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Correlation coefficient between intuitionistic single-valued neutrosophic sets and its applications

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Abstract. In this paper, we redefine an intuitionistic neutrosophic set and study some of its properties. Next, we newly define the correlation coefficient and the cosine similarity measure based on intuitionistic neutrosophic sets and deal with some of their properties. Finally, by using the correlation coefficient and the cosine similarity measure which we propose, we propose an algorithms solving decision-making problems and give examples.

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Keywords: Intuitionistic single-valued neutrosophic set, Information energy, Correlation, Correlation coefficient, Cosine similarity measure, Multicriteria decision-making.

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1. Introduction

In 1965, Zadeh [1] had initially introduced the notion of fuzzy sets as the generalization of classical sets in order to express the real world as it is. After then, numerous mathematicians have been trying to find a mathematical expression of uncertainties and ambiguities which can be applied not only to Mathematics but also to engineering, medicine and social sciences, etc. For examples, an interval-valued fuzzy set (Zadeh [2], 1975 and Gorzalczany [3],1987), a rough set (Pawlak [4], 1982), an intuitionistic fuzzy set (Atanassov [5], 1983), an interval-valued intuitionistic fuzzy set (Atanassov and Gargov [6], 1989), a vague set (Gau and Buchrer [7], 1993), a neutrosophic set (Smarandache [8], 1998), a bipolar fuzzy set (Zhang [9], 1998), a soft set (Molodtsov [10],1999), etc. Wang et al. [11] defined a single-valued neutrosophic set as an instance of neutrosophic set which can be used in real scientific and engineering applications. Moreover, Jun et al. [12] proposed the concept of cubic sets composed of an interval-valued fuzzy set and a fuzzy set. Smarandache et al. [13] extended a cubic set to a neutrosophic cubic set. Jun [14] defined a cubic intuionistic set as a pair of an interval-valued intuitionistic fuzzy set and an intuitionistic fuzzy set. Recently, Lee et al. [15] introduced the notion of octahedron sets composed of three components: an interval-valued fuzzy set, an intuitionistic fuzzy set and a fuzzy set, which will provide more information about ambiguity and uncertainty common in everyday life as the generalization of a cubic set [12] and a cubic interval-valued intuitionistic fuzzy set [16] (See [17, 18, 19] for research articles to which an octahedron set was applied).

Decision-making problems are very important in establishing foreign policy, national defense policy, economic policy, various election strategies, and prevention policy of the recent worldwide coronavirus, etc. So, many mathematicians have dealt with decision-making problems using the above-mentioned various kinds of fuzzy concepts or fused fuzzy concepts. Mainly, they presented algorithms for decision making in three directions: aggregation operators, similarity measures and correlation coefficients based on various fuzzy sets. We can see that many researchers [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] have dealt with decision making problems by using the correlation coefficients and cosine similarity measures. In particular, Broumi et al. [31] defined the distance between n-valued interval neutrosophic sets and applied it to medical diagnosis problems. Deli et al. [32] defined some distances between neutrosophic redefined sets and applied them to medical diagnosis problems. Moreover, Ye [33] dealt with multicriteria decision-making problems by using the correlation coefficient for single valued neutrosophic sets. Broumi and Deli [34] defined the correlation measure for neutrosophic sets and applied it to medical dignosis problems.

The aim of this paper is to define the correlation coefficient and cosine similarity measure based on intuitionistic single-valued neutrosophic sets and propose an algorithm to use multicriteria decision-making problems. In order to accomplish such research, this paper is composed of six sections: In Section 2, we recall some definitions related to intuitionistic fuzzy sets and single-valued neutrosophic sets. Also we recall the definitions of the correlation coefficient between intuitionistic fuzzy sets. In Section 3, we redefine an intuitionistic single-valued neutrosophic set and deal with some of its properties, and give some examples. In Section 4, we define the correlation coefficient and the cosine similarity measure between intuitionistic single-valued neutrosophic sets and study some of their properties. In Section 5, we propose the algorithms of the weighted correlation coefficient and the weighted cosine similarity measure between each alternative and the ideal alternative which are utilized to rank the alternatives and to determine the best one(s). They can handle not only incomplete information but also the indeterminate information and inconsistent information which exist commonly in real situations. And we give an illustrative example as the application of the proposed decision-making method. In Section 6, we propose the algorithm of pattern recognition for the correlation coefficient and cosine similarity measure between intuitionistic single-valued neutrosophic sets and give an illustrative example.

2. Preliminaries

In this section, we list some basic definitions and notations needed in the next sections. Throughout this paper, I denotes the unit closed interval $[0, 1]$ in the set of real numbers $\mathbb{R}$.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\not\in}) \in I \times I : a^{\in} + a^{\not\in} \leq 1\}$. Then each member $\bar{a}$ of $I \oplus I$ is called an *intuitionistic point* or *intuitionistic number*. In particular, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. Refer to [35] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an intuitionistic number, and the infimum and the supremum of any intuitionistic numbers.

For a set X, a mapping $A : X \to I$ is called a *fuzzy set* in X and the set of all fuzzy sets in X is denoted by I^X or $FS(X)$. Refer [1, 36] to basic operations on I^X.

Definition 2.1 ([37]). For a nonempty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X, where for each $x \in X$, $\bar{A}(x) = (A^{\in}(x), A^{\not\in}(x))$, and $A^{\in}(x)$ and $A^{\not\in}(x)$ represent the degree of membership and the degree of nonmembership of an element x to $\bar{A}$, respectively. Let $(I \oplus I)^X$ or $IFS(X)$ denote the set of all IFSs in X and for each $\bar{A} \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\not\in})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

For each $A \in IFS(X)$ and each $x \in X$,

$$\pi(x) = 1 - A^{\in}(x) - A^{\not\in}(x)$$

is called the *intuitionistic index of x in A*. It is hesitancy degree of x to X (See [37, 38, 39, 40]). It is clear that $0 \leq \pi(x) \leq 1$ for each $x \in X$.

Definition 2.2 ([11]). Let X be a nonempty set. Then a mapping $A = \langle A^T, A^I, A^F \rangle : X \to I \times I \times I$ is called a *single-valued neutrosophic set* (briefly, SVNS) in X, where A^T [resp. A^I and A^F] is called the *truth-membership* [resp. *indeterminacy-membership* and *falsity-membership*] *function* of A. We denote the set of all SVNSs in X as $SVNS(X)$.

Definition 2.3 ([11]). Let X be a nonempty set and let $A,\ B \in SVNS(X)$.
(i) We say that A *is contained in* B or A *is subset of* B, denoted by $A \subset B$, if

$$A^T(x) \leq B^T(x),\ A^I(x) \leq B^I(x),\ A^F(x) \geq B^F(x) \text{ for each } x \in X.$$

(ii) We say that A and B are *equal*, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

Definition 2.4 ([11]). Let X be a nonempty set and let $A,\ B \in SVNS(X)$.
(i) The *union* of A and B, denoted by $A \cup B$, is a single-valued neutrosophic set in X defined as: for each $x \in X$,

$$(A \cup B)^T(x) = A^T(x) \vee B^T(x),$$
$$(A \cup B)^I(x) = A^I(x) \vee B^I(x),$$
$$(A \cup B)^F(x) = A^F(x) \wedge B^F(x).$$

(ii) The *intersection* of A and B, denoted by $A \cap B$, is a single-valued neutrosophic set in X defined as: for each $x \in X$,

$$(A \cap B)^T(x) = A^T(x) \wedge B^T(x),$$

$$(A \cap B)^I(x) = A^I(x) \wedge B^I(x),$$
$$(A \cap B)^F(x) = A^F(x) \vee B^F(x).$$

Definition 2.5 ([11]). Let X be a nonempty set and let $A,\ B \in SVNS(X)$.

(i) The *complement* of A, denoted by $c(A)$, is a single-valued neutrosophic set in X defined as: for each $x \in X$,

$$c(A)^T(x) = A^F(x),\ c(A)^I(x) = 1 - A^I(x),\ c(A)^F(x) = A^T(x).$$

(ii) The *difference* of A and B, denoted by $A \setminus B$, is a single-valued neutrosophic set in X defined as: for each $x \in X$,

$$(A \setminus B)^T(x) = A^T(x) \wedge B^F(x),$$
$$(A \setminus B)^I(x) = A^I(x) \wedge (1 - B^I(x)),$$
$$(A \setminus B)^F(x) = A^F(x) \vee B^T(x).$$

It is obvious that $A \setminus B = A \cap c(B)$.

Now we recall some definitions of correlation coefficients for intuitionistic fuzzy sets.

Definition 2.6 ([41, 42]). Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe set and let $A \in IFS(X)$. Then the *average* of A, denoted by $E(A)$, is defined by:

$$E(A) = (\bar{A^\in}, \bar{A^\notin}) = \left(\frac{1}{n}\Sigma_{i=1}^n A^\in(x_i), \frac{1}{n}\Sigma_{i=1}^n A^\notin(x_i)\right).$$

Definition 2.7. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe set and let $A,\ B \in IFS(X)$. Then the *correlation coefficient* of A and B, denoted by $\rho(A, B)$, is defined as:

(i) the correlation coefficient by Gerstenkorn and Mansko (See [20])

$$\rho_{GM}(A, B) = \frac{C(A, B)}{\sqrt{T(A)T(B)}},$$

where $C(A, B) = \Sigma_{i=1}^n [A^\in(x_i)B^\in(x_i) + A^\notin(x_i)B^\notin(x_i)]$,
$T(A) = \Sigma_{i=1}^n [(A^\in(x_i))^2 + (A^\notin(x_i))^2]$,
$T(B) = \Sigma_{i=1}^n [(B^\in(x_i))^2 + (B^\notin(x_i))^2]$,

(ii) the correlation coefficient by Hung (See [43])

$$\rho_H(A, B) = \frac{1}{2}(\rho_1 + \rho_2),$$

where $\rho_1 = \frac{\Sigma_{i=1}^n (A^\in(x_i) - \bar{A^\in})(B^\in(x_i) - \bar{B^\in})}{\sqrt{\Sigma_{i=1}^n (A^\in(x_i) - \bar{A^\in})^2}\sqrt{\Sigma_{i=1}^n (B^\in(x_i) - \bar{B^\in})^2}}$,
$\rho_2 = \frac{\Sigma_{i=1}^n (A^\notin(x_i) - \bar{A^\notin})(B^\notin(x_i) - \bar{B^\notin})}{\sqrt{\Sigma_{i=1}^n (A^\notin(x_i) - \bar{A^\notin})^2}\sqrt{\Sigma_{i=1}^n (B^\notin(x_i) - \bar{B^\notin})^2}}$,

(iii) the correlation coefficient by Park et al. (See [44])

$$\rho_P(A, B) = \frac{1}{2}(\rho_1 + \rho_2 + \rho_3),$$

where $\rho_3 = \frac{\Sigma_{i=1}^n (\pi_A(x_i) - \bar{\pi_A})(\pi_B(x_i) - \bar{\pi_B})}{\sqrt{\Sigma_{i=1}^n (\pi_A(x_i) - \bar{\pi_A})^2}\sqrt{\Sigma_{i=1}^n (\pi_B(x_i) - \bar{\pi_B})^2}}$,

(iv) the correlation coefficient by Xu (See [45])

$$\rho_X(A,B)=\frac{1}{2n}\Sigma_{i=1}^{n}\left[\frac{\triangle A^{\in}_{min}+\triangle A^{\in}_{max}}{\triangle A^{\in}_{i}+\triangle A^{\in}_{max}}+\frac{\triangle A^{\notin}_{min}+\triangle A^{\notin}_{max}}{\triangle A^{\notin}_{i}+\triangle A^{\notin}_{max}}\right],$$

where, $\triangle A^{\in}_i=\mid A^{\in}(x_i)-B^{\in}(x_i)\mid$, $\triangle A^{\notin}_i=\mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid$,
$\bigtriangledown A^{\in}_{max}=maxA^{\in}_i$, $\bigtriangledown A^{\notin}_{max}=maxA^{\notin}_i$,
$\bigtriangledown A^{\in}_{min}=minA^{\in}_i$, $\bigtriangledown A^{\notin}_{min}=minA^{\notin}_i$.

(v) the correlation coefficient by Liu et al. (See [42])

$$\rho_L(A,B)=\frac{C(A,B)}{\sqrt{D(A)D(B)}},$$

where $C(A,B)=\frac{1}{n-1}\Sigma_{i=1}^{n}d_i(A)d_i(B)$,
$D(A)=\frac{1}{n-1}\Sigma_{i=1}^{n}d_i^2(A)$, $D(B)=\frac{1}{n-1}\Sigma_{i=1}^{n}d_i^2(B)$,
$d_i(A)=(A^{\in}(x_i)-\bar{A^{\in}})-(A^{\notin}(x_i)-\bar{A^{\notin}})$,
$d_i(B)=(B^{\in}(x_i)-\bar{B^{\in}})-(B^{\notin}(x_i)-\bar{B^{\notin}})$ for all $i=1,2,\cdots,n$.

Definition 2.8 ([27])**.** Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe set and let $A,\ B\in IFS(X)$. For all $i=1,2,\cdots,n$, let us denote $d_i(A)$ and $d_i(B)$ as follows:

$d_i(A)=(A^{\in}(x_i)-\bar{A^{\in}})-(A^{\notin}(x_i)-\bar{A^{\notin}}),\ d_i(B)=(B^{\in}(x_i)-\bar{B^{\in}})-(B^{\notin}(x_i)-\bar{B^{\notin}})$.

(i) The *variance* of A, denoted by $D(A)$, is defined by:

$$D(A)=\frac{1}{n-1}\Sigma_{i=1}^{n}[(A^{\in}(x_i)-\bar{A^{\in}})^2+(A^{\notin}(x_i)-\bar{A^{\notin}})^2+d_i^2(A)].$$

(ii) The *covariance* of A and B, denoted by $Cov(A,B)$, is defined by:

$$Cov(A,B)=\frac{1}{n-1}\Sigma_{i=1}^{n}[(A^{\in}(x_i)-\bar{A^{\in}})(B^{\in}(x_i)-\bar{B^{\in}})+(A^{\notin}(x_i)-\bar{A^{\notin}})(B^{\notin}(x_i)-\bar{B^{\notin}})+d_i(A)d_i(B)].$$

(iii) The *correlation coefficient* of A and B, denoted by $\rho(A,B)$, is defined by:

$$\rho(A,B)=\frac{Cov(A,B)}{\sqrt{D(A)D(B)}}.$$

3. Intuitionistic single-valued neutrosophic sets

In this section, first of all, we recall the concept of an intuitionistic neutrosophic set proposed by Bhowmik and Pal [46].

Definition 3.1. Let X be a nonempty set. Then a mapping $A=\left\langle A^T,A^I,A^F\right\rangle:X\rightarrow I\times I\times I$ is called an *intuitionistic neutrosophic set* (briefly, INS) in X, if it satisfies the following conditions: for each $x\in X$,

$$A^T(x)\wedge A^F(x)\leq 0.5,\ A^T(x)\wedge A^I(x)\leq 0.5,\ A^F(x)\wedge A^I(x)\leq 0.5.$$

It is obvious that $0\leq A^T(x)+A^I(x)+A^F(x)\leq 1$ for each $x\in X$.

For operations for intuitionistic neutrosophic sets and their properties, see [46].

Now contrary to the definition introduced by Bhowmik and Pal, we define an intuitionistic single-valued neutrosophic set as follows.

Definition 3.2. Let X be a nonempty set. Then a mapping $A = \left\langle A^T, A^I, A^F \right\rangle :$ $X \to (I \oplus I) \times (I \oplus I) \times (I \oplus I)$ is called an *intuitionistic single-valued neutrosophic set* (briefly, ISVNS) in X, where

$$A^T = (A^{T,\in}, A^{T,\not\in}),\ A^I = (A^{I,\in}, A^{I,\not\in}),\ A^F = (A^{F,\in}, A^{F,\not\in}) \in IFS(X)$$

and A^T [resp. A^I and A^F] is called the *intuitionistic truth-membership* [resp. *intuitionistic indeterminacy-membership* and *intuitionistic falsity-membership*] *function* of A. In particular, $\dot{0}$ [resp. $\dot{1}$] denotes the *intuitionistic single-valued neutrosophic empty* [resp. *whole*] *set* in X and defined by: for each $x \in X$,

$$\dot{0}(x) = \langle \bar{0}, \bar{0}, \bar{1} \rangle \ \text{[resp. } \dot{1}(x) = \left\langle \bar{1}, \bar{1}, \bar{0}] \right\rangle].$$

We will denote the set of all ISVNSs in X as $ISVNS(X)$.

When X is continuous, an ISVNS A in X can be written by

$$A = \int_X \left\langle (A^{T,\in}(x), A^{T,\not\in}(x)), (A^{I,\in}(x), A^{I,\not\in}(x)), (A^{F,\in}(x), A^{F,\not\in}(x)) \right\rangle /x,$$

where $x \in X$.

When X is discrete, an ISVNS A in X can be written by: for each $x_i \in X$,

$$A = \Sigma_{i=1}^n \left\langle (A^{T,\in}(x_i), A^{T,\not\in}(x_i)), (A^{I,\in}(x_i), A^{I,\not\in}(x_i)), (A^{F,\in}(x_i), A^{F,\not\in}(x_i)) \right\rangle /x_i,$$

where $x_i \in X$.

In an election, polls of voters about each candidate are mainly conducted from the perspective of a neutrosophic set (i.e., truth-membership, indeterminacy-membership, falsity-membership). The results of these polls and votes may not generally match. It is believed to be due to the choice of indeterminacy-membership on the day of voting. Therefore, it is necessary to understand the voting propensity of indeterminacy-membership by conducting an opinion poll from the perspective of the intuitionistic single-valued netrosophic set proposed by us.

Example 3.3. (1) Suppose $X = \{x_1, x_2, x_3\}$, where x_1 is the capability, x_2 is the trustworthness and x_2 is the price of sementic web services. The membership values of x_1, x_2 and x_3 are in $I \oplus I$. Each member of X is commonly used to define the quality of service of sementic web services (See [47]) and is obtained from the questionnaire of some domain experts. Each option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". Let us consider two ISVNSs A and B given by:

$$A = \langle (0.3, 0.6), (0.4, 0.5), (0.5, 0.4) \rangle /x_1 + \langle (0.5, 0.3), (0.2, 0.7), (0.3, 0.6) \rangle /x_2 + \langle (0.7, 0.2), (0.2, 0.7), (0.2, 0.7) \rangle /x_3,$$

$$B = \langle (0.6, 0.3), (0.1, 0.8), (0.2, 0.7) \rangle /x_1 + \langle (0.3, 0.6), (0.2, 0.7), (0.6, 0.2) \rangle /x_2 + \langle (0.4, 0.5), (0.2, 0.7), (0.5, 0.4) \rangle /x_3.$$

(2) Let $\bar{A} \in IFS(X)$. Then it is clear that $\left\langle \bar{A}, \bar{\mathbf{0}}, \bar{A}^c \right\rangle \in ISVNS(X)$.

(3) Let $A \in ISVNS(X)$. Then it is obvious that $A^T,\ A^I,\ A^F \in IFS(X)$.

(4) Let $A = \left\langle A^T, A^I, A^F \right\rangle$ be a single-valued neutrosophic set in X. Then it is easily check that $\left\langle (A^T, {A^T}^c), (A^I, {A^I}^c), (A^F, {A^F}^c) \right\rangle \in ISVNS(X)$.

(5). Let $X = \{a, b, c\}$ and consider the mapping $A : X \to I \times I \times I$ in X defined as follows:

$$A(a) = \langle 0.7, 0.3, 0.5 \rangle\,,\ A(b) = \langle 0.6, 0.4, 0.3 \rangle\,,\ A(c) = \langle 0.4, 0.5, 0.8 \rangle\,.$$

Then we can easily check that A is an intuitionistic neutrosophis set in X in the sense of Bhowmik and Pal. Moreover, we can see that

$$A = \langle (0.7, 0.3), (0.3, 0.7), (0.5, 0.5) \rangle /a + \langle (0.6, 0.4), (0.4, 0.6), (0.3, 0.7) \rangle /b + \langle (0.4, 0.6), (0.5, 0.5), (0.7, 0.3) \rangle /c$$

is an intuitionistic single-valued neutrosophic set in X.

From (2), (3) and (4), we can consider an ISVNS as the generalization of both an IFS and a SVNS. Also, from (5), we can consider an ISVNS as the generalization of an INS.

Definition 3.4. Let X be a nonempty set and let $A,\ B \in SVNS(X)$.

(i) We say that A *is contained in* B or A *is subset of* B, denoted by $A \subset B$, if for each $x \in X$,

$$A^T(x) \leq B^T(x),\ \text{i.e.,}\ A^{T,\in}(x) \leq B^{T,\in}(x),\ A^{T,\notin}(x) \geq B^{T,\notin}(x),$$

$$A^I(x) \leq B^I(x),\ \text{i.e.,}\ A^{I,\in}(x) \leq B^{I,\in}(x),\ A^{I,\notin}(x) \geq B^{I,\notin}(x),$$

$$A^F(x) \geq B^F(x),\ \text{i.e.,}\ A^{F,\in}(x) \geq B^{F,\in}(x),\ A^{F,\notin}(x) \leq B^{F,\notin}(x).$$

(ii) We say that A and B are *equal*, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

Example 3.5. Let $A,\ B \in ISVNS(X)$ given in Example 3.3 (1). Then we can easily check that $A \not\subset B$ and $B \not\subset A$.

From Definitions 3.2 and 3.4, we get the following.

Proposition 3.6. *For any* $A \in ISVNS(X)$, $\dot{0} \subset A \subset \dot{1}$.

Definition 3.7. Let X be a nonempty set, let $A,\ B \in ISVNS(X)$ and let $(A_j)_{j \in J}$ be a family of ISVNSs in X indexed by J.

(i) The *intersection* of A and B, denoted by $A \cap B$, is an ISVNS in X defined as follows: for each $x \in X$,

$$(A \cap B)(x) = \left\langle A^T(x) \wedge B^T(x), A^I(x) \wedge B^I(x), A^F(x) \vee B^F(x) \right\rangle,$$

where

$$A^T(x) \wedge B^T(x) = (A^{T,\in}(x) \wedge B^{T,\in}(x), A^{T,\notin}(x) \vee B^{T,\notin}(x)),$$

$$A^I(x) \wedge B^I(x) = (A^{I,\in}(x) \wedge B^{I,\in}(x), A^{I,\notin}(x) \vee B^{I,\notin}(x)),$$

$$A^F(x) \vee B^F(x) = (A^{F,\in}(x) \vee B^{F,\in}(x), A^{F,\notin}(x) \wedge B^{F,\notin}(x)).$$

(ii) The *intersection* of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is an ISVNS in X defined as follows: for each $x \in X$,

$$\left(\bigcap_{j \in J} A_j \right)(x) = \left\langle \bigwedge_{j \in J} A_j^T(x), \bigwedge_{j \in J} A_j^I(x), \bigvee_{j \in J} A_j^F(x) \right\rangle,$$

where

$$\bigwedge_{j\in J} A_j^T(x) = \left(\bigwedge_{j\in J} A_j^{T,\in}(x), \bigvee_{j\in J} A_j^{T,\notin}(x)\right),$$

$$\bigwedge_{j\in J} A_j^I(x) = \left(\bigwedge_{j\in J} A_j^{I,\in}(x), \bigvee_{j\in J} A_j^{I,\notin}(x)\right),$$

$$\bigvee_{j\in J} A_j^F(x) = \left(\bigvee_{j\in J} A_j^{F,\in}(x), \bigwedge_{j\in J} A_j^{F,\notin}(x)\right).$$

(iii) The *union* of A and B, denoted by $A \cup B$, is an ISVNS in X defined as follows: for each $x \in X$,

$$(A \cup B)(x) = \left\langle A^T(x) \vee B^T(x), A^I(x) \vee B^I(x), A^F(x) \wedge B^F(x)\right\rangle,$$

where

$$A^T(x) \vee B^T(x) = (A^{T,\in}(x) \vee B^{T,\in}(x), A^{T,\notin}(x) \wedge B^{T,\notin}(x)),$$

$$A^I(x) \vee B^I(x) = (A^{I,\in}(x) \vee B^{I,\in}(x), A^{I,\notin}(x) \wedge B^{I,\notin}(x)),$$

$$A^F(x) \wedge B^F(x) = (A^{F,\in}(x) \wedge B^{F,\in}(x), A^{F,\notin}(x) \vee B^{F,\notin}(x)).$$

(iv) The *union* of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} A_j$, is an ISVNS in X defined as follows: for each $x \in X$,

$$\left(\bigcup_{j\in J} A_j\right)(x) = \left\langle \bigvee_{j\in J} A_j^T(x), \bigvee_{j\in J} A_j^I(x), \bigwedge_{j\in J} A_j^F(x)\right\rangle,$$

where

$$\bigvee_{j\in J} A_j^T(x) = \left(\bigvee_{j\in J} A_j^{T,\in}(x), \bigwedge_{j\in J} A_j^{T,\notin}(x)\right),$$

$$\bigvee_{j\in J} A_j^I(x) = \left(\bigvee_{j\in J} A_j^{I,\in}(x), \bigwedge_{j\in J} A_j^{I,\notin}(x)\right),$$

$$\bigwedge_{j\in J} A_j^F(x) = \left(\bigwedge_{j\in J} A_j^{F,\in}(x), \bigvee_{j\in J} A_j^{F,\notin}(x)\right).$$

Example 3.8. Let $A,\ B \in ISVNS(X)$ given in Example 3.3 (1). Then we have

$$A \cap B = \langle (0.3, 0.6), (0.1, 0.8), (0.5, 0.4)\rangle /x_1 + \langle (0.3, 0.6), (0.2, 0.7), (0.6, 0.2)\rangle /x_2 + \langle (0.4, 0.5), (0.2, 0.7), (0.5, 0.4)\rangle /x_3,$$

$$A \cup B = \langle (0.6, 0.3), (0.4, 0.5), (0.2, 0.7)\rangle /x_1 + \langle (0.5, 0.3), (0.2, 0.7), (0.3, 0.6)\rangle /x_2 + \langle (0.7, 0.2), (0.2, 0.7), (0.2, 0.7)\rangle /x_3.$$

From Definitions 3.4 and 3.7, we get the followings.

Proposition 3.9. *Let $A,\ B,\ C \in ISVNS(X)$. Then*
(1) *if $A \subset B$ and $B \subset C$, then $A \subset C$,*
(2) *$A \subset A \cup B$ and $B \subset A \cup B$,*
(3) *$A \cap B \subset A$ and $A \cap B \subset B$,*
(4) *$A \subset B$ if and only if $A \cap B = A$,*
(5) *$A \subset B$ if and only if $A \cup B = B$.*

Proposition 3.10. *Let $A,\ B,\ C \in ISVNS(X)$ and let $(A_j)_{j\in J}$ be a family of SVNSs in X indexed by J. Then*
(1) (Idempotent laws) $A \cup A = A,\ A \cap A = A$,
(2) (Commutative laws) $A \cup B = B \cup A,\ A \cap B = B \cap A$,
(3) (Associative laws) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap^i B) \cap^i C$,
(4) (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$\bigcup_{j\in J}(A_j \cap B) = (\bigcup_{j\in J} A_j) \cap B$,
$\bigcap_{j\in J}(A_j \cup B) = (\bigcap_{j\in J} A_j) \cup B$,
(5) (Absorption laws) $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$.

Definition 3.11. Let $A,\ B \in ISVNS(X)$. Then
(i) the *complement* of A, denoted by $c(A)$, is an ISVNS in X defined as follows: for each $x \in X$,

$$c(A)(x) = \left\langle A^F(x), A^{I^c}(x), A^T(x) \right\rangle,$$

where $A^{I^c}(x) = (A^{I,\notin}, A^{I,\in})$,
(ii) the *difference* of A and B, denoted by $A \setminus B$, is an ISVNS in X defined as follows: for each $x \in X$,

$$(A \setminus B)(x) = \left\langle A^T(x) \wedge A^F(x), A^I(x) \wedge A^{I^c}(x), A^F(x) \vee A^T(x) \right\rangle.$$

From Definitions 3.7 and 3.11, it is clear that $A \setminus B = A \cap c(B)$.

Example 3.12. Let $A,\ B \in ISVNS(X)$ given in Example 3.3 (1). Then we have
$c(A) = \langle (0.5, 0.4)(0.3, 0.6), (0.5, 0.4), (0.3, 0.6) \rangle /x_1 + \langle (0.3, 0.6), (0.7, 0.2), (0.5, 0.3) \rangle /x_2$
$+ \langle (0.2, 0.7), (0.7, 0.2), (0.7, 0.2) \rangle /x_3$,
$A \setminus B = \langle (0.2, 0.7), (0.4, 0.5), (0.6, 0.3) \rangle /x_1 + \langle (0.5, 0.3), (0.2, 0.7), (0.3, 0.6) \rangle /x_2$
$+ \langle (0.5, 0.4), (0.2, 0.7), (0.4, 0.5) \rangle /x_3$.

From Definitions 3.4, 3.7 and 3.11, we have the following properties.

Proposition 3.13. *Let $A,\ B,\ C \in ISVNS(X)$ and let $(A_j)_{j\in J}$ be a family of SVNSs in X indexed by J. Then*
(1) (DeMorgan's laws) $c(A \cup B) = c(A) \cap c(B),\ c(A \cap B) = c(A) \cup c(B)$,
$c(\bigcup_{j\in J} A_j) = \bigcap_{j\in J} c(A_j),\ c(\bigcap_{j\in J} A_j) = \bigcup_{j\in J} c(A_j)$,
(2) $c(c(A)) = A$,
(3) (3_a) $A \cup \dot{0} = A,\ A \cap \dot{0} = \dot{0}$,
(3_b) $A \cup \dot{1} = \dot{1},\ A \cap \dot{1} = A$,
(3_c) $c(\dot{1}) = \dot{0},\ c(\dot{0}) = \dot{1}$,
(3_d) $A \cup c(A) \neq \dot{1},\ A \cap c(A) \neq \dot{0}$ *in general* (See Example 3.14).

Example 3.14. Let A be the ISVNS in X given in Example 3.3 (1). Then we can easily check that $A \cup c(A) \neq \dot{1},\ A \cap c(A) \neq \dot{0}$.

From Propositions 3.10 and 3.13, we can see that $(ISVNS(X), \cup, \cap, c, \dot{1}, \dot{0})$ forms a Boolean algebra except the condition (3_d).

Definition 3.15. Let $X,\ Y$ be non-empty universe sets, $A \in ISVNS(X),\ B \in ISVNS(Y)$ and let $f : X \to Y$ be a mapping.

(i) The *preimage of B under f*, denoted by $f^{-1}(B)$, is the ISVNS in X defined as: for each $x \in X$,

$$f^{-1}(B)(x) = \left\langle B^T(f(x)), B^I(f(x)), B^F(f(x)) \right\rangle,$$

where $B^T(f(x)) = (B^{T,\in}(f(x)), B^{T,\notin}(f(x))),\ B^I(f(x)) = (B^{I,\in}(f(x)), B^{I,\notin}(f(x))),$
$B^F(f(x)) = (B^{F,\in}(f(x)), B^{F,\notin}(f(x)))$.
In fact, $f^{-1}(B) = \left\langle f^{-1}(B^T), f^{-1}(B^I), f^{-1}(B^F) \right\rangle$.

(ii) The *image of A under f*, denoted by $f(A)$, is the ISVNS in Y defined as: for each $y \in Y$,

$$f(A)(y) = \left\langle f(A^T)(y), f(A^I)(y), f(A^F)(y) \right\rangle,$$

where $f(A^T)(y) = (f(A^{T,\in})(y), 1 - f(1 - A^{T,\notin})(y)),$
$f(A^I)(y) = (f(A^{I,\in})(y), 1 - f(1 - A^{I,\notin})(y)),$
$f(A^F)(y) = (f(A^{F,\in})(y), 1 - f(1 - A^{I,\notin})(y)).$
In fact, $f(A) = \left\langle f(A^T), f(B^I), f(B^F) \right\rangle$.

Example 3.16. Let $X = \{x_1, x_2, x_3\},\ Y = \{y_1, y_2, y_3, y_4\}$ and let $f : X \to Y$ be the mapping given by $f(x_1) = f(x_2) = y_1,\ f(x_3) = y_2$. Let us consider an ISVNS A in X and an ISVNS B in Y respectively given by:

$$A = \langle (0.3, 0.6), (0.4, 0.5), (0.5, 0.4) \rangle / x_1 + \langle (0.5, 0.3), (0.2, 0.7), (0.3, 0.6) \rangle / x_2 + \langle (0.7, 0.2), (0.2, 0.7), (0.2, 0.7) \rangle / x_3,$$

$$B = \langle (0.6, 0.3), (0.1, 0.8), (0.2, 0.7) \rangle / y_1 + \langle (0.3, 0.6), (0.2, 0.7), (0.6, 0.2) \rangle / y_2 + \langle (0.4, 0.5), (0.2, 0.7), (0.5, 0.4) \rangle / y_3 + \langle (0.5, 0.4), (0.7, 0.2), (0.4, 0.5) \rangle / y_4.$$

Then we can easily obtain the followings:

$$f^{-1}(B) = \langle (0.6, 0.3), (0.1, 0.8), (0.2, 0.7) \rangle / x_1 + \langle (0.6, 0.3), (0.1, 0.8), (0.2, 0.7) \rangle / x_2 + \langle (0.3, 0.6), (0.2, 0.7), (0.6, 0.2) \rangle / x_3,$$

$$f(A) = \langle (0.5, 0.3), (0.4, 0.5), (0.5, 0.4) \rangle / y_1 + \langle (0.7, 0.2), (0.2, 0.7), (0.2, 0.7) \rangle / y_2 + \langle (0, 1), (0, 1), (0, 1) \rangle / y_3 + \langle (0, 1), (0, 1), (0, 1) \rangle / y_4.$$

Proposition 3.17. *Let $A,\ A_1,\ A_2 \in ISVNS(X)$, $(A_j)_{j \in J} \subset ISVNS(X)$, let $B,\ B_1,\ B_2 \in ISVNS(Y)$, $(B_j)_{j \in J} \subset ISVNS(Y)$ and let $f : X \to Y$ be a mapping. Then we have*

(1) *if $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$,*
(2) *if $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$,*
(3) *$A \subset f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$,*
(4) *$f(f^{-1}(B)) \subset B$ and if f is surjective, $f(f^{-1}(B)) = B$,*
(5) *$f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$,*
(6) *$f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$,*
(7) *$f(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} f(A_j)$,*
(8) *$f(\bigcap_{j \in J} A_j) \subset \bigcap_{j \in J} f(A_j)$ and if f is injective, then $f(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} f(A_j)$,*

(9) *if* f *is surjective, then* $f(c(A)) \subset f(c(A))$,
(10) $f^{-1}(c(B)) = f^{-1}(c(B))$,
(11) $f^{-1}(\dot{0}) = \dot{0}$, $f^{-1}(\dot{1}) = \dot{1}$,
(12) $f(\dot{0}) = \dot{0}$ *and if* f *is surjective, then* $f(\dot{1}) = \dot{1}$.

Proof. From Definition 3.15, the proofs are straightforward. □

4. A correlation coefficient and cosine measure for ISVNSs

We can think that an ISVNS is a generalization of classic set, fuzzy set, intuitionistic set and single-valued set. In this section, first of all, as the extension of the correlation of intuitionistic fuzzy sets (See Gerstenkorn and Mańko [20] and Ye [22]) and single-valued neutrosophic sets (See Ye [33]), we introduce the concepts of the informational energy of an ISVNS, the correlation of two ISVNSs and the correlation coefficient of two ISVNSs which may be in real scientific and engineering applications. Next, by modifying a cosine measure for IFSs proposed by Ye [29], we define a cosine measure for ISVNSs. Finally, we propose a correlation coefficient between ISVNSs by modifying a correlation coefficient for IFSs introduced by Thao [27].

Definition 4.1 (See [33]). Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set and let $A,\ B \in SVNS(X)$. Then

(i) the *informational energy* of A, denoted by $E_{SVNS}(A)$, is defined as:

$$E_{SVNS}(A) = \Sigma_{i=1}^{n}[(A^T(x_i))^2 + (A^I(x_i))^2 + (A^F(x_i))^2],$$

(ii) the *correlation* between A and B, denoted by $C_{SVNS}(A, B)$, is defined as:

$$C_{SVNS}(A, B) = \Sigma_{i=1}^{n}[A^T(x_i)B^T(x_i) + A^I(x_i)B^I(x_i) + A^F(x_i)B^F(x_i)],$$

(iii) the *correlation coefficient* between A and B, denoted by $\rho_{SVNS}(A, B)$, is defined as:

$$\rho_{SVNS}(A, B) = \frac{C_{SVNS}(A, B)}{\sqrt{E_{SVNS}(A)E_{SVNS}(B)}}. \tag{4.1}$$

Definition 4.2 (See [33]). Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set, let $A,\ B \in SVNS(X)$ and let $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be the weighting vector x_i $(i = 1,\ 2,\ ,\cdots,\ n)$, where $w_i \geq 0$ and $\Sigma_{i=1}^{n} w_i = 1$. Then

(i) the *weighted informational energy* of A, denoted by $E_{WSVNS}(A)$, is defined as:

$$E_{WSVNS}(A) = \Sigma_{i=1}^{n} w_i[(A^T(x_i))^2 + (A^I(x_i))^2 + (A^F(x_i))^2],$$

(ii) the *weighted correlation* between A and B, denoted by $C_{WSVNS}(A, B)$, is defined as:

$$C_{WSVNS}(A, B) = \Sigma_{i=1}^{n} w_i[A^T(x_i)B^T(x_i) + A^I(x_i)B^I(x_i) + A^F(x_i)B^F(x_i)],$$

(iii) the *weighted correlation coefficient* between A and B, denoted by $\rho_{WSVNS}(A,B)$, is defined as:

$$\rho_{WSVNS}(A,B)=\frac{C_{WSVNS}(A,B)}{\sqrt{E_{WSVNS}(A)E_{WSVNS}(B)}}. \tag{4.2}$$

It is clear that if $\mathbf{w}=(1/n,1/n,\cdots,1/n)$, then $\rho_{WSVNS}(A,B)=\rho_{SVNS}(A,B)$.

Now by combining Definitions 3.2 and 4.1, we can define the correlation coefficient between two ISVNSs as follows.

Definition 4.3. Let $X=\{x_1,x_2,\cdots,x_n\}$ be the universe set and let $A,\ B\in ISVNS(X)$. Then

(i) the *informational energy* of A, denoted by $E_{ISVNS}(A)$, is defined as:

$$E_{ISVNS}(A)=\Sigma_{i=1}^{n}[(A^{T,\in}(x_i))^2+(A^{T,\notin}(x_i))^2+(A^{I,\in}(x_i))^2 +(A^{I,\notin}(x_i))^2+(A^{F,\in}(x_i))^2+(A^{F,\notin}(x_i))^2],$$

(ii) the *correlation* between A and B, denoted by $C_{ISVNS}(A,B)$, is defined as:

$$C_{ISVNS}(A,B)=\Sigma_{i=1}^{n}[A^{T,\in}(x_i)B^{T,\in}(x_i)+A^{T,\notin}(x_i)B^{T,\notin}(x_i) +A^{I,\in}(x_i)B^{I,\in}(x_i)+A^{I,\notin}(x_i)B^{I,\notin}(x_i) +A^{F,\in}(x_i)B^{F,\in}(x_i)+A^{F,\notin}(x_i)B^{F,\notin}(x_i)],$$

(iii) the *correlation coefficient* between A and B, denoted by $\rho_{ISVNS}(A,B)$, is defined as:

$$\rho_{ISVNS}(A,B)=\frac{C_{ISVNS}(A,B)}{\sqrt{E_{ISVNS}(A)E_{ISVNS}(B)}}. \tag{4.3}$$

Proposition 4.4. *Let $X=\{x_1,x_2,\cdots,x_n\}$ be the universe set and let $A,\ B\in ISVNS(X)$. Then we have*

(1) $\rho_{ISVNS}(A,A)=E_{ISVNS}(A)$,
(2) $\rho_{ISVNS}(A,B)=\rho_{ISVNS}(B,A)$,
(3) $\rho_{ISVNS}(A,B)=1$, *if* $A=B$
(4) $0\le\rho_{ISVNS}(A,B)\le 1$.

Proof. From Definition 4.3, the proofs of (1), (2) and (3) are obvious.

(4) It is clear that $0\le\rho_{ISVNS}(A,B)$. We recall the Cauchy-Schwarz inequality: for any $(x_1,x_2,\cdots,x_n),\ (y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$,

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\le(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+y_2^2+\cdots+y_n^2). \tag{4.4}$$

Then by (4.4), we get easily the following inequality: for each $x_i\in X$,

$$A^{T,\in}(x_i)B^{T,\in}(x_i)+A^{T,\notin}(x_i)B^{T,\notin}(x_i)+A^{I,\in}(x_i)B^{I,\in}(x_i) + A^{I,\notin}(x_i)B^{I,\notin}(x_i)+A^{F,\in}(x_i)B^{F,\in}(x_i)+A^{F,\notin}(x_i)B^{F,\notin}(x_i)$$
$$\le\sqrt{(A^{T,\in}(x_i))^2+(A^{T,\notin}(x_i))^2+(A^{I,\in}(x_i))^2+(A^{I,\notin}(x_i))^2+(A^{F,\in}(x_i))^2+(A^{F,\notin}(x_i))^2}$$
$$\cdot\sqrt{(B^{T,\in}(x_i))^2+(B^{T,\notin}(x_i))^2+(B^{I,\in}(x_i))^2+(B^{I,\notin}(x_i))^2+(B^{F,\in}(x_i))^2+(B^{F,\notin}(x_i))^2}.$$

Thus we have $C_{ISVNS}(A,B)\le\sqrt{E_{ISVNS}(A)E_{ISVNS}(B)}$. So $\rho_{ISVNS}(A,B)\le 1$. □

In a general way, if decision makers give different weight value to each element in the given universe set, then the result of decision may be different. Thus in particular, it is very important to consider the weight of element in decision-making problems. By modifying Definition 4.2, we have the following definition.

Definition 4.5. Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set, let $A,\ B \in ISVNS(X)$ and let $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be the weighting vector x_i $(i = 1,\ 2,\ ,\cdots,\ n)$, where $w_i \geq 0$ and $\Sigma_{i=1}^n w_i = 1$. Then

(i) the *weighted informational energy* of A, denoted by $E_{WISVNS}(A)$, is defined as:

$$E_{WISVNS}(A) = \Sigma_{i=1}^n w_i[(A^{T,\in}(x_i))^2 + (A^{T,\notin}(x_i))^2 + (A^{I,\in}(x_i))^2 + (A^{I,\notin}(x_i))^2 + (A^{F,\in}(x_i))^2 + (A^{F,\notin}(x_i))^2],$$

(ii) the *weighted correlation* between A and B, denoted by $C_{WISVNS}(A, B)$, is defined as:

$$C_{WISVNS}(A, B) = \Sigma_{i=1}^n w_i[A^{T,\in}(x_i)B^{T,\in}(x_i) + A^{T,\notin}(x_i)B^{T,\notin}(x_i) + A^{I,\in}(x_i)B^{I,\in}(x_i) + A^{I,\notin}(x_i)B^{I,\notin}(x_i) + A^{F,\in}(x_i)B^{F,\in}(x_i) + A^{F,\notin}(x_i)B^{F,\notin}(x_i)],$$

(iii) the *weighted correlation coefficient* between A and B, denoted by $\rho_{WISVNS}(A, B)$, is defined as:

$$\rho_{WISVNS}(A, B) = \frac{C_{WISVNS}(A, B)}{\sqrt{E_{WISVNS}(A)E_{WISVNS}(B)}}. \tag{4.5}$$

It is obvious that if $\mathbf{w} = (1/n, 1/n, \cdots, 1/n)$, then $\rho_{WISVNS}(A, B) = \rho_{ISVNS}(A, B)$.

From Definition 4.5, we have the similar properties of Proposition 4.4.

Proposition 4.6. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set, let $A,\ B \in ISVNS(X)$ and let $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)$ be the weighting vector of x_i $(i = 1,\ 2,\ ,\cdots,\ n)$, where $w_i \geq 0$ and $\Sigma_{i=1}^n w_i = 1$. Then*

(1) $\rho_{WISVNS}(A, A) = E_{WISVNS}(A)$,
(2) $\rho_{WISVNS}(A, B) = \rho_{WISVNS}(B, A)$,
(3) $\rho_{WISVNS}(A, B) = 1$, *if* $A = B$
(4) $0 \leq \rho_{WISVNS}(A, B) \leq 1$.

Next, we define a cosine measure for ISVNSs.

Definition 4.7. Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set and let $A,\ B \in ISVNS(X)$. Then the *cosine measure* between A and B, denoted by $CM_{ISVNS}(A, B)$, is defined by:

$$CM_{ISVNS}(A, B) = \frac{1}{n}\Sigma_{i=1}^n \frac{A^T(x_i)B^T(x_i) + A^I(x_i)B^I(x_i) + A^F(x_i)B^F(x_i)}{\sqrt{A^T(x_i)^2 + A^I(x_i)^2 + A^F(x_i)^2}\sqrt{B^T(x_i)^2 + B^I(x_i)^2 + B^F(x_i)^2}}, \tag{4.6}$$

where

$$A^T(x_i)B^T(x_i) = A^{T,\in}(x_i)B^{T,\in}(x_i) + A^{T,\notin}(x_i)B^{T,\notin}(x_i),$$
$$A^I(x_i)B^I(x_i) = A^{I,\in}(x_i)B^{I,\in}(x_i) + A^{I,\notin}(x_i)B^{I,\notin}(x_i),$$
$$A^F(x_i)B^F(x_i) = A^{F,\in}(x_i)B^{F,\in}(x_i) + A^{F,\notin}(x_i)B^{F,\notin}(x_i),$$
$$A^T(x_i)^2 = A^{T,\in}(x_i)^2 + A^{T,\notin}(x_i)^2, \quad A^I(x_i)^2 = A^{I,\in}(x_i)^2 + A^{I,\notin}(x_i)^2,$$
$$A^F(x_i)^2 = A^{F,\in}(x_i)^2 + A^{F,\notin}(x_i)^2, \quad B^T(x_i)^2 = B^{T,\in}(x_i)^2 + B^{T,\notin}(x_i)^2,$$
$$B^I(x_i)^2 = B^{I,\in}(x_i)^2 + B^{I,\notin}(x_i)^2, \quad B^F(x_i)^2 = B^{F,\in}(x_i)^2 + B^{F,\notin}(x_i)^2.$$

From Definitions 4.2 and 4.7, it is obvious that if $n = 1$ in (4.6), then we have

$$\rho_{WSVNS}(A,B) = CM_{ISVNS}(A,B).$$

$CM_{ISVNS}(A,B)$ has similar properties to Proposition 4.4.

Proposition 4.8. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set and let $A,\ B \in ISVNS(X)$. Then*
(1) $CM_{ISVNS}(A,B) = C_{ISVNS}(B,A)$,
(2) $CM_{ISVNS}(A,B) = 1$, *if* $A = B$
(3) $0 \leq CM_{ISVNS}(A,B) \leq 1$.

Proof. The proof is similar to one of Proposition 4.4. □

Definition 4.9. Let $X = \{x_1,\ x_2,\ \cdots,\ x_n\}$ be the universe set and let $A,\ B \in ISVNS(X)$. Let $\mathbf{w} = (w_1,\ w_2,\ \cdots,\ w_n)$ be the weight vector of the element $A(x_i)$ and $B(x_i)$ $(i = 1,\ 2,\ \cdots,\ n)$ with $w_i \in I$ and $\Sigma_{i=1}^n w_i = 1$. Then the *weighted cosine measure* between A and B, denoted by $CM_{WISVNS}(A,B)$, is defined by:

$$(4.7)\quad CM_{WISVNS}(A,B)$$
$$= \frac{1}{n}\Sigma_{i=1}^n w_i \frac{A^T(x_i)B^T(x_i) + A^I(x_i)B^I(x_i) + A^F(x_i)B^F(x_i)}{\sqrt{A^T(x_i)^2 + A^I(x_i)^2 + A^F(x_i)^2}\sqrt{B^T(x_i)^2 + B^I(x_i)^2 + B^F(x_i)^2}}.$$

$CM_{WISVNS}(A,B)$ has similar properties of Proposition 4.8.

Proposition 4.10. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be the universe set and let $A,\ B \in ISVNS(X)$. Let $\boldsymbol{w} = (w_1,\ w_2,\ \cdots,\ w_n)$ be the weight vector of the element $A(x_i)$ and $B(x_i)$ $(i = 1,\ 2,\ \cdots,\ n)$ with $w_i \in I$ and $\Sigma_{i=1}^n w_i = 1$. Then*
(1) $CM_{WISVNS}(A,B) = C_{WISVNS}(B,A)$,
(2) $CM_{WISVNS}(A,B) = 1$, *if* $A = B$
(3) $0 \leq CM_{WISVNS}(A,B) \leq 1$.

Proof. The proof is similar to one of Proposition 4.8. □

Finally, we define correlation coefficients between ISVNSs different from Definitions 4.3 and 4.5 and study some of their properties.

Definition 4.11. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe set and let $A,\ B \in ISVNS(X)$.
(i) The *average* of A, denoted by $E(A)$, is defined by:

$$E(A) = \left\langle \bar{A^T}, \bar{A^I}, \bar{A^F} \right\rangle,$$

where, $\bar{A^T} = \left(A^{\bar{T},\in}, A^{\bar{T},\notin}\right) = \left(\frac{1}{n}\Sigma_{i=1}^n A^{T,\in}(x_i), \frac{1}{n}\Sigma_{i=1}^n A^{T,\notin}(x_i)\right)$,
$\bar{A^I} = \left(A^{\bar{I},\in}, A^{\bar{I},\notin}\right) = \left(\frac{1}{n}\Sigma_{i=1}^n A^{I,\in}(x_i), \frac{1}{n}\Sigma_{i=1}^n A^{I,\notin}(x_i)\right)$,
$\bar{A^F} = \left(A^{\bar{F},\in}, A^{\bar{F},\notin}\right) = \left(\frac{1}{n}\Sigma_{i=1}^n A^{F,\in}(x_i), \frac{1}{n}\Sigma_{i=1}^n A^{F,\notin}(x_i)\right)$.

(ii) The *variance* of A, denoted by $D(A)$, is defined by:

$$\begin{aligned} D(A) = \tfrac{1}{n-1}\Sigma_{i=1}^n [&(A^{T,\in}(x_i) - A^{\bar{T},\in})^2 + (A^{T,\notin}(x_i) - A^{\bar{T},\notin})^2 \\ &+(A^{I,\in}(x_i) - A^{\bar{I},\in})^2 + (A^{I,\notin}(x_i) - A^{\bar{I},\notin})^2 \\ &+(A^{F,\in}(x_i) - A^{\bar{F},\in})^2 + (A^{F,\notin}(x_i) - A^{\bar{F},\notin})^2 + d_i^2(A)], \end{aligned}$$

where $d_i(A) = (A^{T,\in}(x_i) - A^{\bar{T},\in}) - (A^{T,\notin}(x_i) - A^{\bar{T},\notin})$
$+(A^{I,\in}(x_i) - A^{\bar{I},\in}) - (A^{I,\notin}(x_i) - A^{\bar{I},\notin})$
$+(A^{F,\in}(x_i) - A^{\bar{F},\in}) - (A^{F,\notin}(x_i) - A^{\bar{F},\notin})$ for $i = 1,\ 2,\ \cdots,\ n$.

(iii) The *covariance* of A and B, denoted by $Cov(A, B)$, is defined by:

$$\begin{aligned} Cov(A,B) = \tfrac{1}{n-1}\Sigma_{i=1}^n [&(A^{T,\in}(x_i) - A^{\bar{T},\in})(B^{T,\in}(x_i) - B^{\bar{T},\in}) \\ &+(A^{T,\notin}(x_i) - A^{\bar{T},\notin})(B^{T,\notin}(x_i) - B^{\bar{T},\notin}) \\ &+(A^{I,\in}(x_i) - A^{\bar{I},\in})(B^{I,\in}(x_i) - B^{\bar{I},\in}) \\ &+(A^{I,\notin}(x_i) - A^{\bar{I},\notin})(B^{I,\notin}(x_i) - B^{\bar{I},\notin}) \\ &+(A^{F,\in}(x_i) - A^{\bar{F},\in})(B^{F,\in}(x_i) - B^{\bar{F},\in}) \\ &+(A^{F,\notin}(x_i) - A^{\bar{F},\notin})(B^{F,\notin}(x_i) - B^{\bar{F},\notin}) + d_i(A)d_i(B)]. \end{aligned}$$

(iv) The *correlation coefficient* of A and B, denoted by $\rho(A, B)$, is defined by:

(4.8)
$$\rho(A, B) = \frac{Cov(A, B)}{\sqrt{D(A)D(B)}}.$$

Proposition 4.12. *Let $A,\ B \in ISVNS(X)$. Then we get*
(1) $Cov(A, B) = Cov(B, A)$,
(2) $Cov(A, A) = D(A)$,
(3) $\mid Cov(A, B) \mid \leq \sqrt{D(A)D(B)}$.

Proof. From Definition 4.11, the proofs of (1) and (2) are easy.
(3) From (4.4), we can easily see that the following inequality holds:

$$Cov(A, B)^2 \leq D(A)D(B).$$

Then we get $\mid Cov(A, B) \mid \leq \sqrt{D(A)D(B)}$. □

Example 4.13. Let $X = \{x_1, x_2, x_3\}$ and let $A,\ B \in ISVNS(X)$ given by:
$A = \langle(0.6, 0.3), (0.4, 0.2), (0.1, 0.8)\rangle /x_1 + \langle(0.7, 0.2), (0.8, 0.1), (0.3, 0.6)\rangle /x_2$
$+ \langle(0.8, 0.1), (0.3, 0.5), (0.2, 0.7)\rangle /x_3$,
$B = \langle(0.3, 0.6), (0.5, 0.2), (0.6, 0.3)\rangle /x_1 + \langle(0.5, 0.4), (0.7, 0.2), (0.4, 0.3)\rangle /x_2$
$+ \langle(0.7, 0.2), (0.4, 0.2), (0.3, 0.5)\rangle /x_3$.
Then we have

$$\bar{A^T} = (0.7, 0.2),\ \bar{A^I} = (0.5, 0.27),\ \bar{A^F} = (0.2, 0.7),$$

$$\bar{B^T} = (0.5, 0.4),\ \bar{B^I} = (0.4, 0.53),\ \bar{B^F} = (0.43, 0.37).$$

Thus we get

$$E(A) = \langle (0.7, 0.2), (0.5, 0.27), (0.2, 0.7) \rangle \,,$$
$$E(B) = \langle (0.5, 0.4), (0.4, 0.53), (0.43, 0.37) \rangle \,.$$

On the other hand, we obtain the followings:

$$d_1(A) = -0.43,\ d_2(A) = 0.67,\ d_3(A) = -0.23,$$
$$d_1(B) = -0.2,\ d_2(B) = 0.2,\ d_3(B) = 0,$$
$$D(A) = 0.496666667,\ D(B) = 0.18.$$

So $Cov(A, B) = 0.18$. Hence $\rho(A, B) = 0.602010056$. Furthermore, we have

$$\rho_{ISVNS}(A, B) = 0.872843844,\ CM_{IDSVNS}(A, B) = 0.998534048.$$

Proposition 4.14. *Let* $A,\ B \in ISVNS(X)$ *and let us* $A = kB + b$ *mean that*

$$A^{T,\in} = kB^{T,\in} + b,\ A^{T,\notin} = kB^{T,\notin} + b,$$
$$A^{I,\in} = kB^{I,\in} + b,\ A^{I,\notin} = kB^{I,\notin} + b,$$
$$A^{F,\in} = kB^{F,\in} + b,\ A^{F,\notin} = kB^{F,\notin} + b,$$

where $k,\ b$ *are any real numbers. Then we get*

(1) $\rho(A, B) = \rho(B, A)$,
(2) $-1 \leq \rho(A, B) \leq 1$,
(3) $\rho(A, B) = 1$ [resp. $\rho(A, B) = -1$], *if* $k > 0$ [resp. $k < 0$].

Proof. (1) The proof is straightforward.
(2) From Proposition 4.12 (3), the proof is clear.
(3) Suppose $A = kB + b$. Then by Definition 4.11 ((iii), we have

$$\begin{aligned}
Cov(A, B) = &\tfrac{1}{n-1}\Sigma_{i=1}^{n}[(A^{T,\in}(x_i) - A^{\bar{T},\in})(B^{T,\in}(x_i) - B^{\bar{T},\in}) \\
&+(A^{T,\notin}(x_i) - A^{\bar{T},\notin})(B^{T,\notin}(x_i) - B^{\bar{T},\notin}) \\
&+(A^{I,\in}(x_i) - A^{\bar{I},\in})(B^{I,\in}(x_i) - B^{\bar{I},\in}) \\
&+(A^{I,\notin}(x_i) - A^{\bar{I},\notin})(B^{I,\notin}(x_i) - B^{\bar{I},\notin}) \\
&+(A^{F,\in}(x_i) - A^{\bar{F},\in})(B^{F,\in}(x_i) - B^{\bar{F},\in}) \\
&+(A^{F,\notin}(x_i) - A^{\bar{F},\notin})(B^{F,\notin}(x_i) - B^{\bar{F},\notin}) + d_i(A)d_i(B)] \\
= &\tfrac{1}{n-1}\Sigma_{i=1}^{n}[(kB^{T,\in}(x_i) - kB^{\bar{T},\in})(B^{T,\in}(x_i) - B^{\bar{T},\in}) \\
&+(kB^{T,\notin}(x_i) - kB^{\bar{T},\notin})(B^{T,\notin}(x_i) - B^{\bar{T},\notin}) \\
&+(kB^{I,\in}(x_i) - kB^{\bar{I},\in})(B^{I,\in}(x_i) - B^{\bar{I},\in}) \\
&+(kB^{I,\notin}(x_i) - kB^{\bar{I},\notin})(B^{I,\notin}(x_i) - B^{\bar{I},\notin}) \\
&+(kB^{F,\in}(x_i) - kB^{\bar{F},\in})(B^{F,\in}(x_i) - B^{\bar{F},\in}) \\
&+(kB^{F,\notin}(x_i) - kB^{\bar{F},\notin})(B^{F,\notin}(x_i) - B^{\bar{F},\notin}) + kd_i(B)d_i(B)] \\
= &\tfrac{k}{n-1}\Sigma_{i=1}^{n}[(B^{T,\in}(x_i) - B^{\bar{T},\in})^2 + (A^{B,\notin}(x_i) - B^{\bar{T},\notin})^2 \\
&+(B^{I,\in}(x_i) - B^{\bar{I},\in})^2 + (B^{I,\notin}(x_i) - B^{\bar{I},\notin})^2 \\
&+(B^{F,\in}(x_i) - B^{\bar{F},\in})^2 + (B^{F,\notin}(x_i) - B^{\bar{F},\notin})^2 + d_i^2(B)] \\
= &kD(B),
\end{aligned}$$

$$\begin{aligned}
D(A) = &\tfrac{1}{n-1}\Sigma_{i=1}^{n}[(A^{T,\in}(x_i) - A^{\bar{T},\in})^2 + (A^{T,\notin}(x_i) - A^{\bar{T},\notin})^2 \\
&+(A^{I,\in}(x_i) - A^{\bar{I},\in})^2 + (A^{I,\notin}(x_i) - A^{\bar{I},\notin})^2 \\
&+(A^{F,\in}(x_i) - A^{\bar{F},\in})^2 + (A^{F,\notin}(x_i) - A^{\bar{F},\notin})^2 + d_i^2(A)]
\end{aligned}$$

$$= \frac{1}{n-1}\Sigma_{i=1}^{n}[(kB^{T,\in}(x_i) - kB^{\bar{T},\in})^2 + (kB^{T,\notin}(x_i) - kB^{\bar{T},\notin})^2 \\ +(kB^{I,\in}(x_i) - kB^{\bar{I},\in})^2 + (kB^{I,\notin}(x_i) - kB^{\bar{I},\notin})^2 \\ +(kB^{F,\in}(x_i) - kB^{\bar{F},\in})^2 + (kB^{F,\notin}(x_i) - kB^{\bar{F},\notin})^2 + k^2 d_i^2(B)]$$
$$= \frac{k^2}{n-1}\Sigma_{i=1}^{n}[(B^{T,\in}(x_i) - B^{\bar{T},\in})^2 + (B^{T,\notin}(x_i) - B^{\bar{T},\notin})^2 \\ +(B^{I,\in}(x_i) - B^{\bar{I},\in})^2 + (B^{I,\notin}(x_i) - B^{\bar{I},\notin})^2 \\ +(B^{F,\in}(x_i) - B^{\bar{F},\in})^2 + (B^{F,\notin}(x_i) - B^{\bar{F},\notin})^2 + d_i^2(B)]$$
$$= k^2 D(B).$$

Thus we get

$$\rho(A,B) = \frac{Cov(A,B)}{\sqrt{D(A)D(B)}} = \frac{kD(B)}{\sqrt{k^2 D(B)D(B)}} = \frac{kD(B)}{\mid kD(B) \mid}.$$

So the result holds. □

Definition 4.15. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe set and let $A,\ B \in ISVNS(X)$. Let $\mathbf{w} = (w_1,\ w_2,\ \cdots,\ w_n)$ be the weight vector of the element $A(x_i)$ and $B(x_i)$ $(i = 1,\ 2,\ \cdots,\ n)$ with $w_i \in I$ and $\Sigma_{i=1}^{n} w_i = 1$.

(i) The *weighted average* of A, denoted by $E_w(A)$, is defined by:

$$E_w(A) = \left\langle \bar{A_w^T}, \bar{A_w^I}, \bar{A_w^F} \right\rangle,$$

where,
$$\bar{A_w^T} = \left(A_w^{\bar{T},\in}, A_w^{\bar{T},\notin}\right) = \left(\tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{T,\in}(x_i), \tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{T,\notin}(x_i)\right),$$
$$\bar{A_w^I} = \left(A_w^{\bar{I},\in}, A^{\bar{I},\notin}\right) = \left(\tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{I,\in}(x_i), \tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{I,\notin}(x_i)\right),$$
$$\bar{A_w^F} = \left(A_w^{\bar{F},\in}, A_w^{\bar{F},\notin}\right) = \left(\tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{F,\in}(x_i), \tfrac{1}{n}\Sigma_{i=1}^{n} w_i A^{F,\notin}(x_i)\right).$$

(ii) The *weighted variance* of A, denoted by $D_w(A)$, is defined by:

$$D_w(A) = \frac{1}{n-1}\Sigma_{i=1}^{n}[(w_i A^{T,\in}(x_i) - A_w^{\bar{T},\in})^2 + (w_i A^{T,\notin}(x_i) - A_w^{\bar{T},\notin})^2 \\ +(w_i A^{I,\in}(x_i) - A_w^{\bar{I},\in})^2 + (w_i A^{I,\notin}(x_i) - A_w^{\bar{I},\notin})^2 \\ +(w_i A^{F,\in}(x_i) - A_w^{\bar{F},\in})^2 + (w_i A^{F,\notin}(x_i) - A_w^{\bar{F},\notin})^2 + d_{w,i}^2(A)],$$

where $d_{w,i}(A) = (w_i A^{T,\in}(x_i) - A_w^{\bar{T},\in}) - (w_i A^{T,\notin}(x_i) - A_w^{\bar{T},\notin})$
$+(w_i A^{I,\in}(x_i) - A_w^{\bar{I},\in}) - (w_i A^{I,\notin}(x_i) - A_w^{\bar{I},\notin})$
$+(w_i A^{F,\in}(x_i) - A_w^{\bar{F},\in}) - (w_i A^{F,\notin}(x_i) - A_w^{\bar{F},\notin})$ for $i = 1,\ 2,\ \cdots,\ n$.

(iii) The *weighted covariance* of A and B, denoted by $Cov_w(A,B)$, is defined by:

$$Cov_w(A,B) = \frac{1}{n-1}\Sigma_{i=1}^{n}[(w_i A^{T,\in}(x_i) - A_w^{\bar{T},\in})(w_i B^{T,\in}(x_i) - B_w^{\bar{T},\in}) \\ +(w_i A^{T,\notin}(x_i) - A_w^{\bar{T},\notin})(w_i B^{T,\notin}(x_i) - B_w^{\bar{T},\notin}) \\ +(w_i A^{I,\in}(x_i) - A_w^{\bar{I},\in})(w_i B^{I,\in}(x_i) - B_w^{\bar{I},\in}) \\ +(w_i A^{I,\notin}(x_i) - A_w^{\bar{I},\notin})(w_i B^{I,\notin}(x_i) - B_w^{\bar{I},\notin}) \\ +(w_i A^{F,\in}(x_i) - A_w^{\bar{F},\in})(B^{F,\in}(x_i) - B^{\bar{F},\in}) \\ +(A^{F,\notin}(x_i) - A^{\bar{F},\notin})(w_i B^{F,\notin}(x_i) - B_w^{\bar{F},\notin}) + d_{w,i}(A)d_{w,i}(B)].$$

(iv) The *weighted correlation coefficient* of A and B, denoted by $\rho_w(A, B)$, is defined by:

$$\rho_w(A, B) = \frac{Cov_w(A, B)}{\sqrt{D_w(A)D_w(B)}}. \tag{4.9}$$

$Cov_w(A, B)$ has the similar properties of Proposition 4.12.

Proposition 4.16. *Let* $A,\ B \in ISVNS(X)$, *let* $\boldsymbol{w} = (w_1,\ w_2,\ \cdots,\ w_n)$ *be the weight vector of the element* $A(x_i)$ *and* $B(x_i)$ $(i = 1,\ 2,\ \cdots,\ n)$ *with* $w_i \in I$ *and* $\Sigma_{i=1}^{n} w_i = 1$. *Then we have*
(1) $Cov_w(A, B) = Cov_w(B, A)$,
(2) $Cov_w(A, A) = D_w(A)$,
(3) $\mid Cov_w(A, B) \mid \leq \sqrt{D_w(A)D_w(B)}$.

Also, $\rho_w(A, B)$ has the similar properties of Proposition 4.14.

Proposition 4.17. *Let* $A,\ B \in ISVNS(X)$ *and let* $A = kB + b$ *for any real numbers* $k,\ b.m.$ *Then we get*
(1) $\rho_w(A, B) = \rho_w(B, A)$,
(2) $-1 \leq \rho_w(A, B) \leq 1$,
(3) $\rho_w(A, B) = 1$ [resp. $\rho_w(A, B) = -1$], *if* $k > 0$ [resp. $k < 0$].

In the next two sections, we propose an MADM method by using the correlation coefficient and cosine measur between ISVNSs respectively. Since an ISVNS is a generalization of a classical set, an intuitionistic fuzzy set and single-valued neutrosophic set, it is moe general and can handle not only incomplete informations but also the indeterminate informations and inconsistent informations which are commonly in real world. Then we expect that the intuitionistic single-valued neutrosophic decision-making is more suitable for real scientific and engineering applications.

5. Multicriteria decision-making method via the correlation coefficient and cosine measure between ISVNSs

In this section, we present a handling method for the multicriteria decision-making problem based on intuitionistic single-valued neutrosophic environment by means of the weighted correlation coefficient and the cosine measure between ISVNSs respectively.

Let $A = \{A_1, A_2, \cdots, A_m\}$ be a set of alternatives, let $C = \{C_1, C_2, \cdots, Cn\}$ be a set of criteria and let the weight of the criterion C_j $(j = 1,\ 2,\ \cdots,\ n)$, entered by the decision-maker, is w_j, where $w_j \in I$ and $\Sigma_{j=1}^{n} w_j = 1$. In this case, the characteristic of the alternative A_i $(i = 1,\ 2,\ \cdots,\ m)$ is given by the following ISVNS:

$$A_i = \Sigma_{j=1}^{n} \frac{\left\langle (A_i^{T,\in}, A_i^{T,\not\in}), (A_i^{I,\in}, A_i^{I,\not\in}), (A_i^{F,\in}, A_i^{F,\not\in}) \right\rangle}{C_j},\ C_j \in C, \tag{5.1}$$

where $(A_i^{T,\in}, A_i^{T,\not\in}),\ (A_i^{I,\in}, A_i^{I,\not\in}),\ (A_i^{F,\in}, A_i^{F,\not\in}) \in I \oplus I,\ i = 1,\ 2,\ \cdots,\ m$ and $j = 1,\ 2,\ \cdots,\ n$.

Each member of $(I \oplus I) \times (I \oplus I) \times (I \oplus I)$ will be called an *intuitionistic single-valued neutrosophic number* (briefly, ISVNN) and denoted by

$$a = \left\langle (a^{T,\in}, (a^{T,\notin}), (a^{I,\in}, (a^{I,\notin}), (a^{F,\in}, (a^{F,\notin}) \right\rangle.$$

Here, an ISVNS is usually derived from the evaluation of an alternative A_i with respect to a criterion C_j by means of a score law and data processing in practice. Thus we can elicit an intuitionistic single-valued neutrosophic decision matrix $D = (a_{ij})_{m\times n}$, where a_{ij} is an ISVNN.

In MADM problems, we can use the ideal alternative to select the best one in all alternatives. Although the ideal alternative does not exist in real world, it does provide a useful theoretical construct against which to evaluate alternatives (See [22]). The best alternative is selected according to the degree of correlation between known and ideal solution. The selection process is as follows.

MADM method by the correlation coefficient

Step 1. Construct the intuitionistic single-valued neutrosophic decision matrix $D = (a_{ij})_{m\times n}$ based on decision information.

Step 2. Set up the ideal alternative $A^* = \{A_1^*, A_2^*, \cdots, A_n^*\}$ with respect to a_{ij} $(j = 1,\ 2,\ \cdots,\ n)$ by the following cases: for each j,

$$A_j^* = \left\langle (\max(a_{ij}^{T,\in}), \min(a_{ij}^{T,\notin})), (\min(a_{ij}^{I,\in}), \max(a_{ij}^{I,\notin})), (\min(a_{ij}^{F,\in}), \max(a_{ij}^{F,\notin})) \right\rangle$$

corresponding benefit type of an criterion,

$$A_j^* = \left\langle (\min(a_{ij}^{T,\in}), \max(a_{ij}^{T,\notin})), (\max(a_{ij}^{I,\in}), \min(a_{ij}^{I,\notin})), (\max(a_{ij}^{F,\in}), \min(a_{ij}^{F,\notin})) \right\rangle$$

corresponding benefit type of an criterion.

Step 3. Calculate the weighted correlation coefficient between A_i and A^*,

$$\rho_{WISVNS}(A_i, A^*) \text{ and } \rho_w(A_i, A^*)\ (i = 1,\ 2,\ \cdots,\ m)$$

by using the equations (4.5) and (4.9) respectively.

Step 4. Rank all of the weighted correlation coefficient between ISVNSs and select the best one.

MADM method by the cosine measure

Step 1. Construct the intuitionistic single-valued neutrosophic decision matrix $D = (a_{ij})_{m\times n}$ based on decision information.

Step 2. Set up the ideal alternative $A^* = \{A_1^*, A_2^*, \cdots, A_n^*\}$ with respect to a_{ij} $(j = 1,\ 2,\ \cdots,\ n)$ by the following cases: for each j,

$$A_j^* = \left\langle (\max(a_{ij}^{T,\in}), \min(a_{ij}^{T,\notin})), (\min(a_{ij}^{I,\in}), \max(a_{ij}^{I,\notin})), (\min(a_{ij}^{F,\in}), \max(a_{ij}^{F,\notin})) \right\rangle$$

corresponding benefit type of an criterion,

$$A_j^* = \left\langle (\min(a_{ij}^{T,\in}), \max(a_{ij}^{T,\notin})), (\max(a_{ij}^{I,\in}), \min(a_{ij}^{I,\notin})), (\max(a_{ij}^{F,\in}), \min(a_{ij}^{F,\notin})) \right\rangle$$

corresponding benefit type of an criterion.

Step 3. Calculate the weighted cosine measure between A_i and A^*,

$$CM_{_{WISVNS}}(A_i, A^*)\ (i = 1,\ 2,\ \cdots,\ m)$$

by using the equation (4.7).

Step 4. Rank all of the the weighted cosine measure between ISVNSs and select the best one.

Now we give an example to demonstrate the application of the proposed MADM methods with intuitionistic single-valued neutrosophic information.

Example 5.1. Let us consider the decision-making problem adapted from Lu and Ye [30]. There is an investment company, which wants to invest a sum of money in the best option.

There is a panel with four possible alternatives to invest the money:

(1) A_1 is a textile company, (2) A_2 is an automobile company,

(3) A_3 is a computer company, (4) A_4 is a software company.

The evaluation requirements of the four alternatives are on the basis of three criteria:

(1) C_1 is the risk, (2) C_2 is the growth, (3) C_3 is the environmental impact, where the criteria C_1 and C_2 are benefit types and the criterion C_3 is a cost type.

Let the weight vector of three criteria is given by $\mathbf{w} = (0.32, 0.38, 0.30)$.

For the evaluation of an alternative A_i with respect to a criterion C_j $(i = 1,\ 2,\ 3,\ 4;\ j = 1,\ 2,\ 3)$, it is obtained from the questionnaire of a domain expert. When the four possible alternatives with respect to the above three criteria are evaluated by the expert, suppose the characteristic of the alternative A_i $(i = 1,\ 2,\ 3,\ 4)$ is given by the following ISVNS:

(5.2)

$$A_1 = \{a_{11}, a_{12}, a_{13}\},\ A_2 = \{a_{21}, a_{22}, a_{23}\},\ A_3 = \{a_{31}, a_{32}, a_{33}\},\ A_4 = \{a_{41}, a_{42}, a_{43}\},$$

where $a_{ij} = A_i(C_j)$ is an ISVNS and a_{ij} is given:

$a_{11} = \langle(0.5, 0.4), (0.1, 0.7), (0.2, 0.6)\rangle$, $a_{12} = \langle(0.5, 0.4), (0.1, 0.7), (0.2, 0.6)\rangle$,
$a_{13} = \langle(0.6, 0.2), (0.2, 0.7), (0.1, 0.8)\rangle$, $a_{21} = \langle(0.6, 0.2), (0.1, 0.8), (0.2, 0.7)\rangle$,
$a_{22} = \langle(0.6, 0.3), (0.1, 0.8), (0.2, 0.7)\rangle$, $a_{23} = \langle(0.6, 0.3), (0.3, 0.6), (0.1, 0.8)\rangle$,
$a_{31} = \langle(0.4, 0.4), (0.2, 0.7), (0.1, 0.7)\rangle$, $a_{32} = \langle(0.5, 0.4), (0.2, 0.7), (0.3, 0.6)\rangle$,
$a_{33} = \langle(0.5, 0.3), (0.2, 0.7), (0.3, 0.6)\rangle$, $a_{41} = \langle(0.7, 0.2), (0.1, 0.8), (0.1, 0.8)\rangle$,
$a_{42} = \langle(0.6, 0.3), (0.1, 0.8), (0.1, 0.7)\rangle$, $a_{43} = \langle(0.6, 0.3), (0.3, 0.6), (0.2, 0.7)\rangle$.

Step 1. According to the aforementioned four alternatives, i.e., (5.2), we obtain the following intuitionistic single-valued neutrosophic decision matrix D:

$$D = \begin{bmatrix} & C_1 & C_2 & C_3 \\ A_1 & a_{11} & a_{12} & a_{13} \\ A_2 & a_{21} & a_{22} & a_{23} \\ A_3 & a_{31} & a_{32} & a_{23} \\ A_4 & a_{41} & a_{42} & a_{43} \end{bmatrix} \tag{5.3}$$

Step 2. Set up the ideal alternative A^* with respect to a_{ij} $(j = 1,\ 2,\ 3)$:

$$A^* = \{A_1^*, A_2^*, A_3^*\}, \tag{5.4}$$

where $A_1^* = \langle (0.7, 0.2), (0.1, 0.8), (0.1, 0.8) \rangle$, $A_2^* = \langle (0.6, 0.3), (0.1, 0.8), (0.1, 0.7) \rangle$, $A_3^* = \langle (0.5, 0.3), (0.3, 0.6), (0.3, 0.6) \rangle$.

Step 3. Calculate weight correlation coefficient between A_i and A^*, $\rho_{WISVNS}(A_i, A^*)$, $CM_{WISVNS}(A_i, A^*)$ and $\rho_w(A_i, A^*)$ $(i = 1,\ 2,\ 3,\ 4)$ by using the equations (4.5), (4.7) and (4.9) respectively:

$$\rho_{WISVNS}(A_1, A^*) = 0.967934278, \quad \rho_{WISVNS}(A_2, A^*) = 0.987226876,$$
$$\rho_{WISVNS}(A_3, A^*) = 0.971215363, \quad \rho_{WISVNS}(A_4, A^*) = 0.997336497,$$
$$CM_{WISVNS}(A_1, A^*) = 0.324899044, \quad CM_{WISVNS}(A_2, A^*) = 0.329670651,$$
$$CM_{WISVNS}(A_3, A^*) = 0.324899219, \quad CM_{WISVNS}(A_4, A^*) = 0.332461043,$$
$$\rho_w(A_1, A^*) = 0.06266888, \quad \rho_w A_2, A^*) = 0.783755949,$$
$$\rho_w(A_3, A^*) = 0.580284884, \quad \rho_w(A_4, A^*) = 0.986574767.$$

Step 4. Rank all of the weighted correlation coefficient and the weighted cosine measure between ISVNSs: From **Step 3**, we get the following rank.

$$\rho_{WISVNS}(A_4, A^*) > \rho_{WISVNS}(A_2, A^*) > \rho_{WISVNS}(A_3, A^*) > \rho_{WISVNS}(A_1, A^*),$$
$$CM_{WISVNS}(A_4, A^*) > CM_{WISVNS}(A_2, A^*) > CM_{WISVNS}(A_3, A^*) > CM_{WISVNS}(A_1, A^*),$$
$$\rho_w(A_4, A^*) > \rho_w(A_2, A^*) > \rho_w(A_3, A^*) > \rho_w(A_1, A^*).$$

Then in either case, A_4 is selected as the best alternative.

6. Pattern recognition based on the correlation coefficient and cosine measure between ISVNSs

In this section, we propose a recognition method based on the correlation coefficient [resp. cosine measure] between ISVNSs. According to the maximum correlation principle in mathematical statistics, we assume that if the correlation coefficient [resp. cosine measure] of ideal pattern with sample pattern is greater than or equal to 0.6, we consider that the sample model belongs to a group of ideal model. The algorithm of pattern recognition with respect to the correlation coefficient [resp. cosine measure] is as follows.

Step 1. Construct the ideal ISVNS A^* on a universe set X.

Step 2. Construct ISVNSs A_i, $i = 1, 2, \cdots, n$ as the sample pattern that is recognized.

Step 3. Calculate the correlation coefficient [resp. the cosine measure] between A_i and A^*,

$$\rho_{ISVNS}(A_i, A^*) \text{ [resp. } C_{ISVNS}(A_i, A^*)] \ (i = 1, 2, \cdots, n)$$

by using the equation (4.3) [resp. (4.6)].

Step 4. If $\rho_{ISVNS}(A_i, A^*) \geq 0.7$ [resp. $C_{ISVNS}(A_i, A^*) \geq 0.7$], then A_i belongs to the ideal pattern A^* and if $\rho_{ISVNS}(A_i, A^*) < 0.7$ [resp. $C_{ISVNS}(A_i, A^*) < 0.7$], then A_i does not belong to the ideal pattern A^*.

In the following, we give an examples to illustrate the utility of the correlation coefficient [resp. the cosine measure] between ISVNSs in pattern recognition.

Example 6.1. Let $X = \{x_1, x_2, x_3\}$ be a universe set, let $A^* \in ISVNS(X)$ be an ideal pattern and let $A_i \in ISVNS(X)$, $i = 1, 2, 3$, be a sample pattern.

Step 1. Construct the ideal ISVNS A^* on X as:

$$A^* = \langle (0.2, 0.6), (0.3, 0.5), (0.3, 0.5) \rangle / x_1 + \langle (0.5, 0.3), (0.0, 0.5), (0.2, 0.7) \rangle / x_2 + \langle (0.6, 0.2), (0.0, 0.1), (0.3, 0.6) \rangle / x_3.$$

Step 2. Construct ISVNSs A_i, $i = 1, 2, 3$ on X for the sample patterns as:

$$A_1 = \langle (0.2, 0.5), (0.4, 0.5), (0.3, 0.5) \rangle / x_1 + \langle (0.7, 0.3), (0.1, 0.7), (0.1, 0.7) \rangle / x_2 + \langle (0.6, 0.2), (0.5, 0.4), (0.3, 0.6) \rangle / x_3,$$

$$A_2 = \langle (0.3, 0.3), (0.3, 0.5), (0.3, 0.1) \rangle / x_1 + \langle (0.6, 0.3), (0.1, 0.2), (0.2, 0.7) \rangle / x_2 + \langle (0.6, 0.1), (0.9, 0.0), (0.3, 0.6) \rangle / x_3,$$

$$A_3 = \langle (0.8, 0.1), (0.1, 0.8), (0.8, 0.1) \rangle / x_1 + \langle (0.1, 0.8), (0.8, 0.1), (0.3, 0.1) \rangle / x_2 + \langle (0.5, 0.1), (0.1, 0.0), (0.4, 0.3) \rangle / x_3.$$

Step 3. Calculate the correlation coefficient and the cosine measure between A_i and A^* by using the equation (4.3) and (4.6) respectively:

$$\rho_{ISVNS}(A_1, A^*) = 0.940304326, \quad \rho_{ISVNS}(A_2, A^*) = 0.811214095,$$
$$\rho_{ISVNS}(A_3, A^*) = 0.589901964, \quad C_{ISVNS}(A_1, A^*) = 0.940990597,$$
$$C_{ISVNS}(A_2, A^*) = 0.843526611, \quad C_{ISVNS}(A_3, A^*) = 0.650069089.$$

Step 4. From **Step 3**, we can see that A_1 and A_2 belong to the ideal pattern A^* but A_3 does not belong to A^*.

7. Conclusions

We introduced the concept of intuitionistic neutrosophic sets and dealt with some of its properties. We defined the correlation coefficient and cosine similarity measure be between intuitionistic neutrosophic sets. Also, we proposed the algorithms for the correlation coefficient and cosine similarity measure in order to apply them to decision-making problems and gave examples.

In the future, we expect that one can apply the notion of intuitionistic neutrosophic sets to group and ring theory, BCK-algebra and category theory, decision-making problems, etc. Moreover, we expect that one can define an octahedron neutrosophic set as the generalization of an interval-valued neutrosophic set and an intuitionistic neutrosophic set.

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