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The structure-preservation under consistent functions for interior and closure of fuzzy neighborhood systems

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Abstract. The neighborhood system(NS) is one of the most general structures for granular computing. The concept of consistent function is a tool to study the structure-preservation for closure and interior operators of NS. In this paper, the definition of interior and closure operators are generalized for fuzzy neighborhood system(FNS) via fuzzy implicator and triangular norm, some properties of FNS under consistent functions are investigated. The consistent functions satisfy structural preservation properties on the universe, and surjective functions also preserve properties for FNS on the image universe with left-continuous triangular norm operator. In addition, the Lukasiewicz operators for FNS satisfy the duality.

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1. Introduction

The rough set theory (RST), initiated by Pawlak [1, 2], is a mathematical tool to deal with inexact or uncertain knowledge. The basic structure of RST is an approximation space which is constituted by a universe and an equivalence relation on the universe. The related equivalence classes are induced by the equivalent relation, they are definable subsets. It can be used to approximate subsets of universe by the lower and upper approximations. The lower approximation is the greatest definable set contained in the given set of objects, while the upper approximation is the smallest definable set containing the given set.

Besides the RST, there are many other useful tools to deal with uncertain problems. Soft set theory is a mathematical method, it was proposed by Molodtsov [3], and makes up for the lack of parameter set in traditional fuzzy set. There are increasing researches on the properties and applications of soft sets in recent years, such as the theory of soft topological space generated by L-soft sets [4], soft bitopological space theory [5, 6], and soft topological subspaces [7].

Over the years, many generalizations and applications of rough set model have been developed. Information system is one of the applications of rough sets, which is a formalism for representing knowledge about some objects in terms of attributes and their values [8]. RST is a useful tool to deal with single information system and the communication between two information systems [9, 10, 11, 12]. The communication, namely, the information transmission or mapping between information systems. Sometimes the information transmission is required for some practical reason. For example, the equivalent attribute reduction and rule extraction [13]. Therefore, it is worthwhile to study the communication between information systems.

Wang et al. [9] first proposed the notion of consistent function, and it was used to investigate the communication between two information systems [9]. Zhu [14] presented two equivalent definitions of the consistent functions in [9] respectively, and improved some important properties of consistent functions with respect to the lower and upper approximations under relation mappings. But those conclusions are limited on the general binary relation.

In addition, as a generalization of RST, neighborhood system associates each element of a universe with a family of neighborhoods. The concept of NS was initiated by Lin as a concrete model of granular computing(GrC), and it is one of the most general structures for GrC [15]. Focus on the properties and applications of NS, there has been much work in recent years [16, 17, 18]. The concept of consistent function was extended to the framework of NS to conclude some important properties in [19]. Liau et al. [20] viewed it as granule-based consistent functions and proposed the notion of system-consistent(sys-consistent) function for NS. Combined with interior and closure operators on NS [21, 22], it is proved that sys-consistent functions are structure-preserving mappings. In addition, the concept of NS has also been extended into a fuzzy setting by allowing neighborhoods to be fuzzy [23]. Therefore, are these conclusions before also valid for fuzzy neighborhood systems? That is the problem we want to solve in this paper.

In fact, the conclusions for fuzzy neighborhood system based on the interior and closure also have been investigated in [20]. However, the conclusions for FNS are based on the interior and closure with Godel operator in [20]. Godel operator is a kind of special implication operator, the case for more general operators has not been studied. Therefore, this paper does provide a new research direction for FNS and makes a new contribution the research of FNS. The general operators for interior and closure of FNS are introduced in this paper, they are defined by general fuzzy implicator and triangle norm. Some properties of consistent functions with respect to the interior and closure are discussed, and for the special Lukasiewicz operators, the conclusions are presented there.

2. Preliminaries

The theory of fuzzy sets initiated by Zadeh [24] provides an appropriate framework for representing and processing vague concepts by allowing partial memberships. Let U be a nonempty set, called universe. A fuzzy set μ of U is defined by a membership function $\mu : U \to [0, 1]$. For any $x \in U$, the membership value $\mu(x)$ specifies the degree to which x belongs to the fuzzy set μ. We denote by $P(U)$ and $F(U)$ the set of all subsets and fuzzy sets of U, respectively.

Let U and V be two universes and $f : U \to V$ a mapping. We note that $[x]_f = \{x' \in U | f(x') = f(x)\}$ and $U_y = \{x \in U | f(x) = y\} = f^{-1}(\{y\})$. By Zadeh extension principle for fuzzy sets, the mapping f can be extended to fuzzy power sets $f : F(U) \to F(V)$ and $f^{-1} : F(V) \to F(U)$ given by: $\forall y \in V$ and $\forall x \in U$

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} \mu(x), & f^{-1}(\{y\}) \neq \varnothing \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$f^{-1}(\lambda)(x) = \lambda(f(x))$$

for any $\mu \in F(U)$ and $\lambda \in F(V)$.

Lemma 2.1. *For any $\mu, \nu \in F(U)$ and $\omega, \lambda \in F(V)$, it is obvious that the following properties hold:*

(1) $\mu \subseteq f^{-1}(f(\mu))$,
(2) $f(f^{-1}(\lambda)) \subseteq \lambda$,
(3) $f(\mu) \subseteq f(\nu)$ *if* $\mu \subseteq \nu$,
(4) $f^{-1}(\omega) \subseteq f^{-1}(\lambda)$ *if* $\omega \subseteq \lambda$.

In semantics, an fuzzy implication operator is a binary operation $\to: [0,1]^2 \to [0,1]$ satisfying the following conditions [25]:
(1) $0 \to 0 = 0 \to 1 = 1 \to 1 = 1, 1 \to 0 = 0$.
(2) $a \to b$ is increasing with respect to b and decreasing with respect to a.

Triangular norms (t-norms) are closely related to fuzzy implication operators. A function $\otimes : [0,1]^2 \to [0,1]$ is said to be a t-norm if $\otimes$ is associative, commutative and satisfy the conditions $a \otimes 1 = a$ and that $a \leq b$ implies $a \otimes c \leq b \otimes c$ for all $a, b, c \in [0,1]$. If a t-norm $\otimes$ is left-continuous, then $a \otimes \sup\limits_{i \in I} b_i = \sup\limits_{i \in I}(a \otimes b_i)$ holds where $a, b_i \in [0,1] (i \in I)$, and I is a nonempty set of indices. Let $\otimes$ be a t-norm and $\to_\otimes: [0,1]^2 \to [0,1]$ be defined as

$$a \to_\otimes b = sup\{x \in [0,1] | a \otimes x \leq b\} \tag{2.2}$$

for all $a, b \in [0,1]$. $\to_\otimes$ is called the *adjoint implication operator induced by* $\otimes$.

Theorem 2.2 ([26])**.** *Suppose that $\otimes$ is a left-continuous t-norm. Then $\to_\otimes$ is an fuzzy implication operator and*

(1) $a \to_\otimes b = 1$ *if and only if* $a \leq b$,
(2) $a \leq b \to_\otimes c$ *if and only if* $b \leq a \to_\otimes c$,
(3) $a \to_\otimes (b \to_\otimes c) = b \to_\otimes (a \to_\otimes c)$,
(4) $1 \to_\otimes a = a$,
(5) $a \otimes b \leq c$ *if and only if* $a \leq (b \to_\otimes c)$,
(6) $a \to_\otimes \inf\{b_i | i \in I\} = \inf\{a \to_\otimes b_i | i \in I\}$,
(7) $\sup\{b_i | i \in I\} \to_\otimes a = \inf\{b_i \to_\otimes a | i \in I\}$.

Example 2.3. The following are three left-continuous t-norms:

$$a \otimes_L b = (a + b - 1) \vee 0, \tag{2.3}$$

$$a \otimes_G b = a \wedge b,$$

$$a \otimes_\pi b = ab.$$

The related fuzzy implication operators $\to_L$, $\to_G$ and $\to_\pi$ (they are called Lukasiewicz operator, Godel operator, ans product operator, respectively) are given by:

$$a \to_L b = (1 - a + b) \wedge 1, \tag{2.4}$$

$$a \to_G b = \begin{cases} 1, & \text{if} \quad a \leq b \\ b, & \text{if} \quad a > b, \end{cases}$$

$$a \to_\pi b = \begin{cases} 1, & \text{if} \quad a = 0 \\ \frac{b}{a} \wedge 1, & \text{if} \quad a > 0. \end{cases}$$

The neighborhood system is a general structure for GrC. It has been extended to fuzzy setting by Lin [23] as follows.

Definition 2.4 ([23])**.** Let U be a universal set. A *fuzzy neighborhood system* (FNS) on U is a mapping $\widetilde{N} : U \to P(F(U))$, which associates each element x of U with a family $\widetilde{N}(x)$ of fuzzy subsets of U.

Every element of $\widetilde{N}(x)$ is called a *fuzzy neighborhood* of x. A neighborhood is also called a *granule in terms* of GrC. Fuzzy neighborhood system can be constructed by using fuzzy relations or fuzzy coverings on U.

Example 2.5. (1) Let U be a universe and $R : U \times U \to [0, 1]$ an fuzzy relation on U. Each $x \in U$ is associated with both a fuzzy predecessor neighborhood R^x_p and a fuzzy successor neighborhood R^x_s, where $R^x_p(y) = R(y, x)$ and $R^x_s(y) = R(x, y)$ for any $y \in U$. Consequently, we have three fuzzy neighborhood systems given by:

$$\widetilde{N}_p(x) = \{R^x_p\},\ \widetilde{N}_s(x) = \{R^x_s\} \text{ and } \widetilde{N}_R(x) = \{R^x_p, R^x_s\} \text{ for any } x \in U.$$

(2) Let U be a universe and $\Delta = \{C_i | i \in I\} \subseteq F(U)$ be a fuzzy covering of U, that is $\sum\limits_{i \in I} C_i(x) \geq 1$ for any $x \in U$. Δ induces an fuzzy neighborhood system N_Δ given by $N_\Delta(x) = \{C_i | i \in I \wedge C_i(x) > 0\}$ for any $x \in U$.

The notion of consistent functions with respect to fuzzy neighborhood systems is introduced to reduce information systems while preserving their basic functions.

Definition 2.6 ([19])**.** Let U and V be two universal sets, $f : U \to V$ is a mapping and $\widetilde{N}$ is a fuzzy neighborhood system on U. f is called a *consistent function with respect to* $\widetilde{N}$, if for any $x, y \in U$, $f(x) = f(y)$ implies that $M(x) = M(y)$ for any $M \in \bigcup\limits_{z \in U} \widetilde{N}(z)$.

In view of GrC, $\bigcup\limits_{z \in U} \widetilde{N}(z)$ is the granular structure induced by $\widetilde{N}$. Consistent function is also called granular consistent function in [20]. In order to investigate the closure and interior with respect to neighborhood systems, Liau et al. [20] introduced the notion of weakly system consistent function and system consistent function.

Definition 2.7 ([20]). Let U and V be two nonempty universes, and let $\widetilde{N}: U \to P(F(U))$ be an fuzzy neighborhood system on U. A mapping $f: U \to V$ is called a *weakly system-consistent function* (in short, weakly sys-consistent function) *with respect to* $\widetilde{N}$ if for any $x, y \in U$, $f(x) = f(y)$ implies that $f(\widetilde{N}(x)) = f(\widetilde{N}(y))$.

Definition 2.8 ([20]). Let U and V be two nonempty universes, and $\widetilde{N}$ be a fuzzy neighborhood system on U. Then, a mapping $f: U \to V$ is called a *system consistent function* (in short, sys-consistent function) *with respect to* $\widetilde{N}$ if for any $x, y \in U$, $f(x) = f(y)$ implies that $\widetilde{N}(x) = \widetilde{N}(y)$.

3. The interior and closure operators in FNS

The notions of closure and interior in neighborhood systems are introduced in [21] for representing the lower and upper approximations of a concept respectively. These notions are generalized to fuzzy neighborhood systems by Liau et al. [20]. In this section, we consider a further generalization form of closure and interior in fuzzy neighborhood systems on the universe.

Definition 3.1. Let $\widetilde{N}: U \to P(F(U))$ be an FNS on U and $\mu \in F(U)$. The interior $\underline{\mu}_{\widetilde{N}}$ and closure $\overline{\mu}_{\widetilde{N}}$ of μ with respect to $\widetilde{N}$ are fuzzy sets given by

$$\underline{\mu}_{\widetilde{N}}(x) = \sup_{\nu \in \widetilde{N}(x)} \inf_{y \in U} (\nu(y) \to \mu(y)) \tag{3.1}$$

$$\overline{\mu}_{\widetilde{N}}(x) = \inf_{\nu \in \widetilde{N}(x)} \sup_{y \in U} (\nu(y) \otimes \mu(y)) \tag{3.2}$$

for any $x \in U$, where $\to$ is a fuzzy implication operator and $\otimes$ is a t-norm.

We note that if $\otimes = \wedge$ and $\to$ is Godel implication operator, then (3.1) and (3.2) will degenerate to the closure and interior defined in [20].

We consider the transformation of an FNS by a mapping. Let U and V be two universes, $f: U \to V$ be a mapping and $\widetilde{N}: U \to P(F(U))$ be an FNS on U. Then $f(\widetilde{N}): V \to P(F(V))$ is an FNS on V defined by:

$$f(\widetilde{N})(y) = \cup\{f(\widetilde{N}(x)) | x \in U, f(x) = y\} \tag{3.3}$$

for any $y \in V$. It is trivial to verify that $f(\widetilde{N})(y) = \{f(A) | A \in \bigcup_{x \in f^{-1}(\{y\})} \widetilde{N}(x)\}$. We note that $f(\widetilde{N})(y) = \varnothing$ if $f^{-1}(\{y\}) = \varnothing$.

Theorem 3.2. *Let U and V be two nonempty universes, let $\widetilde{N}: U \to P(F(U))$ be an FNS, and let $f: U \to V$ ba a mapping. Then for any fuzzy subsets $\mu \in F(U)$, we have*

(1) $f(\underline{\mu}_{\widetilde{N}}) \subseteq \underline{f(\mu)}_{f(\widetilde{N})}$ *if f is consistent with respect to $\widetilde{N}$,*
(2) $\underline{\mu}_{\widetilde{N}} \subseteq f^{-1}(\underline{f(\mu)}_{f(\widetilde{N})})$ *if f is consistent with respect to $\widetilde{N}$,*
(3) $f(\overline{\mu}_{\widetilde{N}}) \subseteq \overline{f(\mu)}_{f(\widetilde{N})}$ *if f is weakly sys-consistent with respect to $\widetilde{N}$,*
(4) $\overline{\mu} \subseteq f^{-1}(\overline{f(\mu)}_{f(\widetilde{N})})$ *if f is weakly sys-consistent with respect to $\widetilde{N}$.*

Proof. (1) For any $y \in V$, if $y \in V \backslash f(U)$, $f(\underline{\mu}_{\widetilde{N}})(y) = 0$, then the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following steps.

$$\underline{f(\mu)}_{f(\widetilde{N})}(y) = \sup_{\omega \in f(\widetilde{N})(y)} \inf_{y' \in V} (\omega(y') \to f(\mu)(y')).$$

According to the definition of $f(\widetilde{N})(y)$ and $y \in f(U)$, for any $\omega \in f(\widetilde{N})(y)$, there exists $x \in f^{-1}(\{y\})$ and $\nu \in \widetilde{N}(x)$ such that $\omega = f(\nu)$. Thus we have

$$\begin{aligned} \underline{f(\mu)}_{f(\widetilde{N})}(y) &= \sup_{\nu \in \bigcup\limits_{x \in U_y} \widetilde{N}(x)} \inf_{y' \in V} (f(\nu)(y') \to f(\mu)(y')) \\ &= \sup_{x \in f^{-1}(\{y\})} \sup_{\nu \in \widetilde{N}(x)} \inf_{y' \in V} (f(\nu)(y') \to f(\mu)(y')). \end{aligned}$$

If $y' \in V \backslash f(U)$, then by Zadeh extension principle, we have $f(\nu)(y') = 0$. Thus we get

$$\begin{aligned} \inf_{y' \in V} (f(\nu)(y') \to f(\mu)(y')) &= (\inf_{y' \in f(U)} (f(\nu)(y') \to f(\mu)(y'))) \wedge (0 \to f(\mu)(y')) \\ &= (\inf_{y' \in f(U)} (f(\nu)(y') \to f(\mu)(y'))) \wedge 1 \\ &= \inf_{y' \in f(U)} (f(\nu)(y') \to f(\mu)(y')) \\ &= \inf_{x' \in U} (f(\nu)(f(x')) \to f(\mu)(f(x'))). \end{aligned}$$

So $\underline{f(\mu)}_{f(\widetilde{N})}(y) = \sup_{x \in f^{-1}(\{y\})} \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (f(\nu)(f(x')) \to f(\mu)(f(x')))$.

In addition, we have

$$f(\underline{\mu}_{\widetilde{N}})(y) = \sup_{x \in f^{-1}(\{y\})} \underline{\mu}_{\widetilde{N}}(x) = \sup_{x \in f^{-1}(\{y\})} \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to \mu(x')).$$

According to Zadeh extension principle and f is consistent with respect to $\widetilde{N}$, we have

$$f(\nu)(f(x')) = \sup_{x'' \in [x']_f} \nu(x'') = \nu(x'),$$

and

$$f(\mu)(f(x')) = \sup_{x'' \in [x']_f} \mu(x'') \geq \mu(x').$$

Hence $\nu(x') \to \mu(x') \leq \nu(x') \to f(\mu)(f(x'))$. Therefore $f(\underline{\mu}_{\widetilde{N}})(y) \leq \underline{f(\mu)}_{f(\widetilde{N})}(y)$ and the conclusion holds.

(2) By Lemma 2.1 (1) and (4), it follows immediately from (1).

(3) For any $y \in V$, if $y \in V \backslash f(U)$, $f(\overline{\mu}_{\widetilde{N}})(y) = 0$, then the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following.

$$\overline{f(\mu)}_{f(\widetilde{N})}(y) = \inf_{\omega \in f(\widetilde{N})(y)} \sup_{y' \in V} (\omega(y') \otimes f(\mu)(y')).$$

According to the definition of $f(\widetilde{N})(y)$ and $y \in f(U)$, for any $\omega \in f(\widetilde{N})(y)$, there exists $x'' \in f^{-1}(\{y\})$ and $\nu \in \widetilde{N}(x'')$, such that $\omega = f(\nu)$. Thus

$$\overline{f(\mu)}_{f(\widetilde{N})}(y) = \inf_{\nu \in \bigcup\limits_{x'' \in U_y} \widetilde{N}(x'')} \sup_{y' \in V} (f(\nu)(y') \otimes f(\mu)(y')).$$

Because f is weakly sys-consistent with respect to $\widetilde{N}$, for any $x'' \in U_y$,

$$\overline{f(\mu)}_{f(\widetilde{N})}(y) = \inf_{\nu \in \widetilde{N}(x'')} \sup_{y' \in V} (f(\nu)(y') \otimes f(\mu)(y')).$$

For any $y' \in V$, if $y' \in V \backslash f(U)$, then by Zadeh extension principle, $f(\nu)(y') = 0$. Thus

$$\begin{aligned}\sup_{y' \in V} (f(\nu)(y') \otimes f(\mu)(y')) &= (\sup_{y' \in f(U)} (f(\nu)(y') \otimes f(\mu)(y'))) \vee (0 \otimes f(\mu)(y')) \\ &= (\sup_{x' \in U} (f(\nu)(f(x')) \otimes f(\mu)(f(x')))) \vee 0 \\ &= \sup_{x' \in U} (f(\nu)(f(x')) \otimes f(\mu)(f(x'))).\end{aligned}$$

So we have for any $x'' \in U_y$,

$$\overline{f(\mu)}_{f(\widetilde{N})}(y) = \inf_{\nu \in \widetilde{N}(x'')} \sup_{x' \in U} (f(\nu)(f(x')) \otimes f(\mu)(f(x'))).$$

In addition, we get

$$f(\overline{\mu}_{\widetilde{N}})(y) = \sup_{x \in U_y} \overline{\mu}_{\widetilde{N}}(x) = \sup_{x \in U_y} \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (\nu(x') \otimes \mu(x')).$$

For any $x \in U_y$, namely, $f(x) = y$, because $\nu(x') \leq f(\nu)(f(x'))$, $\mu(x') \leq f(\mu)(f(x'))$

$$\nu(x') \otimes \mu(x') \leq f(\nu)(f(x')) \otimes f(\mu)(f(x')).$$

Hence we have

$$\inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (\nu(x') \otimes \mu(x')) \leq \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (f(\nu)(f(x')) \otimes f(\mu)(f(x'))),$$

because $f(x) = y$, $\inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (f(\nu)(f(x')) \otimes f(\mu)(f(x'))) = \overline{f(\mu)}_{f(\widetilde{N})}(y)$. Therefore, by the arbitrariness of x, we get

$$f(\overline{\mu}_{\widetilde{N}})(y) = \sup_{x \in U_y} \overline{\mu}_{\widetilde{N}}(x) \leq \overline{f(\mu)}_{f(\widetilde{N})}(y)$$

and the conclusion holds.

(4) By Lemma 2.1 (1) and (4), it follows immediately from (3). □

Theorem 3.3. *Let U and V be two nonempty universes, let $\widetilde{N} : U \to P(F(U))$ be an FNS, and let $f : U \to V$ ba a mapping. Then for any fuzzy subset $\lambda \in F(V)$, we have*

(1) $\underline{\lambda}_{f(\widetilde{N})} \subseteq f(\underline{f^{-1}(\lambda)}_{\widetilde{N}})$,

(2) $f^{-1}(\underline{\lambda}_{f(\widetilde{N})}) \subseteq \underline{f^{-1}(\lambda)}_{\widetilde{N}}$ *if f is weakly sys-consistent with respect to $\widetilde{N}$,*

(3) $f(\overline{f^{-1}(\lambda)}_{\widetilde{N}}) \subseteq \overline{\lambda}_{f(\widetilde{N})}$ *if f is weakly sys-consistent with respect to $\widetilde{N}$,*

(4) $\overline{f^{-1}(\lambda)}_{\widetilde{N}} \subseteq f^{-1}(\overline{\lambda}_{f(\widetilde{N})})$ *if f is weakly sys-consistent with respect to $\widetilde{N}$.*

Proof. (1) For any $y \in V$, if $y \in V \backslash f(U)$, $\underline{\lambda}_{f(\widetilde{N})}(y) = 0$, then the conclusion is obvious. We just need to prove the case $y \in f(U)$ in the following.

$$\underline{\lambda}_{f(\widetilde{N})}(y) = \sup_{\omega \in f(\widetilde{N})(y)} \inf_{y' \in V} (\omega(y') \to \lambda(y')).$$

According to the definition of $f(\widetilde{N})(y)$, for any $\omega \in f(\widetilde{N})(y)$, there exists $x \in U$ and $\nu \in \widetilde{N}(x)$ such that $f(x) = y$ and $\omega = f(\nu)$. Thus

$$\begin{aligned}\underline{\lambda}_{f(\widetilde{N})}(y) &= \sup_{\nu \in \bigcup\limits_{x \in U_y} \widetilde{N}(x)} \inf_{y' \in V} (f(\nu)(y') \to \lambda(y')) \\ &= \sup_{x \in U_y} \sup_{\nu \in \widetilde{N}(x)} \inf_{y' \in V} (f(\nu)(y') \to \lambda(y')).\end{aligned}$$

For any $y' \in V$, if $y' \in V \backslash f(U)$, then by Zadeh extension principle, $f(\nu)(y') = 0$. Thus

$$\begin{aligned}\inf_{y' \in V} (f(\nu)(y') \to \lambda(y')) &= (\inf_{y' \in f(U)} (f(\nu)(y') \to \lambda(y'))) \wedge (0 \to \lambda(y')) \\ &= (\inf_{x' \in U} (f(\nu)(f(x')) \to \lambda(f(x')))) \wedge 1 \\ &= \inf_{x' \in U} (f(\nu)(f(x')) \to \lambda(f(x'))).\end{aligned}$$

So $\underline{\lambda}_{f(\widetilde{N})}(y) = \sup\limits_{x \in U_y} \sup\limits_{\nu \in \widetilde{N}(x)} \inf\limits_{x' \in U} (f(\nu)(f(x')) \to \lambda(f(x')))$.
In addition, we have

$$\begin{aligned}f(\underline{f^{-1}(\lambda)}_{\widetilde{N}})(y) &= \sup_{x \in U_y} \underline{f^{-1}(\lambda)}_{\widetilde{N}}(x) \\ &= \sup_{x \in U_y} \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (v(x') \to \lambda(f(x'))).\end{aligned}$$

Because $\nu(x') \leq f(\nu)(f(x'))$ and fuzzy implication operator is decreasing with respect to the first variable, we get

$$v(x') \to \lambda(f(x')) \geq f(\nu)(f(x')) \to \lambda(f(x')).$$

Hence $f(\underline{f^{-1}(\lambda)}_{\widetilde{N}})(y) \geq \underline{\lambda}_{f(\widetilde{N})}(y)$ and the conclusion holds.

(2) For any $x \in U$,

$$\begin{aligned}f^{-1}(\underline{\lambda}_{f(\widetilde{N})})(x) &= \underline{\lambda}_{f(\widetilde{N})}(f(x)) \\ &= \sup_{\omega \in f(\widetilde{N})(f(x))} \inf_{y \in V} (\omega(y) \to \lambda(y)).\end{aligned}$$

According to the definition of $f(\widetilde{N})(f(x))$, for any $\omega \in f(\widetilde{N})(f(x))$, there exists $x'' \in U$ and $\nu \in \widetilde{N}(x'')$ such that $x'' \in [x]_f$ (namely,$f(x'') = f(x)$) and $\omega = f(\nu)$. Then we have

$$f^{-1}(\underline{\lambda}_{f(\widetilde{N})})(x) = \sup_{\nu \in \bigcup\limits_{x'' \in [x]_f} \widetilde{N}(x'')} \inf_{y \in V} (f(\nu)(y) \to \lambda(y)).$$

Because f is weakly sys-consistent with respect to $\widetilde{N}$,

$$f^{-1}(\underline{\lambda}_{f(\widetilde{N})})(x) = \sup_{\nu \in \widetilde{N}(x)} \inf_{y \in V} (f(\nu)(y) \to \lambda(y)).$$

For any $y \in V$, if $y \in V \backslash f(U)$, then by the Zadeh extension principle, $f(\nu)(y) = 0$. Thus

$$\begin{aligned}\inf_{y\in V}(f(\nu)(y) \to \lambda(y)) &= (\inf_{y\in f(U)}(f(\nu)(y) \to \lambda(y))) \wedge (0 \to \lambda(y)) \\ &= (\inf_{x'\in U}(f(\nu)(f(x')) \to \lambda(f(x')))) \wedge 1 \\ &= \inf_{x'\in U}(f(\nu)(f(x')) \to \lambda(f(x'))).\end{aligned}$$

So $f^{-1}(\underline{\lambda}_{f(\widetilde{N})})(x) = \sup_{\nu\in\widetilde{N}(x)} \inf_{x'\in U}(f(\nu)(f(x')) \to \lambda(f(x')))$.

In addition, $\underline{f^{-1}(\lambda)}_{\widetilde{N}}(x) = \sup_{\nu\in\widetilde{N}(x)} \inf_{x'\in U}(\nu(x') \to \lambda(f(x')))$.

Because $f(\nu)(f(x')) \geq \nu(x')$ and fuzzy implication operator is decreasing with respect to the first variable,

$$f(\nu)f(x')) \to \lambda(f(x')) \leq \nu(x') \to \lambda(f(x')).$$

Hence $f^{-1}(\underline{\lambda}_{f(\widetilde{N})})(x) \leq \underline{f^{-1}(\lambda)}_{\widetilde{N}}(x)$. By the arbitrariness of x, the conclusion holds.

(3) The conclusion can be proved similarly.

(4) By Lemma 2.1 (3) and (4), it follows immediately from (3). □

Theorem 3.4. *Let U and V be two nonempty universes, let $\widetilde{N} : U \to P(F(U))$ be an FNS, and let $f : U \to V$ ba a function. Then for any fuzzy subset $\lambda \in F(V)$, if f is consistent with respect to $\widetilde{N}$, then we have*

(1) $\underline{\lambda}_{f(\widetilde{N})} \supseteq f(\underline{f^{-1}(\lambda)}_{\widetilde{N}})$,

(2) $f^{-1}(\underline{\lambda}_{f(\widetilde{N})}) \supseteq \underline{f^{-1}(\lambda)}_{\widetilde{N}}$,

(3) $f(\overline{f^{-1}(\lambda)}_{\widetilde{N}}) \supseteq \overline{\lambda}_{f(\widetilde{N})}$,

(4) $\overline{f^{-1}(\lambda)}_{\widetilde{N}} \supseteq f^{-1}(\overline{\lambda}_{f(\widetilde{N})})$.

Proof. The theorem can be proved similarly. □

A common corollary of the two theorems above shows the transformation of the consistent functions with respect to interior and closure.

Corollary 3.5. *With the common notations of U, V, and $\widetilde{N}$, and f in the theorems above, then for any fuzzy subset $\lambda \in F(V)$, we have*

(1) $\underline{\lambda}_{f(\widetilde{N})} = f(\underline{f^{-1}(\lambda)}_{\widetilde{N}})$ *if f is consistent with respect to $\widetilde{N}$,*

(2) $f^{-1}(\underline{\lambda}_{f(\widetilde{N})}) = \underline{f^{-1}(\lambda)}_{\widetilde{N}}$ *if f is consistent and weakly sys-consistent with respect to $\widetilde{N}$,*

(3) $f(\overline{f^{-1}(\lambda)}_{\widetilde{N}}) = \overline{\lambda}_{f(\widetilde{N})}$ *if f is consistent and weakly sys-consistent with respect to $\widetilde{N}$,*

(4) $\overline{f^{-1}(\lambda)}_{\widetilde{N}} = f^{-1}(\overline{\lambda}_{f(\widetilde{N})})$ *if f is consistent and weakly sys-consistent with respect to $\widetilde{N}$.*

Proof. It follows immediately from Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *Let U and V be two nonempty universes, let $\widetilde{N} : U \to P(F(U))$ be an FNS, and let $f : U \to V$ ba a function. If the fuzzy implication operator and the*

t-norm form an adjoint pair, then for any fuzzy subsets $\mu \in F(U)$ and $\lambda \in F(V)$, $\mu^c \in F(U)$ (for any $x \in U$, $\mu^c(x) = \mu(x) \to 0$), we have
(1) $\overline{\mu}_{\widetilde{N}} \subseteq (\underline{\mu^c}_{\widetilde{N}})^c$,
(2) $\underline{\mu}_{\widetilde{N}} \subseteq (\overline{\mu^c}_{\widetilde{N}})^c$,
(3) $\overline{\lambda}_{f(\widetilde{N})} \subseteq (\underline{\lambda^c}_{f(\widetilde{N})})^c$,
(4) $\underline{\lambda}_{f(\widetilde{N})} \subseteq (\overline{\lambda^c}_{f(\widetilde{N})})^c$.

Proof. We prove the first and second item of the theorem, and the other ones can be proved similarly.

(1) For any $x \in U$,

$$\begin{aligned}
(\underline{\mu^c}_{\widetilde{N}})^c(x) &= \underline{\mu^c}_{\widetilde{N}}(x) \to 0 \\
&= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to \mu^c(x')) \to 0 \\
&= \inf_{\nu \in \widetilde{N}(x)} (\inf_{x' \in U} (\nu(x') \to \mu^c(x')) \to 0) \\
&\geq \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} ((\nu(x') \to \mu^c(x')) \to 0) \\
&= \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} ((\nu(x') \to (\mu(x') \to 0)) \to 0) \\
&\geq \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (\nu(x') \otimes \mu(x')) \\
&= \overline{\mu}_{\widetilde{N}}(x).
\end{aligned}$$

Then the conclusion holds.

(2) For any $x \in U$,

$$\begin{aligned}
(\overline{\mu^c}_{\widetilde{N}})^c(x) &= \overline{\mu^c}_{\widetilde{N}}(x) \to 0 \\
&= \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (\nu(x') \otimes \mu^c(x')) \to 0 \\
&\geq \sup_{\nu \in \widetilde{N}(x)} (\sup_{x' \in U} (\nu(x') \otimes \mu^c(x')) \to 0) \\
&= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} ((\nu(x') \otimes \mu^c(x')) \to 0) \\
&= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to (\mu^c(x') \to 0)) \\
&= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to ((\mu(x') \to 0) \to 0)) \\
&\geq \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to \mu(x')) \\
&= \underline{\mu}_{\widetilde{N}}(x).
\end{aligned}$$

Then the conclusion holds. □

Theorem 3.7. *Let U and V be two nonempty universes, let $\widetilde{N} : U \to P(F(U))$ be an FNS, and let $f : U \to V$ ba a mapping. If the fuzzy implication operator and the t-norm are Lukasiewicz operators as we said in (2.3) and (2.4). Then for* *any fuzzy subsets* $\mu \in F(U)$ *and* $\lambda \in F(V)$; $\mu^c \in F(U)$, *and for any* $x \in U$, $\mu^c(x) = \mu(x) \to 0 = 1 - \mu(x)$, *we have*

(1) $\overline{\mu}_{\widetilde{N}} = (\underline{\mu^c}_{\widetilde{N}})^c$,

(2) $\underline{\mu}_{\widetilde{N}} = (\overline{\mu^c}_{\widetilde{N}})^c$,

(3) $\overline{\lambda}_{f(\widetilde{N})} = (\underline{\lambda^c}_{f(\widetilde{N})})^c$,

(4) $\underline{\lambda}_{f(\widetilde{N})} = (\overline{\lambda^c}_{f(\widetilde{N})})^c$.

Proof. We prove the first and second item of the theorem , and the other ones can be proved similarly.

(1) For any $x \in U$,

$$\begin{aligned}(\underline{\mu^c}_{\widetilde{N}})^c(x) &= 1 - \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to \mu^c(x')) \\ &= \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (1 - (\nu(x') \to (1 - \mu(x')))) \\ &= \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} ((\nu(x') + \mu(x')) \vee 0) \\ &= \overline{\mu}_{\widetilde{N}}(x).\end{aligned}$$

Then the conclusion holds.

(2) For any $x \in U$,

$$\begin{aligned}(\overline{\mu^c}_{\widetilde{N}})^c(x) &= 1 - \inf_{\nu \in \widetilde{N}(x)} \sup_{x' \in U} (\nu(x') \otimes \mu^c(x')) \\ &= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (1 - (\nu(x') \otimes \mu^c(x'))) \\ &= \sup_{\nu \in \widetilde{N}(x)} \inf_{x' \in U} (\nu(x') \to \mu(x')) \\ &= \underline{\mu}_{\widetilde{N}}(x).\end{aligned}$$

Then the conclusion holds. □

4. The interior and closure operators in FNS on the image universe

Similarly, let $f : U \to V$ be a mapping and $\widetilde{N} : V \to P(F(V))$ be an FNS on V. Then, $f^{-1}(\widetilde{N}) : U \to P(F(U))$ is an FNS on U given by:

$$f^{-1}(\widetilde{N})(x) = f^{-1}(\widetilde{N}(f(x))) = \{f^{-1}(A) | A \in \widetilde{N}(f(x))\} \tag{4.1}$$

for any $x \in U$. Clearly, f is sys-consistent with respect to $f^{-1}(\widetilde{N})$.

Some preservation results about closure and interior with respect to $\widetilde{N}$ and $f(\widetilde{N})$ are presented before. In this section, we consider the related properties with respect to N and $f^{-1}(\widetilde{N})$. Let $\widetilde{N} : U \to P(F(U))$ be an FNS on U and let $W \subseteq U$. We define the restriction of $\widetilde{N}$ to W as an FNS on W, that is $\widetilde{N}|_W : W \to P(F(W))$, such that $\widetilde{N}|_W(x) = \{\mu|_W | \mu \in \widetilde{N}(x)\}$ for each $x \in W$.

Theorem 4.1. *Let* $\widetilde{N} : V \to P(F(V))$ *be an FNS on* V, $f : U \to V$ *and* $\lambda \in F(V)$.

(1) $f^{-1}(\underline{\lambda}_{\widetilde{N}}) \subseteq \underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}$.

(2) $\underline{\lambda}_{\widetilde{N}}|_{f(U)} \subseteq f(\underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})})|_{f(U)}$.

3) $f(\overline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}) \subseteq \overline{\lambda}_{\widetilde{N}}$.
(4) $\overline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})} \subseteq f^{-1}(\overline{\lambda}_{\widetilde{N}})$.
(5) $f^{-1}(\underline{\lambda}_{\widetilde{N}}) = \underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}$ *and* $\underline{\lambda}_{\widetilde{N}} = f(\underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})})$ *if* f *is surjective.*

Proof. We prove (1) and (5) of the theorem, and the other items can be proved similarly.

(1) For each $x \in U$, we have

$$f^{-1}(\underline{\lambda}_{\widetilde{N}})(x) = \underline{\lambda}_{\widetilde{N}}(f(x)) = \sup_{\nu \in \widetilde{N}(f(x))} \inf_{y \in V} (\nu(y) \to \lambda(y)).$$

In addition, according to (4.1),

$$\begin{aligned}\underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}(x) &= \sup_{\mu \in f^{-1}(\widetilde{N})(x)} \inf_{x' \in U} (\mu(x) \to f^{-1}(\lambda)(x)) \\ &= \sup_{\nu \in \widetilde{N}(f(x))} \inf_{x' \in U} (f^{-1}(\nu)(x) \to f^{-1}(\lambda)(x)) \\ &= \sup_{\nu \in \widetilde{N}(f(x))} \inf_{x' \in U} (\nu(f(x)) \to \lambda(f(x))) \\ &= \sup_{\nu \in \widetilde{N}(f(x))} \inf_{y \in f(U)} (\nu(y) \to \lambda(y)).\end{aligned}$$

Because $\inf_{y \in V} (\nu(y) \to \lambda(y)) \leq \inf_{y \in f(U)} (\nu(y) \to \lambda(y))$,

$$f^{-1}(\underline{\lambda}_{\widetilde{N}})(x) \leq \underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}(x).$$

Then the conclusion holds.

(5) For any $x \in U$ and $\nu \in \widetilde{N}(f(x))$, since f is surjective, we have

$$\inf_{y \in f(U)} (\nu(y) \to \lambda(y)) = \inf_{y \in V} (\nu(y) \to \lambda(y)).$$

By (1), we know that

$$f^{-1}(\underline{\lambda}_{\widetilde{N}})(x) = \sup_{\nu \in \widetilde{N}(f(x))} \inf_{y \in V} (\nu(y) \to \lambda(y))$$

$$\underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}(x) = \sup_{\nu \in \widetilde{N}(f(x))} \inf_{y \in f(U)} (\nu(y) \to \lambda(y)).$$

Then $f^{-1}(\underline{\lambda}_{\widetilde{N}})(x) = \underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}(x)$. Thus $f^{-1}(\underline{\lambda}_{\widetilde{N}}) = \underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})}$. In addition, we have $\underline{\lambda}_{\widetilde{N}} = f(f^{-1}(\underline{\lambda}_{\widetilde{N}})) = f(\underline{f^{-1}(\lambda)}_{f^{-1}(\widetilde{N})})$. □

Theorem 4.2. *Let* $\widetilde{N} : V \to P(F(V))$ *be an FNS on* V, $f : U \to V$ *and* $\mu \in F(U)$.
(1) $f(\underline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)} \subseteq \underline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}$.
(2) $\underline{\mu}_{f^{-1}(\widetilde{N})} \subseteq f^{-1}(\underline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}})$.
(3) $f(\underline{\mu}_{f^{-1}(\widetilde{N})}) \subseteq \underline{f(\mu)}_{\widetilde{N}}$ *and* $\underline{\mu}_{f^{-1}(\widetilde{N})} \subseteq f^{-1}(\underline{f(\mu)}_{\widetilde{N}})$ *if* f *is surjective.*
(4) $f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)} \subseteq \overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}$.
(5) $\overline{\mu}_{f^{-1}(\widetilde{N})} \subseteq f^{-1}(\overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}})$.

Proof. (1)According (4.1), for any $y \in f(U)$, $x \in U_y$ and $\nu \in f^{-1}(\widetilde{N})(x)$, we have

$$\begin{aligned}\underline{\mu}_{f^{-1}(\widetilde{N})}(x) &= \sup_{\nu \in f^{-1}(\widetilde{N})(x)} \inf_{z\in U}(\nu(z) \to \mu(z)) \\ &= \sup_{\lambda \in \widetilde{N}(y)} \inf_{z\in U}(f^{-1}(\lambda)(z) \to \mu(z)) \\ &= \sup_{\lambda \in \widetilde{N}(y)} \inf_{z\in U}(\lambda(f(z)) \to \mu(z)).\end{aligned}$$

It follows that

$$\begin{aligned}f(\underline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)}(y) &= \sup_{x\in f^{-1}(\{y\})} \underline{\mu}_{f^{-1}(\widetilde{N})}(x) \\ &= \sup_{\lambda \in \widetilde{N}(y)} \inf_{z\in U}(\lambda(f(z)) \to \mu(z)).\end{aligned}$$

In addition, we have

$$\begin{aligned}\underline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y) &= \sup_{\lambda \in \widetilde{N}(y)} \inf_{u\in f(U)}(\lambda(u) \to f(\mu)|_{f(U)}(u)) \\ &= \sup_{\lambda \in \widetilde{N}(y)} \inf_{z\in U}(\lambda(f(z)) \to f(\mu)|_{f(U)}(f(z))).\end{aligned}$$

Because $f(\mu)|_{f(U)}(f(z)) = \sup\limits_{x\in[z]_f} \mu(x) \geq \mu(z)$,

$$\lambda(f(z)) \to f(\mu)|_{f(U)}(f(z)) \geq \lambda(f(z)) \to \mu(z).$$

Then $f(\underline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)}(y) \leq \underline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y)$.

(2) It can be proved similarly.

(3) follows directly from (1) and (2).

(4) According (4.1), for any $y \in f(U)$, $x \in U_y$ and $\nu \in f^{-1}(\widetilde{N})(x)$, we have

$$\begin{aligned}\overline{\mu}_{f^{-1}(\widetilde{N})}(x) &= \inf_{\nu \in f^{-1}(\widetilde{N})(x)} \sup_{z\in U}(\nu(z) \otimes \mu(z)) \\ &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{z\in U}(f^{-1}(\lambda)(z) \otimes \mu(z)) \\ &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{z\in U}(\lambda(f(z)) \otimes \mu(z)).\end{aligned}$$

It follows that

$$\begin{aligned}f(\overline{\mu}_{f^{-1}(\widetilde{N})})(y) &= \sup_{x\in f^{-1}(\{y\})} \overline{\mu}_{f^{-1}(\widetilde{N})}(x) \\ &= \sup_{x\in f^{-1}(\{y\})} \inf_{\lambda \in \widetilde{N}(y)} \sup_{z\in U}(\lambda(f(z)) \otimes \mu(z)) \\ &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{z\in U}(\lambda(f(z)) \otimes \mu(z)).\end{aligned}$$

In addition, we have

$$\begin{aligned}\overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y) &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{v\in f(U)}(\lambda(v) \otimes f(\mu)(v)) \\ &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{z\in U}(\lambda(f(z)) \otimes f(\mu)(f(z))).\end{aligned}$$

Because $\mu(z) \leq f(\mu)(f(z))$, $\lambda(f(z)) \otimes \mu(z) \leq \lambda(f(z)) \otimes f(\mu)(f(z))$. Then we get

$$f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)}(y) \leq \overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y).$$

(5) It can be proved similarly. □

Theorem 4.3. *Let $\widetilde{N} : V \to P(F(V))$ be an FNS on V, $f : U \to V$, $\mu \in F(U)$ and $\otimes$ be a left-continuous t-norm.*

(1) $f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)} = \overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}$.

(2) $\overline{\mu}_{f^{-1}(\widetilde{N})} = f^{-1}(\overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}})$.

(3) $f(\overline{\mu}_{f^{-1}(\widetilde{N})}) = \overline{f(\mu)}_{\widetilde{N}}$ *and* $\overline{\mu}_{f^{-1}(\widetilde{N})} = f^{-1}(\overline{f(\mu)}_{\widetilde{N}})$ *if f is surjective.*

Proof. (1) For any $y \in f(U)$, by Theorem 4.2 (4), we have

$$\begin{aligned} f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)}(y) &= \sup_{x \in f^{-1}(\{y\})} \overline{\mu}_{f^{-1}(\widetilde{N})}(x) \\ &= \inf_{\lambda \in \widetilde{N}(y)} \sup_{z \in U}(\lambda(f(z)) \otimes \mu(z)), \end{aligned}$$

$$\overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y) = \inf_{\lambda \in \widetilde{N}(y)} \sup_{v \in f(U)} (\lambda(v) \otimes f(\mu)(v)).$$

For any $\lambda \in \widetilde{N}(y)$ and $v \in f(U)$, we have

$$\begin{aligned} \lambda(v) \otimes f(\mu)(v) &= \lambda(v) \otimes \sup_{x' \in f^{-1}(\{v\})} \mu(x') \\ &= \sup_{x' \in f^{-1}(\{v\})} (\lambda(v) \otimes \mu(x')) \\ &= \sup_{x' \in f^{-1}(\{v\})} (\lambda(f(x')) \otimes \mu(x')) \\ &\leq \sup_{z \in U}(\lambda(f(z)) \otimes \mu(z)). \end{aligned}$$

Then we have $\sup_{v \in f(U)} (\lambda(v) \otimes f(\mu)(v)) \leq \sup_{z \in U}(\lambda(f(z)) \otimes \mu(z))$. This gives

$$\overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}(y) \leq f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)}(y).$$

Thus $f(\overline{\mu}_{f^{-1}(\widetilde{N})})|_{f(U)} = \overline{f(\mu)|_{f(U)}}_{\widetilde{N}|_{f(U)}}$ by Theorem 4.2 (4).

(2) It can be proved similarly.

(3) It follows directly from (1) and (2). □

5. Conclusions

In the preceding account, we defined a generalized notion of interior and closure for FNS, where generalized t-norms and fuzzy implication operator. Some preservation results on the interior and closure of fuzzy neighborhood system under consistent functions are presented. We note that (granular) consistent function and weakly sys-consistent functions satisfy structural-preservation properties with respect to interior and closure operators of fuzzy neighborhood system on the universe. However, we also show that preservations are related to surjective functions in fuzzy neighborhood systems on the image universe. Compared with neighborhood systems, the basic preservations remain similar, this also confirms that our conclusion is reasonable in the sense, and some definitions and results for neighborhood systems can be generalized to the fuzzy case. Furthermore, if the t-norms and fuzzy implication operator are Lukasiewicz operators, the interior and closure can obtain special properties like the duality of upper and lower approximation operator in rough set theory.

However, due to the limitation of space, we just investigate the duality for fuzzy neighborhood system, the more similar properties can be explored in the future. For example, the upper approximation keeps the union operation, and the lower keeps the intersection. More research on properties of operators for fuzzy neighborhood systems will be completed in the future. It may be similar to the generalized rough set model, explore the properties of operators from the binary relations of reflexivity, symmetry, and transitivity. In addition, the requirements for operators may be more stringent, this adds to the difficulty of the problem.

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