Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On the convergence of series of fuzzy numbers

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Received 20 September 2021; Revised 7 October 2021; Accepted 17 November 2021

Abstract. In this paper, we study some criteria for three types of convergence of series of fuzzy numbers, using the usual order relation extended to intervals and fuzzy numbers. Then, we prove a fuzzy version of Abel theorem for fuzzy series, we introduce the fuzzy Cauchy product of two fuzzy series, which we apply to prove the main properties of the exponential of a fuzzy number. To illustrate our theoretical results, we exhibit some examples namely the fuzzy geometric and telescopic series.

2020 AMS Classification: 26E50, 46S40, 47N20, 47S40

Keywords: Fuzzy series , Fuzzy Abel theorem, Absolute convergence, Fuzzy comparison test, Fuzzy ratio test, Fuzzy root test.

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1. Introduction

The study of the sequences of fuzzy numbers was initiated by Matloka [1], who proved many results similar in form to the classical literature and established that every convergent sequence was bounded and proved the elementary operations on fuzzy convergent sequences. Then Nanda studied the spaces of bounded and convergent sequences of fuzzy numbers and showed that every Cauchy sequence of fuzzy numbers is convergent [2]. In [3], Talo and Çakan determined necessary and sufficient Tauberian conditions under which convergence follows from Cesàro-convergence of sequences of fuzzy numbers. Talo and Başar introduced the concept of the slow decreasing sequence for fuzzy numbers and established that Cesàro summable sequence is convergent if it is slowly decreasing in [4]. Furthermore, Yavuz and Talo introduced the Abel convergence of sequences and series of fuzzy numbers in [5] and generalized some tauberian results in classical analysis to fuzzy analysis. In [6], Kadak and Başar studied the the convergence of power series of fuzzy numbers and introduced the notion of a power series of fuzzy numbers with fuzzy coefficients. Çanak investigated the Riesz mean of sequences of fuzzy real numbers in [7]. Önder and Çanak in [8], then Önder et al. in [9], established some Tauberian theorems for Cesaro summability of sequences of fuzzy numbers, and proved some Tauberian type theorems for the weighted mean method of summability of sequences of fuzzy numbers. Sezer and Çanak introduced the power series method of summability for a series of fuzzy numbers and established some Tauberian conditions to obtain convergence of a fuzzy series from its summability by power series method [10].

Recently, Mursaleen and Başar summarized the literature on some sets of fuzzy valued sequences and series in their interesting book [11]. They introduced the classes consisting of all bounded, convergent, null and absolutely p-summable fuzzy valued sequences. Motivated by the later works, we establish that the absolute convergence implies the convergence of fuzzy series. Then, we give a necessary and sufficient condition for the convergence of a telescopic fuzzy series and we exhibit some criteria and tests for the convergence, the convergence in norm and the absolute convergence of series of fuzzy numbers: comparison test, ratio test and root test. Moreover, we prove a fuzzy version of Abel theorem using a fuzzy Abel transform. As first application, we extend the results on convergence and the sum's expression of the geometric series $\sum u^n$ of fuzzy numbers, given in [10] only for $\widetilde{0} < u < \widetilde{1}$, to general case $-\widetilde{1} < u < \widetilde{1}$ as in the classical theory. And as second application, we define the fuzzy Cauchy product of two fuzzy series, then we introduce the exponential of a fuzzy number and establish some of its main properties.

Please notice that our main purpose here is to enrich the theory of fuzzy series with tools, which we can exploit in next works to study fuzzy differential equations and develop their solutions as sum of fuzzy power series.

The main obstacle of dealing with fuzzy series (and fuzzy differential equations) is the fact that the spaces of fuzzy numbers, fuzzy sequences and fuzzy-valued functions are not linear spaces. In particular, they are not groups with respect to addition and the scalar multiplication is not, in general, distributive with respect to usual addition of scalars. To overcome all these difficulties we will take advantage of the nice properties of the Hausdorff distances, which make all these sets complete metric spaces [12].

The remainder of this paper is organized as follows. Section 2 is devoted for some preliminaries. And section 3 is reserved to study convergence and absolute convergence of series of fuzzy numbers. In section 4 some tests of absolute convergence for series of fuzzy numbers are given and in section 5 some criteria of convergence for fuzzy series are investigated. Then, section 6 treats a fuzzy version of Abel theorem. Section 7 deals with the geometric series of fuzzy numbers (introduced in [6]), its convergence and the calculus of its sum are extended to general case $-\widetilde{1} < u < \widetilde{1}$. In section 8 we introduce the fuzzy Cauchy product of two fuzzy series and in section 9, we define the exponential of a fuzzy number and prove some of its properties. In the last section, we present conclusion and a further research topic.

2. Preliminaries

Let $P_K(\mathbb{R})$ be the family of all nonempty compact convex subsets of $\mathbb{R}$. The distance between two nonempty bounded subsets A and B of $\mathbb{R}$, is defined by the Hausdorff metric

$$d(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\Big\}.$$

Then it is clear that $(\mathcal{P}_K(\mathbb{R}), d)$ becomes a complete and separable metric space [13].

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \longrightarrow [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e. there exits an element $x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y))$, for any x, $y \in \mathbb{R}$ and $\lambda \in [0,1]$,
(iii) u is upper semi-continuous,
(iv) The closure $[u]^0 = \overline{\{x \in \mathbb{R} | u(x) > 0\}}$ of the support of u is compact.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E or E^1 and called it as the *space of fuzzy numbers*. For $0 < \lambda \leq 1$, denote $[u]_\lambda = \{x \in \mathbb{R} \mid u(x) \geq \lambda\}$.
Then from (i)-(iv), it follows that the λ-level set $[u]_\lambda \in P_K(\mathbb{R})$, i.e., $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for all $0 \leq \lambda \leq 1$.

The following properties hold true, for all $u, v \in E, k \in \mathbb{R}$ and $\lambda \in [0,1]$,

$$[u+v]_\lambda = [u]_\lambda + [v]_\lambda\,, \quad [ku]_\lambda = k\,[u]_\lambda \text{ and } [uv]_\lambda = [u]_\lambda\,[v]_\lambda\,,$$

Theorem 2.1 (Representation Theorem [14]). *Let $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for all $u \in E$ and $0 \leq \lambda \leq 1$. Then the following statements hold:*

(1) *$u^-(\lambda)$ is a bounded non-decreasing left continuous function in $(0,1]$,*
(2) *$u^+(\lambda)$ is a bounded non-increasing left continuous function in $(0,1]$,*
(3) *$u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$,*
(4) *$u^-(1) \leq u^+(1)$.*

Conversely, if the pair of functions α and β satisfies the conditions (i)-(iv), then there exists a unique $u \in E$ such that $[u]_\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $0 \leq \lambda \leq 1$. Moreover, u is defined by

$$u : \mathbb{R} \longrightarrow [0,1],\ u(x) = \sup\{\lambda; \alpha(\lambda) \leq x \leq \beta(\lambda)\}.$$

A crisp number k is simply represented by $u^-(\lambda) = u^+(\lambda) = k$ for each $0 \leq \lambda \leq 1$.
We denote $\widetilde{0}$ the fuzzy number defined by its membership function as follows

$$\widetilde{0}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

We define the Hausdorff metric D on E by means of the Hausdorff distance d as follows

$$D(u,v) = \sup_{0 \leq \lambda \leq 1} d\Big([u]_\lambda\,,\,[v]_\lambda\Big) = \sup_{0 \leq \lambda \leq 1} \max\{|u^-(\lambda) - v^+(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}. \tag{2.1}$$

In particular,

$$D(u, \widetilde{0}) = \sup_{0 \leq \lambda \leq 1} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{2.2}$$

Proposition 2.2 ([13, 15]). *Let $u, v, w, z \in E$ and $k \in \mathbb{R}$. Then*

(1) *(E, D) is a complete metric space,*
(2) *$D(u+z, v+z) = D(u,v)$,*
(3) *$D(ku, kv) = |k| D(u,v)$,*
(4) *$D(u+v, w+z) \leq D(u,w) + D(v,z)$.*

The natural order relation on the real line can be extended to intervals as follows:

$$A \preceq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B},$$

where $A = [\underline{A}, \overline{A}]$ and $B = [\underline{B}, \overline{B}]$. The partial ordering relation on E is defined as follows:
$u \preceq v \Leftrightarrow [u]_\lambda \preceq [v]_\lambda$ for all $\lambda \in [0,1]$, i.e., $u^-(\lambda) \leq v^-(\lambda)$ and $u^+(\lambda) \leq v^+(\lambda)$ $\forall \lambda \in [0,1]$.

Definition 2.3. An *absolute value* $|u|$ of a fuzzy number u is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\} & (t \geq 0) \\ 0 & (t < 0). \end{cases}$$

λ-level set $[|u|]_\lambda$ of $|u|$ is in the form $[|u|]_\lambda = [|u|^-(\lambda), |u|^+(\lambda)]$, where

$$|u|^-(\lambda) = \max\{0, u^-(\lambda), -u^+(\lambda)\}, |u|^+(\lambda) = \max\{|u^-(\lambda)|, |u^+(\lambda)|\}.$$

Proposition 2.4 ([16]). *Let $u, v, m \in E$ with $m \succeq \widetilde{0}$ and $k \in \mathbb{R}$. Then*

(1) $|u| = \begin{cases} u & u \succeq \widetilde{0} \\ -u & u \prec \widetilde{0}, \end{cases}$
(2) $|u + v| \preceq |u| + |v|$,
(3) $|ku| = |k||u|$,
(4) $|u| = \widetilde{0}$ *if and only if* $u = \widetilde{0}$,
(5) $|u| \preceq m$ *if and only if* $-m \preceq u \preceq m$.

Lemma 2.5 ([17]). *The following statements hold:*

(1) *For all $u \in E$ and any $a, b \in \mathbb{R}$ such that $ab \geq 0$ or $ab \leq 0$, we have $(a + b)u = au + bu$.*
(2) *For all $u, v \in E$ and any $a \in \mathbb{R}$, we have $a(u + v) = au + av$.*
(3) *For all $u \in E$ and any $a, b \in \mathbb{R}$, we have $a(bu) = (ab)u$.*

Definition 2.6 ([18]). Let u be any non-negative fuzzy number. We define u^n for non-zero real number n by

$$u^n(x) = \begin{cases} u(x^{1/n}) & (x > 0) \\ 0 & (x \leq 0). \end{cases}$$

The λ-level set of the fuzzy number u^n with $[u]_\lambda = [u^-_\lambda, u^+_\lambda]$ is determined as follows:

$$[u^n]_\lambda = \{x : u^n(x) \geq \lambda\} = \{x : u(x^{1/n}) \geq \lambda\} = [(u^-_\lambda)^n, (u^+_\lambda)^n].$$

In the case $n = 0$, we define u^0 by

$$u^0(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0). \end{cases}$$

Lemma 2.7 (Basic Lemma [6]). *Let $u \in E$ and $n \in \mathbb{N}$. Then the following four statements hold: for u^n,*

(1) *If $u^-_\lambda, u^+_\lambda > 0$, then $[u]^n_\lambda = [(u^-_\lambda)^n, (u^+_\lambda)^n]$.*
(2) *If $u^-_\lambda < 0$ and $u^+_\lambda > 0$, then $[u]^n_\lambda = [(u^+_\lambda)^{n-1}u^-_\lambda, (u^+_\lambda)^n]$.*
(3) *If $u^-_\lambda, u^+_\lambda < 0$, then $[u]^n_\lambda = [(u^+_\lambda)^n, (u^-_\lambda)^n]$, if n is even and $[u]^n_\lambda = [(u^-_\lambda)^n, (u^+_\lambda)^n]$, if n is odd.*
(4) *If $u^-_\lambda < 0$ and $u^+_\lambda = 0$, then $[u]^n_\lambda = [0, (u^-_\lambda)^n]$, if n is even and $[u]^n_\lambda = [(u^-_\lambda)^n, 0]$, if n is odd.*

Now, we give some background materials concerning sequences and series of fuzzy numbers, required throughout this work.

Definition 2.8 ([1]). A *sequence $u = (u_k)$ of fuzzy numbers* is a function u from the set $\mathbb{N}$ into E. The fuzzy number u_k is the value of the function at $k \in \mathbb{N}$ and is called as the *kth term* of the sequence.

By $w(F)$ we denote the set of all sequences of fuzzy numbers.

Definition 2.9. A sequence $(u_k) \in w(F)$ is said to be *convergent with limit $u \in E$*, if for every $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_k, u) < \varepsilon$ for all $k \geq n_0$.

If the sequence $(u_k) \in w(F)$ converges to a fuzzy number u then the sequences of functions $\{u^-_k(\lambda)\}$ and $\{u^+_k(\lambda)\}$ are uniformly convergent to $u^-(\lambda)$ and $u^+(\lambda)$ in $[0, 1]$ respectively.

Lemma 2.10 ([4]). *The following statements hold:*

(1) $D(uv, \widetilde{0}) \leq D(u, \widetilde{0})D(v, \widetilde{0})$ *for all $u, v \in E$.*

(2) *If* $(u_k) \in w(F)$ *such that* $u_k \longrightarrow u$ *as* $k \to \infty$*, then* $D(u_k, \widetilde{0}) \longrightarrow D(u, \widetilde{0})$ *as* $k \to \infty$*.*

Definition 2.11. A sequence $(u_k) \in w(F)$ is said to be *bounded*, if the set of fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set, i.e., there exist two fuzzy numbers m and M such that $m \preceq u_k \preceq M$ for all $k \in \mathbb{N}$. This means that $m^-(\lambda) \le u_k^-(\lambda) \le M^-(\lambda)$ and $m^+(\lambda) \le u_k^+(\lambda) \le M^+(\lambda)$ for all $\lambda \in [0,1]$.

Then the boundedness of the sequence $(u_k) \in w(F)$ is equivalent to the uniform boundedness of the functions $u_k^-(\lambda)$ and $u_k^+(\lambda)$ on $[0,1]$.

Definition 2.12. For $u, v \in E$, if there exists $w \in E$ such that $u = v + w$, then w is called the *Hukuhara difference of* u *and* v, and denoted by $u \ominus v$.

Definition 2.13 ([10])**.** Let $(u_k) \in w(F)$ Then the expression $\sum u_k$ is called a *series of fuzzy numbers* and denotes $s_n = \sum\limits_{k=0}^{n} u_k$ for all $n \in \mathbb{N}$.

If the sequence (s_n) converges to a fuzzy number u, then we say that the series $\sum u_k$ of fuzzy numbers *converges to* u and write $u = \sum\limits_{k=0}^{\infty} u_k$ which implies that

$$\textstyle\sum_{k=0}^{n} u_k^-(\lambda) \longrightarrow u^-(\lambda) \quad \text{and} \quad \sum_{k=0}^{n} u_k^+(\lambda) \longrightarrow u^+(\lambda), \text{ uniformly in } \lambda \in [0,1], \text{ as } n \to \infty.$$

Conversely, if $\sum u_k^-(\lambda) = u^-(\lambda)$ and $\sum u_k^+(\lambda) = u^+(\lambda)$ converge uniformly in λ, then

$$u = \{(u^-(\lambda), u^+(\lambda)); \lambda \in [0,1]\}$$

defines a fuzzy number such that $u = \sum\limits_{k=0}^{\infty} u_k$.

We say otherwise the series of fuzzy numbers *diverges*.

Lemma 2.14 ([4])**.** *If the fuzzy numbers* $u = \{(u^-(\lambda), u^+(\lambda)); \lambda \in [0,1]\}$, $\sum u_k^-(\lambda) = u^-(\lambda)$ *and* $\sum u_k^+(\lambda) = u^+(\lambda)$ *converge uniformly in* λ*, then* $u = \{(u^-(\lambda), u^+(\lambda)); \lambda \in [0,1]\}$ *defines a fuzzy number such that* $u = \sum\limits_{k=0}^{\infty} u_k$*.*

Theorem 2.15 ([4])**.** *If* $\sum u_k$ *and* $\sum v_k$ *converge, then* $D\left(\sum\limits_{k=0}^{\infty} u_k, \sum\limits_{k=0}^{\infty} v_k\right) \le \sum\limits_{k=0}^{\infty} D(u_k, v_k)$*.*

3. Series of fuzzy numbers

Now we give new definitions and properties concerning fuzzy series. Set $\|u\| = D(u, \widetilde{0})$, which we call "norm" of $u \in E$. Please notice that since E is not a vector space, then the mapping $u \mapsto \|u\|$ is not an effective norm on E, even if it verifies all the axioms of a norm.
Some results concerning the convergence in norm were studied in [11], and we suggest different proofs for some of them.

Lemma 3.1. *Let* $(u_k), (v_k) \in w(F)$ *such that* $u_k \longrightarrow a, v_k \longrightarrow b$ *and* $u_k \ominus v_k$ *exists for all* $k \in \mathbb{N}$*. Then* $a \ominus b$ *exists and* $u_k \ominus v_k \longrightarrow a \ominus b$ *as* $k \to \infty$*.*

Proof. Let $k, p \in \mathbb{N}$ and set $w_k = u_k \ominus v_k$, i.e., $u_k = v_k + w_k$. Then there exists $k_0 \in \mathbb{N}$ for all $k \ge k_0$,

$$D(u_{k+p}, u_k) \le \varepsilon/2 \text{ and } D(v_k, v_{k+p}) \le \varepsilon/2.$$

Thus for all $k \ge k_0$, we have

$$D(w_{k+p}, w_k) = D(w_{k+p} + v_{k+p} + v_k, w_k + v_{k+p} + v_k)$$

$= D(u_{k+p} + v_k, u_k + v_{k+p})$
$\leq D(u_{k+p}, u_k) + D(v_k, v_{k+p})$
$\leq \varepsilon.$

So (w_k) is a Cauchy sequence in the complete metric space (E, D). Hence (w_k) converges to a certain $c \in E$. Furthermore $u_k = v_k + w_k \longrightarrow b + c$ as $k \to \infty$. From the uniqueness of the limit of (u_k), we get $a = b + c$. Therefore $a \ominus b$ exists and $c = a \ominus b$. □

Theorem 3.2 (Necessary condition [11])**.** *If the series $\sum u_k$ converges, then $u_k \longrightarrow \widetilde{0}$ as $k \to \infty$.*

Proof. For this result, we suggest a different proof to that in [11].

Let $\sum u_k$ be a convergent series and set $s_k = \sum_{i=0}^{k} u_i$. One can remark that

$$s_k = s_{k-1} + u_k \quad \text{for all } k \geq 1.$$

Then the Hukuhara difference $s_k \ominus s_{k-1}$ exists and $s_k \ominus s_{k-1} = u_k$. Since $\sum u_k$ converges, (s_k) converges to $u \in E$. Thus by Lemma 3.1, $s_k \ominus s_{k-1} \longrightarrow u \ominus u = \widetilde{0}$ as $k \to \infty$, i.e., $u_k \longrightarrow \widetilde{0}$ as $k \to \infty$. □

Definition 3.3. We say that the series $\sum u_k$ of fuzzy numbers is *absolutely convergent*, if the series $\sum |u_k|$ (of nonnegative fuzzy numbers) converges.

Remark 3.4. Notice that our new definition of the absolute convergence is similar to the same definition in the crisp case, it is different from the definition according to the authors in [11], their notion of the absolute convergence is here called the convergence in norm.

Theorem 3.5 ([11])**.** . *If the series $\sum u_k$ of fuzzy numbers is absolutely convergent, then $\sum u_k$ is convergent and*

$$\left| \sum_{k=0}^{\infty} u_k \right| \preceq \sum_{k=0}^{\infty} |u_k|. \tag{3.1}$$

Proof. For the first part of this result, we suggest a different proof to that in [11].

Assume that $\sum u_k$ is absolutely convergent, i.e., $\sum |u_k|$ converges. Then $\sum |u_k|^+(0)$ converges. Set $s_n = \sum_{k=0}^{n} u_k$ and $t_n = \sum_{k=0}^{n} |u_k|$ for all $n \in \mathbb{N}$. Then for each $n, p \in \mathbb{N}$, we have

$$D(s_{n+p}, s_n) = D\left(s_n + \sum_{k=n+1}^{n+p} u_k, s_n\right) = D\left(\sum_{k=n+1}^{n+p} u_k, \widetilde{0}\right) \leq \sum_{k=n+1}^{n+p} D(u_k, \widetilde{0}) = \sum_{k=n+1}^{n+p} |u_k|^+(0).$$

Since the numeric series $\sum |u_k|^+(0)$ converges, i.e., (t_n) converges,

$$\sum_{k=n+1}^{n+p} |u_k|^+(0) \longrightarrow 0 \text{ as } n \to \infty.$$

Thus $D(s_{n+p}, s_n) \longrightarrow 0$ as $n \to \infty$, i.e., (s_n) is a Cauchy sequence in the complete metric space (E, D). So (s_n) converges, i.e., $\sum u_k$ is convergent. By induction and from Proposition 2.4, we obtain $|s_n| \preceq t_n$. Hence for all $\lambda \in [0, 1]$,

$$\left| \sum_{k=0}^{n} u_k^- \right| (\lambda) \leq \sum_{k=0}^{n} |u_k|^-(\lambda) \text{ and } \left| \sum_{k=0}^{n} u_k^+ \right| (\lambda) \leq \sum_{k=0}^{n} |u_k|^+(\lambda).$$

By tending $n \to \infty$, we get

$$\left|\sum_{k=0}^{\infty} u_k^-\right|(\lambda) \leq \sum_{k=0}^{\infty} |u_k|^-(\lambda) \text{ and } \left|\sum_{k=0}^{\infty} u_k^+\right|(\lambda) \leq \sum_{k=0}^{\infty} |u_k|^+(\lambda).$$

This yields the required inequality (3.1), which is not given in [11]. □

Definition 3.6. We say that the series $\sum u_k$ of fuzzy numbers is *convergent in norm*, if the numeric series $\sum \|u_k\| = \sum D(u_k, \widetilde{0})$ converges.

Theorem 3.7. *If the series $\sum u_k$ of fuzzy numbers is convergent in norm, then $\sum u_k$ is convergent and we have*

$$D\left(\sum_{k=0}^{\infty} u_k, \widetilde{0}\right) \leq \sum_{k=0}^{\infty} D(u_k, \widetilde{0}), \quad \textit{i.e.,} \quad \left\|\sum_{k=0}^{\infty} u_k\right\| \leq \sum_{k=0}^{\infty} \|u_k\|. \tag{3.2}$$

Proof. Assume that $\sum u_k$ is convergent in norm, i.e., $\sum D(u_k, \widetilde{0})$ converges. Set $s_n = \sum_{k=0}^{n} u_k$ and $t_n = \sum_{k=0}^{n} D(u_k, \widetilde{0})$ for all $n \in \mathbb{N}$. Then for each $n, p \in \mathbb{N}$, we have

$$D(s_{n+p}, s_n) = D\left(s_n + \sum_{k=n+1}^{n+p} u_k, s_n\right) = D\left(\sum_{k=n+1}^{n+p} u_k, \widetilde{0}\right) \leq \sum_{k=n+1}^{n+p} D(u_k, \widetilde{0}) = t_{n+p} - t_n.$$

Since the numeric series $\sum D(u_k, \widetilde{0})$ converges, i.e., (t_n) converges, $t_{n+p} - t_n \longrightarrow 0$ as $n \to \infty$. Thus $D(s_{n+p}, s_n) \longrightarrow 0$ as $n \to \infty$. So the Cauchy sequence (s_n) converges in the complete metric space (E, D), i.e., $\sum u_k$ is convergent. The required inequality (3.2) follows immediately from Theorem 2.15. □

Theorem 3.8. *If the series $\sum u_k$ of fuzzy numbers is absolutely convergent, then $\sum u_k$ is convergent in norm and*

$$D\left(\sum_{k=0}^{\infty} u_k, \widetilde{0}\right) \leq \sum_{k=0}^{\infty} D(u_k, \widetilde{0}) = \sum_{k=n+1}^{n+p} |u_k|^+(0). \tag{3.3}$$

Proof. This is a rapid consequence of the identity $D(u_k, \widetilde{0}) = |u_k|^+(0)$. □

Example 3.9. Let $x \in E \setminus \{\widetilde{0}\}$ and $u_k = \frac{1}{k}x$ for all $k \geq 1$ the general term of fuzzy harmonic series, so

$$D(s_n, \widetilde{0}) = D\left(\sum_{k=1}^{n} \frac{1}{k}x, \widetilde{0}\right) = \sum_{k=1}^{n} \frac{1}{k} D(x, \widetilde{0}) \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Then (s_n) is not bounded and then, (s_n) is not a Cauchy sequence. Thus $\sum u_k$ diverges.

Example 3.10. Let $x \in E, \alpha \in \mathbb{R}$ and $u_k = \frac{1}{k^\alpha}x$ for all $k \geq 1$ the general term of the fuzzy Riemann series.

If $\alpha > 1$, then $\sum D(u_k, \widetilde{0}) = \sum \frac{1}{k^\alpha} D(x, \widetilde{0})$ converges. Thus $\sum u_k$ converges, since it is convergent in norm by Theorem 3.7.

If $\alpha \leq 0$, then $D(u_k, \widetilde{0}) \nrightarrow 0$ as $k \longrightarrow \infty$. Thus $\sum u_k$ diverges.

If $0 < \alpha < 1$, then it is well known that

$$\sum_{k=1}^{n} \frac{1}{k^\alpha} \to \infty \text{ as } k \longrightarrow \infty.$$

Thus if $x \neq \widetilde{0}$, then we get

$$D(s_n, \widetilde{0}) = D\left(\sum_{k=1}^{n} \frac{1}{k^\alpha} x, \widetilde{0}\right) = \sum_{k=1}^{n} \frac{1}{k^\alpha} D(x, \widetilde{0}) \longrightarrow \infty \text{ as} \, n \longrightarrow \infty.$$

So (s_n) is not bounded. Hence (s_n) is not a Cauchy sequence. Therefore $\sum u_k$ diverges.

Definition 3.11. By a *fuzzy telescopic series*, we mean a fuzzy series $\sum u_k$ with general term in the form $u_k = v_k \ominus v_{k-1}$ for all k, where $(v_k) \in w(F)$ such that the Hukuhara difference $v_k \ominus v_{k-1}$ exists for all $k \geq 1$.

Lemma 3.12. *Under the same notations and assumptions as in Definition 3.11. The telescopic series $\sum (v_k \ominus v_{k-1})$ converges if and only if the sequence (v_k) converges. And in this case, we have*

$$\sum_{k=1}^{\infty} (v_k \ominus v_{k-1}) = \lim_{k \to \infty} v_k \ominus v_0. \tag{3.4}$$

Proof. By induction, we show that for all $k \geq 1$, $s_k = \sum\limits_{i=1}^{k} u_i = \sum\limits_{i=1}^{k} (v_i \ominus v_{i-1}) = v_k \ominus v_0$.
Then (s_k) converges if and only if (v_k) converges, i.e., $\sum u_k$ converges if and only if (v_k) converges. By tending $k \to \infty$ in the previous identity, we get

$$\sum_{k=1}^{\infty} (v_k \ominus v_{k-1}) = \lim_{k \to \infty} s_k = \lim_{k \to \infty} v_k \ominus v_0.$$

□

Example 3.13. Let $x \in E$ and $u_k = \frac{1}{k(k-1)} x$ all $k \geq 2$. Note $\frac{1}{k(k-1)} x + \frac{1}{k} x = (\frac{1}{k(k-1)} + \frac{1}{k}) x = \frac{1}{k-1} x$. Then the H-difference $v_{k-1} \ominus v_k = \frac{1}{k(k-1)} x \ominus \frac{1}{k-1} x$ exits and $u_k = \frac{1}{k-1} x \ominus \frac{1}{k} x$. Since $D(\frac{1}{k} x, \widetilde{0}) = \frac{1}{k} D(x, \widetilde{0}) \longrightarrow 0$ as $k \to \infty$, the sequence $(\frac{1}{k} x)$ converges to $\widetilde{0}$. Thus from Lemma 3.12, we deduce that $\sum u_k$ converges and $\sum\limits_{k=2}^{\infty} \frac{1}{k(k-1)} x = v_2 \ominus \lim\limits_{k \to \infty} v_k = x$.

4. Criteria of convergence in norm for series of fuzzy numbers

Theorem 4.1 (Comparison Test 1)**.** *Let $(u_k), (v_k) \in w(F)$ such that $D(u_k, \widetilde{0}) \leq D(v_k, \widetilde{0})$ for all $k \in \mathbb{N}$. If $\sum v_k$ is convergent in norm, then $\sum u_k$ is convergent in norm and*

$$\sum_{k=0}^{\infty} D(u_k, \widetilde{0}) \leq \sum_{k=0}^{\infty} D(v_k, \widetilde{0}).$$

Proof. Set $t_n = \sum\limits_{k=0}^{n} D(u_k, \widetilde{0})$ and $q_n = \sum\limits_{k=0}^{n} D(v_k, \widetilde{0})$ for all $n \in \mathbb{N}$. Then it is clear that

$$0 \leq t_n \leq q_n \leq \sum_{k=1}^{\infty} D(v_k, \widetilde{0}) < \infty.$$

Thus (t_n) is a nondecreasing bounded sequence, i.e., (t_n) converges. So $\sum u_k$ is convergent in norm and by tending $n \longrightarrow \infty$, we get $\sum_{k=0}^{\infty} D(u_k, \widetilde{0}) \leq \sum_{k=0}^{\infty} D(v_k, \widetilde{0})$. □

Theorem 4.2 (Ratio Test 1 [11])**.** *Let $(u_k) \in w(F)$ such that $D(u_k, \widetilde{0}) > 0$ and*

$$\lim_{k\to\infty} D(u_{k+1}, \widetilde{0})/D(u_k, \widetilde{0}) = l, \text{ where } l \in \mathbb{R}_+ \cup \{\infty\}.$$

If $l < 1$, then $\sum u_k$ is convergent in norm and if $l > 1$, then $\sum u_k$ is divergent.

Proof. This theorem is an immediate consequence of the classical "Ratio Test" for numeric series. □

Remark 4.3. Under the same hypothesis of Theorem 4.2, we deduce that

(1) If $l < 1$ then, $\lim_{k\to\infty} u_k = \widetilde{0}$.
(2) If $l > 1$ then, $\lim_{k\to\infty} D(u_k, \widetilde{0}) = \infty$ and the sequence (u_k) is divergent.

Example 4.4. Let $a \in E$ and $u_k = \frac{1}{(2k)!} \prod_{i=1}^{k} (a + i\widetilde{1})$ for all $k \geq 1$, where $\widetilde{1}(\lambda) = [\lambda, 2 - \lambda]$ for each $\lambda \in [0, 1]$. Set $v_k = D(u_{k+1}, \widetilde{0})/D(u_k, \widetilde{0})$. Then utilizing Lemma 2.10, we get

$$\begin{aligned} v_k &= \frac{D(\prod_{i=1}^{k}(a+i\widetilde{1})(a+(k+1)\widetilde{1}), \widetilde{0})}{(2k+2)(2k+1)D(\prod_{i=1}^{k}(a+i\widetilde{1}), \widetilde{0})} \\ &\leq \frac{D(\prod_{i=1}^{k}(a+i\widetilde{1}), \widetilde{0})\, D((a+(k+1)\widetilde{1}), \widetilde{0})}{(2k+2)(2k+1)D(\prod_{i=1}^{k}(a+i\widetilde{1}), \widetilde{0})} \\ &= \frac{D((a+(k+1)\widetilde{1}), \widetilde{0})}{(2k+2)(2k+1)} \\ &\leq \frac{1}{(2k+2)(2k+1)} D(a, \widetilde{0}) + \frac{(k+1)D(\widetilde{1}, \widetilde{0})}{(2k+2)(2k+1)}. \end{aligned}$$

It is clear that $\frac{D(a,\widetilde{0})}{(2k+2)(2k+1)} + \frac{(k+1)D(\widetilde{1},\widetilde{0})}{(2k+2)(2k+1)}$ tends to 0 as $k \to \infty$. Thus $\lim_{k\to\infty} D(u_{k+1}, \widetilde{0})/D(u_k, \widetilde{0}) = 0$. So by applying the Ratio Test 1, we deduce that $\sum u_k$ is convergent in norm.

Theorem 4.5 (Root Test 1 [11])**.** *Let $(u_k) \in w(F)$ such that $D(u_k, \widetilde{0}) > 0$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \sqrt[k]{D(u_k, \widetilde{0})} = L, \text{ where } L \in \mathbb{R}_+ \cup \{\infty\}.$$

If $L < 1$, then $\sum u_k$ is convergent in norm and if $L > 1$ or $L = \infty$, then $\sum u_k$ is divergent.

Proof. This theorem follows from the classical "Root Test" for series of real numbers. □

Remark 4.6. Under the same hypothesis of Theorem 4.5, we deduce that

(1) If $L < 1$ then, $\lim_{k\to\infty} u_k = \widetilde{0}$.
(2) If $L > 1$ then, $\lim_{k\to\infty} D(u_k, \widetilde{0}) = \infty$ and the sequence (u_k) is divergent.

5. Criteria of convergence for series of fuzzy numbers

Theorem 5.1 (Comparison Test 2)**.** *Let $(u_k),(v_k)\in w(F)$ such that $\widetilde{0}\leq u_k\leq v_k$ for all $k\in\mathbb{N}$.*

1. *If $\sum v_k$ is convergent, then $\sum u_k$ is convergent and $\widetilde{0}\leq\sum\limits_{k=0}^{\infty}u_k\leq\sum\limits_{k=0}^{\infty}v_k$.*
2. *If $\sum u_k$ is divergent, then $\sum v_k$ is divergent.*

Proof. Set $s_n^-(\lambda)=\sum\limits_{k=0}^{n}u_k^-(\lambda), s_n^+(\lambda)=\sum\limits_{k=0}^{n}u_k^+(\lambda)$, $t_n^-(\lambda)=\sum\limits_{k=0}^{n}v_k^-(\lambda)$ and $t_n^+(\lambda)=\sum\limits_{k=0}^{n}v_k^+(\lambda)$ for all $n\in\mathbb{N}$. From the inequalities $0\leq u_k^-(\lambda)\leq v_k^-(\lambda)$ and $0\leq u_k^+(\lambda)\leq v_k^+(\lambda)$, we deduce that

$$0\leq\sum_{k=n+1}^{n+p}u_k^-(\lambda)\leq\sum_{k=n+1}^{n+p}v_k^-(\lambda)\text{ and }0\leq\sum_{k=n+1}^{n+p}u_k^+(\lambda)\leq\sum_{k=n+1}^{n+p}v_k^+(\lambda),\text{ i.e.,}$$

$$0\leq s_{n+p}^-(\lambda)-s_n^-(\lambda)\leq t_{n+p}^-(\lambda)-t_n^-(\lambda)\quad\text{and}\quad 0\leq s_{n+p}^+(\lambda)-s_n^+(\lambda)\leq t_{n+p}^+(\lambda)-t_n^+(\lambda).\tag{5.1}$$

(1) Assume that $\sum v_k$ is convergent. Then using Definition 2.13, we get that

$$t_n^-(\lambda)\longrightarrow v^-(\lambda)\quad\text{and}\quad t_n^+(\lambda)\longrightarrow v^+(\lambda)$$

uniformly in $\lambda\in[0,1]$. Thus $t_{n+p}^-(\lambda)-t_n^-(\lambda)\longrightarrow 0$ and $t_{n+p}^+(\lambda)-t_n^+(\lambda)\longrightarrow 0$ uniformly in $\lambda\in[0,1]$. So the inequalities (5.1) yield that $s_{n+p}^-(\lambda)-s_n^-(\lambda)\longrightarrow 0$ and $s_{n+p}^+(\lambda)-s_n^+(\lambda)\longrightarrow 0$ uniformly in $\lambda\in[0,1]$. Hence $\{s_n^-(\lambda)\}$ and $\{s_n^+(\lambda)\}$ are Cauchy sequences uniformly in $\lambda\in[0,1]$. Since $\mathbb{R}$ is complete, there exist $u^-(\lambda),u^+(\lambda)\in\mathbb{R}$ such that $s_n^-(\lambda)\longrightarrow u^-(\lambda)$ and $s_n^+(\lambda)\longrightarrow v^+(\lambda)$, uniformly in $\lambda\in[0,1]$. Therefore by reutilizing Definition 2.13, we conclude that $\sum u_k$ converges.

(2) The second assertion (2) is obtained by contra-posited of (1). □

Theorem 5.2 (Ratio Test 2)**.** *Let $(u_k)\in w(F)$ such that $u_k>\widetilde{0}$ and there exists $v_k\in E$ such that $u_{k+1}=u_kv_k$ for all $k\in\mathbb{N}$ and $\lim\limits_{k\to\infty}v_k=l$, where $l\in E$.*

1. *If $l<\widetilde{1}$, then $\sum u_k$ is convergent.*
2. *if $l>\widetilde{1}$, then $\sum u_k$ is divergent.*

Where $\widetilde{1}(t)=1$ if $t=1$ and $\widetilde{1}(t)=0$ otherwise.

Proof. Let $\varepsilon>0$. Then there exists $k_0\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ with $k\geq k_0$, $D(v_k,l)\leq\varepsilon$, i.e.,

$$\text{for all }k\geq k_0,\ \sup_{0\leq\lambda\leq 1}\max\{|v_k^-(\lambda)-l^-(\lambda)|,|v_k^+(\lambda)-l^+(\lambda)|\}\leq\varepsilon.$$

Thus we have for all $k\geq k_0$,

$$l^-(0)-\varepsilon\leq l^-(\lambda)-\varepsilon\leq v_k^-(\lambda)\leq l^-(\lambda)+\varepsilon\leq l^+(0)+\varepsilon$$

and

$$l^-(0)-\varepsilon\leq l^+(\lambda)-\varepsilon\leq v_k^+(\lambda)\leq l^+(\lambda)+\varepsilon\leq l^+(0)+\varepsilon.$$

One can verify that $v_k^-(\lambda)=u_{k+1}^-(\lambda)/u_k^-(\lambda)$ and $v_k^+(\lambda)=u_{k+1}^+(\lambda)/u_k^+(\lambda)$ (See [**?**]). So by induction, we get for all $k\geq k_0$,

$$a^{k-k_0}u_{k_0}^-(0)\leq u_k^-(\lambda)\leq b^{k-k_0}u_{k_0}^+(0),\quad\text{and}\quad a^{k-k_0}u_{k_0}^+(0)\leq u_k^+(\lambda)\leq b^{k-k_0}u_{k_0}^+(0),\tag{5.2}$$

where $a=l^-(0)-\varepsilon, b=l^+(0)+\varepsilon$.

(1) Assume that $l<\widetilde{1}$ and let $\varepsilon>0$ verifying $b=l^+(0)+\varepsilon<1$. Then from (5.2), the geometric series $\sum a^{k-k_0}$ and $\sum b^{k-k_0}$ converge. This implies that $\sum u_k^-(\lambda)$ and $\sum u_k^+(\lambda)$ converge uniformly in $\lambda\in[0,1]$. Thus by Definition 2.13, we deduce that $\sum u_k$ is convergent.

(2) Suppose that $l > \widetilde{1}$ and let $\varepsilon > 0$ verifying $a = l^-(0) - \varepsilon > 1$. Then using (5.2) and the fact that $\lim\limits_{k\to\infty} a^{k-k_0} u^+_{k_0}(0) = \infty$, we get $\lim\limits_{k\to\infty} D(u_k, \widetilde{0}) = \lim\limits_{k\to\infty} u^+_k(0) = \infty$. Thus by application of Theorem 3.2, we conclude that $\sum u_k$ is divergent. □

Remark 5.3. Under the same hypothesis of Theorems 4.2 and 5.2, one can not conclude in the case $l = 1$. Indeed, we reconsider the fuzzy Riemann series $\sum u_k = \sum \frac{1}{k^\alpha} x$ studied in Example 3.10, where the fuzzy number x is chosen such that $x > \widetilde{0}$. One can show that it verifies all the hypothesis of both Theorems 4.2 and 5.2, but $\sum u_k$ converges if $\alpha > 1$ and diverges otherwise.

Definition 5.4. For a nonnegative fuzzy number u , if there exists a nonnegative fuzzy number v such that $u^k = v$, we say that v is the *k-th root of* u, which we denote by $\sqrt[k]{u}$ or $(u_k)^{1/k}$. The uniqueness of v is obvious according to Definition 2.6.

Theorem 5.5 (Root Test 2)**.** *Let $(u_k) \in w(F)$ such that $u_k > \widetilde{0}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty}(u_k)^{1/k} = L$, where $L \in E$.*

(1) *If $L < \widetilde{1}$ then, $\sum u_k$ is convergent.*
(2) *if $L > \widetilde{1}$ then, $\sum u_k$ is divergent.*

Proof. Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq k_0$, $D((u_k)^{1/k}, L) \leq \varepsilon$. Thus for all $k \geq k_0$, we have

$$\sup_{0\leq\lambda\leq1} \max\{|[u^-_k(\lambda)]^{1/k} - L^-(\lambda)|, |[u^+_k(\lambda)]^{1/k} - L^+(\lambda)|\} \leq \varepsilon.$$

So for all $k \geq k_0$, we get

$$L^-(0) - \varepsilon \leq L^-(\lambda) - \varepsilon \leq [u^-_k(\lambda)]^{1/k} \leq L^-(\lambda) + \varepsilon \leq L^+(0) + \varepsilon$$

and

$$L^-(0) - \varepsilon \leq L^+(\lambda) - \varepsilon \leq [u^+_k(\lambda)]^{1/k} \leq L^+(\lambda) + \varepsilon \leq L^+(0) + \varepsilon$$

Hence for all $k \geq k_0$, we obtain

$$(L^-(0) - \varepsilon)^k \leq u^-_k(\lambda) \leq (L^+(0) + \varepsilon)^k \quad \text{and} \quad (L^-(0) - \varepsilon)^k \leq u^+_k(\lambda) \leq (L^+(0) + \varepsilon)^k. \tag{5.3}$$

(1) Assume that $L < \widetilde{1}$ and let $\varepsilon > 0$ verifying $L^+(0) + \varepsilon < 1$. Then by using (5.3), the geometric series $\sum(L^-(0) + \varepsilon)^k$ and $\sum(L^+(0) + \varepsilon)^k$ converge. This implies that $\sum u^-_k(\lambda)$ and $\sum u^+_k(\lambda)$ converge uniformly in $\lambda \in [0, 1]$. Thus by Definition 2.13, we deduce that $\sum u_k$ is convergent.

(2) Suppose that $L > \widetilde{1}$ and let $\varepsilon > 0$ verifying $L^-(0) - \varepsilon > 1$. Since $\lim\limits_{k\to\infty}(L^-(0) - \varepsilon)^k = \infty$, from the inequalities (5.3), $u_k \nrightarrow \widetilde{0}$. By application of Theorem 3.2, we conclude that $\sum u_k$ is divergent. □

Remark 5.6. Under the same hypothesis of Theorems 4.5 and 5.5, one can not conclude in the case $L = 1$. Indeed, we reconsider the fuzzy Riemann series $\sum u_k = \sum \frac{1}{k^\alpha} x$ studied in Example 3.10, where the fuzzy number x is chosen such that $x > \widetilde{0}$. One can show that it verifies all the hypothesis of both Theorems 4.5 and 5.5, but $\sum u_k$ converges if $\alpha > 1$ and diverges otherwise.

6. Fuzzy Abel theorem

Theorem 6.1 (Fuzzy Abel theorem)**.** *Let $(a_k) \in w(F)$ and (v_k) be a decreasing sequence of non-negative real numbers such that $\lim\limits_{k\to\infty} v_k = 0$. Set $A_n = \sum\limits_{k=0}^{n} a_k$ and assume that the sequence (A_k) is bounded, i.e.,*

$$\exists M > 0,\ D(A_k, \widetilde{0}) \leq M,\ \text{for}\, k \in \mathbb{N}.$$

Then the series $\sum v_k a_k$ is convergent.

Proof. Set $s_n = \sum\limits_{k=0}^{n} v_k a_k$. It is obvious that the Hukuhara difference $A_k \ominus A_{k-1}$ exists and is equal to a_k for all $k \geq 1$. Then for each $n \geq 1$ and $p \in \mathbb{N}$, one can obtain the following Abel transformation

$$\begin{aligned}\textstyle\sum_{k=n+1}^{n+p} v_k a_k &= \textstyle\sum_{k=n+1}^{n+p} v_k (A_k \ominus A_{k-1}) \\ &= \textstyle\sum_{k=n+1}^{n+p} (v_k A_k) \ominus (v_k A_{k-1}) \\ &= \left(\textstyle\sum_{k=n+1}^{n+p} v_k A_k\right) \ominus \left(\textstyle\sum_{k=n+1}^{n+p} v_k A_{k-1}\right).\end{aligned}$$

That is, we have

$$\begin{aligned}\textstyle\sum_{k=n+1}^{n+p} v_k a_k &= \left(\textstyle\sum_{k=n+1}^{n+p} v_k A_k\right) \ominus \left(\textstyle\sum_{k=n}^{n+p-1} v_{k+1} A_k\right) \\ &= \left(v_{n+p}A_{n+p} + \textstyle\sum_{k=n+1}^{n+p-1} v_k A_k\right) \ominus \left(v_{n+1}A_n + \textstyle\sum_{k=n+1}^{n+p-1} v_{k+1} A_k\right).\end{aligned}$$

Thus we get

$$\begin{aligned} D(s_{n+p}, s_n) &= D\left(\sum_{k=0}^{n} v_k a_k + \sum_{k=n+1}^{n+p} v_k a_k, \sum_{k=0}^{n} v_k a_k\right) = D\left(\sum_{k=n+1}^{n+p} v_k a_k, \widetilde{0}\right) \\ &= D\left(\left(v_{n+p}A_{n+p} + \sum_{k=n}^{n+p-1} v_k A_k\right) \ominus \left(v_{n+1}A_n + \sum_{k=n}^{n+p-1} v_{k+1} A_k\right), \widetilde{0}\right) \\ &= D\left(v_{n+p}A_{n+p} + \sum_{k=n}^{n+p-1} v_k A_k, v_{n+1}A_n + \sum_{k=n}^{n+p-1} v_{k+1} A_k\right) \\ &\leq D\left(v_{n+p}A_{n+p}, \widetilde{0}\right) + D\left(\widetilde{0}, v_{n+1}A_n\right) + D\left(\sum_{k=n+1}^{n+p-1} v_k A_k, \sum_{k=n+1}^{n+p-1} v_{k+1} A_k\right). \end{aligned}$$

Since the sequence (v_k) is decreasing, $v_k = v_{k+1} + w_k$ with $w_k = v_k - v_{k+1} \geq 0$. Since $v_{k+1} \geq 0$ and $w_k \geq 0$, $v_k A_k = (v_{k+1} + w_k)A_k = v_{k+1}A_k + w_k A_k$. So the H-difference $v_k A_k \ominus v_{k+1} A_k$ exists and is equal to $(v_k - v_{k+1})A_k$. Hence the H-difference $\left(\sum\limits_{k=n+1}^{n+p-1} v_k A_k\right) \ominus \left(\sum\limits_{k=n+1}^{n+p-1} v_{k+1} A_k\right)$ exists and

$$\left(\sum_{k=n+1}^{n+p-1} v_k A_k\right) \ominus \left(\sum_{k=n+1}^{n+p-1} v_{k+1} A_k\right) = \sum_{k=n+1}^{n+p-1} (v_k - v_{k+1}) A_k.$$

And so, we obtain

$$D\left(\sum_{k=n+1}^{n+p-1} v_k A_k, \sum_{k=n+1}^{n+p-1} v_{k+1} A_k\right) = D\left(\sum_{k=n+1}^{n+p-1} v_k A_k \ominus \sum_{k=n+1}^{n+p-1} v_{k+1} A_k, \widetilde{0}\right) = D\left(\sum_{k=n+1}^{n+p-1} (v_k - v_{k+1}) A_k, \widetilde{0}\right).$$

Therefore we get

$$\begin{aligned} D(s_{n+p},s_n) &\leq |v_{n+p}|D\left(A_{n+p},\widetilde{0}\right)+|v_{n+1}|D\left(A_n,\widetilde{0}\right)+D\left(\sum_{k=n+1}^{n+p-1}(v_k-v_{k+1})A_k,\widetilde{0}\right)\\ &\leq M(|v_{n+p}|+|v_{n+1}|)+\sum_{k=n+1}^{n+p-1}(v_k-v_{k+1})D\left(A_k,\widetilde{0}\right)\\ &\leq M\left(|v_{n+p}|+|v_{n+1}|+\sum_{k=n+1}^{n+p-1}(v_k-v_{k+1})\right)\\ &\leq M\left(|v_{n+p}|+|v_{n+1}|+v_{n+1}-v_{n+p}\right). \end{aligned}$$

Since $|v_{n+p}|+|v_{n+1}|+v_{n+1}-v_{n+p}\longrightarrow 0$ as $n\to\infty$, $D(s_{n+p},s_n)\longrightarrow 0$ as $n\to\infty$. That is, (s_n) is a Cauchy sequence in the complete space E. Hence the series $\sum v_k a_k$ is convergent. □

Example 6.2. Consider the series $\sum\frac{1}{k+1}u^k$, where

$$u(t)=\begin{cases}1 & (1/3\leq t\leq 1/2)\\ 0 & \text{otherwise,}\end{cases}\quad \text{i.e., } [u]_\lambda=[1/3,1/2] \text{ for all } \lambda\in[0,1].$$

Then by using the Basic Lemma 2.7,

$$[A_n]_\lambda=\sum_{k=0}^{n}[u^k]_\lambda=\left[\frac{3(1-(1/3)^{n+1})}{2},2(1-(1/2)^{n+1})\right].$$

Thus we get

$$D(A_n,\widetilde{0})\leq\frac{3(1-(1/3)^{n+1})}{2}+2(1-(1/2)^{n+1})\leq 7/2.$$

Furthermore, the sequence $(v_k)=(1/(k+1))$ is decreasing. So, by utilizing Fuzzy Abel Theorem 6.1, we deduce that the fuzzy series $\sum\frac{1}{k+1}u^k$ is convergent.

7. Geometric series of fuzzy numbers

Now, we extend the results on convergence and sum of the geometric series $\sum u^n$ of fuzzy numbers given in [6] only for $\widetilde{0}<u<\widetilde{1}$ to general case $-\widetilde{1}<u<\widetilde{1}$.

Theorem 7.1. *The geometric fuzzy series $\sum u^n$ is convergent if and only if $-\widetilde{1}<u<\widetilde{1}$, i.e., $|u|<\widetilde{1}$. Moreover, the following four statements hold for $\sum\limits_{n=0}^{\infty}u^n$,*

1. *If $0<u_\lambda^-\leq u_\lambda^+<1$, then $\sum\limits_{n=0}^{\infty}[u]_\lambda^n=\left[\frac{1}{1-u_\lambda^-},\frac{1}{1-u_\lambda^+}\right]$.*
2. *If $-1<u_\lambda^-<0<u_\lambda^+<1$, then $\sum\limits_{n=0}^{\infty}[u]_\lambda^n=\left[\frac{1+u_\lambda^--u_\lambda^+}{1-u_\lambda^+},\frac{1}{1-u_\lambda^+}\right]$.*
3. *If $-1<u_\lambda^-\leq u_\lambda^+<0$, then $\sum\limits_{n=0}^{\infty}[u]_\lambda^n=\left[\frac{1}{1-(u_\lambda^+)^2}+\frac{u_\lambda^-}{1-(u_\lambda^-)^2},\frac{1}{1-(u_\lambda^-)^2}+\frac{u_\lambda^+}{1-(u_\lambda^+)^2}\right]$.*
4. *If $-1<u_\lambda^-<0=u_\lambda^+$, the, $\sum\limits_{n=0}^{\infty}[u]_\lambda^n=\left[\frac{1+u_\lambda^--(u_\lambda^-)^2}{1-(u_\lambda^-)^2},\frac{1}{1-(u_\lambda^-)^2}\right]$.*

Proof. If $-\widetilde{1}<u<\widetilde{1}$, then $D(u,\widetilde{0})=\max\{|u^-(0)|,|u^+(0)|\}<1$. Thus the classic geometric series $\sum(D(u,\widetilde{0}))^k$ converges. On the other hand, we have $D(u^k,\widetilde{0})\leq(D(u,\widetilde{0}))^k$ for all $k\in\mathbb{N}$. So the fuzzy geometric series $\sum u^k$ is convergent in norm, and by Theorem 3.7, it is convergent.

If $u \geq \widetilde{1}$ or $u \leq -\widetilde{1}$, then $\max\{|u^-(0)|, |u^+(0)|\} \geq 1$ and from the Basic Lemma,

$$D(u^k, \widetilde{0}) = \max\{|u^-(0)|^k, |u^+(0)|^k\} \nrightarrow 0 \,\text{as}\, k \to \infty.$$

Thus $u^k \nrightarrow \widetilde{0}$ as $k \to \infty$. So from Theorem 3.2, we deduce that $\sum u^k$ diverges.

Assume that $-\widetilde{1} < u < \widetilde{1}$. By application of the Basic Lemma, we obtain:

(1) Suppose $0 < u^-_\lambda \leq u^+_\lambda < 1$. Then we have

$$\sum_{n=0}^{\infty} [u]^n_\lambda = \left[\sum_{n=0}^{\infty} (u^-_\lambda)^n, \sum_{n=0}^{\infty} (u^+_\lambda)^n\right] = \left[\frac{1}{1-u^-_\lambda}, \frac{1}{1-u^+_\lambda}\right].$$

(2) Suppose $-1 < u^-_\lambda < 0 < u^+_\lambda < 1$. Then we get

$$\sum_{n=0}^{\infty} [u]^n_\lambda = \left[1 + \sum_{n=1}^{\infty} (u^+_\lambda)^{n-1} u^-_\lambda, \sum_{n=0}^{\infty} (u^+_\lambda)^n\right] = \left[\frac{1 + u^-_\lambda - u^+_\lambda}{1-u^+_\lambda}, \frac{1}{1-u^+_\lambda}\right].$$

(3) Suppose $-1 < u^-_\lambda \leq u^+_\lambda < 0$. Then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} [u]^n_\lambda &= \left[\sum_{n=0}^{\infty} (u^+_\lambda)^{2n}, \sum_{n=0}^{\infty} (u^-_\lambda)^{2n}\right] + \left[\sum_{n=0}^{\infty} (u^-_\lambda)^{2n+1}, \sum_{n=0}^{\infty} (u^+_\lambda)^{2n+1}\right] \\ &= \left[\frac{1}{1-(u^+_\lambda)^2} + \frac{u^-_\lambda}{1-(u^-_\lambda)^2}, \frac{1}{1-(u^-_\lambda)^2} + \frac{u^+_\lambda}{1-(u^+_\lambda)^2}\right].\end{aligned}$$

(4) Suppose $-1 < u^-_\lambda < 0 = u^+_\lambda$. Then we get

$$\sum_{n=0}^{\infty} [u]^n_\lambda = \left[1, \sum_{n=0}^{\infty} (u^-_\lambda)^{2n}\right] + \left[\sum_{n=0}^{\infty} (u^-_\lambda)^{2n+1}, 0\right] = \left[\frac{1 + u^-_\lambda - (u^-_\lambda)^2}{1-(u^-_\lambda)^2}, \frac{1}{1-(u^-_\lambda)^2}\right].$$

□

8. Fuzzy Cauchy product of two fuzzy series

Definition 8.1. Let (u_k) and (v_k) be two sequences of fuzzy numbers and define the sequence (w_k) by $w_k = \sum\limits_{i=0}^{k} u_i v_{k-i}$ for all $k \in \mathbb{N}$. $\sum w_k$ is called the *fuzzy Cauchy product* of the two fuzzy series $\sum u_k$ and $\sum v_k$.

Theorem 8.2. *If the fuzzy series $\sum u_k$ and $\sum v_k$ are convergent in norm, then $\sum w_k$ is also convergent in norm. Additionally, if $\sum u_k$ is a series of positive real numbers, we have*

$$\left(\sum_{k=0}^{\infty} u_k\right)\left(\sum_{k=0}^{\infty} v_k\right) = \sum_{k=0}^{\infty} w_k. \tag{8.1}$$

Proof. By application of Lemma 2.10, we have for each $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{n} D(w_k, \widetilde{0}) &= \sum_{k=0}^{n} D\Big(\sum_{i=0}^{k} u_i v_{k-i}, \widetilde{0}\Big) \leq \sum_{k=0}^{n}\sum_{i=0}^{k} D(u_i, \widetilde{0})\, D(v_{k-i}, \widetilde{0}) = \sum_{k=0}^{n} \sum_{i+j=k} D(u_i, \widetilde{0})\, D(v_j, \widetilde{0}) \\ &\leq \sum_{i+j \leq n} D(u_i, \widetilde{0})\, D(v_j, \widetilde{0}) = A_n \leq \sum_{\substack{i \leq n \\ j \leq n}} D(u_i, \widetilde{0})\, D(v_j, \widetilde{0}) \\ &\leq \left(\sum_{i=0}^{n} D(u_i, \widetilde{0})\right)\left(\sum_{j=0}^{n} D(v_j, \widetilde{0})\right) \leq \left(\sum_{i=0}^{\infty} D(u_i, \widetilde{0})\right)\left(\sum_{j=0}^{\infty} D(v_j, \widetilde{0})\right) < \infty.\end{aligned}$$

Then the sequence $\{\sum_{k=0}^{n} D(w_k, \widetilde{0})\}$ is convergent, since it is nondecreasing and upper bounded. Thus the series $\sum w_k$ is convergent in norm and the sequence (A_n) is also convergent. Furthermore, assume that $u_k \geq 0$ for all $k \in \mathbb{N}$. Set $s_n = \sum_{k=0}^{n} u_k, t_n = \sum_{k=0}^{n} v_k$ and $q_n = \sum_{k=0}^{n} w_k$. Recall that if $a, b \in \mathbb{R}$ such that $ab \geq 0$ and $x \in E$, $(a+b)x = ax + bx$ (see Bede). This result can be easily generalized by induction to a finite family of positive real numbers. Denote $C_n = \{(i,j) \in \mathbb{N}^2; i \leq n, j \leq n\}$ and $T_n = \{(i,j) \in \mathbb{N}^2; i + j \leq n\}$. Then

$$\begin{aligned} D(s_n t_n, q_n) &= D\left((\sum_{i=0}^{n} u_i)(\sum_{j=0}^{n} v_j), \sum_{i=0}^{n} u_i v_{n-i}\right) = D\left(\sum_{i=0}^{n}(\sum_{j=0}^{n} u_i v_j), \sum_{i+j=n} u_i v_j\right) \\ &= D\left(\sum_{(i,j)\in C_n \setminus T_n} u_i v_j + \sum_{i+j=n} u_i v_j, \sum_{i+j=n} u_i v_j\right) = D\left(\sum_{(i,j)\in C_n \setminus T_n} u_i v_j, \widetilde{0}\right) \\ &\leq \sum_{(i,j)\in C_n \setminus T_n} D\left(u_i v_j, \widetilde{0}\right) \leq \sum_{(i,j)\in C_n \setminus T_n} D\left(u_i, \widetilde{0}\right) D\left(v_j, \widetilde{0}\right) \\ &\leq \sum_{(i,j)\in T_{2n} \setminus T_n} D\left(u_i, \widetilde{0}\right) D\left(v_j, \widetilde{0}\right) = A_{2n} - A_n, \end{aligned}$$

where we used the inclusion $C_n \subset T_{2n}$. Since (A_n) is convergent, $A_{2n} - A_n \longrightarrow 0$. Thus $D(s_n t_n, q_n) \longrightarrow 0$ as $n \longrightarrow \infty$. So

$$D\left(\Big(\sum_{k=0}^{\infty} u_k\Big)\Big(\sum_{k=0}^{\infty} v_k\Big), \sum_{k=0}^{\infty} w_k\right) = \lim_{n\to\infty} D(s_n t_n, q_n) = 0.$$

By consequence

$$\Big(\sum_{k=0}^{\infty} u_k\Big)\Big(\sum_{k=0}^{\infty} v_k\Big) = \sum_{k=0}^{\infty} w_k.$$

□

Remark 8.3. Unfortunately, equation (8.1) doesn't hold for each couple of fuzzy series, because in general, distributivity of multiplication is not valid in the set of fuzzy numbers. Moreover, it is well known that if $u, v, w \in E$ such that $vw \geq \widetilde{0}$, i.e., $ab \geq 0$ for all $a \in v, a \in w$, then $u(v + w) = uv + uw$.
So by a similar way, one can prove that equation (8.1) is in particular true, if we assume that $v_k v_{k+1} \geq \widetilde{0}$ for all $k \in \mathbb{N}$.

9. Exponential of a fuzzy number

Definition 9.1. Let $u \in E$ and define the fuzzy exponential of u by $\exp(u) = \sum_{k=0}^{\infty} \frac{1}{k!} u^k$.

Lemma 9.2. $\exp(u)$ *is well defined for all* $u \in E$ *and verifies*

$$D(\exp(u), \widetilde{0}) \leq \exp\left(D(u, \widetilde{0})\right).$$

Proof. One can verify by induction and using Lemma 2.10, that for all $k \in \mathbb{N}$,

$$D(u^k, \widetilde{0}) \leq \left(D(u, \widetilde{0})\right)^k.$$

Then $D\left(\frac{1}{k!}u^k, \widetilde{0}\right) \leq \frac{1}{k!}\left(D(u, \widetilde{0})\right)^k$. Since the series of real numbers $\sum \frac{1}{k!}\left(D(u, \widetilde{0})\right)^k$ converges, by the Comparison Test 1, we conclude that $\sum \frac{1}{k!}u^k$ is convergent in norm. Thus $\exp(u)$ exists as sum of a convergent series and we get

$$D(\exp(u), \widetilde{0}) = D\left(\sum_{k=0}^{\infty} \frac{1}{k!}u^k, \widetilde{0}\right) \leq \sum_{k=0}^{\infty} \frac{1}{k!}\left(D(u, \widetilde{0})\right)^k = \exp\left(D(u, \widetilde{0})\right).$$

□

Lemma 9.3. *For all $u, v \in E$ such that $uv \geq \widetilde{0}$, the fuzzy binomial formula holds: for all $k \in \mathbb{N}$,*

$$(u+v)^k = \sum_{i=0}^{k} \binom{k}{i} u^{k-i} v^i.$$

Proof. This formula is obvious for $k = 0$ and $k = 1$. Assume, for a fixed $k \in \mathbb{N}$, that

$$(u+v)^k = \sum_{i=0}^{k} \binom{k}{i} u^{k-i} v^i.$$

Since $uv \geq \widetilde{0}$, we have

$$\begin{aligned}
(u+v)^{k+1} &= (u+v)^k(u+v) = (u+v)^k u + (u+v)^k v \\
&= \sum_{i=0}^{k} \binom{k}{i} u^{k+1-i}\, v^i + \sum_{i=0}^{k} \binom{k}{i} u^{k-i}\, v^{i+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} u^{k+1-i}\, v^i + \sum_{i=1}^{k+1} \binom{k}{i-1} u^{k+1-i}\, v^i \\
&= u^{k+1} + \sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1}\right] u^{k+1-i}\, v^i + v^{k+1}.
\end{aligned}$$

It is well known that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$. Then

$$(u+v)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} u^{k+1-i} v^i.$$

Thus the fuzzy binomial formula holds true, for all $k \in \mathbb{N}$. □

Theorem 9.4. *For all $u, v \in E$ such that $uv \geq \widetilde{0}$, we have*

$$\exp(u+v) = \exp(u)\, \exp(v).$$

Furthermore, if $u \geq v$ and the Hukuhara difference $u \ominus v$ exists then the Hukuhara quotient $\exp(u) \oslash \exp(v)$ *of u and v exists and we have*

$$\exp(u \ominus v) = \exp(u) \oslash \exp(v).$$

Proof. Due to the convergence in norm of the following series and to the assumption $uv \geq \widetilde{0}$, we have

$$\exp(u)\ \exp(v) = \Big(\sum_{k=0}^{\infty}\frac{1}{k!}u^k\Big)\Big(\sum_{k=0}^{\infty}\frac{1}{k!}v^k\Big) = \sum_{k=0}^{\infty} w_k,$$

where

$$w_k = \sum_{i=0}^{k}\frac{1}{i!}u^i\frac{1}{(k-i)!}u^{k-i} = \frac{1}{k!}\sum_{i=0}^{k}\left(\begin{array}{c} k \\ i \end{array}\right) u^i\, v^{k-i}.$$

Then from Lemma 9.3, we get $w_k = \frac{1}{k!}(u+v)^k$ for all $k \in \mathbb{N}$. Thus

$$\exp(u)\ \exp(v) = \exp(u+v).$$

Now, assume that $u \geq v$ and the Hukuhara difference $w = u \ominus v$ exists, i.e., $u = v + w$. Then

$$\exp(u) = \exp(v+w) = \exp(v)\ \exp(w).$$

Thus the Hukuhara quotient $\exp(u) \oslash \exp(v)$ exists and

$$\exp(u) \oslash \exp(v) = \exp(w) = \exp(u \ominus v).$$

□

For more details about the Hukuhara quotient of u and v one can see [19].

Conclusion

In this work, we give the fuzzy comparison, some tests and criteria for the (simple) convergence, the convergence in norm and the absolute convergence of series of fuzzy numbers. We investigate the relationship between this three kinds of convergence, and introduce a fuzzy version of Abel theorem and the Cauchy product for fuzzy series. Finally, we extend the convergence of the fuzzy geometric series $\sum u^k$ (studied in [4] only for $\widetilde{0} < u < \widetilde{1}$) to the fuzzy open interval $-\widetilde{1} < u < \widetilde{1}$, then we define the exponential of a fuzzy number and present some of its properties.
For future research we will use these results to study function series and power series of fuzzy numbers.

References

[1] M. Matloka, Sequences of fuzzy numbers, BUSEFAL 28 (1986) 28–37.
[2] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123-126.
[3] Ö. Talo and C. Çakan, On the Cesaro convergence of sequences of fuzzy numbers, Applied Mathematics Letters 25 (2012) 676–681.
[4] Ö. Talo and F. Başar, On the slowly decreasing sequences of fuzzy numbers, Abstr. Appl. Ana, l. Art (2013) 7 pages, Article ID 891986.
[5] E. Yavuz and Ö. Talo, Abel summability of sequences of fuzzy numbers, Soft Computing (2014) Doi: 10.1007/s00500-01.
[6] U. Kadak and F. Başar, Power series of fuzzy numbers with real or fuzzy coefficients, Filomat 26 (3) (2012) 519–528.
[7] I. Çanak, On the Riesz mean of sequences of fuzzy real numbers, Journal of Intelligent and Fuzzy Systems 26 (6) (2014) 2685–2688.
[8] Z. Onder and I. Çanak, Tauberian conditions under which convergence follows from the weighted mean summability and its statistical extension for fuzzy numbers, Ukranian Mathematical Journal 73 (8) (2021) 1085–1101.
[9] Z. Onder, S. A. Sezer and I. Çanak, A Tauberian theorem for the weighted mean method of summability of sequences of fuzzy numbers, Journal of Intelligent and Fuzzy Systems 28 (2015) 1403–1409.
[10] S. A. Sezer and I. Çanak, Power series methods of summability for series of fuzzy numbers and related tauberian theorems, Soft Computing (2015) Doi: 10.1007/s00500-015-1840-0.

[11] M. Mursaleen and F. Başar, Sequence Spaces, Topics in Modern Summability Theory, Taylor & Francis Group, CRC Press 2020.
[12] C. G. Gal and S. G. Gal, Semigroups of operators on spaces of fuzzy-number-valued functions with applications to fuzzy differential equations, J. Fuzzy Math. 13 (2005) 647–682.
[13] M. L. Puri and D. A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.
[14] R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31–43.
[15] Ö. Talo and F. Başar, Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, Comput. Math. Appl. 58 (2009) 717–733.
[16] Z. Gong and C. Wu, Bounded variation, absolute continuity and absolute integrability for fuzzy-number-valued functions, Fuzzy Sets and Systems 129 (2002) 83–94.
[17] G. A. Anastassiou, On H-fuzzy differentiation, Math. Balkanica 16 (2002) 155–193.
[18] M. Mares, Weak arithmetics of fuzzy numbers, Fuzzy Sets and Systems 91 (1997) 143–153.
[19] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems 161 (2010) 1564–1584.

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