^y(\top^*_z))^* \odot (\mathcal{N}^x(\top^*_y))^*) \\ &= (\mathcal{N}^x(\top^*_z))^* = e_{\mathcal{N}}(x,z). \end{aligned}$$

($\Leftarrow$) It is similarly proved as in (1).

(3) It is obvious that $\mathcal{W}_{\mathcal{N}}$ satisfies (F1) and (F2). Moreover, for each $u \in L^{X\times X}$,

$$\begin{aligned} \mathcal{W}_{\mathcal{N}}(u) &= \bigwedge_{x\in X}\mathcal{N}^x(u(x,-)) \\ &\leq \bigwedge_{x\in X} u(x,-)(x) \\ &= \bigwedge_{x\in X} u(x,x) \\ &= \bigwedge_{x\in X}[(x,x)](u). \end{aligned}$$

For all $u(x,-) \in L^{X\times X}$, since $u(x,-) = \bigwedge_{y\in X}(u^*(x,y) \to \top^*_y)$, by Theorem 2.12 (F1) and (F2), we have

$$\begin{aligned}\mathcal{W}_{\mathcal{N}}(u) &= \bigwedge_{x\in X} \mathcal{N}^x(u(x,-)) \\ &= \bigwedge_{x\in X} \mathcal{N}^x(\bigwedge_{y\in X}(u^*(x,y)\to \top^*_y)) \\ &= \bigwedge_{x,y\in X}(u^*(x,y)\to \mathcal{N}^x(\top^*_y)) \\ &= \bigwedge_{x,y\in X}((\mathcal{N}^x(\top^*_y))^*\to u(x,y)).\end{aligned}$$

If $\{\mathcal{N}^x \mid x\in X\}$ is topological, since $\mathcal{W}_{\mathcal{N}}(\top^*_{(x,y)}) = \mathcal{N}^x(\top^*_y)$, by (2),

$$\begin{aligned}\bigvee_{y\in X}((\mathcal{W}_{\mathcal{N}}(\top^*_{(x,y)}))^*\odot(\mathcal{W}_{\mathcal{N}}(\top^*_{(y,z)}))^*) &= \bigvee_{y\in X}((\mathcal{N}^x(\top^*_y))^*\odot(\mathcal{N}^y(\top^*_z))^*) \\ &= (\mathcal{N}^x(\top^*_z))^* \\ &= (\mathcal{W}_{\mathcal{N}}(\top^*_{(x,z)}))^*.\end{aligned}$$

Then $\mathcal{W}_{\mathcal{N}}$ is an Alexandrov L-quasiuniform filter on $X\times X$.

(4) Let $\mathcal{W}$ be an Alexandrov L-quasiuniform filter on $X\times X$. Then $\mathcal{N}^x_{\mathcal{W}}$ satisfies (F1) and (F2). Moreover, for each $A\in L^X$,

$$\begin{aligned}\mathcal{N}^x_{\mathcal{W}}(A) &= \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)}))\to A(y)) \\ &\le \mathcal{W}^*(\top^*_{(x,x)}))\to A(x) \\ &= A(x),\end{aligned}$$

$$\begin{aligned}\mathcal{N}^x_{\mathcal{W}}(\mathcal{N}^-_{\mathcal{W}}(A)) &= \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)}))\to \mathcal{N}^y_{\mathcal{W}}(A)) \\ &= \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)}))\to \bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)}))\to A(z))) \\ &= \bigwedge_{z\in X}(\bigvee_{y\in X}((\mathcal{W}^*(\top^*_{(x,y)}))\odot \mathcal{W}^*(\top^*_{(y,z)}))\to A(z)) \\ &= \bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(x,z)})\to A(z)) \\ &= \mathcal{N}^x_{\mathcal{W}}(A).\end{aligned}$$

Since $\mathcal{N}^x_{\mathcal{W}}(\top^*_y) = \mathcal{W}(\top^*_{(x,y)})$,

$$\begin{aligned}\mathcal{W}_{\mathcal{N}_{\mathcal{W}}}(u) &= \bigwedge_{x,y\in X}((\mathcal{N}^x_{\mathcal{W}}(\top^*_y))^*\to u(x,y)) \\ &= \bigwedge_{x,y\in X}((\mathcal{W}(\top^*_{(x,y)}))^*\to u(x,y)) \\ &= \bigwedge_{x,y\in X}(u^*(x,y)\to \mathcal{W}(\top^*_{(x,y)})) \\ &= \mathcal{W}(\bigwedge_{x,y\in X}(u^*(x,y)\to \top^*_{(x,y)})) \\ &= \mathcal{W}(u).\end{aligned}$$

(5) For $A\in L^X$,

$$\begin{aligned}\mathcal{N}^x_{\mathcal{W}_{\mathcal{N}}}(A) &= \bigwedge_{y\in X}(\mathcal{W}^*_{\mathcal{N}}(\top^*_{(x,y)})\to A(y)) \\ &= \bigwedge_{y\in X}((\mathcal{N}^x(\top^*_y))^*\to A(y)) \\ &= \bigwedge_{y\in X}(A^*\to \mathcal{N}^x(\top^*_y)) \\ &= \mathcal{N}^x(A).\end{aligned}$$

□

Remark 3.3. For $\triangle = \{(x,x)\mid x\in X\}\subset D$, we define $\mathcal{W} = \bigwedge_{(x,y)\in D}[(x,y)]$ is an Alexandrov L-preuniform filter on $X\times X$. If $D\circ D = D$, then $\bigwedge_{(x,y)\in D}[(x,y)]$ is an Alexandrov L-quasiuniform filter on $X\times X$. Since

$$e_{\mathcal{W}}(y,z) = (\bigwedge_{(x,y)\in D}[(x,y)])^*(\top^*_{(y,z)}) = \bigvee_{(x,y)\in D}\top_{(y,z)}(x,y),$$

$$e_{\mathcal{W}}(x,y) = \begin{cases}\top, & \text{if } (x,y)\in D,\\ \bot, & \text{if } (x,y)\notin D.\end{cases}$$

By Theorem 3.2 (4), we obtain an Alexandrov L-neighborhood filter $\mathcal{N}^x_{\mathcal{W}}$ on X such that $\mathcal{N}^x_{\mathcal{W}}(A) = \bigwedge_{(x,y)\in D}(e_{\mathcal{W}}(x,y)\to A(y))$.

(1) If $D=\triangle$, then we have

$$\mathcal{N}^x_{\mathcal{W}}(A) = \bigwedge_{(x,y)\in\triangle}(e_{\mathcal{W}}(x,y)\to A(y)) = A(x) = [x](A).$$

Since $\mathcal{N}^x_{\mathcal{W}}(\top^*_y) = \top^*_y(x)$,

$$\mathcal{W}_{\mathcal{N}_{\mathcal{W}}}(u) = \bigwedge_{x,y\in X}((\mathcal{N}^x_{\mathcal{W}}(\top^*_y))^* \to u(x,y)) = \bigwedge_{x\in X} u(x,x) = \bigwedge_{x\in X}[(x,x)](u).$$

(2) If $D = X \times X$, then we get

$$\mathcal{N}^x_{\mathcal{W}}(A) = \bigwedge_{(x,y)\in X\times X}(e_{\mathcal{W}}(x,y) \to A(y)) = \bigwedge_{x\in X} A(x) = \bigwedge_{x\in X}[x](A).$$

Since $\mathcal{N}^x_{\mathcal{W}}(\top^*_y) = \bigwedge_{x\in X}\top^*_y(x) = \bot$,

$$\begin{aligned}\mathcal{W}_{\mathcal{N}_{\mathcal{W}}}(u) &= \bigwedge_{x,y\in X}((\mathcal{N}^x_{\mathcal{W}}(\top^*_y))^* \to u(x,y))\\ &= \bigwedge_{(x,y)\in X\times X} u(x,y)\\ &= \bigwedge_{x,y\in X}[(x,y)](u).\end{aligned}$$

Theorem 3.4. (1) *Let $\mathcal{W} : L^{X\times X} \to L$ be an Alexandrov L-preuniform filter on $X \times X$. Define $\tau_{\mathcal{W}} = \{A \in L^X \mid \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})) \to A(y)) = A\}$. Then $\tau_{\mathcal{W}}$ is an Alexandrov L-topology on X. If $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X \times X$, $\tau_{\mathcal{W}} = \{\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})) \to A(y)) \mid A \in L^X\}$.*

(2) *Let τ be an Alexandrov L-topology on X. Define $\mathcal{W}_\tau : L^{X\times X} \to L$ as*

$$\mathcal{W}_\tau(u) = \bigwedge_{x,y\in X}(\bigwedge_{A\in\tau}(A(x) \to A(y)) \to u(x,y)).$$

Then $\mathcal{W}_\tau$ is an Alexandrov L-quasiuniform filter on $X \times X$ with $\tau_{\mathcal{W}_\tau} = \tau$.

(3) $\bigwedge_{A\in\tau_{\mathcal{W}}}(A(x) \to A(y)) \geq \bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)}) \to \mathcal{W}^*(\top^*_{(x,z)}))$. *Moreover, if $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X \times X$, then*

$$\bigwedge_{A\in\tau_{\mathcal{W}}}(A(x) \to A(y)) = \bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)}) \to \mathcal{W}^*(\top^*_{(x,z)})) = \mathcal{W}^*(\top^*_{(x,y)}).$$

(4) *If $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X \times X$, then $\mathcal{W}_{\tau_{\mathcal{W}}} = \mathcal{W}$.*

Proof. (1) (AT1) Since

$$\begin{aligned}\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)})) \to \alpha_X(y)) &\leq \mathcal{W}^*(\top^*_{(x,x)})) \to \alpha_X(x))\\ &\leq \bigvee_{x\in X}[(x,x)]^*(\top^*_{(x,x)}) \to \alpha_X(x)\\ &= \alpha\end{aligned}$$

and

$$\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)})) \to \alpha_X(y)) \geq \alpha,$$

we have $\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})) \to \alpha_X(y)) = \alpha_X$, i.e., $\alpha_X \in \tau_{\mathcal{W}}$.

(AT2) If $A_i = \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to A_i(y))$ for all $i \in \Gamma$, we get

$$\begin{aligned}\bigvee_{i\in\Gamma} A_i &= \bigvee_{i\in\Gamma}(\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to A_i(y)))\\ &\leq \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to \bigvee_{i\in\Gamma} A_i(y))\\ &\leq \bigvee_{i\in\Gamma} A_i,\\ \bigwedge_{i\in\Gamma} A_i &= \bigwedge_{i\in\Gamma}\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to A_i(y))\\ &= \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to \bigwedge_{i\in\Gamma} A_i(y))\\ &= \bigwedge_{i\in\Gamma} A_i.\end{aligned}$$

Thus $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{W}}$.

(AT3) If $A = \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to A(y))$, then we have

$$\begin{aligned}\alpha \to A &= \alpha \to \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to A(y))\\ &= \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}) \to (\alpha \to A)(y)),\end{aligned}$$

$$\begin{aligned}\alpha\odot A &= \alpha\odot \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})\to A(y))\\ &\le \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})\to \alpha\odot A(y))\\ &\le \alpha\odot A.\end{aligned}$$

Thus $\alpha\odot A, \alpha\to A\in\tau_{\mathcal{W}}$. So $\tau_{\mathcal{W}}$ is an Alexandrov L-topology on X.

Suppose $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X\times X$. Put

$$\tau=\{\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}))\to A(y))\mid A\in L^X\}.$$

Let $B\in\tau_{\mathcal{W}}$. Then $B\in\tau$. Let $B=\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)}))\to A(y))\in\tau$. Then we get

$$\begin{aligned}&\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(-,z)}))\to B(z))\\ &=\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(-,z)}))\to \bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(z,y)}))\to A(y)))\\ &=\bigwedge_{y\in X}(\bigvee_{z\in X}(\mathcal{W}^*(\top^*_{(-,z)}))\odot(\mathcal{W}^*(\top^*_{(z,y)}))\to A(y))\\ &=\bigwedge_{y\in X}(\mathcal{W}^*(\top^*_{(-,y)})\to A(y))\\ &=B\in\tau_{\mathcal{W}}.\end{aligned}$$

(2) For $A\in L^X$, since $\mathcal{W}_\tau(\alpha\to u)=\alpha\to\mathcal{W}_\tau(u)$,

$$\mathcal{W}_\tau(\bigwedge_{i\in\Gamma}u_i)=\bigwedge_{i\in\Gamma}\mathcal{W}_\tau(u_i)$$

and

$$\begin{aligned}\mathcal{W}_\tau(u) &= \bigwedge_{x,y\in X}(\bigwedge_{A\in\tau}(A(x)\to A(y))\to u(x,y))\\ &\le \bigwedge_{x\in X}(\bigwedge_{A\in\tau}(A(x)\to A(x))\to u(x,x))\\ &=\bigwedge_{x\in X}[(x,x)](u).\end{aligned}$$

Since $\mathcal{W}^*_\tau(\top^*_{(x,y)})=\bigwedge_{A\in\tau}(A(x)\to A(y))$, we have

$$\begin{aligned}&\bigvee_{y\in X}(\mathcal{W}^*_\tau(\top^*_{(x,y)})\odot\mathcal{W}^*_\tau(\top^*_{(x,y)}))\\ &=\bigvee_{y\in X}(\bigwedge_{A\in\tau}(A(x)\to A(y))\odot\bigwedge_{A\in\tau}(A(y)\to A(z)))\\ &\le\bigwedge_{A\in\tau}(A(x)\to A(z))\\ &=\mathcal{W}^*_\tau(\top^*_{(x,z)}),\\ &\bigvee_{y\in X}(\mathcal{W}^*_\tau(\top^*_{(x,y)})\odot\mathcal{W}^*_\tau(\top^*_{(y,z)}))\ge\mathcal{W}^*_\tau(\top^*_{(x,x)})\odot\mathcal{W}^*_\tau(\top^*_{(x,z)})\\ &\qquad\qquad=\mathcal{W}^*_\tau(\top^*_{(x,z)}).\end{aligned}$$

Then $\mathcal{W}_\tau$ is an Alexandrov L-quasiuniform filter on $X\times X$.

Let $B\in\tau$. Then

$$\bigwedge_{y\in X}(\bigwedge_{A\in\tau}(A(x)\to A(y))\to B(y))\le(A(x)\to A(x))\to B(x)=B(x).$$

Thus we have

$$B(x)\odot(\bigwedge_{A\in\tau}(A(x)\to A(y)))\le B(x)\odot(B(x)\to B(y))\le B(y),$$

$$B(x)\le\bigwedge_{y\in X}(\bigwedge_{A\in\tau}(A(x)\to A(y))\to B(y)).$$

So $B=\bigwedge_{y\in X}(\mathcal{W}^*_\tau(\top^*_{(-,y)})\to B(y))\in\tau_{\mathcal{W}_\tau}$.

Now let $B\in\tau_{\mathcal{W}_\tau}$. Since $\bigvee_{A\in\tau}(A^*(y)\odot A(-)))\in\tau$ and $\bigwedge_{y\in X}(B^*(y)\to\bigvee_{A\in\tau}(A^*(y)\odot A(-)))\in\tau$, we get

$$\begin{aligned}B&=\bigwedge_{y\in X}(\mathcal{W}^*_\tau(\top^*_{(-,y)})\to B(y))\\ &=\bigwedge_{y\in X}(\bigwedge_{A\in\tau}(A(-)\to A(y))\to B(y))\\ &=\bigwedge_{y\in X}(B^*(y)\to\bigvee_{A\in\tau}(A^*(y)\odot A(-)))\in\tau.\end{aligned}$$

(3) Since $A=\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(-,z)})\to A(z))\in\tau_{\mathcal{W}}$,

$$\begin{aligned}&\bigwedge_{A\in\tau_{\mathcal{W}}}(A(x)\to A(y))\\ &=\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(x,z)})\to A(z))\to\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to A(z))\\ &\geq\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)})).\end{aligned}$$

If $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X\times X$, then

$$\bigvee_{y\in X}(\mathcal{W}^*(\top^*_{(x,y)})\odot\mathcal{W}^*(\top^*_{(y,z)}))\leq\mathcal{W}^*(\top^*_{(x,z)}).$$

Thus $\mathcal{W}^*(\top^*_{(x,y)})\leq\bigwedge_{z\in X}((\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)}))$. Moreover, we have

$$\begin{aligned}\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)}))&\leq\mathcal{W}^*(\top^*_{(y,y)})\to\mathcal{W}^*(\top^*_{(x,y)})\\ &\leq(\bigvee_{x\in X}[(x,x)]^*)(\top^*_{(y,y)})\to\mathcal{W}^*(\top^*_{(x,y)})\\ &=\mathcal{W}^*(\top^*_{(x,y)}).\end{aligned}$$

So $\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)}))=\mathcal{W}^*(\top^*_{(x,y)})$. Since $A(-)=\mathcal{W}^*(\top^*_{(x,-)})\in\tau_{\mathcal{W}}$, we get

$$\begin{aligned}\bigwedge_{A\in\tau_{\mathcal{W}}}(A(x)\to A(y))&\leq\bigwedge_{x\in X}(\mathcal{W}^*(\top^*_{(x,x)})\to\mathcal{W}^*(\top^*_{(x,y)}))\\ &\leq(\bigvee_{y\in X}[(y,y)]^*)(\top^*_{(x,x)})\to\mathcal{W}^*(\top^*_{(x,y)})\\ &=\mathcal{W}^*(\top^*_{(x,y)})\\ &=\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)})).\end{aligned}$$

(4) If $\mathcal{W}$ is an Alexandrov L-quasiuniform filter on $X\times X$, then by (3),

$$\begin{aligned}\mathcal{W}_{\tau_{\mathcal{W}}}(u)&=\bigwedge_{x,y\in X}(\bigwedge_{z\in X}(\mathcal{W}^*(\top^*_{(y,z)})\to\mathcal{W}^*(\top^*_{(x,z)}))\to u(x,y))\\ &\leq\bigwedge_{x,y\in X}(\mathcal{W}^*(\top^*_{(x,y)})\to u(x,y))\\ &=\mathcal{W}(u).\end{aligned}$$

□

Example 3.5. (1) For $A\in L^X$, we define $e_A(x,y)=A(x)\to A(y)$. Then e_A is an L-fuzzy preordered set on X. By a similar way in Theorem 3.4 (1), we obtain

$$\tau_X=\{\bigwedge_{y\in X}(e_A(-,y)\to B(y))\mid B\in L^X\}.$$

Define $e_{\tau_X}(x,y)=\bigwedge_{B\in\tau_X}(B(x)\to B(y))$. Since $A\in\tau_X$, $e_{\tau_X}(x,y)\leq e_A(x,y)$,

$$\begin{aligned}e_{\tau_X}(x,y)&=\bigwedge_{B\in\tau_X}(B(x)\to B(y))\\ &=\bigwedge_{C\in L^X}(\bigwedge_{z\in X}(e_A(x,z)\to C(z))\to\bigwedge_{z\in X}(e_A(y,z)\to C(z)))\\ &\geq\bigwedge_{z\in X}(e_A(y,z)\to e_A(x,z))\geq A(x)\to A(y).\end{aligned}$$

Then $e_{\tau_X}(x,y)=e_A(x,y)$. Thus by Theorem 3.2 (1), $\mathcal{W}_{\tau_X}$ is an Alexandrov L-quasiuniform filter on $X\times X$ such that

$$\mathcal{W}_{\tau_X}(u)=\bigwedge_{x,y\in X}(e_A(x,y)\to u(x,y)).$$

Moreover, since $\mathcal{W}^*_{\tau_X}(\top^*_{(-,y)})=e_A(-,y)$,

$$\tau_{\mathcal{W}_{\tau_X}}=\{\bigwedge_{y\in X}(e_A(-,y)\to B(y))\mid B\in L^X\}=\tau_X.$$

(2) Let $\tau_X=L^X$. Then $e_{\tau_X}(x,y)=\bigwedge_{B\in\tau_X}(B(x)\to B(y))$. For $\top_x\in\tau_X$ and $x\neq y$, $e_{\tau_X}(x,y)=\bigwedge_{B\in\tau_X}(B(x)\to B(y))\leq\top_x(x)\to\top_x(y)=\perp$. Thus for $e_X=\triangle_{X\times X}$ with

$$\triangle_{X\times X}(x,y)=\begin{cases}\top, & \text{if } y=x,\\ \perp, & \text{otherwise,}\end{cases}$$

$$\mathcal{W}_{\tau_X}(u)=\bigwedge_{x,y\in X}(\triangle_{X\times X}(x,y)\to u(x,y))=\bigwedge_{x\in X}u(x,x).$$

By Theorem 3.2 (1), $\mathcal{W}_{\tau_X}$ is an Alexandrov L-quasiuniform filter on $X \times X$. Moreover, since $\mathcal{W}^*_{\tau_X}(\top^*_{(-,y)}) = \triangle_{X\times X}(-,y)$,

$$\tau_{\mathcal{W}_{\tau_X}} = \{B \in L^X \mid \bigwedge_{y\in X}(\triangle_{X\times X}(-,y) \to B(y)) = B\} = L^X = \tau_X.$$

(3) Let $\tau_X = \{\alpha_X \mid \alpha \in L\}$. Then

$$e_{\tau_X}(x,y) = \bigwedge_{B\in\tau_X}(B(x) \to B(y)) = \top,$$

$$\mathcal{W}_{\tau_X}(u) = \bigwedge_{x,y\in X}(\top_{X\times X}(x,y) \to u(x,y)) = \bigwedge_{x,y\in X} u(x,y).$$

Thus by Theorem 3.2 (1), $\mathcal{W}_{\tau_X}$ is an Alexandrov L-quasiuniform filter on $X \times X$. Moreover, since $\mathcal{W}^*_{\tau_X}(\top^*_{(-,y)}) = \top_{X\times X}(-,y)$,

$$\begin{aligned}\tau_{\mathcal{W}_{\tau_X}} &= \{B \in L^X \mid \bigwedge_{y\in X}(\top_{X\times X}(-,y) \to B(y)) = B\}\\ &= \{B \in L^X \mid \bigwedge_{y\in X} B(y) = B\}\\ &= \{\alpha_X \mid \alpha \in L\}\\ &= \tau_X.\end{aligned}$$

Example 3.6. Let $e_X \in L^{X\times X}$ be a reflexive L-fuzzy relation.

(1) Define $\mathcal{W}_{e_X} : L^{X\times X} \to L$ as

$$\mathcal{W}_{e_X}(u) = \bigwedge_{x,y\in X}(e_X(x,y) \to u(x,y)).$$

By Theorem 3.2 (1), $\mathcal{W}_{e_X}$ is an Alexandrov L-preuniform filter on $X \times X$. If e_X is an L-fuzzy preorder on X, then we have

$$\begin{aligned}\bigvee_{y\in X}(\mathcal{W}^*_{e_X}(\top^*_{(x,y)}) \odot \mathcal{W}^*_{e_X}(\top^*_{(y,z)})) &= \bigvee_{y\in X}(e_X(x,y) \odot e_X(y,z))\\ &= e_X(x,z)\\ &= \mathcal{W}^*_{e_X}(\top^*_{(x,z)}).\end{aligned}$$

Thus $\mathcal{W}_{e_X}$ is an Alexandrov L-quasiuniform filter on $X \times X$.

(2) Define $\mathcal{N}^x_{e_X} : L^X \to L$ as

$$\mathcal{N}^x_{e_X}(A) = \bigwedge_{y\in X}(e_X(x,y) \to A(y)).$$

By Theorem 3.2 (2), $\mathcal{N}_{e_X} = \{\mathcal{N}^x_{e_X} \mid x \in X\}$ is an Alexandrov L-neighborhood system on X. If e_X is an L-fuzzy preorder on X, then $\mathcal{N}_{e_X}$ is topological.

(3) From (2) and Theorem 3.2 (3),

$$\mathcal{W}_{\mathcal{N}_{e_X}}(u) = \bigwedge_{x\in X}\mathcal{N}^x_{e_X}(u(x,-)) = \bigwedge_{x,y\in X}(e_X(x,y) \to u(x,y)) = \mathcal{W}_{e_X}(u).$$

Then $\mathcal{W}_{\mathcal{N}_{e_X}}$ is an Alexandrov preuniform L-filter on $X \times X$.

(4) Since $\mathcal{W}_{e_X}(\top^*_{(x,y)}) = e^*_X(x,y)$,

$$\mathcal{N}^x_{\mathcal{W}_{e_X}}(A) = \bigwedge_{y\in X}(\mathcal{W}^*_{e_X}(\top^*_{(x,y)}) \to A(y)) = \bigwedge_{y\in X}(e_X(x,y) \to A(y)) = \mathcal{N}^x_{e_X}(A).$$

(5) By (3) and (4), $\mathcal{W}_{\mathcal{N}_{\mathcal{W}_{e_X}}} = \mathcal{W}_{\mathcal{N}_{e_X}} = \mathcal{W}_{e_X}$. By (3) and Theorem 3.2 (5),

$$\mathcal{N}^x_{\mathcal{W}_{\mathcal{N}_{e_X}}} = \mathcal{N}^x_{\mathcal{W}_{e_X}} = \mathcal{N}^x_{e_X}.$$

(6) By Theorem 3.2, let $\mathcal{W}: L^{X\times X} \to L$ be an Alexandrov preuniform L-filter on $X\times X$ with a reflexive relation $e_{\mathcal{W}} \in L^{X\times X}$ such that $e_{\mathcal{W}}(x,z) = \mathcal{W}^*(\top^*_{(x,y)})$. Then

$$\mathcal{W}(u) = \bigwedge_{x,z\in X}(e_{\mathcal{W}}(x,z) \to u(x,z)) = \mathcal{W}_{e_{\mathcal{W}}}(u).$$

Since $\mathcal{W}_{e_X}(u) = \bigwedge_{x,z\in X}(e_X(x,z)\to u(x,z))$ and $\mathcal{W}_{e_X}(\top^*_{(x,y)}) = e^*_X(x,z)$,

$$e_{\mathcal{W}_{e_X}}(x,y) = \mathcal{W}^*_{e_X}(\top^*_{(x,y)}) = e_X(x,y).$$

(7) By Theorem 3.2, since $\mathcal{W}_{e_X}(\top^*_{(-,y)}) = e^*_X(-,y)$,

$$\begin{aligned}\tau_{\mathcal{W}_{e_X}} &= \{A\in L^X \mid A = \textstyle\bigwedge_{y\in X}(\mathcal{W}^*_{e_X}(\top^*_{(-,y)}) \to A(y))\\ &= \textstyle\bigwedge_{y\in X}(e_X(-,y)\to A(y))\},\\ \mathcal{W}_{\tau_{\mathcal{W}_{e_X}}}(u) &= \textstyle\bigwedge_{x,y\in X}(\bigwedge_{A\in\tau_{\mathcal{W}_{e_X}}}(A(x)\to A(y)) \to u(x,y))\\ &= \textstyle\bigwedge_{x,y\in X}(\bigwedge_{z\in X}(\mathcal{W}^*_{e_X}(\top^*_{(y,z)}) \to \mathcal{W}^*_{e_X}(\top^*_{(x,z)})) \to u(x,y))\\ &= \textstyle\bigwedge_{x,y\in X}(\bigwedge_{z\in X}(e_X(y,z)\to e_X(x,z)) \to u(x,y))\\ &\geq \textstyle\bigwedge_{x,y\in X}((e_X(y,y)\to e_X(x,y)) \to u(x,y))\\ &= \mathcal{W}_{e_X}(u).\end{aligned}$$

If e_X is an L-fuzzy preorder on X, then $\mathcal{W}_{\tau_{\mathcal{W}_{e_X}}} = \mathcal{W}_{e_X}$.

Example 3.7. Let $X = \{h_i \mid i = \{1,...,3\}\}$ with h_i=house and $Y = \{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings. Let $([0,1],\odot,\to,^*,0,1)$ be a complete residuated lattice (See [6, 8, 27]) as

$$x\odot y = \max\{0, x+y-1\},\ x\to y = \min\{1-x+y, 1\},\ \ x^* = 1-x.$$

Let $R\in[0,1]^{X\times Y}$ be a fuzzy information as follows:

R	e	b	w	c	i
h_1	0.7	0.6	0.5	0.9	0.2
h_2	0.6	0.8	0.4	0.3	0.5
h_3	0.4	0.9	0.8	0.6	0.6

Define an L-fuzzy preorder $e^{\{e,b\}}_X, e^Y_X \in [0,1]^{X\times X}$ by

$$e^{\{e,b\}}_X(h_i,h_j) = \bigwedge_{y\in\{e,b\}}(R(h_i,y)\to R(h_j,y)),$$

$$e^Y_X(h_i,h_j) = \bigwedge_{y\in Y}(R(h_i,y)\to R(h_j,y)).$$

Then we have

$$e^{\{e,b\}}_X = \begin{pmatrix} 1 & 0.9 & 0.7\\ 0.8 & 1 & 0.8\\ 0.7 & 0.9 & 1\end{pmatrix} e^Y_X = \begin{pmatrix} 1 & 0.4 & 0.7\\ 0.7 & 1 & 0.8\\ 0.6 & 0.6 & 1\end{pmatrix}$$

(1) We obtain Alexandrov L-quasiuniform filters $\mathcal{W}_{e^{\{e,b\}}_X}, \mathcal{W}_{e^Y_X} : L^{X\times X}\to L$ as

$$\begin{aligned}\mathcal{W}_{e^{\{e,b\}}_X}(u) &= \textstyle\bigwedge_{i,j\in\{1,2,3\}}(e^{\{e,b\}}_X(h_i,h_j)\to u(h_i,h_j)),\\ \mathcal{W}_{e^Y_X}(u) &= \textstyle\bigwedge_{i,j\in\{1,2,3\}}(e^Y_X(h_i,h_j)\to u(h_i,h_j)).\end{aligned}$$

(2) By Theorem 3.2, since $\mathcal{W}_{e_X}(\top^*_{(-,y)}) = e^*_X(-,y)$ for each $e_X\in\{e^{\{e,b\}}_X, e^Y_X\}$,

$$\begin{aligned}\tau_{\mathcal{W}_{e_X}} &= \{\bigwedge_{j\in\{1,2,3\}}(\mathcal{W}^*_{e_X}(\top^*_{(-,h_j)}) \to A(h_j)) \mid A \in L^X\}\\ &= \bigwedge_{j\in\{1,2,3\}}(e_X(-,h_j) \to A(h_j)) \mid A\in L^X\},\end{aligned}$$

$$\bigwedge_{j\in\{1,2,3\}}(e_X^{\{e,b\}}(-,h_j) \to A(h_j)) = \left(\begin{array}{c} A(h_1)\wedge(0.1+A(h_2))\wedge(0.3+A(h_3))\\ (0.2+A(h_1))\wedge A(h_2)\wedge(0.2+A(h_3))\\ (0.3+A(h_1))\wedge(0.1+A(h_2))\wedge A(h_3)\end{array}\right)$$

$$\bigwedge_{j\in\{1,2,3\}}(e_X^{Y}(-,h_j) \to A(h_j)) = \left(\begin{array}{c} A(h_1)\wedge(0.6+A(h_2))\wedge(0.3+A(h_3))\\ (0.3+A(h_1))\wedge A(h_2)\wedge(0.2+A(h_3))\\ (0.4+A(h_1))\wedge(0.4+A(h_2))\wedge A(h_3)\end{array}\right)$$

(3) For each $e_X \in \{e_X^{\{e,b\}}, e_X^Y\}$,

$$\begin{aligned}\bigwedge_{A\in\tau_{\mathcal{W}_{e_X}}}(A(x)\to A(y)) &= \bigwedge_{z\in X}(\mathcal{W}^*_{e_X}(\top^*_{(y,z)}) \to \mathcal{W}^*_{e_X}(\top^*_{(x,z)}))\\ &= \mathcal{W}^*_{e_X}(\top^*_{(x,y)})\\ &= \bigwedge_{z\in X}(e_X(y,z)\to e_X(x,z))\\ &= \mathcal{W}^*_{e_X}(\top^*_{(x,y)})\\ &= e_X(x,z)\end{aligned}$$

and

$$\mathcal{W}_{\tau_{\mathcal{W}_{e_X}}} = \mathcal{W}_{e_X}.$$

(4) For each $e_X \in \{e_X^{\{e,b\}}, e_X^Y\}$, since $\mathcal{N}^{h_1}_{e_X} : L^X \to L$ as

$$\mathcal{N}^{h_1}_{\mathcal{W}_{e_X}}(A) = \bigwedge_{y\in X}(e_X(h_1,h_2)\to A(h_2)),$$

$$\begin{aligned}\mathcal{N}^{h_1}_{\mathcal{W}_{e_X^{\{e,b\}}}}(A) &= \bigwedge_{y\in X}(e_X^{\{e,b\}}(h_1,h_2)\to A(h_2))\\ &= A(h_1)\wedge(0.1+A(h_2))\wedge(0.3+A(h_3)),\\ \mathcal{N}^{h_1}_{\mathcal{W}_{e_X^{Y}}}(A) &= \bigwedge_{y\in X}(e_X^{Y}(h_1,h_2)\to A(h_2))\\ &= A(h_1)\wedge(0.6+A(h_2))\wedge(0.3+A(h_3)).\end{aligned}$$

4. Conclusion

In this paper, we investigate the relations among Alexandrov L-neighborhood filters, Alexandrov L-topologies and Alexandrov L-preuniform filters as a viewpoint for fuzzy rough sets. The relations among Alexandrov L-neighborhood spaces, Alexandrov L-topological spaces and Alexandrov L-preunitorm filter spaces are studied. As a very important point of view for fuzzy information systems, Alexandrov L-neighborhood filters, Alexandrov L-topologies and Alexandrov L-preuniform filters can be fined in Example 3.7.

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