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Dokdo commutative ideals of BCK-algebras

Jeong-Gon Lee, Samy M. Mostafa, Kul Hur and Young Bae Jun

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Abstract. The purpose of this paper is to study by applying Dokdo structure to commutative ideal in BCK-algebras. The notion of Dokdo commutative ideal is introduced, and their properties are investigated. The relationship between Dokdo ideal and Dokdo commutative ideal is discussed. Example to show that a Dokdo ideal may not be a Dokdo commutative ideal is provided, and then the conditions under which a Dokdo ideal can be a Dokdo commutative ideal are explored. Conditions for a Dokdo structure to be a Dokdo commutative ideal are provided, and characterizations of a Dokdo commutative ideal are displayed. Finally, the extension property for a Dokdo commutative ideal is established.

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Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

Fuzzy and soft set theory are useful tools to solve problems that retain uncertainty in everyday life. As an extension of an existing set using fuzzy logic, fuzzy sets are introduced by Zadeh. As a useful tool for considering positive information and negative information at the same time, bipolar fuzzy set is considered. Interval-valued fuzzy sets, whose membership degree range is a subinterval of $[0, 1]$, are also considered as the extension of fuzzy sets. Fuzzy set theory, bipolar fuzzy set theory, interval-valued fuzzy set theory and soft set theory are good mathematical tools for dealing with uncertainty in a parametric manner, and have many applications in medical diagnosis and decision making etc. In the information age, there is an increasing need to use hybrid structures in various fields. To present the mathematical tools needed to meet these needs, it became necessary to study mixed structures based on logical algebra. Hybrid structures dealing with two or more different concepts at the same time have the advantage of reducing the loss of information when addressing uncertainty issues. To address these backgrounds and needs, Jun introduced a new type of hybrid structure called *Dokdo structure*, where “Dokdo” is the name of the most beautiful island in Korea, using the concept of bipolar fuzzy set, soft set, and interval value fuzzy set and first applied it to BCK/BCI-algebras (See [1]).

In this paper, we apply Dokdo structure to commutative ideal of BCK-algebras. We introduce the notion of Dokdo commutative ideal, and investigate their properties. We discuss the relationship between Dokdo ideal and Dokdo commutative ideal. We provide example to show that any Dokdo ideal may not be a Dokdo commutative ideal, and then we explore the conditions under which Dokdo ideal can be Dokdo commutative ideal. We provide conditions for a Dokdo structure to be a Dokdo commutative ideal. We explore the characterization of Dokdo commutative ideal and establish an extension property for Dokdo commutative ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (See [2] and [3]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [4, 5] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I_1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,
(I_2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,
(I_3) $(\forall a \in X)\ (a * a = 0)$,
(I_4) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*.

The order relation “$\leq$” in a BCK/BCI-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b\ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions (See [4, 5]):

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2.2}$$

$$(\forall a, b, c \in X)\,(a \leq b\ \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a)\,, \tag{2.3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{2.4}$$

Every BCI-algebra X satisfies (See [4]):

$$(\forall a, b \in X)\,(a * (a * (a * b)) = a * b)\,, \tag{2.5}$$

$$(\forall a, b \in X)\,(0 * (a * b) = (0 * a) * (0 * b))\,. \tag{2.6}$$

A BCK-algebra X is said to be *commutative* (See [5]), if $a * (a * b) = b * (b * a)$ for all $a, b \in X$. We will abbreviate commutative BCK-algebra to cBCK-algebra.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (See [4, 5]), if it satisfies:

$$0 \in A, \tag{2.7}$$

$$(\forall a, b \in X)(a * b \in A,\ b \in A \ \Rightarrow\ a \in A). \tag{2.8}$$

A subset A of a BCK-algebra X is called a *commutative ideal* of X (See [6]), if it satisfies (2.7) and

$$(\forall a, b, c \in X)((a * b) * c \in A,\ c \in A \ \Rightarrow\ a * (b * (b * a)) \in A). \tag{2.9}$$

Let X be a set. A *bipolar fuzzy set* in X (see [7]) is an object of the following type

$$\mathring{\xi} = \{(a, \mathring{\xi}^-(a), \mathring{\xi}^+(a)) \mid a \in X\} \tag{2.10}$$

where $\mathring{\xi}^- : X \to [-1, 0]$ and $\mathring{\xi}^+ : X \to [0, 1]$ are mappings. The bipolar fuzzy set which is described in (2.10) is simply denoted by $\mathring{\xi} := (X; \mathring{\xi}^-, \mathring{\xi}^+)$.

A bipolar fuzzy set can be reinterpreted as a function:

$$\mathring{\xi} : X \to [-1, 0] \times [0, 1],\ a \mapsto (\mathring{\xi}^-(a), \mathring{\xi}^+(a)).$$

Denote by $BF(X)$ the set of all bipolar fuzzy sets in X. We define a binary relation "$\leq_b$" on $BF(X)$ as follows:

$$(\forall \mathring{\xi}, \mathring{\eta} \in BF(X)) \left(\mathring{\xi} \leq_b \mathring{\eta} \ \Leftrightarrow \left\{ \begin{array}{l} \mathring{\xi}^-(a) \geq \mathring{\eta}^-(a) \\ \mathring{\xi}^+(a) \leq \mathring{\eta}^+(a) \end{array} \right. \quad \text{for all } a \in X \right). \tag{2.11}$$

Then $(\mathrm{BF(X)}, \leq_b)$ is a poset.

Let U be an initial universe set and X be a set of parameters. For any subset A of X, a pair (ξ^s, A) is called a *soft set* over U (See [8, 9]), where ξ^s is a mapping described as follows:

$$\xi^s : A \to 2^U$$

where 2^U is the power set of U. If $A = X$, the soft set (ξ^s, A) over U is simply denoted by ξ^s only.

A mapping $\tilde{\xi} : X \to [[0, 1]]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X (See [10, 11]) where $[[0, 1]]$ is the set of all closed subintervals of $[0, 1]$, and members of $[[0, 1]]$ are called *interval numbers* and are denoted by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, etc., where $\tilde{a} = [a_L, a_R]$ with $0 \leq a_L \leq a_R \leq 1$.

For every two interval numbers $\tilde{a}$ and $\tilde{b}$, we define

$$\tilde{a} \preceq \tilde{b} \text{ (or } \tilde{b} \succeq \tilde{a}) \ \Leftrightarrow\ a_L \leq b_L,\ a_R \leq b_R, \tag{2.12}$$

$$\tilde{a} = \tilde{b} \ \Leftrightarrow\ \tilde{a} \preceq \tilde{b},\ \tilde{b} \preceq \tilde{a}, \tag{2.13}$$

$$\mathrm{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a_L, b_L\}, \min\{a_R, b_R\}]. \tag{2.14}$$

Let U be an initial universe set and X a set of parameters. A triple $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is called a *Dokdo structure* (see [1]) in (U, X) if $\mathring{\xi} : X \to [-1, 0] \times [0, 1]$ is a bipolar fuzzy set in X, $\xi^s : X \to 2^U$ is a soft set over U and $\tilde{\xi} : X \to [[0, 1]]$ is an interval-valued fuzzy set in X.

The Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in (U, X) can be represented as follows:

$$Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi}) : X \to ([-1,0]\times[0,1]) \times 2^U \times [[0,1]],\quad x \mapsto \left(\mathring{\xi}(x), \xi^s(x), \tilde{\xi}(x)\right) \tag{2.15}$$

where $\mathring{\xi}(x) = (\mathring{\xi}^-(x), \mathring{\xi}^+(x))$ and $\tilde{\xi}(x) = [\tilde{\xi}_L(x), \tilde{\xi}_R(x)]$.

Given a Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in a Dokdo universe (U, X), we consider the following sets:

$$\mathring{\xi}(\max,\min) := \left\{ \tfrac{x}{(y,z)} \in \tfrac{X}{X\times X} \,\middle|\, \begin{array}{l} \mathring{\xi}^-(x) \le \max\{\mathring{\xi}^-(y), \mathring{\xi}^-(z)\} \\ \mathring{\xi}^+(x) \ge \min\{\mathring{\xi}^+(y), \mathring{\xi}^+(z)\} \end{array} \right\}.$$

In what follows, let U be an initial universe set and X a set of parameters unless otherwise specified. We say that the pair (U, X) is a BCK-Dokdo universe (resp., BCI-Dokdo universe), if X is a BCK-algebra (resp., BCI-algebra). If X is a BCK-algebra or a BCI-algebra, the pair (U, X) is simply called *Dokdo* universe.

Definition 2.1 ([1])**.** Let (U, X) be a Dokdo universe. A Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in (U, X) is called a *Dokdo subalgebra* of X, if it satisfies:

$$(\forall x, y \in X)\left(\tfrac{x*y}{(x,y)} \in \mathring{\xi}(\max,\min)\right), \tag{2.16}$$

$$(\forall x, y \in X)\,(\xi^s(x*y) \supseteq \xi^s(x) \cap \xi^s(y)), \tag{2.17}$$

$$(\forall x, y \in X)\left(\tilde{\xi}(x*y) \succeq \operatorname{rmin}\{\tilde{\xi}(x), \tilde{\xi}(y)\}\right). \tag{2.18}$$

Definition 2.2 ([1])**.** Let (U, X) be a Dokdo universe. A Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in (U, X) is called a *Dokdo ideal* of X, if it satisfies:

$$(\forall x \in X)\left(\begin{array}{l} \tfrac{0}{(x,x)} \in \mathring{\xi}(\max,\min), \\ \xi^s(0) \supseteq \xi^s(x), \\ \tilde{\xi}(0) \succeq \tilde{\xi}(x) \end{array}\right). \tag{2.19}$$

$$(\forall x, y \in X)\left(\begin{array}{l} \tfrac{x}{(x*y,\,y)} \in \mathring{\xi}(\max,\min), \\ \xi^s(x) \supseteq \xi^s(x*y) \cap \xi^s(y), \\ \tilde{\xi}(x) \succeq \operatorname{rmin}\{\tilde{\xi}(x*y), \tilde{\xi}(y)\} \end{array}\right). \tag{2.20}$$

Lemma 2.3 ([1])**.** *Every Dokdo ideal $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ of X satisfies:*

$$(\forall x, y \in X)\left(x \le y \;\Rightarrow\; \left\{\begin{array}{l} \tfrac{x}{(y,y)} \in \mathring{\xi}(\max,\min) \\ \xi^s(x) \supseteq \xi^s(y) \\ \tilde{\xi}(x) \succeq \tilde{\xi}(y) \end{array}\right.\right). \tag{2.21}$$

3. Dokdo commutative ideals

In this section, we define a Dokdo commutative ideal in a BCK-algebra, and investigate related properties. The symbol X and (U, X) in this section represent a BCK-algebra and a BCK-Dokdo universe, respectively, unless otherwise specified.

Definition 3.1. A Dokdo structure $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ in (U, X) is called a *Dokdo commutative ideal* of X, if it satisfies (2.19) and

$$(3.1)\qquad (\forall x, y, z \in X)\left(\begin{array}{l} \frac{x*(y*(y*x))}{((x*y)*z,\, z)} \in \mathring{\xi}(\max, \min), \\ \xi^s(x*(y*(y*x))) \supseteq \xi^s((x*y)*z) \cap \xi^s(z), \\ \tilde{\xi}(x*(y*(y*x))) \succeq \operatorname{rmin}\{\tilde{\xi}((x*y)*z), \tilde{\xi}(z)\} \end{array}\right).$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation "$*$" which is given in the following Cayley Table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then X is a BCK-algebra (See [5]). Consider a Dokdo structure $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ in (U, X) defined as below.

X	$\mathring{\xi}(x)$	$\xi^s(x)$	$\tilde{\xi}(x)$
0	$(-0.77, 0.88)$	α_3	$[0.49, 0.86]$
1	$(-0.57, 0.73)$	α_2	$[0.33, 0.73]$
2	$(-0.57, 0.73)$	α_2	$[0.33, 0.73]$
3	$(-0.67, 0.58)$	α_1	$[0.25, 0.57]$

where $\varnothing \neq \alpha_1 \subsetneq \alpha_2 \subsetneq \alpha_3$ in 2^U. It is routine to verify that $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo commutative ideal of X.

We discuss the relationship between Dokdo ideal and Dokdo commutative ideal.

Theorem 3.3. *Every Dokdo commutative ideal is a Dokdo ideal.*

Proof. Let $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ be a Dokdo commutative ideal of X. If we put $y = 0$ in (3.1) and use (K) and (2.2), then

$$\frac{x}{(x*z,\, z)} = \frac{x*(0*(0*x))}{((x*0)*z,\, z)} \in \mathring{\xi}(\max, \min),$$
$$\xi^s(x) = \xi^s(x*(0*(0*x))) \supseteq \xi^s((x*0)*z) \cap \xi^s(z) = \xi^s(x*z) \cap \xi^s(z),$$
$$\tilde{\xi}(x) = \tilde{\xi}(x*(0*(0*x))) \succeq \operatorname{rmin}\{\tilde{\xi}((x*0)*z), \tilde{\xi}(z)\} = \operatorname{rmin}\{\tilde{\xi}(x*z), \tilde{\xi}(z)\}$$

for all $x, z \in X$. Therefore $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo ideal of X. □

The example below informs the existence of the Dokdo ideal, not the Dokdo commutative ideal.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation "$*$" which is given in the following Cayley Table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then X is a BCK-algebra (see [5]). Consider a Dokdo structure $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ in $(U := \mathbb{Z}, X)$ defined as below.

X	$\mathring{\xi}(x)$	$\xi^s(x)$	$\tilde{\xi}(x)$
0	$(-0.65, 0.82)$	$\mathbb{Z}$	$[0.47, 0.89]$
1	$(-0.65, 0.54)$	$2\mathbb{N}$	$[0.42, 0.79]$
2	$(-0.51, 0.82)$	$4\mathbb{N}$	$[0.33, 0.71]$
3	$(-0.51, 0.54)$	$8\mathbb{N}$	$[0.28, 0.57]$
4	$(-0.51, 0.54)$	$8\mathbb{N}$	$[0.28, 0.57]$

It is routine to verify that $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo ideal of X. But it is not a Dokdo commutative ideal of X since $\frac{2*(3*(3*2))}{((2*3)*0,\, 0)} = \frac{2}{((2*3)*0,\, 0)} = \frac{2}{(0,\, 0)} \notin \mathring{\xi}(\max, \min)$, and/or $\xi^s(2*(3*(3*2))) = \xi^s(2) = 4\mathbb{N} \not\supseteq \mathbb{Z} = \xi^s((2*3)*0) \cap \xi^s(0)$.

We explore the conditions under which Dokdo ideal can be Dokdo commutative ideal.

Lemma 3.5 ([1]). *Every Dokdo ideal $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ of X satisfies:*

$$(\forall x, y, z \in X)\left(x * y \leq z \;\Rightarrow\; \begin{cases} \frac{x}{(y,z)} \in \mathring{\xi}(\max, \min) \\ \xi^s(x) \supseteq \xi^s(y) \cap \xi^s(z) \\ \tilde{\xi}(x) \succeq \mathrm{rmin}\{\tilde{\xi}(y), \tilde{\xi}(z)\} \end{cases} \right). \tag{3.2}$$

Theorem 3.6. *In a commutative BCK-algebra, every Dokdo ideal is a Dokdo commutative ideal.*

Proof. Let X be a commutative BCK-algebra and let $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ be a Dokdo ideal of X. Using (I_1), (I_3), (2.4) and the commutativity of X, we have

$$\begin{aligned} &((x*(y*(y*x)))*((x*y)*z))*z = ((x*(y*(y*x)))*z)*((x*y)*z) \\ &\leq (x*(y*(y*x)))*(x*y) = (x*(x*y))*(y*(y*x)) = 0, \end{aligned}$$

that is, $(x*(y*(y*x)))*((x*y)*z) \leq z$ for all $x, y, z \in X$. It follows from Lemma 3.5 that

$$\begin{aligned} &\tfrac{x*(y*(y*x))}{((x*y)*z,\, z)} \in \mathring{\xi}(\max, \min) \\ &\xi^s(x*(y*(y*x))) \supseteq \xi^s((x*y)*z) \cap \xi^s(z) \\ &\tilde{\xi}(x*(y*(y*x))) \succeq \mathrm{rmin}\{\tilde{\xi}((x*y)*z), \tilde{\xi}(z)\}. \end{aligned}$$

Therefore $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo commutative ideal of X. □

Corollary 3.7. *If a BCK-algebra X satisfies any of the following conditions:*

(1) $x*(x*y) \leq y*(y*x)$,
(2) $x*(x*y) = y*(y*(x*(x*y)))$,
(3) $x \leq y$ *implies* $x = y*(y*x)$,

for all $x, y \in X$, then every Dokdo ideal is a Dokdo commutative ideal.

Corollary 3.8. *If a BCK-algebra X is a lower semilattice with respect to the order relation "$\leq$", then every Dokdo ideal is a Dokdo commutative ideal.*

Proof. Let X be a BCK-algebra which is a lower semilattice with respect to the order relation "$\leq$". Let $x, y \in X$. Then $x*(x*y)$ is a common lower bound of x and y and $y*(y*x)$ is the greatest lower bound of x and y. Hence $x*(x*y) \leq y*(y*x)$, and so every Dokdo ideal is a Dokdo commutative ideal by Corollary 3.7. $\square$

Theorem 3.9. *If a Dokdo ideal* $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ *of* X *satisfies:*

$$(3.3)\qquad (\forall x, y, z \in X)\left(\begin{array}{l} \frac{(x*z)*(y*(y*x))}{((x*y)*z,\,(x*y)*z)} \in \mathring{\xi}(\max, \min) \\ \xi^s((x*z)*(y*(y*x))) \supseteq \xi^s((x*y)*z) \\ \tilde{\xi}((x*z)*(y*(y*x))) \succeq \tilde{\xi}((x*y)*z) \end{array}\right),$$

then $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ *is a Dokdo commutative ideal of* X.

Proof. Let $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ be a Dokdo ideal of X that satisfies the condition (3.3). The combination of (2.4), (2.20), and (3.3) leads to

$$\begin{aligned} \mathring{\xi}^-(x*(y*(y*x))) &\leq \max\{\mathring{\xi}^-((x*(y*(y*x)))*z), \mathring{\xi}^-(z)\} \\ &= \max\{\mathring{\xi}^-((x*z)*(y*(y*x))), \mathring{\xi}^-(z)\} \\ &\leq \max\{\mathring{\xi}^-((x*y)*z), \mathring{\xi}^-(z)\}, \end{aligned}$$

$$\begin{aligned} \mathring{\xi}^+(x*(y*(y*x))) &\geq \min\{\mathring{\xi}^+((x*(y*(y*x)))*z), \mathring{\xi}^+(z)\} \\ &= \min\{\mathring{\xi}^+((x*z)*(y*(y*x))), \mathring{\xi}^+(z)\} \\ &\geq \min\{\mathring{\xi}^+((x*y)*z), \mathring{\xi}^+(z)\}, \end{aligned}$$

that is, $\frac{x*(y*(y*x))}{((x*y)*z,\, z)} \in \mathring{\xi}(\max, \min)$, and

$$\begin{aligned} \xi^s(x*(y*(y*x))) &\supseteq \xi^s((x*(y*(y*x)))*z) \cap \xi^s(z) \\ &= \xi^s((x*z)*(y*(y*x))) \cap \xi^s(z) \\ &\supseteq \xi^s((x*y)*z) \cap \xi^s(z), \end{aligned}$$

$$\begin{aligned} \tilde{\xi}(x*(y*(y*x))) &\succeq \text{rmin}\{\tilde{\xi}((x*(y*(y*x)))*z), \tilde{\xi}(z)\} \\ &= \text{rmin}\{\tilde{\xi}((x*z)*(y*(y*x))), \tilde{\xi}(z)\} \\ &\succeq \text{rmin}\{\tilde{\xi}((x*y)*z), \tilde{\xi}(z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ is a Dokdo commutative ideal of X. $\square$

We consider characterizations of a Dokdo commutative ideal.

Theorem 3.10. *A Dokdo structure* $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ *in* (U, X) *is a Dokdo commutative ideal of* X *if and only if it is a Dokdo ideal of* X *that satisfies the condition below.*

$$(3.4)\qquad (\forall x, y \in X)\left(\begin{array}{l} \frac{x*(y*(y*x))}{(x*y,\,x*y)} \in \mathring{\xi}(\max, \min) \\ \xi^s(x*(y*(y*x))) \supseteq \xi^s(x*y) \\ \tilde{\xi}(x*(y*(y*x))) \succeq \tilde{\xi}(x*y) \end{array}\right).$$

Proof. Assume that $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in (U, X) is a Dokdo commutative ideal of X. Then it is a Dokdo ideal of X by Theorem 3.3. If we put $z = 0$ in (3.1), and use (2.2) and (2.19), then (3.4) is derived.

Conversely, let $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ be a Dokdo ideal of X that satisfies the condition (3.4). For any $x, y, z \in X$, we have

$$\mathring{\xi}^-(x*(y*(y*x))) \le \mathring{\xi}^-(x*y) \le \max\{\mathring{\xi}^-((x*y)*z), \mathring{\xi}^-(z)\},$$
$$\mathring{\xi}^+(x*(y*(y*x))) \ge \mathring{\xi}^+(x*y) \ge \min\{\mathring{\xi}^+((x*y)*z), \mathring{\xi}^+(z)\},$$

that is, $\frac{x*(y*(y*x))}{((x*y)*z,\, z)} \in \mathring{\xi}(\max, \min)$, and

$$\xi^s(x*(y*(y*x))) \supseteq \xi^s(x*y) \supseteq \xi^s((x*y)*z) \cap \xi^s(z),$$
$$\tilde{\xi}(x*(y*(y*x))) \succeq \tilde{\xi}(x*y) \succeq \text{rmin}\{\tilde{\xi}((x*y)*z), \tilde{\xi}(z)\}.$$

Therefore $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ is a Dokdo commutative ideal of X. □

Given a Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ in (U, X), we consider the following sets:

$$\mathring{\xi}(t^-) := \{x \in X \mid \mathring{\xi}^-(x) \le t^-\},$$
$$\mathring{\xi}(t^+) := \{x \in X \mid \mathring{\xi}^+(x) \ge t^+\},$$
$$\mathring{\xi}(t^-, t^+) := \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+),$$
$$\xi^s_\alpha := \{x \in X \mid \xi^s(x) \supseteq \alpha\},$$
$$\tilde{\xi}_{\tilde{a}} := \{x \in X \mid \tilde{\xi}(x) \succeq \tilde{a}\},$$

where $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.

We consider the characterization of a Dokdo (commutative) ideal.

Lemma 3.11. *A Dokdo structure $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ is a Dokdo ideal of X if and only if the nonempty sets $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.*

Proof. Assume that $Dok_\xi := (\mathring{\xi},\, \xi^s,\, \tilde{\xi})$ is a Dokdo ideal of X and $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are nonempty for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. It is clear that $0 \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$ by (2.19). Let $x, y \in X$ be such that $y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$ and $x*y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$. Then $\mathring{\xi}^-(y) \le t^-$, $\mathring{\xi}^+(y) \ge t^+$, $\xi^s(y) \supseteq \alpha$, $\tilde{\xi}(y) \succeq \tilde{a}$, $\mathring{\xi}^-(x*y) \le t^-$, $\mathring{\xi}^+(x*y) \ge t^+$, $\xi^s(x*y) \supseteq \alpha$, and $\tilde{\xi}(x*y) \succeq \tilde{a}$. It follows from (2.20) that

$$\mathring{\xi}^-(x) \le \max\{\mathring{\xi}^-(x*y), \mathring{\xi}^-(y)\} \le t^-,$$
$$\mathring{\xi}^+(x) \ge \min\{\mathring{\xi}^+(x*y), \mathring{\xi}^+(y)\} \ge t^+,$$
$$\xi^s(x) \supseteq \xi^s(x*y) \cap \xi^s(y) \supseteq \alpha,$$
$$\tilde{\xi}(x) \succeq \text{rmin}\{\tilde{\xi}(x*y), \tilde{\xi}(y)\} \succeq \tilde{a}.$$

Hence $x \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$, and therefore $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are ideals of X.

Conversely, suppose that $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are nonempty ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. Then they are subalgebras of X. Let $x, y \in X$ be such that $Dok_\xi(x) := (\mathring{\xi}(x),\ \xi^s(x),\ \tilde{\xi}(x)) = ((t^-_x, t^+_x), \alpha_x, \tilde{a}_x)$ and $Dok_\xi(y) := (\mathring{\xi}(y),\ \xi^s(y),\ \tilde{\xi}(y)) = ((t^-_y, t^+_y), \alpha_y, \tilde{a}_y)$. If we take

$$((t^-, t^+), \alpha, \tilde{a}) = ((\max\{t^-_x, t^-_y\}, \min\{t^+_x, t^+_y\}), \alpha_x \cap \alpha_y, \mathrm{rmin}\{\tilde{a}_x, \tilde{a}_y\}),$$

then $x, y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$ and so $x * y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$. If we put $x = y$ and use (I_3), then $\frac{0}{(x,x)} \in \mathring{\xi}(\max, \min)$, $\xi^s(0) \supseteq \xi^s(x)$ and $\tilde{\xi}(0) \succeq \tilde{\xi}(x)$ for all $x \in X$. Let $x, y \in X$ be such that $Dok_\xi(c) := (\mathring{\xi}(c),\ \xi^s(c),\ \tilde{\xi}(c)) = ((t^-_c, t^+_c), \alpha_c, \tilde{a}_c)$ and $Dok_\xi(y) := (\mathring{\xi}(y),\ \xi^s(y),\ \tilde{\xi}(y)) = ((t^-_y, t^+_y), \alpha_y, \tilde{a}_y)$ where $c := x * y$. Taking

$$((t^-, t^+), \alpha, \tilde{a}) = ((\max\{t^-_c, t^-_y\}, \min\{t^+_c, t^+_y\}), \alpha_c \cap \alpha_y, \mathrm{rmin}\{\tilde{a}_c, \tilde{a}_y\})$$

implies that $c := x * y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$ and $y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$. It follows that $x \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$. Hence

$$\begin{aligned} &\tfrac{x}{(x*y,\ y)} \in \mathring{\xi}(\max, \min), \\ &\xi^s(x) \supseteq \xi^s(x * y) \cap \xi^s(y), \\ &\tilde{\xi}(x) \succeq \mathrm{rmin}\{\tilde{\xi}(x * y), \tilde{\xi}(y)\}. \end{aligned}$$

Therefore $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ is a Dokdo ideal of X. □

Lemma 3.12 ([5]). *A subset A of X is a commutative ideal of X if and only if it is an ideal of X that satisfies:*

$$(\forall x, y \in X)(x * y \in A \ \Rightarrow\ x * (y * (y * x)) \in A). \tag{3.5}$$

Theorem 3.13. *A Dokdo structure $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ is a Dokdo commutative ideal of X if and only if the nonempty sets $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are commutative ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.*

Proof. Suppose that $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ is a Dokdo commutative ideal of X. Then it is a Dokdo ideal of X by Theorem 3.3, and so the nonempty sets $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$ by Lemma 3.11. Let $x, y \in X$ be such that $x * y \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$. Then $\mathring{\xi}^-(x * y) \leq t^-$, $\mathring{\xi}^+(x * y) \geq t^+$, $\xi^s(x * y) \supseteq \alpha$, and $\tilde{\xi}(x * y) \succeq \tilde{a}$. It follows from (3.4) that

$$\begin{aligned} &\mathring{\xi}^-(x * (y * (y * x))) \leq \max\{\mathring{\xi}^-(x * y), \mathring{\xi}^-(x * y)\} \leq t^-, \\ &\mathring{\xi}^+(x * (y * (y * x))) \geq \min\{\mathring{\xi}^+(x * y), \mathring{\xi}^+(x * y)\} \geq t^+, \\ &\xi^s(x * (y * (y * x))) \supseteq \xi^s(x * y) \supseteq \alpha, \\ &\tilde{\xi}(x * (y * (y * x))) \succeq \tilde{\xi}(x * y) \succeq \tilde{a}. \end{aligned}$$

Hence $x * (y * (y * x)) \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}$, and therefore $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are commutative ideals of X by Lemma 3.12.

Assume that $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are nonempty commutative ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. Then they are ideals of X, and hence $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo ideal of X by Lemma 3.11. Let $x, y \in X$ be such that $Dok_\xi(c) := (\mathring{\xi}(c), \xi^s(c), \tilde{\xi}(c)) = ((t_c^-, t_c^+), \alpha_c, \tilde{a}_c)$ where $c := x * y$. Then $x*y \in \mathring{\xi}(t_c^-) \cap \mathring{\xi}(t_c^+) \cap \xi^s_{\alpha_c} \cap \tilde{\xi}_{\tilde{a}_c}$, which implies from Lemma 3.12 that $x*(y*(y*x)) \in \mathring{\xi}(t_c^-) \cap \mathring{\xi}(t_c^+) \cap \xi^s_{\alpha_c} \cap \tilde{\xi}_{\tilde{a}_c}$. It follows that

$$\mathring{\xi}^-(x*(y*(y*x))) \leq t_c^- = \max\{t_c^-, t_c^-\} = \max\{\mathring{\xi}^-(x*y), \mathring{\xi}^-(x*y)\},$$
$$\mathring{\xi}^+(x*(y*(y*x))) \geq t_c^+ = \min\{t_c^+, t_c^+\} = \min\{\mathring{\xi}^+(x*y), \mathring{\xi}^+(x*y)\},$$

that is, $\frac{x*(y*(y*x))}{(x*y, x*y)} \in \mathring{\xi}(\max, \min)$, and $\xi^s(x*(y*(y*x))) \supseteq \alpha_c = \xi^s(x*y)$ and $\tilde{\xi}(x*(y*(y*x))) \succeq \tilde{a}_c = \tilde{\xi}(x*y)$. It follows from Theorem 3.10 that $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo commutative ideal of X. □

Corollary 3.14. *If $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo commutative ideal of X, then the nonempty sets $\mathring{\xi}(t^-, t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are commutative ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.*

Proof. Straightforward. □

The convese of Corollary 3.14 is not ture as seen in the following example.

Example 3.15. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation "$*$" which is given in the following Cayley Table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Then X is a BCK-algebra (see [5]). Consider a Dokdo structure $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ in $(U := \mathbb{N}, X)$ defined as below.

X	$\mathring{\xi}(x)$	$\xi^s(x)$	$\tilde{\xi}(x)$
0	$(-0.77, 0.83)$	$\mathbb{N}$	$[0.48, 0.87]$
1	$(-0.63, 0.55)$	$8\mathbb{N}$	$[0.42, 0.79]$
2	$(-0.51, 0.76)$	$4\mathbb{N}$	$[0.32, 0.59]$
3	$(-0.46, 0.64)$	$2\mathbb{N}$	$[0.38, 0.67]$
4	$(-0.32, 0.43)$	$16\mathbb{N}$	$[0.28, 0.47]$

It is routine to verify that $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is a Dokdo ideal of X and the nonempty sets $\mathring{\xi}(t^-, t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are commutative ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. We have $\frac{2*(4*(4*2))}{(2*4, 2*4)} = \frac{2}{(0,0)} \notin \mathring{\xi}(\max, \min)$, $\xi^s(3*(4*(4*3))) = \xi^s(3) = 2\mathbb{N} \not\supseteq \mathbb{N} = \xi^s(0) = \xi^s(3*4)$, and/or $\tilde{\xi}(1*(4*(4*1))) = \tilde{\xi}(1) = [0.42, 0.79] \not\succeq [0.48, 0.87] = \tilde{\xi}(0) = \tilde{\xi}(1*4)$. Hence $Dok_\xi := (\mathring{\xi}, \xi^s, \tilde{\xi})$ is not a Dokdo commutative ideal of X by Theorem 3.10.

Note that a Dokdo ideal might not be a Dokdo commutative ideal (See Example 3.4). But we can consider the extension property for a Dokdo commutative ideal.

Lemma 3.16 ([5]). *Let A and B be ideals of X such that $A \subseteq B$. If A is a commutative ideal of X, then so is B.*

Theorem 3.17. *Let $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ and $Dok_\eta := (\mathring{\eta},\ \eta^s,\ \tilde{\eta})$ be Dokdo ideals of X such that $\mathring{\xi}(0) = \mathring{\eta}(0)$, $\xi^s(0) = \eta^s(0)$, $\tilde{\xi}(0) = \tilde{\eta}(0)$, $\mathring{\xi} \leq_b \mathring{\eta}$, $\eta^s(x) \supseteq \xi^s(x)$ and $\tilde{\eta}(x) \succeq \tilde{\xi}(x)$ for all $x(\neq 0) \in X$. If $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ is a Dokdo commutative ideal of X, then so is $Dok_\eta := (\mathring{\eta},\ \eta^s,\ \tilde{\eta})$.*

Proof. Assume that $Dok_\xi := (\mathring{\xi},\ \xi^s,\ \tilde{\xi})$ is a Dokdo commutative ideal of X. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. Then $\mathring{\xi}(t^-)$, $\mathring{\xi}(t^+)$, ξ^s_α and $\tilde{\xi}_{\tilde{a}}$ are commutative ideals of X whenever they are nonempty by Theorem 3.13, and it is clear that the given condition induces $\mathring{\xi}(t^-) \subseteq \mathring{\eta}(t^-)$, $\mathring{\xi}(t^+) \subseteq \mathring{\eta}(t^+)$, $\xi^s_\alpha \subseteq \eta^s_\alpha$ and $\tilde{\xi}_{\tilde{a}} \subseteq \tilde{\eta}_{\tilde{a}}$. Since $Dok_\eta := (\mathring{\eta},\ \eta^s,\ \tilde{\eta})$ is a Dokdo ideal of X, the nonempty sets $\mathring{\eta}(t^-)$, $\mathring{\eta}(t^+)$, η^s_α and $\tilde{\eta}_{\tilde{a}}$ are ideals of X for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$ by Lemma 3.11. Let $x, y \in X$ be such that $x * y \in \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}$. Using (I_3) and (2.4), we have

$$(x * (x * y)) * y = (x * y) * (x * y) = 0 \in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}}.$$

It follows from (2.4) and Lemma 3.12 that

$$\begin{aligned}&(x * (y * (y * (x * (x * y))))) * (x * y) = (x * (x * y)) * (y * (y * (x * (x * y))))\\&\in \mathring{\xi}(t^-) \cap \mathring{\xi}(t^+) \cap \xi^s_\alpha \cap \tilde{\xi}_{\tilde{a}} \subseteq \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}.\end{aligned}$$

Since $x * y \in \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}$, we have

$$x * (y * (y * (x * (x * y)))) \in \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}$$

by (2.8). Combinations of (I_1), (I_3), (K) and (2.4) derive the following:

$$\begin{aligned}&(x * (y * (y * x))) * (x * (y * (y * (x * (x * y)))))\\&\leq (y * (y * (x * (x * y)))) * (y * (y * x))\\&\leq (y * x) * (y * (x * (x * y)))\\&\leq (x * (x * y)) * x = 0 \in \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}.\end{aligned}$$

Hence $x * (y * (y * x)) \in \mathring{\eta}(t^-) \cap \mathring{\eta}(t^+) \cap \eta^s_\alpha \cap \tilde{\eta}_{\tilde{a}}$, and thus $\mathring{\eta}(t^-)$, $\mathring{\eta}(t^+)$, η^s_α and $\tilde{\eta}_{\tilde{a}}$ are commutative ideals of X by Lemma 3.12. Therefore $Dok_\eta := (\mathring{\eta},\ \eta^s,\ \tilde{\eta})$ is a Dokdo commutative ideal of X by Theorem 3.13. □

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References

[1] Y. B. Jun, Dokdo structure and its application in *BCK*/*BCI*-algebras, TWMS J. Pure Appl. Math. (submitted).
[2] K. Iséki, On *BCI*-algebras, Math. Seminar Notes 8 (1980) 125–130.
[3] K. Iséki and S. Tanaka, An introduction to the theory of *BCK*-algebras, Math. Japon. 23 (1978) 1–26.
[4] Y. Huang, *BCI*-algebra, Science Press: Beijing, China 2006.
[5] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co. Seoul, Korea 1994.

[6] J. Meng, Commutative ideals in BCK-algebras, Pure Appl. Math. (in Chinese) 9 (1991) 49–53.
[7] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok Thailand (2000) 307–312.
[8] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[9] D. Molodtsov, Soft set theory - first results, Comput. Math. Appl. 37 (1999) 19–31.
[10] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.
[11] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

Jeong Gon Lee (jukolee@wku.ac.kr)
Division of Applied Mathematics,
Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Samy M. Mostafa (samymostafa@yahoo.com)
Department of Mathematics,
Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt

Kul Hur (kulhur@wku.ac.kr)
Division of Applied Mathematics,
Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea