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Interior and closure operators on generalized co-residuated lattices

Ju-mok Oh, Yong Chan Kim

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Abstract. In this paper, we introduce the notions and properties of generalized co-residuated lattices as an non-commutative algebraic structure. We define right and left distance functions. In particular, we study the relations between various operators and various connections. We give their examples.

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Corresponding Author: Yong Chan Kim (yck@gwnu.ac.kr)

1. Introduction

Ward and Dilworth [1] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2, 3] investigate the properties of fuzzy Galois connections and fuzzy closure operators on a residuated lattice which supports part of foundation of theoretic computer science. On the other hand, Georgescu and Popescue [4, 5] introduced fuzzy Galois connection in a generalized residuated lattice as an non-commutative algebraic structure which is induced by two implications.

As a dual sense of complete residuated lattice, Zheng and Wang [6] introduced a complete co-residuated lattice as the generalization of t-conorm. For an extension of Pawlak's rough sets [7, 8], Junsheng and Qing [9] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \odot, \&)$, where $(L, \&)$ is a complete residuated lattice and $(L, \odot)$ is complete coresiduated lattice. Ko and Kim [10] introduced the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces in complete co-residuated lattices. Moreover, Oh and kim [11] investigated the properties of Alexandrov fuzzy topologies, distance functions, join preserving maps and join approximation maps in complete co-residuated lattices.

The aim of this paper is to study the notions and properties of generalized co-residuated lattices as an non-commutative algebraic structure. Using right (resp. left) distance spaces instead of fuzzy partially ordered spaces, we define various connections, right (resp. left) isotone (antitone) maps and rough sets on a generalized co-residuated lattice.

We investigate the properties of right and left closure on a generalized co-residuated lattice. In particular, we obtain right (left) closure (interior) operators and rough sets from various connections. We give their examples.

2. Preliminaries

As an extension of Zheng's co-residuated lattices [6], we define generalized co-residuated lattices as an non-commutative algebraic structure.

Definition 2.1. A structure $(L, \vee, \wedge, \oplus, \ominus, \oslash, \perp, \top)$ is called a *generalized co-residuated lattice*, if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \perp, \top)$ is lattice, where $\top$ is the upper bound and $\top$ denotes the universal lower bound,

(GR2) $x \oplus \top = x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y, z \in L$,

(GR3) it satisfies a co-residuation, i.e.,

$$a \oplus b \geq c \text{ iff } a \geq c \ominus b \text{ iff } b \geq c \oslash a.$$

A generalized co-residuated lattice is called *co-residuated lattice*, if $x \oplus y = y \oplus x$ for each $x, y \in L$.

For $\alpha \in L, A \in L^X$, we denote $(A \ominus \alpha), (\alpha \oplus A), \alpha_X \in L^X$ as

$$(A \ominus \alpha)(x) = A(x) \ominus \alpha,\ (\alpha \oplus A)(x) = \alpha \oplus A(x),\quad \alpha_X(x) = \alpha.$$

Put $n_1(x) = \top \ominus x$ and $n_2(x) = \top \oslash x$. The condition $n_1(n_2(x)) = n_2(n_1(x)) = x$ for each $x \in L$ is called a *double negative law.*

Example 2.2 ([10, 11])**.** (1) If a generalized co-residuated lattice $(L, \vee, \wedge, \oplus, \ominus, \oslash, \perp, \top)$ is a co-residuated lattice, then $\ominus = \oslash$ and $n_1 = n_2$.

(2) An infinitely distributive lattice $(L, \vee, \wedge, \oplus = \vee, \perp, \top)$ is a co-residuated lattice. In particular, the unit interval $([0,1], \vee, \wedge, \oplus = \vee, 0, 1)$ is a co-residuated lattice, where

$$x \ominus y = \bigwedge\{z \in L \mid y \vee z \geq x\} = \begin{cases} 0, & \text{if } y \geq x, \\ x, & \text{if } y \not\geq x. \end{cases}$$

Put $n(x) = 1 \ominus x = 1$ for $x \neq 1$ and $n(1) = 0$. Then $n(n(x)) = 0$ for $x \neq 1$ and $n(n(1)) = 1$. Thus n does not satisfy a double negative law.

(3) Let $([0,1], \vee, \wedge, \oplus, 0, 1)$ be a co-residuated lattice, where

$$\begin{aligned} x \oplus y &= (x^p + y^p)^{\frac{1}{p}} \wedge 1,\ \ 1 \leq p < \infty, \\ x \ominus y &= \bigwedge\{z \in [0,1] \mid (z^p + y^p)^{\frac{1}{p}} \geq x\} \\ &= \bigwedge\{z \in [0,1] \mid z \geq (x^p - y^p)^{\frac{1}{p}}\} \\ &= (x^p - y^p)^{\frac{1}{p}} \vee 0. \end{aligned}$$

Put $n(x) = 1 \ominus x = (1 - x^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$. Then $n(n(x)) = x$ for $x \in [0,1]$. Thus n satisfies a double negative law.

(4) Let $L \subset \{(x,y)\in R^2 \mid x>0\}$ be a set and for $(x_1,y_1),(x_2,y_2)\in L$, we define

$$(x_1,y_1)\le (x_2,y_2)\text{if and only if } \ x_1<x_2 \ \text{ or } \ x_1=x_2, y_1\le y_2.$$

Then the structure $(L,\vee,\wedge,\oplus,\ominus,\oslash,(\frac{1}{2},1),(1,0))$ is a generalized co-residuated lattice with a double negative law where $\bot=(\frac{1}{2},1)$ is the least element and $\top=(1,0)$ is the greatest element from the following statements:

$$\begin{array}{ll} (x_1,y_1)\oplus(x_2,y_2) & =(2x_1x_2, 2x_2y_1+y_2-2x_2)\wedge(1,0),\\ (x_1,y_1)\ominus(x_2,y_2) & =(\frac{x_1}{2x_2}, 1+\frac{y_1-y_2}{2x_2})\vee(\frac{1}{2},1),\\ (x_1,y_1)\oslash(x_2,y_2) & =(\frac{x_1}{2x_2}, y_1+\frac{x_1}{x_2}(1-y_2))\vee(\frac{1}{2},1). \end{array}$$

Furthermore, we have $(x,y)=n_2(n_1(x,y))=n_1(n_2(x,y))$ from:

$$\begin{array}{l} n_1(x,y)=(1,0)\ominus(x,y)=(\frac{1}{2x},1-\frac{y}{2x}),\\ n_2(x,y)=(1,0)\oslash(x,y)=(\frac{1}{2x},\frac{1}{x}(1-y)),\\ n_2(n_1(x,y))=(1,0)\oslash(\frac{1}{2x},1-\frac{y}{2x})=(x,y),\\ n_1(n_2(x,y))=(1,0)\ominus(\frac{1}{2x},\frac{1}{x}(1-y))=(x,y). \end{array}$$

Let $A=\{(\frac{2}{3},y)\mid y\in R\}$ be given. Then $\bigvee A$ and $\bigwedge A$ do not exist. Thus L is not complete.

In this paper, we assume $(L,\vee,\wedge,\oplus,\ominus,\oslash,\bot,\top)$ is a generalized co-residuated lattice with a double negative law and and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

3. Residuated and Galois connections

In this section, we study notions of residuated and Galois connections on generalized co-residuated lattices. Moreover, we investigate the relations between various connections and operators.

Lemma 3.1. *For each $x,y,z,x_i,y_i\in L$, we have the following properties.*

(1) *If $y\le z$, $(x\oplus y)\le(x\oplus z)$, $x\ominus y\le x\ominus z$ and $z\ominus x\le y\ominus x$ for $\ominus\in\{\ominus,\oslash\}$.*

(2) *$y\oplus(x\oslash y)\ge x$ and $(x\ominus y)\oplus y\ge x$.*

(3) *$x\ominus(\bigwedge_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\ominus y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\ominus y=\bigvee_{i\in\Gamma}(x_i\ominus y)$ for $\ominus\in\{\ominus,\oslash\}$.*

(4) *$x\ominus(\bigvee_{i\in\Gamma}y_i)\le\bigwedge_{i\in\Gamma}(x\ominus y_i)$ and $(\bigvee_{i\in\Gamma}x_i)\ominus y=\bigwedge_{i\in\Gamma}(x_i\ominus y)$ for $\ominus\in\{\ominus,\oslash\}$.*

(5) *$x\oplus(y\ominus z)\ge(x\oplus y)\ominus z$ and $(x\oslash y)\oplus z=(x\oplus z)\oslash y$.*

(6) *$x\ominus(y\oplus z)=(x\ominus z)\ominus y$ and $(x\oslash y)\oslash z=x\oslash(y\oplus z)$.*

(7) *$(x\ominus y)\oslash z=(x\oslash z)\ominus y$.*

(8) *$(y\oslash z)\oplus(x\oslash y)\ge x\oslash z$ and $(x\ominus y)\oplus(y\ominus z)\ge x\ominus z$.*

(9) *$(x\oslash z)\ge(y\oplus x)\oslash(y\oplus z)$ and $(x\ominus z)\ge(x\oplus y)\ominus(z\oplus y)$.*

(10) *$y\oslash z\ge(x\oslash z)\ominus(x\oslash y)$ and $x\oslash y\ge(x\oslash z)\oslash(y\oslash z)$.*

(11) *$x\ominus y\ge(x\ominus z)\ominus(y\ominus z)$ and $y\ominus z\ge(x\ominus z)\oslash(x\ominus y)$.*

(12) *$x\ominus x=x\oslash x=\bot$, $x\ominus\bot=x\oslash\bot=x$ and $\bot\ominus x=\bot\oslash x=\bot$.*

(13) *$x\ominus y=\bot$ iff $x\le y$ iff $x\oslash y=\bot$.*

(14) *$x\oplus y=\bot$ iff $x=\bot$ and $y=\bot$.*

(15) *$x\ominus y=n_1(y)\oslash n_1(x)$ and $x\oslash y=n_2(y)\ominus n_2(x)$.*

(16) *$n_1(y\oplus z)=n_1(z)\ominus y$ and $n_2(y\oplus z)=n_2(y)\oslash z$. Moreover, $n_2(x\ominus y)=y\oplus n_2(x)$ and $n_1(x\oslash y)=n_1(x)\oplus y$.*

(17) *For each $k=1,2$, $n_k(\bigwedge_{i\in\Gamma}x_i)=\bigvee_{i\in\Gamma}n_k(x_i)$ and $n_k(\bigvee_{i\in\Gamma}x_i)=n_k(\bigwedge_{i\in\Gamma}x_i)$.*

Proof. (1) Since $y = y \wedge z$, $x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$. Then $(x \oplus y) \leq (x \oplus z)$. Since $y \leq z \leq (z \ominus x) \oplus x$, $y \ominus x \leq z \ominus x$. Since $x \leq (x \ominus y) \oplus y \leq (x \ominus y) \oplus z$, $x \ominus z \leq x \ominus y$. The cases of $\oslash$ are similarly proved.

(2) Since $x \oslash y \geq x \oslash y$, $y \oplus (x \oslash y) \geq x$. Since $x \ominus y \geq x \ominus y$, $(x \ominus y) \oplus y \geq x$.

(3) By (1), $(\bigvee_{i\in\Gamma} x_i) \ominus y \geq \bigvee_{i\in\Gamma}(x_i \ominus y)$. Since $(\bigvee_{i\in\Gamma}(x_i \ominus y)) \oplus y \geq \bigvee_{i\in\Gamma}((x_i \ominus y) \oplus y) \geq \bigvee_{i\in\Gamma} x_i$, $(\bigvee_{i\in\Gamma} x_i) \ominus y \leq \bigvee_{i\in\Gamma}(x_i \ominus y)$.

By (1), $x \ominus (\bigwedge_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \ominus y_i)$. Since $\bigvee_{i\in\Gamma}(x \ominus y_i) \oplus (\bigwedge_{i\in\Gamma} y_i) \geq \bigwedge_{i\in\Gamma}((x \ominus y_i) \oplus y_i) \geq x$, $\bigvee_{i\in\Gamma}(x \ominus y_i) \geq x \ominus (\bigwedge_{i\in\Gamma} y_i)$.

(4) It follows from (1).

(5) Since $x \oplus ((y \ominus z) \oplus z) \geq x \oplus y$, $x \oplus (y \ominus z) \geq (x \oplus y) \ominus z$.

Since $y \oplus ((x \oslash y) \oplus z) = (y \oplus (x \oslash y)) \oplus z \geq x \oplus z$, $(x \oslash y) \oplus z \geq (x \oplus z) \oslash y$.

(6) Since $(x \ominus (y \oplus z)) \oplus (y \oplus z) \geq x$ iff $(x \ominus (y \oplus z)) \oplus y \geq x \ominus z$, $x \ominus (y \oplus z) \geq (x \ominus z) \ominus y$. Since $((x \ominus z) \ominus y) \oplus y \oplus z \geq (x \ominus z) \oplus z \geq x$, $(x \ominus z) \ominus y \geq x \ominus (y \oplus z)$. Then $x \ominus (y \oplus z) = (x \ominus z) \ominus y$.

Since $y \oplus z \oplus (x \oslash (y \oplus z)) \geq x$ iff $z \oplus (x \oslash (y \oplus z)) \geq x \oslash y$, $x \oslash (y \oplus z) \geq (x \oslash y) \oslash z$. Since $y \oplus z \oplus ((x \oslash y) \oslash z) \geq y \oplus (x \oslash y) \geq x$, $(x \oslash y) \oslash z \geq x \oslash (y \oplus z)$. Then $(x \oslash y) \oslash z = x \oslash (y \oplus z)$

(7) Since $(z \oplus ((x \ominus y) \oslash z)) \oplus y \geq (x \ominus y) \oplus y \geq x$, $((x \ominus y) \oslash z) \oplus y \geq x \oslash z$. Then $(x \ominus y) \oslash z \geq (x \oslash z) \ominus y$. Since $z \oplus (((x \oslash z) \ominus y)) \oplus y) \geq z \oplus (x \oslash z) \geq x$, $z \oplus ((x \oslash z) \ominus y) \geq x \ominus y$. Thus $(x \oslash z) \ominus y \geq (x \ominus y) \oslash z$.

(8) Since $x \ominus y \geq x \ominus y$, $y \oplus (x \ominus y) \geq x$. Moreover, $y \geq x \ominus (x \ominus y)$. Since $(x \ominus y) \oplus (y \ominus z) \oplus z \geq (x \ominus y) \oplus y \geq x$, $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.

(9) Since $(z \oplus (y \oslash z)) \oplus (x \oslash y) \geq y \oplus (x \oslash y) \geq x$, $(y \oslash z) \oplus (x \oslash y) \geq x \oslash z$.

(10) Since $(y \oplus z) \oplus (x \oslash z) = y \oplus (z \oplus (x \oslash z)) \geq y \oplus x$, $(x \oslash z) \geq (y \oplus x) \oslash (y \oplus z)$.

Since $(x \ominus z) \oplus (z \oplus y) = ((x \ominus z) \oplus z) \oplus y) \geq x \oplus y$, $(x \ominus z) \geq (x \oplus y) \ominus (z \oplus y)$.

(11) Since $(y \oplus z) \oplus (x \ominus y) \geq x \oplus z$, $x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$. Since $x \oplus (y \ominus x) \oplus (z \ominus y) \geq z$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$. Since $x \oplus y \leq z \oplus (x \ominus z) \oplus w \oplus (y \ominus w)$, $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.

(12) Let $x \oplus \bot = \bot \oplus x = x$. Then $x \ominus x = x \oslash x = \bot$. $x \ominus \bot = \bigwedge\{z \in L \mid z \oplus \bot \geq x\} = x$, and $x \oslash \bot = \bigwedge\{z \in L \mid \bot \oplus z \leq x\} = x$.

(13) Let $x \ominus y = \bot$. Then $y = \bot \oplus y = (x \ominus y) \oplus y = \bigwedge\{z \in L \mid z \oplus y \geq x\} \oplus y = \bigwedge\{z \oplus y \in L \mid z \oplus y \geq x\} \geq x$. Thus $x \leq y$.

Let $x \leq y$. Then $x \ominus y = \bigwedge\{z \in L \mid z \oplus y \geq x\} = \bot$. Other cases are similarly proved.

(14) Let $x \oplus y = \bot$. Then $y = y \ominus (x \oplus y) = (y \ominus y) \ominus x = \bot \ominus x = \bot$ and $x = x \oslash (x \oplus y) = (x \oslash x) \oslash y = \bot \oslash y = \bot$. Conversely, $\bot \oplus \bot = \bot$.

(15) By (11), $x \ominus y \geq (\top \ominus y) \oslash (\top \ominus x) = n_1(y) \oslash n_1(x)$. By (10),

$$x \oslash y \geq (\top \oslash y) \ominus (\top \oslash x) = n_2(y) \ominus n_2(x).$$

Moreover, we have

$$x \ominus y = n_2(n_1(x)) \ominus n_2(n_1(y)) \leq n_1(y) \oslash n_1(x)$$

and

$$x \oslash y = n_1(n_2(x)) \oslash n_1(n_2(y)) \leq n_2(y) \ominus n_2(x).$$

Thus $x \ominus y = n_1(y) \oslash n_1(x)$ and $x \oslash y = n_2(y) \ominus n_2(x)$.

(16) By (6), $n_1(y \oplus z) = \top \ominus (y \oplus z) = (\top \ominus z) \ominus y = n_1(z) \ominus y$ and $n_2(y \oplus z) = \top \oslash (y \oplus z) = (\top \oslash y) \oslash z = n_2(y) \oslash z$.

(17) By (3), $n_k(\bigwedge_i x_i) = \bigvee_i n_k(x_i)$ for each $k = 1, 2$. Since $\bigwedge_i x_i = n_2(n_1(\bigwedge_i x_i)) = n_2(\bigvee_i n_1(x_i))$, $\bigwedge_i n_2(x_i) = n_2(\bigvee_i n_1(n_2(x_i))) = n_2(\bigvee_i x_i)$. Other cases are similarly proved. □

Definition 3.2. Let X be a set. A function $d^r_X : X \times X \to L$ is called a *right distance function*, if it satisfies the following conditions:
(D1) $d^r_X(x, x) = \bot$ for all $x \in X$,
(D2) If $d^r_X(x, y) = d^r_X(y, x) = \top$, then $x = y$,
(R) $d^r_X(x, y) \oplus d^r_X(y, z) \geq d^r_X(x, z)$, for all $x, y, z \in X$.
A function $d^l_X : X \times X \to L$ is called a *left distance function*, if it satisfies (D1), (D2) and
(L) $d^l_X(y, z) \oplus d^l_X(x, y) \geq d^l_X(x, z)$, for all $x, y, z \in X$.
The triple (X, d^r_X, d^l_X) is a bi-distance space.

Example 3.3. (1) We define a function $d^r_L, d^l_L : L \times L \to L$ as

$$d^r_L(x, y) = x \ominus y, \ d^l_L(x, y) = x \oslash y.$$

By Lemma 3.1 (8), (L, d^r_L, d^l_L) is a bi-distance space.

(2) We define a function $d^r_{L^X}, d^l_{L^X} : L^X \times L^X \to L$ as

$$d^r_{L^X}(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x)), \ \ d^l_{L^X}(A, B) = \bigvee_{x \in X}(A(x) \oslash B(x)).$$

By Lemma 3.1 (8), $(L^X, d^r_{L^X}, d^l_{L^X})$ is a bi-distance space.

Definition 3.4. Let X and Y be two sets. Let $F, H : L^X \to L^Y$ and $G, K : L^Y \to L^X$ be operators.

(1) The pair (F, G) is called a *residuated connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $F(A) \leq B$ iff $A \leq G(B)$.

(2) The pair (H, K) is called a *Galois connection* between X and Y, if for $A \in L^X$ and $B \in L^Y$, $B \leq H(A)$ iff $A \leq K(B)$.

(3) The pair (H, K) is called a *dual Galois connection* between X and Y, if for $A \in L^X$ and $B \in L^Y$, $H(A) \leq B$ iff $K(B) \leq A$.

(4) A map $F : L^X \to L^Y$ is a *right isotone map*, if for all $A, \ B \in L^X$, $d^r_{L^X}(A, B) \geq d^r_{L^Y}(F(A), F(B))$.

(5) A map $F : L^X \to L^Y$ is a *left isotone map*, if for all $A, \ B \in L^X$, $d^l_{L^X}(A, B) \geq d^l_{L^Y}(F(A), F(B))$.

(6) A map $F : L^X \to L^Y$ is a *right antitone map*, if for all $A, \ B \in L^X$, $d^l_{L^X}(A, B) \geq d^r_{L^Y}(F(B), F(A))$.

(7) A map $F : L^X \to L^Y$ is a *left antitone map*, if for all $A, \ B \in L^X$, $d^r_{L^X}(A, B) \geq d^l_{L^Y}(F(B), F(A))$.

Theorem 3.5. *Let $G : L^X \to L^Y$ be a map.*

(1) *A map $G : L^X \to L^Y$ is a right isotone map iff $\alpha \oplus G(A) \geq G(\alpha \oplus A)$ and $G(A) \leq G(B)$ for $A \leq B$ iff $G(A \oslash \alpha) \geq G(A) \oslash \alpha$ and $G(A) \leq G(B)$ for $A \leq B$.*

(2) *A map $G : L^X \to L^Y$ is a left isotone map iff $G(A) \oplus \alpha \geq G(A \oplus \alpha)$ and $G(A) \leq G(B)$ for $A \leq B$ iff $G(A \ominus \alpha) \geq G(A) \ominus \alpha$ and $G(A) \leq G(B)$ for $A \leq B$.*

(3) *A map* $G : L^X \to L^Y$ *is a left antitone map iff* $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ *and* $G(B) \leq G(A)$ *for* $A \leq B$ *iff* $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ *and* $G(B) \leq G(A)$ *for* $A \leq B$.

(4) *A map* $G : L^X \to L^Y$ *is a right antitone map iff* $G(A \oplus \alpha) \geq G(A) \oslash \alpha$ *and* $G(B) \leq G(A)$ *for* $A \leq B$ *iff* $\alpha \oplus G(A) \geq G(A \ominus \alpha)$ *and* $G(B) \leq G(A)$ *for* $A \leq B$.

(5) *If* $G : L^X \to L^Y$ *is a left isotone map, then* $n_2 G : L^X \to L^Y$ *is a right antitone map.*

(6) *If* $G : L^X \to L^Y$ *is a right isotone map, then* $n_1 G : L^X \to L^Y$ *is a left antitone map.*

(7) *If* $G : L^X \to L^Y$ *is a right antitone map, then* $n_1 G : L^X \to L^Y$ *is a left isotone map.*

(8) *If* $G : L^X \to L^Y$ *is a left antitone map, then* $n_2 G : L^X \to L^Y$ *is a right isotone map.*

Proof. (1) Let $d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B))$. Put $A = \alpha \oplus B$. Then

$$\alpha \geq d^r_{L^X}(\alpha \oplus B, B) \geq d^r_{L^Y}(G(\alpha \oplus B), G(B)).$$

Thus $\alpha \oplus G(B) \geq G(\alpha \oplus B)$.

Conversely, put $\alpha = d^r_{L^X}(A, B)$. Since $d^r_{L^X}(A, B) \oplus B \geq A$,

$$d^r_{L^X}(A, B) \oplus G(B) \geq G(d^r_{L^X}(A, B) \oplus B) \geq G(A).$$

So $d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B))$.

Second, let $\alpha \oplus G(A) \geq G(\alpha \oplus A)$ and $G(A) \leq G(B)$ for $A \leq B$. Since $\alpha \oplus G(A \oslash \alpha) \geq G(\alpha \oplus (A \oslash \alpha)) \geq G(A)$, $G(A \oslash \alpha) \geq G(A) \oslash \alpha$.

Conversely, let $G(A \oslash \alpha) \geq A \oslash \alpha$ and $G(A) \leq G(B)$ for $A \leq B$. Since $G((\alpha \oplus A) \oslash \alpha) \geq G(\alpha \oplus A) \oslash \alpha$ iff $\alpha \oplus G((\alpha \oplus A) \oslash \alpha) \geq G(\alpha \oplus A)$, we have

$$G(\alpha \oplus A) \leq \alpha \oplus G((\alpha \oplus A) \oslash \alpha) \leq \alpha \oplus G(A).$$

(3) Let $G : L^X \to L^Y$ be a left antitone map. Then $d^r_{L^X}(A, B) \geq d^l_{L^Y}(G(B), G(A))$. Put $A = \alpha \oplus B$. Then $\alpha \geq d^r_{L^X}(\alpha \oplus B, B) \geq d^l_{L^Y}(G(B), G(\alpha \oplus B))$. Thus $G(\alpha \oplus B) \geq G(B) \ominus \alpha$.

Conversely, since $G(d^r_{L^X}(A, B) \oplus B) \geq G(B) \ominus d^r_{L^X}(A, B)$ and $G(d^r_{L^X}(A, B) \oplus B) \leq G(A)$ for $d^r_{L^X}(A, B) \oplus B \geq A$, we have

$$d^r_{L^X}(A, B) \geq G(B) \oslash G(d^r_{L^X}(A, B) \oplus B) \geq G(B) \oslash G(A).$$

Second, we show that $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ and $G(B) \leq G(A)$ for $A \leq B$ iff $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$.

Let $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(\alpha \oplus A) \oplus \alpha \geq G(A)$. Thus we get

$$G(A) \oplus \alpha \geq G(\alpha \oplus (A \oslash \alpha)) \oplus \alpha \geq G(A \oslash \alpha).$$

Let $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(A) \geq G(A \oslash \alpha) \ominus \alpha$. Put $A = \alpha \oplus B$. Then $G(\alpha \oplus B) \geq G((\alpha \oplus B) \oslash \alpha) \ominus \alpha \geq G(B) \ominus \alpha$.

(5) Let $G : L^X \to L^Y$ be a left isotone map. Then by Lemma 3.1 (15), we have

$$d^l_{L^X}(A, B) \leq d^l_{L^Y}(G(A), G(B)) = d^r_{L^Y}(n_2 G(B), n_2 G(A)).$$

Thus $n_2 G : L^X \to L^Y$ is a right antitone map.

(6) Let $G : L^X \to L^Y$ be a right isotone map. Then by Lemma 3.1 (15), we get

$$d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B)) = e^l_{L^Y}(n_1 G(B), n_1 G(A)).$$

Thus $n_1G : L^X \to L^Y$ is a left antitone map.

(2), (4), (7) and (8) are proved similar methods as (1), (3), (5) and (6) respectively. □

Definition 3.6. A map $C : L^X \to L^X$ is called a *right* (resp. *left*) *closure operator, if it satisfies the following conditions:*

(C1) $A \le C(A)$, *for all* $A \in L^X$.

(C2) $C(C(A)) = C(A)$, *for all* $A \in L^X$.

(C3) C *is a right (resp. left) isotone map.*

A map $I : L^X \to L^X$ *is called a right (resp. left) interior operator*, if it satisfies the following conditions:

(I1) $I(A) \le A$ for all $A \in L^X$,

(I2) $I(I(A)) = I(A)$ for all $A \in L^X$

(I3) I is a right (resp. left) isotone map.

The pair $(I(A), C(A))$ ic called a *rough set* of A.

Theorem 3.7. (1) *Let* $C : L^X \to L^X$ *be a right closure operator. Define a map* $I : L^X \to L^X$ *as* $I(A) = n_1(C(n_2(A)))$. *Then* I *is a left interior operator where* $(I(A), C(A))$ *is a rough set of* A.

(2) *Let* $C : L^X \to L^X$ *be a left closure operator. Define a map* $I : L^X \to L^X$ *as* $I(A) = n_2(C(n_1(A)))$. *Then* I *is a right interior operator where* $(I(A), C(A))$ *is a rough set of* A.

Proof. (1) (I1) Since $n_2(A) \le C(n_2(A))$, $I(A) = n_1(C(n_2(A))) \le n_1(n_2(A)) = A$.

(I2) b$I(I(A)) = n_1(C(n_2(n_1(C(n_2(A)))))) = n_1(C(C(n_2(A)))) = n_1(n_2(A)) = A$.

(I3) I is a left isotone map from:

$$\begin{aligned}&d^l_{L^X}(A,B) = d^r_{L^X}(n_2(B), n_2(A)) \ge d^r_{L^X}(C(n_2(B)), C(n_2(A)))\\&= d^l_{L^X}(n_1(C(n_2(A))), n_1(C(n_2(B)))) = d^l_{L^X}(I(A), I(B)).\end{aligned}$$

(2) It is proved by a similar method as (1). □

Theorem 3.8. *Let* $G : L^X \to L^Y$ *and* $H : L^Y \to L^X$ *be two maps.*

(1) *A pair* (G, H) *is a residuated connection with two right isotone maps* G *and* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^r_{L^Y}(G(A), B) = d^r_{L^X}(A, H(B))$.

(2) *A pair* (G, H) *is a residuated connection with two left isotone maps* G *and* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^l_{L^Y}(G(A), B) = d^l_{L^X}(A, H(B))$.

(3) *A pair* (G, H) *is a Galois connection with right antitone map* G *and left antitone map* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^l_{L^X}(A, H(B)) = d^r_{L^Y}(B, G(A))$.

(4) *A pair* (G, H) *is a Galois connection with left antitone map* G *and right antitone map* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^r_{L^X}(A, H(B)) = d^l_{L^Y}(B, G(A))$.

(5) *A pair* (G, H) *is a dual Galois connection with right antitone map* G *and left antitone map* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^l_{L^X}(H(B), A) = d^r_{L^Y}(G(A), B)$.

(6) *A pair* (G, H) *is a dual Galois connection with left antitone map* G *and right antitone map* H *iff for all* $A \in L^X$ *and* $B \in L^Y$, $d^r_{L^X}(H(B), A) = d^l_{L^Y}(G(A), B)$.

Proof. (1) Let (G, H) be a residuated connection. Then $G(A) \le G(A)$ iff $A \le H(G(A))$ and $H(B) \le H(B)$ iff $G(H(B)) \le B$. Thus $d^r_{L^Y}(G(A), B) = d^r_{L^X}(A, H(B))$

from

$$d^r_{L^Y}(G(A),B) \geq d^r_{L^X}(A,H(G(A))) \oplus d^r_{L^X}((H(G(A)),H(B)) \geq d^r_{L^X}(A,H(B)),$$
$$d^r_{L^X}(A,H(B)) \geq d^r_{L^Y}(G(A),G(H(B))) \oplus d^r_{L^Y}(G(H(B)),B) \geq e^r_{L^Y}(G(A),B).$$

Conversely, since $d^r_{L^Y}(G(A),B) = d^r_{L^X}(A,H(B))$ for all $A \in L^X$ and $B \in L^Y$, $d^r_{L^X}(A,H(B)) = \perp$ $A \leq H(B)$ iff $G(A) \leq B$ iff $d^r_{L^Y}(G(A),B) = \perp$. Then (G,H) is a residuated connection. Put $B = G(A)$. Then $\perp = d^r_{L^Y}(G(A),G(A)) = d^r_{L^X}(A,H(G(A)))$. Thus $A \leq H(G(A))$. Put $A = H(B)$. Then we similarly obtain $G(H(B)) \leq B$. Thus we obtain two right isotone maps G and H from:

$$\begin{aligned}
&d^r_{L^X}(A,B) = d^r_{L^X}(A,B) \oplus d^r_{L^X}(B,H(G(B)))\\
&\geq d^r_{L^X}(A,H(G(B))) = d^r_{L^Y}(G(A),G(B)),\\
&d^r_{L^Y}(A,B) = d^r_{L^Y}(G(H(A)),A) \oplus d^r_{L^Y}(A,B)\\
&\geq d^r_{L^Y}(G(H(A),B) = d^r_{L^X}(H(A),H(B)).
\end{aligned}$$

(3) Let (G,H) be a Galois connection. Then $G(A) \leq G(A)$ iff $A \leq H(G(A))$ and $H(B) \leq H(B)$ iff $B \leq G(H(B))$. Moreover, since G is a right antitone map and H is a left antitone map, we have

$$d^r_{L^Y}(B,G(A)) \geq d^l_{L^X}(H(G(A)),H(B)) \geq d^l_{L^X}(A,H(B)),$$
$$d^l_{L^X}(A,H(B)) \geq d^r_{L^Y}(G(H(B)),G(A)) \geq d^r_{L^Y}(B,G(A)).$$

Thus $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$.

Conversely, since $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$, $A \leq H(B)$ iff $B \leq G(A)$. Moreover,

$$d^r_{L^Y}(G(A),G(B)) = d^l_{L^X}(B,H(G(A))) \leq d^l_{L^X}(B,A),$$
$$d^l_{L^X}(H(A),H(B)) = d^r_{L^Y}(B,G(H(A))) \leq d^r_{L^Y}(B,A).$$

(5) Let (G,H) be a dual Galois connection. Then $G(A) \leq G(A)$ iff $H(G(A)) \leq A$ and $H(B) \leq H(B)$ iff $G(H(B)) \leq B$. Moreover, since G is a right antitone map and H is a left antitone map, we have

$$d^r_{L^Y}(G(A),B) \geq d^l_{L^X}(H(B),H(G(A))) \geq d^l_{L^X}(H(B),A),$$
$$d^l_{L^X}(H(B),A) \geq d^r_{L^Y}(G(A),G(H(B))) \geq d^r_{L^Y}(G(A),B).$$

Thus $d^l_{L^X}(H(B),A) = d^r_{L^Y}(G(A),B)$.

Conversely, since $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$, $H(B) \leq A$ iff $G(A) \leq B$. Moreover,

$$d^r_{L^Y}(G(A),G(B)) = d^l_{L^X}(H(G(B)),A) \leq d^l_{L^X}(B,A),$$
$$d^l_{L^X}(H(A),H(B)) = d^r_{L^Y}(G(H(B)),A) \leq d^r_{L^Y}(B,A).$$

(2), (4) and (6) are similarly proved as (1), (3) and (5) respectively. □

Theorem 3.9. *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be right isotone maps with a residuated connection (G,H). Then the following statements hold:*

(1) *$H \circ G : L^X \to L^X$ is a right closure operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a right interior operator,*

(3) *if $X = Y$, then $(G(H(A)),H(G(A)))$ is a rough set of A.*

Proof. (1) Since $A \le H(G(A))$, $H(G(A)) \le H(G(H(G(A))))$ for all $A \in L^X$. Since $B \ge G(H(B))$, $G(A) \ge G(H(G(A)))$ and $H(G(A)) \ge H(G(H(G(A))))$. Then $H(G(A)) = H(G(H(G(A))))$. Since G and H are right isotone maps, $d^r_{L^X}(A,B) \ge d^r_{L^X}(H(G(A)), H(G(B)))$.

(2) and (3) are similarly proved as (1) and the definition of rough set. □

Corollary 3.10. *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be left isotone maps with a residuated connection (G,H). Then the following statements hold:*

(1) *$H \circ G : L^X \to L^X$ is a left closure operator,*
(2) *$G \circ H : L^Y \to L^Y$ is a left interior operator,*
(3) *If $X = Y$, then $(G(H(A)), H(G(A)))$ is a rough set of A.*

Theorem 3.11. *Let $G : L^X \to L^Y$ be a right antitone map and $H : L^Y \to L^X$ be a left antitone map with a Galois connection (G,H). Then*

(1) *$H \circ G : L^X \to L^X$ is a left closure operator,*
(2) *$G \circ H : L^Y \to L^Y$ is a right closure operator.*

Proof. (1) Since $A \le H(G(A))$, $H(G(A)) \le H(G(H(G(A))))$ for all $A \in L^X$. Since $B \le G(H(B))$ then $G(A) \le G(H(G(A)))$ and $H(G(A)) \ge H(G(H(G(A))))$, because H is a left antitone map. Then $H(G(A)) = H(G(H(G(A))))$.

Since G is a right antitone map and H is a left antitone map, $d^l_{L^X}(A,B) \ge d^r_{L^X}(G(B), G(A)) \ge d^l_{L^X}(H(G(A)), H(G(B)))$.

(2) is similarly proved as (1). □

Corollary 3.12. *Let $G : L^X \to L^Y$ be a left antitone map and $H : L^Y \to L^X$ be a right antitone map with a Galois connection (G,H). Then*

(1) *$H \circ G : L^X \to L^X$ is a right closure operator,*
(2) *$G \circ H : L^Y \to L^Y$ is a left closure operator.*

Theorem 3.13. *Let $G : L^X \to L^Y$ be a right antitone map and $H : L^Y \to L^X$ be a left antitone map with a dual Galois connection (G,H). Then*

(1) *$H \circ G : L^X \to L^X$ is a left interior operator,*
(2) *$G \circ H : L^Y \to L^Y$ is a right interior operator.*

Proof. (1) Since $H(G(A)) \le A$, $H(G(H(G(A)))) \le H(G(A))$ for all $A \in L^X$. Since $G(H(B)) \le B$, $G(H(G(A))) \le G(A)$ and $H(G(A)) \le H(G(H(G(A))))$, because H is a left antitone map. Then $H(G(A)) = H(G(H(G(A))))$.

Since G is a right antitone map and H is a left antitone map, $d^l_{L^X}(A,B) \ge d^r_{L^X}(G(B), G(A)) \ge d^l_{L^X}(H(G(A)), H(G(B)))$.

(2) is similarly proved as (1). □

Corollary 3.14. *Let $G : L^X \to L^Y$ be a left antitone map and $H : L^Y \to L^X$ be a right antitone map with a Galois connection (G,H). Then*

(1) *$H \circ G : L^X \to L^X$ is a right interior operator,*
(2) *$G \circ H : L^Y \to L^Y$ is a left interior operator.*

Theorem 3.15. *Let $F, G : L^X \to L^X$ be maps such that $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$. Then the following statements are equivalent.*

(1) *F is a right interior operator.*
(2) *G is a right closure operator.*

(3) $F \circ G = F$.
(4) $G \circ F = G$.

Proof. Since $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$, by Theorem 3.8 (1), F and G are right isotone maps.

(1) $\Rightarrow$(2). Since $\perp = d^r_{L^X}(F(A), A) = d^r_{L^X}(A, G(A))$, $A \le G(A)$. Then we have

$$\begin{aligned} d^r_{L^X}(G(G(A)), G(A)) &= d^r_{L^X}(F(G(G(A))), A) \\ &= d^r_{L^X}(F(F(G(G(A)))), A) \\ &= d^r_{L^X}(F(G(G(A))), G(A)) \\ &= d^r_{L^X}(G(G(A)), G(G(A))) \\ &= \perp. \end{aligned}$$

Thus G is a right closure operator.

(2) $\Rightarrow$(3). Since F is a right isotone map, $\perp = d^r_{L^X}(A, G(A)) \ge d^r_{L^X}(F(A), F(G(A)))$. Then $F(A) \le F(G(A))$. Moreover, $F(A) = F(G(A))$ from:

$$\begin{aligned} d^r_{L^X}(F(G(A)), F(A)) &= d^r_{L^X}(G(A), G(F(A))) \\ &= d^r_{L^X}(G(A), G(G(F(A)))) \\ &\le d^r_{L^X}(A, G(F(A))) \\ &= \perp. \text{ [Since } G \text{ is a right isotone map]} \end{aligned}$$

(3) $\Rightarrow$(4). Let $F \circ G = F$. Then $G \circ F \circ G = G \circ F$. Since

$$\perp = d^r_{L^X}(F(G(A)), F(G(A))) = d^r_{L^X}(G(A), G(F(G(A))))$$

and

$$\perp = d^r_{L^X}(G(A), G(A)) = d^r_{L^X}(F(G(A)), A),$$

$G \circ F \circ G \ge G$ and $F \circ G(A) \le A$. Thus $G \circ F \circ G(A) \le G(A)$. So $G \circ F = G \circ F \circ G = G$.

(4) $\Rightarrow$(3). It follows from $F \circ G \circ F = F$.

(3) and (4) $\Rightarrow$(1).

$$\begin{aligned} &d^r_{L^X}(F(A), A) \le d^r_{L^X}(F(A), F(G(A))) \oplus d^r_{L^X}(F(G(A)), A) = \perp, \\ &d^r_{L^X}(F(A), F(F(A))) = d^r_{L^X}(A, G(F(F(A)))) = d^r_{L^X}(A, G(F(A))) = \perp. \end{aligned}$$

□

Corollary 3.16. *Let $F, G : L^X \to L^X$ be maps such that $d^l_{L^X}(F(A), B) = d^l_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a left interior operator.*
(2) *G is a left closure operator.*
(3) $F \circ G = F$.
(4) $G \circ F = G$.

Corollary 3.17. *Let $F, G : L^X \to L^X$ be maps such that $d^l_{L^X}(F(A), B) = d^l_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a left closure operator.*
(2) *G is a left interior operator.*
(3) $G \circ F = F$.
(4) $F \circ G = G$.

Corollary 3.18. *Let $F, G : L^X \to L^X$ be maps such that $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a right closure operator.*
(2) *G is a right interior operator.*

(3) $G \circ F = F$.
(4) $F \circ G = G$.

4. Examples of interior and closure operators

In this section, we investigate interior, closure operators and rough sets for an information $(X, Y, R \in L^{X\times Y})$, where X is a set of objects and Y is a set of attributes.

Definition 4.1. For each $A \in L^X$ and $B \in L^Y$ and $R \in L^{X\times Y}$, $R^{\oplus}, {}^{\oplus}R, R^{\ominus}, R^{\oslash}, {}^{\ominus}R, \oslash R : L^X \to L^Y$ are defined as:

$$R^{\oplus}(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x)),\ {}^{\oplus}R(A)(y) = \bigwedge_{x\in X}(A(x) \oplus R(x,y)),$$
$$R^{\ominus}(A)(y) = \bigvee_{x\in X}(R(x,y) \ominus A(x)),\ R^{\oslash}(A)(y) = \bigvee_{x\in X}(R(x,y) \oslash A(x)),$$
$${}^{\ominus}R(A)(y) = \bigvee_{x\in X}(A(x) \ominus R(x,y)),\ {}^{\oslash}R(A)(y) = \bigvee_{x\in X}(A(x) \oslash R(x,y)),$$
$$R^{1\oplus}(A)(y) = \bigwedge_{x\in X}(n_1R(x,y) \oplus n_1A(x)),\ {}^{2\oplus}R(A)(y) = \bigwedge_{x\in X}(n_2A(x) \oplus n_2R(x,y)).$$

Theorem 4.2. (1) *$R^{\oplus}$ and ${}^{\oslash}R^{-1}$ are left isotone maps with a residuated connection $({}^{\oslash}R^{-1}, R^{\oplus})$.*
(2) *$R^{\oplus} \circ {}^{\oslash} R^{-1} : L^Y \to L^Y$ is a left closure operator.*
(3) *${}^{\oslash}R^{-1} \circ R^{\oplus} : L^X \to L^X$ is a left interior operator.*
(4) *$({}^{\oslash}R^{-1} \circ R^{\oplus}(A), R^{-1\oplus} \circ {}^{\oslash} R(A))$ is a rough set of $A \in L^X$.*
(5) *If $X = Y$, then $({}^{\oslash}R^{-1} \circ R^{\oplus}(A), R^{\oplus} \circ {}^{\oslash} R^{-1}(A))$ is a rough set of $A \in L^X$.*

Proof. (1) Since $R(x,y) \oplus B(x) \oplus (A(x) \oslash B(x)) \geq R(x,y) \oplus A(x)$, $d^l_{L^X}(A,B) \geq d^l_{L^Y}(R^{\oplus}(A), R^{\oplus}(B))$. Since $(B(y) \oslash R(x,y)) \oplus (A(y) \oslash B(y)) \leq A(y) \oslash R(x,y)$, $d^l_{L^Y}(A,B) \geq d^l_{L^X}({}^{\oslash}R^{-1}(A), {}^{\oslash} R^{-1}(B))$. Then by Theorem 3.8 (2), we only show the following statement:

$$\begin{aligned} d^l_{L^Y}(B, R^{\oplus}(A)) &= \bigvee_{y\in Y}(B(y) \oslash R^{\oplus}(A)(y)) \\ &= \bigvee_{y\in Y}\Big(B(y) \oslash \bigwedge_{x\in X}(R(x,y) \oplus A(x))\Big) \\ &= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((B(y) \oslash R(x,y)) \oslash A(x)\Big) \\ &= \bigvee_{x\in X}\Big(\bigvee_{y\in Y}(B(y) \oslash R(x,y)) \oslash A(x)\Big) \\ &= \bigvee_{x\in X}\Big({}^{\oslash}R^{-1}(B)(x) \oslash A(x)\Big) \\ &= d^l_{L^X}({}^{\oslash}R^{-1}(B), A). \end{aligned}$$

(2), (3) and (4), (5) follow from Corollary 3.10 and the definition of a rough set. □

Theorem 4.3. (1) *${}^{\oplus}R$ and ${}^{\ominus}R^{-1}$ are right isotone maps with a residuated connection $({}^{\ominus}R^{-1}, {}^{\oplus} R)$.*
(2) *${}^{\oplus}R \circ {}^{\ominus} R^{-1} : L^Y \to L^Y$ is a right closure operator.*
(3) *${}^{\ominus}R^{-1} \circ {}^{\oplus} R : L^X \to L^X$ is a right interior operator.*
(4) *$({}^{\ominus}R^{-1} \circ {}^{\oplus} R(A), {}^{-1\oplus} R \circ {}^{\ominus} R(A))$ is a rough set of A.*
(5) *If $X = Y$, then $({}^{\ominus}R^{-1} \circ {}^{\oplus} R(A), {}^{\oplus} R \circ {}^{\ominus} R^{-1}(A))$ is a rough set of A.*

Proof. (1) Since $(A(x) \ominus B(x)) \oplus B(x) \oplus R(x,y) \geq A(x) \oplus R(x,y)$, $d^r_{L^X}(A,B) \geq d^r_{L^Y}({}^{\oplus}R(A), {}^{\oplus} R(B))$. Since $(A(y) \ominus B(y)) \oplus (B(x) \ominus R(x,y)) \geq A(x) \ominus R(x,y)$, $d^r_{L^Y}(A,B) \geq d^r_{L^X}({}^{\ominus}R^{-1}(A), {}^{\ominus} R^{-1}(B))$. Then by Theorem 3.8 (1), we only show the following statement:

$$\begin{aligned} d^r_{L^Y}(B,^{\oplus} R(A)) &= \bigvee_{y\in Y}(B(y)\ominus^{\oplus} R(A)(y)) \\ &= \bigvee_{y\in Y}\Big(B(y)\ominus \bigwedge_{x\in X}(A(x)\oplus R(x,y))\Big) \\ &= \bigvee_{y\in Y}\bigvee_{x\in X}\Big(B(y)\ominus (A(x)\oplus (R(x,y))\Big) \\ &= \bigvee_{x\in X}\Big(\bigvee_{y\in Y}(B(y)\ominus R(x,y))\ominus A(x)\Big) \\ &= d^r_{L^X}(^{\ominus}R^{-1}(B),A). \end{aligned}$$

(2), (3) and (4), (5) follow from Theorem 3.9 and the definition of a rough set. □

Theorem 4.4. *Let (X,d^l_X) be a left distance function. Let $^{\oslash}(d^l_X)^{-1},(d^l_X)^{\oplus},:L^X\to L^X$ be maps in above theorem with $R^{-1}=(d^l_X)^{-1}$. Then the following statements hold.*

(1) *$^{\oslash}(d^l_X)^{-1}$ is a left closure operator.*

(2) *$(d^l_X)^{\oplus}$ is a left interior operator with a rough set $((d^l_X)^{\oplus}(A),^{\oslash}(d^l_X)^{-1}(A))$ for each $A\in L^X$.*

(3) *$(d^l_X)^{\oplus}=^{\oslash}(d^l_X)^{-1}\circ(d^l_X)^{\oplus}$.*

(4) *$(d^l_X)^{\oplus}\circ^{\oslash}(d^l_X)^{-1}=^{\oslash}(d^l_X)^{-1}$.*

(5) *Define $I:L^X\to L^X$ as $I(A)=n_2(^{\oslash}(d^l_X)^{-1}(n_1(A)))$. Then I is a right interior operator such that $I(A)=\bigwedge_{z\in X}(A(z)\oplus n_2(n_2(d^l_X(y,z))))$.*

(6) *Define $C:L^X\to L^X$ as $C(A)=n_2((d^l_X)^{\oplus}(n_1(A)))$. Then C is a right closure operator such that $C(A)(y)=\bigvee_{x\in X}(A(x)\ominus n_2(n_2(d^l_X(x,y))))$.*

Proof. (1) Since $(B(y)\oslash d^l_X(x,y))\oplus(A(y)\oslash B(y))\geq A(y)\oslash d^l_X(x,y)$,

$$d^l_{L^X}(A,B)\geq d^l_{L^X}(^{\oslash}(d^l_X)^{-1}(A),^{\oslash}(d^l_X)^{-1}(B)).$$

Since d^l_X is a left distance function, $\bigvee_{y\in X}((d^l_X(y,z)\oplus d^l_X(x,y))=d^l_X(x,z)$. Thus

$$^{\oslash}(d^l_X)^{-1}(A)(y)=\bigvee_{z\in X}(B(z)\oslash d^l_X(y,z))\geq B(y)\oslash d^l_X(y,y)=A(y)$$

and

$$\begin{aligned} ^{\oslash}(d^l_X)^{-1}(^{\oslash}(d^l_X)^{-1}(B))(x) &= \bigvee_{y\in X}(^{\oslash}(d^l_X)^{-1}(B)(y)\oslash d^l_X(x,y)) \\ &= \bigvee_{y\in X}\Big(\bigvee_{z\in X}(B(z)\oslash d^l_X(y,z))\oslash d^l_X(x,y)\Big) \\ &= \bigvee_{z\in X}\Big(B(z)\oslash \bigwedge_{y\in X}(d^l_X(y,z)\oplus d^l_X(x,y))\Big) \\ &= \bigvee_{z\in X}(B(z)\oslash d^l_X(x,z)) \\ &=^{\oslash}(d^l_X)^{-1}(B)(x). \end{aligned}$$

Thus $^{\oslash}(d^l_X)^{-1}$ is a left closure operator.

(2) Since $d^l_X(x,y)\oplus B(x)\oplus(A(x)\oslash B(x))\geq d^l_X(x,y)\oplus A(y)$,

$$d^l_{L^X}(A,B)\geq d^l_{L^X}((d^l_X)^{\oplus}(A),(d^l_X)^{\oplus}(B)).$$

Since d^l_X is a left isotone map, $\bigwedge_{y\in X}((d^l_X(y,z)\oplus d^l_X(x,y))=d^l_X(x,z)$. Then

$$(d^l_X)^{\oplus}(A)\leq A$$

and

$$(d^l_X)^{\oplus}((d^l_X)^{\oplus}(A))(y)=\bigwedge_{z\in X}(d^l_X(z,y)\oplus(d^l_X)^{\oplus}(A)(z))$$

$$= \bigwedge_{z\in X}((d^l_X(z,y)\oplus \bigwedge_{x\in X}(d^l_X(x,z)\oplus A(x)))$$
$$= \bigwedge_{z\in X}(\bigwedge_{x\in X}((d^l_X(z,y)\oplus d^l_X(x,z))\oplus A(x)))$$
$$= \bigwedge_{x\in X}(\bigwedge_{z\in X}((d^l_X(z,y)\oplus d^l_X(x,z))\oplus A(x)))$$
$$= \bigwedge_{x\in X}((d^l_X(x,y)\oplus A(x))$$
$$= (d^l_X)^{\oplus}(A)(y).$$

(3) Since $(d^l_X)^{\oplus} \leq^{\oslash} (d^l_X)^{-1}\circ (d^l_X)^{\oplus}$, by (1), we only show

$$(d^l_X)^{\oplus} \geq^{\oslash} (d^l_X)^{-1}\circ (d^l_X)^{\oplus}$$

from:

$$\begin{aligned}\perp &= d^r_{L^X}((d^l_X)^{\oplus}(A),(d^l_X)^{\oplus}(A)) = d^r_{L^X}((d^l_X)^{\oplus}(A),(d^l_X)^{\oplus}((d^l_X)^{\oplus}(A)))\\ &= d^r_{L^X}((^{\oslash}(d^l_X)^{-1}((d^l_X)^{\oplus}(A)),(d^l_X)^{\oplus}(A)).\end{aligned}$$

(2) and (4) are similarly proved as (1) and (3).

(5) By Theorem 3.8 (2), I is a right interior operator. Moreover,

$$\begin{aligned}&I(A)(x) = n_2(^{\oslash}(d^l_X)^{-1}(n_1(A))(x) = n_2(\bigvee_{y\in X}(n_1(A)(y)\oslash d^l_X(x,y)))\\ &= \bigwedge_{y\in X} n_2((n_1(A)(y)\oslash d^l_X(x,y))) = \bigwedge_{y\in X} n_2(n_2(d^l_X(x,y))\ominus A(y))\\ &= \bigwedge_{y\in X}(A(y)\oplus n_2(n_2(d^l_X(x,y)))).\ \text{[By Lemma 3.1 (15), 16)]}\end{aligned}$$

(6) It is similarly proved as (5). □

Corollary 4.5. *Let (X, d^r_X) be a right distance function. Let $^{\ominus}(d^r_X)^{-1}, ^{\oplus}(d^r_X),: L^X \to L^X$ be maps in above theorem with $R^{-1} = (d^r_X)^{-1}$. Then the following statements hold.*

(1) *$^{\ominus}(d^r_X)^{-1}$ is a right closure operator.*

(2) *$^{\oplus}(d^r_X)$ is a right interior operator with a rough set $(^{\oplus}(d^r_X)(A), ^{\ominus}(d^r_X)^{-1}(A))$ for each $A \in L^X$.*

(3) *$^{\oplus}(d^r_X) = ^{\ominus}(d^r_X)^{-1} \circ ^{\oplus}(d^r_X)$.*

(4) *$^{\oplus}(d^r_X) \circ ^{\ominus}(d^r_X)^{-1} = ^{\ominus}(d^r_X)^{-1}$.*

Theorem 4.6. (1) *$R^{\oslash}$ is a right antitone map and $R^{-1\ominus}$ is a left antitone map.*

(2) *$R^{\ominus}$ is a left antitone map and $R^{-1\oslash}$ is a right antitone map.*

(3) *The pair $(R^{\oslash}, R^{-1\ominus})$ is a dual Galois connection.*

(4) *The pair $(R^{\ominus}, R^{-1\oslash})$ is a dual Galois connection.*

(5) *$R^{-1\ominus}\circ R^{\oslash} : L^X \to L^X$ is a left interior operator and $R^{\oslash}\circ R^{-1\ominus} : L^Y \to L^Y$ is a right interior operator.*

(6) *$R^{-1\oslash}\circ R^{\ominus} : L^X \to L^X$ is a right interior operator and $R^{\ominus}\circ R^{-1\oslash} : L^Y \to L^Y$ is a left interior operator.*

Proof. (1) Since $(A(x)\oslash B(x))\oplus (R(x,y)\oslash A(x)) \geq R(x,y)\oslash B(x)$,

$$d^l_{L^X}(A,B) \geq d^r_{L^Y}(R^{\oslash}(B), R^{\oslash}(A)).$$

Since $(R(x,y)\ominus A(y))\oplus (A(y)\ominus B(y)) \geq R(x,y)\ominus B(y)$,

$$d^r_{L^X}(A,B) \geq d^l_{L^Y}(R^{-1\ominus}(B), R^{-1\ominus}(A)).$$

(2) It is similarly proved as (1).

(3) From Theorem 3.8 (1), $d^l_{L^X}(R^{-1\ominus}(B),A)=d^r_{L^Y}(R^{\oslash}(A),B)$ from:

$$\begin{aligned}
&d^l_{L^X}(R^{-1\ominus}(B),A)\\
&=\bigvee_{x\in X}(R^{-1\ominus}(B)(x)\oslash A(x))\\
&=\bigvee_{x\in X}(\bigvee_{y\in X}(R(x,y)\ominus B(y))\oslash A(x))\\
&=\bigvee_{x\in X}\bigvee_{y\in X}((R(x,y)\oslash A(x))\ominus B(y))\\
&=\bigvee_{y\in X}(R^{\oslash}(A)\ominus B(y))\\
&=d^r_{L^Y}(R^{\oslash}(A),B).
\end{aligned}$$

(2) and (4) are similarly proved as (1) and (3) respectively.

(5) and (6) are proved from Theorem 3.11 and Corollary 3.12 respectively. □

Theorem 4.7. (1) *$R^{1\oplus}$ is a right antitone map and ${}^{2\oplus}R^{-1}$ is a left antitone map.*

(2) *The pair $(R^{1\oplus},{}^{2\oplus}R^{-1})$ is a Galois connection.*

(3) *${}^{2\oplus}R^{-1}\circ R^{1\oplus}:L^X\to L^X$ is a left closure operator and $R^{1\oplus}\circ{}^{2\oplus}R^{-1}:L^Y\to L^Y$ is a right closure operator.*

Proof. (1) Since $(n_1R(x,y)\oplus n_1A(x))\oplus(n_1B(x)\oslash n_1A(x))\geq R(x,y)\oplus n_1B(x)$, $(A(x)\ominus B(x))=(n_1B(x)\oslash n_1A(x))\geq(n_1R(x,y)\oplus n_1B(x))\oslash(n_1R(x,y)\oplus n_1A(x))$. Then we have

$$d^r_{L^X}(A,B)\geq d^l_{L^Y}(R^{1\oplus}(B),R^{1\oplus}(B)).$$

Since $(n_2B(y)\ominus n_2A(y))\oplus(n_2A(y)\oplus n_2R^{-1}(x,y)\geq n_2B(y)\oplus n_2R^{-1}(x,y)$,

$$(A(y)\oslash B(y))=(n_2B(y)\ominus n_2A(y))\geq(n_2B(y)\oplus n_2R^{-1}(x,y))\ominus(n_2A(y)\oplus n_2R^{-1}(x,y)).$$

Thus we get

$$d^l_{L^Y}(A,B)\geq d^r_{L^X}({}^{2\oplus}R^{-1}(B),{}^{2\oplus}R^{-1}(A)).$$

(2) From Theorem 3.8 (4), $d^l_{L^X}(R^{-1\ominus}(B),A)=d^r_{L^Y}(R^{\oslash}(A),B)$ from:

$$\begin{aligned}
&d^l_{L^Y}(B,R^{1\oplus}(A))=\bigvee_{y\in Y}(B(y)\oslash R^{1\oplus}(A)(y))\\
&=\bigvee_{x\in X}(B(y)\oslash\bigwedge_{y\in X}(n_1R(x,y)\oplus n_1(A)(y)))\\
&=\bigvee_{x\in X}\bigvee_{y\in X}((B(y)\oslash n_1R(x,y))\oslash n_1(A)(y))\\
&=\bigvee_{x\in X}\bigvee_{y\in X}(A(y)\ominus n_2((B(y)\oslash n_1R(x,y))))\\
&=\bigvee_{x\in X}\bigvee_{y\in X}(A(y)\ominus n_2(R(x,y)\ominus n_2B(y)))\\
&=\bigvee_{x\in X}\bigvee_{y\in X}(A(y)\ominus(n_2(B)(y)\oplus n_2R(x,y)))\\
&=d^r_{L^Y}(A,{}^{2\oplus}R^{-1}(B)).
\end{aligned}$$

(3) It follows from Theorem 3.11. □

Example 4.8. Let $(L,\oplus,\oplus,\ominus,\oslash,(\frac{1}{2},1),(1,0))$ be a generalized co-residuated lattice with a double negative law, where $\perp=(\frac{1}{2},1)$ is the least element and $\top=(1,0)$ is the greatest element in Example 2.2 (7).

Let $X=\{a,b,c\}$ be a set. Define $d_X,d^l_X:X\times X\to L$ as

$$d_X=\begin{pmatrix}(\frac{1}{2},1) & (\frac{4}{5},-1) & (\frac{3}{5},0)\\ (\frac{7}{10},-2) & (\frac{1}{2},1) & (\frac{4}{5},0)\\ (\frac{1}{2},3) & (\frac{7}{10},-\frac{4}{3}) & (\frac{1}{2},1)\end{pmatrix}$$

$$d^l_X=\begin{pmatrix}(\frac{1}{2},1) & (\frac{3}{4},\frac{1}{4}) & (\frac{3}{5},\frac{2}{5})\\ (\frac{7}{10},-\frac{11}{10}) & (\frac{1}{2},1) & (\frac{7}{12},-\frac{4}{3})\\ (\frac{3}{5},\frac{8}{5}) & (\frac{2}{3},-\frac{1}{3}) & (\frac{1}{2},1)\end{pmatrix}$$

Then we easily show that d_X is a right and left distance function and d^l_X is a left distance function. But d^l_X is not a right distance function, because

$$d^l_X(b,c)\oplus d^l_X(c,a)=(\tfrac{7}{12},-\tfrac{4}{3})\oplus(\tfrac{3}{5},\tfrac{8}{5})=(\tfrac{7}{10},-\tfrac{6}{5})\not\geq d^l_X(b,a)=(\tfrac{7}{10},-\tfrac{11}{10})\ .$$

By Theorem 4.4 and Corollary 4.5, we have various rough sets as follows, for each $A\in L^X$,

$$(d^{\oplus}_X(A),{}^{\oslash}(d_X)^{-1}(A)),({}^{\oplus}d_X(A),{}^{\ominus}d^{-1}_X(A)),((d^l_X)^{\oplus}(A),{}^{\oslash}(d^l_X)^{-1}(A)).$$

Since

$$\begin{array}{l}{}^{\oplus}(d^l_X)(A)(y)=\bigwedge_{x\in X}(A(x)\oplus d^l_X(x,y))\\ {}^{\ominus}(d^l_X)^{-1}(A)(x)=\bigvee_{y\in X}(A(y)\ominus d^l_X(x,y))\end{array}$$

for $D=((\frac{3}{4},\frac{1}{4}),(\frac{5}{6},\frac{11}{6}),(\frac{1}{2},\frac{3}{2}))$,

$$\begin{array}{l}{}^{\oplus}(d^l_X)(D)=((\frac{3}{5},\frac{8}{5}),(\frac{2}{3},\frac{1}{3}),(\frac{1}{2},\frac{3}{2}))\\ {}^{\ominus}(d^l_X)^{-1}(D)=((\frac{3}{4},\frac{1}{4}),(\frac{5}{6},\frac{11}{6}),(\frac{5}{8},\frac{21}{8}))\\ {}^{\ominus}(d^l_X)^{-1}({}^{\oplus}(d^l_X)(D))=((\frac{3}{5},\frac{8}{5}),(\frac{4}{5},-\frac{1}{2}),(\frac{1}{2},\frac{3}{2}))\\ {}^{\oplus}(d^l_X)({}^{\ominus}(d^l_X)^{-1}(D))=((\frac{3}{4},\frac{1}{4}),(\frac{5}{6},-\frac{25}{12}),(\frac{5}{8},\frac{21}{8}))\end{array}$$

Since d^l_X is not a right distance function, in general, ${}^{\oplus}(d^l_X)(D)\neq{}^{\ominus}(d^l_X)^{-1}({}^{\oplus}(d^l_X)(D))$ and ${}^{\ominus}(d^l_X)^{-1}(D)\neq{}^{\oplus}(d^l_X)({}^{\ominus}(d^l_X)^{-1}(D))$. Moreover, ${}^{\oplus}(d^l_X)$ is not a right interior operator because , for $\perp_b$ with $\perp_b(b)=\perp$ and $\perp_b(x)=\top$ for $x\neq b$,

$$\begin{array}{l}{}^{\oplus}(d^l_X)(\perp_b)(-)=\bigwedge_{x\in X}(\perp_b(x)\oplus d^l_X(x,-))=d^l_X(b,-)=((\frac{7}{10},-\frac{11}{10}),(\frac{1}{2},1),(\frac{7}{12},-\frac{4}{3}))\\ {}^{\oplus}(d^l_X)((-)^{\oplus}(d^l_X)(\perp_b))(-)=((\frac{7}{10},-\frac{6}{5}),(\frac{1}{2},1),(\frac{7}{12},-\frac{4}{3})).\end{array}$$

As a information system $(X,Y,R\in L^{X\times Y})$, let $X=\{a,b,c\}$ be a set of objects and $Y=\{u,v\}$ be a set of attributes with an information $R\in L^{X\times Y}$ as

$$R=\left(\begin{array}{cc}(1,0)&(\frac{5}{8},\frac{5}{2})\\(\frac{2}{3},-1)&(\frac{3}{4},\frac{1}{4})\\(\frac{1}{2},2)&(\frac{1}{2},1)\end{array}\right)$$

For $A=((\frac{2}{3},1),(\frac{1}{2},2),(\frac{3}{4},-1))$,

$$\begin{array}{rl}R^{\oplus}(A)=((\frac{2}{3},0),(\frac{3}{4},-1)), & {}^{\oplus}R(A)=((\frac{2}{3},-\frac{7}{3}),(\frac{3}{4},-1)),\\ R^{\ominus}(A)=((\frac{3}{4},\frac{1}{4}),(\frac{3}{4},-\frac{3}{4})), & R^{\oslash}(A)=((\frac{3}{4},0),(\frac{3}{4},-\frac{5}{4})),\\ {}^{\ominus}R(A)=((\frac{3}{4},-2),(\frac{3}{4},-1)), & {}^{\oslash}R(A)=((\frac{3}{4},-\frac{5}{2}),(\frac{3}{4},-1)),\\ R^{1\oplus}(A)=((\frac{3}{4},\frac{1}{4}),(1,0)), & {}^{2\oplus}R(A)=((\frac{3}{4},0),(1,0)).\end{array}$$

For $B=((\frac{3}{5},2),(\frac{2}{3},-1))$,

$$\begin{array}{rl}R^{-1\oplus}(B)=((\frac{5}{6},\frac{5}{6}),(\frac{4}{5},\frac{14}{5}),(\frac{3}{5},\frac{16}{5})), & {}^{\oplus}R^{-1}(B)=((\frac{5}{6},0),(\frac{4}{5},\frac{1}{3}),(\frac{3}{5},3)),\\ R^{-1\ominus}(B)=((\frac{5}{6},-\frac{2}{3}),(\frac{5}{9},-\frac{3}{2}),(\frac{1}{2},1)), & R^{-1\oslash}(B)=((\frac{5}{6},-\frac{5}{3}),(\frac{5}{9},-\frac{19}{9}),(\frac{1}{2},1)),\\ {}^{\ominus}R^{-1}(B)=((\frac{8}{15},-\frac{9}{5}),(\frac{1}{2},1),(\frac{2}{3},-1)), & {}^{\oslash}R^{-1}(B)=((\frac{8}{15},-\frac{13}{5}),(\frac{1}{2},1),(\frac{2}{3},-1)),\\ (R^{-1})^{1\oplus}(B)=((\frac{5}{6},-\frac{2}{3}),(1,0),(1,0)), & {}^{2\oplus}R^{-1}(B)=((\frac{5}{6},-\frac{5}{3}),(1,0),(1,0)).\end{array}$$

Rough sets of $A\in L^X$ are

$$\begin{array}{l}({}^{\oslash}R^{-1}\circ R^{\oplus}(A),R^{-1\oplus}\circ^{\oslash}R(A)),\\ ({}^{\ominus}R^{-1}\circ^{\oplus}R(A),{}^{-1\oplus}R\circ^{\ominus}R(A)),\\ ({}^{\oslash}R^{-1}(R^{\oplus}(A)),{}^{2\oplus}R^{-1}(R^{1\oplus}(A))),\end{array}$$

where

$$\begin{array}{l} R^{-1\oplus}({}^{\oslash}R(A)) = ((\frac{15}{16},\frac{5}{4}),(1,-\frac{11}{2}),(\frac{3}{4},-1)),\\ {}^{\oplus}R^{-1}({}^{\ominus}R(A)) = ((\frac{15}{16},0),(1,-5),(\frac{3}{4},-1)),\\ {}^{\ominus}R^{-1}({}^{\oplus}R(A)) = ((\frac{3}{5},-\frac{9}{5}),(\frac{1}{2},1),(\frac{3}{4},-1)),\\ {}^{\oslash}R^{-1}(R^{\oplus}(A)) = ((\frac{3}{5},-\frac{29}{20}),(\frac{1}{2},2),(\frac{3}{4},-1)),\\ {}^{2\oplus}R^{-1}(R^{1\oplus}(A)) = ((\frac{2}{3},1),(\frac{2}{3},\frac{9}{8}),(1,0)). \end{array}$$

5. Conclusion

In this paper, we are interested distance spaces instead of fuzzy partially ordered sets on generalized co-residuated lattices as an non-commutative algebraic structure. Using distance functions, we have investigated the relations between various closure (interior) operators and various connections. Moreover, as an application, we give various rough sets for a information system in Section 4.

In the future, we plan to investigate fuzzy concepts, information systems and decision rules by using the concepts of distance spaces in generalized co-residuated lattices.

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References

[1] M. Ward and R. P. Dilworth, Residuated lattices, Trans. Amer. Math. Soc. 45 (1939) 335–354.
[2] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York 2002.
[3] R. Bělohlávek, Fuzzy Galois connection, Math. Log. Quart. 45 (2000) 497–504.
[4] G. Georgescu and A. Popescue, Non-commutative Galois connections, Soft Computing 7 (2003) 458–467.
[5] G. Georgescu and A. Popescue, Non-dual fuzzy connections, Arch. Math. Log. 43 (2004) 1009–1039.
[6] M. C. Zheng and G. J. Wang, Coresiduated lattice with applications, Fuzzy systems and Mathematics 19 (2005) 1–6.
[7] Z. Pawlak, Rough sets, Internat. J. Comput. Inform. Sci. 11 (1982) 341–356.
[8] Z. Pawlak, Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, Dordrecht, The Netherlands 1991.
[9] Q. Junsheng and Hu. Bao Qing, On $(\odot,\&)$-fuzzy rough sets based on residuated and co-residuated lattices, Fuzzy Sets and Systems 336 (2018) 54–86.
[10] J. M. Ko and Y. C. Kim, Preserving maps and approximation operators in complete co-residuated lattices, Journal of the Korean Insitutute of Intelligent Systems 30 (5)(2020) 389–398.
[11] J. M. Oh and Y. C. Kim, Distance functions, upper approximation operators and Alexandrov fuzzy topologies, Journal of Intelligent and Fuzzy Systems 40 (6) (2021) 11915–11925.

Ju-mok Oh (jumokoh@gwnu.ac.kr)
Department of mathematics, faculty of natural science, university of Gangneung-Wonju, Gangneung, Gangwondo, postal code 25457, Korea

Yong Chan Kim (yck@gwnu.ac.kr)
Department of mathemathics, faculty of natural science, university of Gangneung-Wonju, Gangneung, Gangwondo, postal code 25457, Korea