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On necessary conditions of interval-valued optimization problem with the gH-Gâteaux derivative and the gH-Fréchet derivative

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Abstract. The optimization problem with interval-valued function is discussed in this paper. On the basis of the gH-Gâteaux derivative and the gH-Fréchet derivative proposed by D. Ghosh et al, we study the optimization conditions for interval-valued optimization problems. Finally, the necessary conditions for determining the optimal solution point of the interval-valued function are obtained, and examples are given to illustrate the correctness of the theorem.

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1. Introduction

The theory of interval arithmetic was first systematically introduced by Moore in 1966, marking the birth of interval analysis [1]. Interval analysis, unlike classical mathematics which takes point variables as its object of study, is computed on intervals. It was originally developed on the basis of the theory of computational error [1, 2], which arises from data errors, truncation errors and rounding errors that arise during computation [3]. Interval analysis differs from traditional probabilistic methods in that it attempts to bring the results of the calculation within the required or given accuracy range [1, 2]. However, in practical problems, it is often necessary to speculate on the accuracy of the calculation results or to improve the accuracy of the calculation as much as possible and to reduce the error rate in the calculation process so as to ensure the accuracy of the calculation results [4, 5]. As the errors arising from the calculation process are superimposed on each other, they may cause the calculation results to deviate significantly and lose their meaning. To address this problem, interval analysis provides a simple method which takes into account the various errors in the calculation process [1, 2, 6, 7]. Since the 1970s, interval analysis has grown exponentially and has been used in a variety of applications, particularly in dealing with ambiguities and incompleteness in engineering [8, 9, 10, 11, 12].

With the development and progress of the times, more and more scholars are involved in the study of interval analysis and apply it to several aspects[7, 11, 13]. For example, interval iteration can be used to discuss and study the existence uniqueness of solutions of nonlinear equations and the convergence of interval iteration sequences [14, 15, 16, 17, 18]. This method has significant advantages over traditional classical mathematical methods and is not possible with classical point-variable iterative methods [19, 20]. In addition, interval analysis has been widely used in a number of applications such as interval interpolation problems [21], linear equation problems [22, 23], nonlinear programming problems [24], and differential equation problems [25, 26, 27]. In 2019, Senol et al. [27] proposed a perturbative-iterative algorithm (PIA) for solving numerical solutions of certain types of fuzzy partial differential equations (FFPDEs) with generalized Hukuhara derivatives. The convergence of this method in the solution process is also discussed. The advantage of this method over other solution methods is that over calculation is eliminated in the process of solving partial differential equations of fractional order, and the effectiveness of the method is illustrated and verified with practical arithmetic examples.

Moreover, The interval language, similar to traditional mathematical languages, can be used directly as a computer language. In most cases, decision information is often uncertain due to the increasing complexity of the environment and the inherent subjective ambiguity of the human mind. For example, data in many practical engineering and economic problems are inaccurate [28]. It is difficult to quantify their opinions accurately with clear numbers [1, 2]. And interval numbers, as an important branch of fuzzy numbers, can be used to deal with the uncertainty and imprecision of information. Therefore, the introduction of interval analysis to deal with various uncertain phenomena, or ambiguous problems in reality is very necessary, as well as of practical importance [29, 30, 31]. Solving interval-valued optimization problems is an important theoretical basis and method for solving uncertainty in optimization problems [1, 2, 32, 33, 34, 35].

Derivatives are a local property of functions. The essence of derivatives is instantaneous rate of change. The derivative of a function at a point describes its rate of change near that point. It represents the slope of tangent line at a point in function curve, instantaneous velocity in physical displacement time relationship, instantaneous acceleration in velocity time relationship, and marginal cost in economy.

Optimization conditions are based on the derivative definitions [36, 37, 38, 39]. The concept of generalized *Hukuhara* differentiablity is a more general concept than *Hukuhara* differentiability [25]. In order to extend the application of generalized *Hukuhara* derivative, the definition of sub-derivative is given in [40]. Based on the above definitions, we introduce the definitions of directional sub-derivative and partial sub-derivative. Considering the sub-differentiability of interval-valued objective functions, the necessary optimal conditions can be obtained. In the convex environment, the necessary optimization conditions are also sufficient.

The paper includes four parts. In Section 2, we introduce some basic concepts which will be used throughout the paper and give new definitions of directional sub-derivative and partial sub-derivative of interval-valued functions. In Section 3, there are theorems to explain the optimality conditions of interval-valued functions and examples to show that LU-minimum points of interval-valued functions. Section 4 shows some conclusions in this paper.

2. Preliminaries

Recently, DGhosh et al. [16] proposed the gH-directional derivative, the gH-Gâteaux derivative and the gH-Fr**é**chet derivative of interval-valued functions, and applied them to the characterizations of efficient points of interval-valued optimization problems. In this paper, we present some new theorems related to efficient solutions based on the original interval number order relationships, and the definition of efficient solutions to interval optimization problems [16], and give some examples to justify the new theorems. For the sake of convenience, we review some basic definitions and related conclusions firstly.

Let $\mathbb{R}$ be the set of real numbers, $I(\mathbb{R})$ be the set that with all bounded and closed intervals. The elements of $I(\mathbb{R})$ will be represented by the capital bold letters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

For any two elements $\mathbf{A}$=$[\underline{a},\overline{a}]$, $\mathbf{B}$=$[\underline{b},\overline{b}]$ of $I(\mathbb{R})$, we define the following operations. The addition of $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{A} \oplus \mathbf{B}$=$[\underline{a}+\underline{b},\overline{a}+\overline{b}]$;
the multiplication between a real number λ and an interval is denoted by $\lambda \odot \mathbf{A}$, which is defined as

$$\lambda \odot \mathbf{A} = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}] & \text{if } \lambda \geq 0 \\ [\lambda\overline{a}, \lambda\underline{a}] & \text{if } \lambda < 0. \end{cases}$$

Particularly, if $\lambda = -1$, then $\mathbf{A} \oplus (-1) \odot \mathbf{B}$=$\mathbf{A} \ominus \mathbf{B}$=$[\underline{a}\text{-}\overline{b},\overline{a}\text{-}\underline{b}]$.

Definition 2.1 ([16])**.** For any two elements $\mathbf{A}$, $\mathbf{B} \in I(\mathbb{R})$, there exists an interval $\mathbf{C}$ belonging to $I(\mathbb{R})$ that satisfies $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$ or $\mathbf{B} = \mathbf{A} \ominus \mathbf{C}$. Here, $\mathbf{C}$ is called the *gH-difference between* $\mathbf{A}$ *and* $\mathbf{B}$, written as $\mathbf{C} = \mathbf{A} \ominus_{gH} \mathbf{B}$.

In particular, if $\mathbf{A}$, $\mathbf{B}$ are expressed as $\mathbf{A} = [\underline{a}, \overline{a}], \mathbf{B} = [\underline{b}, \overline{b}]$, then

$$\mathbf{A} \ominus_{gH} \mathbf{B} = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}].$$

Definition 2.2 ([16])**.** For any two elements $\mathbf{A} = [\underline{a}, \overline{a}], \mathbf{B} = [\underline{b}, \overline{b}]$ in $I(\mathbb{R})$, the following order relations between $\mathbf{A}$ and $\mathbf{B}$ can be defined. We re-express the intervals $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} = [\underline{a}, \overline{a}] = \{a(t)|a(t) = \underline{a} + t(\overline{a} - \underline{a}), 0 \leq t \leq 1\},$$
$$\mathbf{B} = [\underline{b}, \overline{b}] = \{b(t)|b(t) = \underline{b} + t(\overline{b} - \underline{b}), 0 \leq t \leq 1\}.$$

(i) $\mathbf{B}$ is said to be *dominated by* $\mathbf{A}$, if $a(t) \leq b(t)$ for all $t \in [0, 1]$, and then we write $\mathbf{A} \leq \mathbf{B}$.

(ii) $\mathbf{B}$ is said to be *strictly dominated by* $\mathbf{A}$, if $\mathbf{A} \leq \mathbf{B}$ and there exists a $t_0 \in [0, 1]$ such that $a(t_0) \neq b(t_0)$, and then we write $\mathbf{A} < \mathbf{B}$.

(iii) $\mathbf{B}$ is said to be *not dominated by* $\mathbf{A}$, if $a(t) > b(t)$ for at least one $t \in [0, 1]$, and then we write $\mathbf{A} \not\leq \mathbf{B}$.

(iv) If neither $\mathbf{A} \leq \mathbf{B}$ nor $\mathbf{B} \leq \mathbf{A}$, we say that *none of* $\mathbf{A}$ *and* $\mathbf{B}$ *are comparable.*

Lemma 2.3. *For any two elements* $\mathbf{A}, \mathbf{B}$ *of* $\mathrm{I}(\mathbb{R})$*, the following propositions hold.*

(1) $\mathbf{A} \leq \mathbf{B}$ *if and only if* $\begin{cases} \underline{a} \leq \underline{b} \\ \overline{a} \leq \overline{b}. \end{cases}$

(2) $\mathbf{A} < \mathbf{B}$ *if and only if* $\left\{ \begin{array}{l} \underline{a} < \underline{b} \\ \overline{a} \leq \overline{b} \end{array} \right.$ *or* $\left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} < \overline{b}. \end{array} \right.$

(3) $\mathbf{A} \not\leq \mathbf{B}$ *if and only if* $\underline{a} > \underline{b}$ *or* $\overline{a} > \overline{b}$.

(4) $\mathbf{A}$ *and* $\mathbf{B}$ *are not comparable if and only if* $\left\{ \begin{array}{l} \underline{a} > \underline{b} \\ \overline{a} < \overline{b} \end{array} \right.$ *or* $\left\{ \begin{array}{l} \underline{a} < \underline{b} \\ \overline{a} > \overline{b}. \end{array} \right.$

Proof. Since the proofs of (3) and (4) are similar to the proofs of (1) and (2), we only prove (1) and (2). Let $f(t) = a(t) - b(t)$. Then by Definition 2.2, it is easy to see that

$$\begin{aligned} f(t) &= a(t) - b(t) \\ &= (\underline{a} + t(\overline{a} - \underline{a})) - (\underline{b} + t(\overline{b} - \underline{b})) \\ &= (\overline{a} - \overline{b} - (\underline{a} - \underline{b}))t + (\underline{a} - \underline{b}) \\ &= (1-t)(\underline{a} - \underline{b}) + t(\overline{a} - \overline{b}). \end{aligned}$$

(1) Suppose $\mathbf{A} \leq \mathbf{B}$, i.e., $a(t) \leq b(t)$ for every $t \in [0,1]$. Then $f(t) = a(t) - b(t) \leq 0$ for all $t \in [0,1]$. Thus we have

$$f(0) = \underline{a} - \underline{b} \leq 0 \text{ and } f(1) = \overline{a} - \overline{b} \leq 0, \text{ i.e., } \left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} \leq \overline{b}. \end{array} \right.$$

On the contrary, suppose $\left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} \leq \overline{b}. \end{array} \right.$ Then $f(t) = a(t) - b(t) \leq 0$ for all $t \in [0,1]$, i.e., $a(t) \leq b(t)$. Thus $\mathbf{A} \leq \mathbf{B}$.

(2) Assume $\mathbf{A} < \mathbf{B}$. Then according to Definition 2.2 (ii), $\mathbf{A} \leq \mathbf{B}$. Thus there is $t_0 \in [0,1]$ such that $a(t_0) \neq b(t_0)$. Consequently,

$$\left\{ \begin{array}{l} f(t) = (\underline{a} + t(\overline{a} - \underline{a}) - (\underline{b} + t(\overline{b} - \underline{b})) \leq 0 \text{ for every } t \in [0,1], \\ f(t_0) = (\underline{a} + t_0(\overline{a} - \underline{a}) - (\underline{b} + t_0(\overline{b} - \underline{b})) \neq 0 \ t_0 \in [0,1]. \end{array} \right.$$

From (1), since $\mathbf{A} \leq \mathbf{B}$, we get that $\left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} \leq \overline{b}. \end{array} \right.$ If $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$, then $f(t) = 0$ for all $t \in [0,1]$, but it contradicts with $f(t_0) \neq 0$. So $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$ do not hold concurrently. Hence $\left\{ \begin{array}{l} \underline{a} < \underline{b} \\ \overline{a} \leq \overline{b} \end{array} \right.$ or $\left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} < \overline{b}. \end{array} \right.$

On the contrary, suppose $\left\{ \begin{array}{l} \underline{a} < \underline{b} \\ \overline{a} \leq \overline{b}. \end{array} \right.$ Then we can obviously get that

$$f(0) = \underline{a} - \underline{b} < 0 \text{ and } f(t) \leq 0 \text{ for all } t \in [0,1],$$

which implies that $a(0) < b(0)$ and $a(t) \leq b(t)$ for all $t \in [0,1]$, i.e., $\mathbf{A} < \mathbf{B}$.

Similarly, suppose $\left\{ \begin{array}{l} \underline{a} \leq \underline{b} \\ \overline{a} < \overline{b}. \end{array} \right.$ Then it is easy to know

$$f(1) = \overline{a} - \overline{b} < 0 \text{ and } f(t) \leq 0 \text{ for all } t \in [0,1],$$

which implies that $a(1) < b(1)$ and $a(t) \leq b(t)$ for all $t \in [0,1]$, i.e., $\mathbf{A} < \mathbf{B}$. □

Lemma 2.4 ([16]). *For any two elements* $\mathbf{A}$, $\mathbf{B}$ *in* $\mathrm{I}(\mathbb{R})$,

(1) $\mathbf{A} \leq \mathbf{B}$ *if and only if* $\mathbf{A} \ominus_{gH} \mathbf{B} \leq \mathbf{0}$,

(2) $\mathbf{A} \not\leq \mathbf{B}$ *if and only if* $\mathbf{A} \ominus_{gH} \mathbf{B} \not\leq \mathbf{0}$.

Proof. Since the proof of (2) is similar to the proof of (1), we only prove (1). Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$ in $I(\mathbb{R})$ and let $\mathbf{A} \leq \mathbf{B}$. Then from the Lemma 2.3 (1), we have

$$\begin{cases} \underline{a} \leq \underline{b} \\ \overline{a} \leq \overline{b}, \end{cases}$$

which means $\underline{a} - \underline{b} \leq 0$ and $\overline{a} - \overline{b} \leq 0$. Thus by the Definition 2.1 and Lemma 2.3, we get

$$\mathbf{A} \ominus_{gH} \mathbf{B} = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}] \leq \mathbf{0}.$$

□

Definition 2.5 ([16])**.** Let $\mathbf{X}$ be a convex subset of $\mathbb{R}^n$. An interval-valued function $\mathbf{F} : \mathbf{X} \to I(\mathbb{R})$ is said to be *convex* on $\mathbf{X}$, if

$$\mathbf{F}(\lambda_1 x + \lambda_2 y) \leq \lambda_1 \odot \mathbf{F}(x) \oplus \lambda_2 \odot \mathbf{F}(y),$$

for all $x, y \in \mathbf{X}$ and for all $\lambda_1, \lambda_2 \in [0, 1]$, where $\lambda_1 + \lambda_2 = 1$.

Definition 2.6 ([16])**.** Let $\mathbf{X}$ be a nonempty subset of $\mathbb{R}^n$ and $\mathbf{F} : \mathbf{X} \to I(\mathbb{R})$ be an interval-valued function. A point $\overline{x} \in \mathbf{X}$ is said to be an *effective point* of an interval-valued optimization problem:

$$\min_{x \in X} \mathbf{F}(x),$$

if $\mathbf{F}(x) \not< \mathbf{F}(\overline{x})$ for all $x \in \mathbf{X}$.

Definition 2.7 ([16])**.** Let $\mathbf{F}$ be an interval-valued function on a nonempty subset $\mathbf{X}$ of $\mathbb{R}^n$. If the limit

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot (\mathbf{F}(\overline{x} + \lambda h) \ominus_{gH} \mathbf{F}(\overline{x}))$$

exists, then the limit is said to be *gH-directional derivative of* $\mathbf{F}$ *at point* $\overline{x}$ *in the direction* h, and it is denoted by $\mathbf{F}_{\mathscr{D}}(\overline{x})(h)$.

Lemma 2.8 ([16])**.** *Let* $\mathbf{X}$ *be an real linear subspace of* $\mathbb{R}^n$ *and* $\mathbf{F} : \mathbf{X} \to \mathrm{I}(\mathbb{R})$ *be a convex function on* $\mathbf{X}$*. Then at any* $\overline{x} \in \mathbf{X}$*, gH-directional derivative* $\mathbf{F}_{\mathscr{D}}(\overline{x})(h)$ *exists for every direction* $h \in \mathbf{X}$*.*

Definition 2.9 ([16])**.** Let $\mathbf{X} \in \mathbb{R}^n$ be a nonempty open subset, and $\mathbf{F}$ be an interval-valued function on $\mathbf{X}$. If at $\overline{x} \in \mathbf{X}$, the limit

$$\mathbf{F}_{\mathscr{G}}(x)(h) := \lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda h) \ominus_{gH} \mathbf{F}(x))$$

exists for all $h \in \mathbb{R}^n$ and $\mathbf{F}_{\mathscr{G}}(x)(h)$ is a gH-continuous linear interval-valued function from $\mathbb{R}^n$ to $I(\mathbb{R})$, then $\mathbf{F}_{\mathscr{G}}(x)(h)$ is said be *gH-Gâteaux derivative of* $\mathbf{F}$ *at* $\overline{x}$. If the function $\mathbf{F}$ has gH-Gâteaux derivative at $\overline{x}$, then $\mathbf{F}$ is said to be *gH-Gâteaux differentiable at* $\overline{x}$.

3. Main Results

In this section, we will give some new valid solution-related theorems and give some examples to illustrate their reliability. First of all, we define a new order relation between intervals.

Definition 3.1. For any two elements $\mathbf{A} = [\underline{a}, \overline{a}], \mathbf{B} = [\underline{b}, \overline{b}]$ in $I(\mathbb{R})$, $\mathbf{B}$ is said to be *much less than dominated by* $\mathbf{A}$, if $\begin{cases} \overline{a} < \overline{b} \\ \underline{a} < \underline{b} \end{cases}$ and then we write $\mathbf{A} \ll \mathbf{B}$.

Theorem 3.2. *For any two elements* $\mathbf{A}$, $\mathbf{B}$ *in* $\mathrm{I}(\mathbb{R})$, $\mathbf{A} \ll \mathbf{B}$ *if and only if*

$$\mathbf{A} \ominus_{gH} \mathbf{B} \ll \mathbf{0}.$$

Proof. Suppose $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$. Then by the Definition 3.1, it follows that if $\mathbf{A} \ll \mathbf{B}$, then we get

$$\begin{cases} \overline{a} < \overline{b} \\ \underline{a} < \underline{b} \end{cases}$$

which means $\overline{a} - \overline{b} < 0$ and $\underline{a} - \underline{b} < 0$. Thus from the Definition 2.1, it is easy to know

$$\mathbf{A} \ominus_{gH} \mathbf{B} \ll \mathbf{0}.$$

□

Theorem 3.3. *Let* $\mathbf{X}$ *be a nonempty subset of* $\mathbb{R}^n$, *and* $\mathbf{F} : \mathbf{X} \to \mathrm{I}(\mathbb{R})$ *be an interval-valued function.* $\overline{x}$ *is an efficient point for the interval-valued optimization problem* (1). *If for any* $x \in \mathbf{X}$, *the gH-directional derivative* $\mathbf{F}_{\mathscr{D}}(\overline{x})(x - \overline{x})$ *of function* $\mathbf{F}$ *exists at* $\overline{x}$ *in the direction* $x - \overline{x}$, *then there is no* $x \in \mathbf{X}$ *such that*

$$\mathbf{F}_{\mathscr{D}}(\overline{x})(x - \overline{x}) \ll \mathbf{0}.$$

Proof. Let $\overline{x}$ be an efficient point of the interval optimization problem (1). Then from Definition 2.6, we get that for every $x \in X$,

$$\mathbf{F}(\overline{x} + \lambda(x - \overline{x})) \nless \mathbf{F}(\overline{x}).$$

According to Lemma 2.4,

$$\mathbf{F}(\overline{x} + \lambda(x - \overline{x})) \ominus_{gH} \mathbf{F}(\overline{x}) \nless \mathbf{0}.$$

We re-express the function $\mathbf{F}(x)$ as $\mathbf{F} = [\underline{f}(x), \overline{f}(x)]$. Thus by Definition 2.1, we have that

$$\begin{aligned} \mathbf{F}(\overline{x} + \lambda(x - \overline{x})) \ominus_{gH} F(\overline{x}) = [&\min\{\underline{f}(\overline{x} + \lambda(x - \overline{x})) - \underline{f}(\overline{x}), \overline{f}(\overline{x} + \lambda(x - \overline{x})) + \overline{f}(\overline{x})\}, \\ &\max\{\underline{f}(\overline{x} + \lambda(x - \overline{x})) - \underline{f}(\overline{x}), \overline{f}(\overline{x} + \lambda(x - \overline{x})) + \overline{f}(\overline{x})\}]. \end{aligned}$$

So it can be deduced from (3) of Lemma 2.3 that

$$\max\{\underline{f}(\overline{x} + \lambda(x - \overline{x})) - \underline{f}(\overline{x}), \overline{f}(\overline{x} + \lambda(x - \overline{x})) + \overline{f}(\overline{x})\} > 0.$$

Since $\lambda > 0$, the inequality can be transformed as

$$\frac{1}{\lambda} \max\{\underline{f}(\overline{x} + \lambda(x - \overline{x})) - \underline{f}(\overline{x}), \overline{f}(\overline{x} + \lambda(x - \overline{x})) + \overline{f}(\overline{x})\} > 0.$$

Hence we have

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda}\max\{\underline{f}(\overline{x}+\lambda(x-\overline{x}))-\underline{f}(\overline{x}),\overline{f}(\overline{x}+\lambda(x-\overline{x}))+\overline{f}(\overline{x})\}\geq 0.$$

Therefore there is no $x\in\mathbf{X}$ such that $\mathbf{F}_{\mathscr{D}}(\overline{x})(x-\overline{x})\ll\mathbf{0}$. □

Example 3.4. Consider the interval optimization problem

$$\min_{x\in\mathbb{R}}\mathbf{F}(x),$$

where $\mathbf{F}(x)=[x^2,x^2+1]$. Then from the Definition 2.1, we get that

$$\begin{aligned}\mathbf{F}(\overline{x}+\lambda(x-\overline{x}))\ominus_{gH}F(\overline{x})&=[\min\{\underline{f}(\overline{x}+\lambda(x-\overline{x}))-\underline{f}(\overline{x}),\overline{f}(\overline{x}+\lambda(x-\overline{x}))+\overline{f}(\overline{x})\},\\&\quad\max\{\underline{f}(\overline{x}+\lambda(x-\overline{x}))-\underline{f}(\overline{x}),\overline{f}(\overline{x}+\lambda(x-\overline{x}))+\overline{f}(\overline{x})\}]\\&=[\min\{(\overline{x}+\lambda(x-\overline{x}))^2-\overline{x}^2\},\\&\quad\max\{((\overline{x}+\lambda(x-\overline{x}))^2+1)-(\overline{x}^2+1)\}],\\&=[\lambda(x-\overline{x})(\lambda x+(2-\lambda)\overline{x}),\lambda(x-\overline{x})(\lambda x+(2-\lambda)\overline{x})].\end{aligned}$$

Thus we have

$$\begin{aligned}F_{\mathscr{D}}(\overline{x})(x-\overline{x})&=\lim_{\lambda\to 0^+}\frac{1}{\lambda}\odot(\mathbf{F}(\overline{x}+\lambda(x-\overline{x}))\ominus_{gH}\mathbf{F}(\overline{x})),\\&=\lim_{\lambda\to 0^+}\frac{1}{\lambda}\odot[\lambda(x-\overline{x})(\lambda x+(2-\lambda)\overline{x}),\lambda(x-\overline{x})(\lambda x+(2-\lambda)\overline{x})],\\&=\lim_{\lambda\to 0^+}[(x-\overline{x})(\lambda x+(2-\lambda)\overline{x}),(x-\overline{x})(\lambda x+(2-\lambda)\overline{x})],\\&=[2\overline{x}(x-\overline{x}),2\overline{x}(x-\overline{x})],\\&=2\overline{x}(x-\overline{x}).\end{aligned}$$

Clearly, $\overline{x}=0$ is an efficient point of this interval optimization problem, and we have

$$\begin{aligned}F_{\mathscr{D}}(0)(h)&=\lim_{\lambda\to 0^+}\frac{1}{\lambda}\odot(\mathbf{F}(0+\lambda h)\ominus_{gH}\mathbf{F}(0))\\&=\lim_{\lambda\to 0^+}\frac{1}{\lambda}\odot([\lambda^2h^2,\lambda^2h^2+1]\ominus_{gH}[0,1])\\&=[0,0]\\&\leq[0,0].\end{aligned}$$

Obviously, there is no x in X such that $\mathbf{F}_{\mathscr{D}}(\overline{x})(x-\overline{x})\ll\mathbf{0}$.

Theorem 3.5. *Let $\mathbf{X}\subseteq\mathbb{R}^n$ be a nonempty real linear subspace, and $\mathbf{F}$ be a convex interval-valued function on $\mathbf{X}$. Then for all $x,y\in\mathbf{X}$,*

$$\mathbf{F}_{\mathscr{D}}(x)(y-x)\leq\mathbf{F}(y)\ominus_{gH}\mathbf{F}(x).$$

Proof. According to Definition 2.1, we get that

$$\begin{aligned}\mathbf{F}(x+\lambda(y-x))\ominus_{gH}\mathbf{F}(x)&=[\underline{f}(x+\lambda(y-x)),\overline{f}(x+\lambda(y-x))]\ominus_{gH}[\underline{f}(x),\overline{f}(x)]\\&=[\min\{\underline{f}(x+\lambda(y-x))-\underline{f}(x),\overline{f}(x+\lambda(y-x))-\overline{f}(x)\},\\&\quad\max\{\underline{f}(x+\lambda(y-x))-\underline{f}(x),\overline{f}(x+\lambda(y-x))-\overline{f}(x)\}],\end{aligned}$$

and

$$\begin{aligned}(\lambda \odot \mathbf{F}(y) \oplus (1-\lambda) \odot \mathbf{F}(x)) \ominus_{gH} \mathbf{F}(x) &= [\lambda \underline{f}(y) + (1-\lambda)\underline{f}(x), \lambda \overline{f}(y) + (1-\lambda)\overline{f}(x)] \\ &\quad \ominus_{gH} [\underline{f}(x), \overline{f}(x)] \\ &= [\min\{\lambda \underline{f}(y) + \lambda' \underline{f}(x) - \underline{f}(x), \\ &\quad \lambda \overline{f}(y) + \lambda' \overline{f}(x) - \overline{f}(x)\}, \\ &\quad \max\{\lambda \underline{f}(y) + \lambda' \underline{f}(x) - \underline{f}(x), \\ &\quad \lambda \overline{f}(y) + \lambda' \overline{f}(x) - \overline{f}(x)\}] \\ &= [\min\{\lambda(\underline{f}(y) - \underline{f}(x)), \lambda(\overline{f}(y) - \overline{f}(x))\}, \\ &\quad \max\{\lambda(\underline{f}(y) - \underline{f}(x)), \lambda(\overline{f}(y) - \overline{f}(x))\}],\end{aligned}$$

where $\lambda' = 1-\lambda$.
Since $\mathbf{F}$ is convex on $\mathbf{X}$, we can get that for each $x, y \in \mathbf{X}$, $\lambda \in [0,1]$,

$$\begin{aligned}\mathbf{F}(x + \lambda(y-x)) &= \mathbf{F}(\lambda(y) + (1-\lambda)x) \\ &\leq \lambda \odot \mathbf{F}(y) \oplus (1-\lambda) \odot \mathbf{F}(x).\end{aligned}$$

Then by (1) of Lemma 2.4, we have

$$\begin{cases} \overline{f}(x + \lambda(y-x)) \leq \lambda \overline{f}(y) + (1-\lambda)\overline{f}(x), \\ \underline{f}(x + \lambda(y-x)) \leq \lambda \underline{f}(y) + (1-\lambda)\underline{f}(x). \end{cases}$$

Thus we get

$$\begin{cases} \overline{f}(x + \lambda(y-x)) - \overline{f}(x) \leq \lambda \overline{f}(y) + (1-\lambda)\overline{f}(x) - \overline{f}(x), \\ \underline{f}(x + \lambda(y-x)) - \underline{f}(x) \leq \lambda \underline{f}(y) + (1-\lambda)\underline{f}(x) - \underline{f}(x). \end{cases}$$

Namely,

$$\begin{cases} \overline{f}(x + \lambda(y-x)) - \overline{f}(x) \leq \lambda(\overline{f}(y) - \overline{f}(x)), \\ \underline{f}(x + \lambda(y-x)) - \underline{f}(x) \leq \lambda(\overline{f}(y) - \overline{f}(x)). \end{cases}$$

So

$$\begin{aligned}\mathbf{F}(x + \lambda(y-x)) \ominus_{gH} \mathbf{F}(x) &\leq (\lambda \odot \mathbf{F}(y) \oplus (1-\lambda) \odot \mathbf{F}(x)) \ominus_{gH} \mathbf{F}(x) \\ &= \lambda(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)),\end{aligned}$$

which implies

$$\frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda(y-x)) \ominus_{gH} \mathbf{F}(x)) \leq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).$$

We also know from Lemma 2.8 that for every $x, y \in \mathbf{X}$, the directional derivative $\mathbf{F}_{\mathscr{D}}(x)(y-x)$ along the direction $y-x$ exists. Hence we have that

$$\begin{aligned}\mathbf{F}_{\mathscr{D}}(x)(y-x) &= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda(y-x)) \ominus_{gH} \mathbf{F}(x)) \\ &\leq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).\end{aligned}$$

□

Example 3.6. Consider the interval-value function $\mathbf{F}(x) = [0, 1]$. Then obviously, $\mathbf{F}$ is convex on $\mathbb{R}^n$. For any $x, y \in \mathbf{X}$, let $h = y - x \in \mathbb{R}^n$. Then we get

$$\begin{aligned}\mathbf{F}_{\mathscr{D}}(x)(h) &= \mathbf{F}_{\mathscr{D}}(x)(y-x)\\ &= \lim_{\lambda\to 0^+} \frac{1}{\lambda} \odot (F(x+\lambda(y-x))\ominus_{gH} F(x))\\ &= \lim_{\lambda\to 0^+} \frac{1}{\lambda} \odot ([0,1]\ominus_{gH}[0,1])\\ &= \mathbf{0}.\end{aligned}$$

Thus it can be obtained from Definition 2.7 that the function $\mathbf{F}$ is gH-Gâteaux derivative at each $x \in \mathbf{X}$. And it is easy to know that $\mathbf{F}_{\mathscr{D}}(x)(h) \leq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)$ for all x in X.

Theorem 3.7. *Let* $\mathbf{F}$ *be gH-directional differentiable and convex at* $\overline{x}$*. If there exists no* $h \in \mathbb{R}^n$ *satisfying*

$$\mathbf{F}_{\mathscr{D}}(\overline{x})(h) \ll \mathbf{0},$$

then $\overline{x}$ *is an efficient point of* $\mathbf{F}$*.*

Proof. We suppose that there exists an $x \in \mathbb{R}^n$ such that

$$\mathbf{F}(x) \ll \mathbf{F}(\overline{x}).$$

Then from Lemma 3.2, it follows that

$$\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\overline{x}) \ll \mathbf{0}.$$

Thus according to Theorem 3.5, we get that

$$\mathbf{F}_{\mathscr{D}}(\overline{x})(x-\overline{x}) \leq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\overline{x}) \ll \mathbf{0},$$

which means

$$\mathbf{F}_{\mathscr{D}}(\overline{x})(x-\overline{x}) \ll \mathbf{0}.$$

This contradicts with the Theorem 3.3, if we let $h = x - \overline{x}$. □

Example 3.8. Suppose $\mathbf{F}(x) = [x^2 - 5, x^2 - 2x]$, where $x \in [0, 2]$. Then it is easy to get that $\overline{x} = 0$ is an efficient point of $\mathbf{F}$. Obviously, we have

$$\mathbf{F}(\lambda_1 x + \lambda_2 y) = [(\lambda_1 x + \lambda_2 y)^2 - 5, (\lambda_1 x + \lambda_2 y)^2 - 2(\lambda_1 x + \lambda_2 y)],$$

$$\lambda_1 \odot F(x) \oplus \lambda_{2\odot} F(y) = \lambda_1 \odot [x^2 - 5, x^2 - 2x] \oplus \lambda_2 \odot [y^2 - 5, y^2 - 2y].$$

Thus we get

$$F(\lambda_1 x + \lambda_2 y) \leq \lambda_1 \odot F(x) \oplus \lambda_2 \odot F(y)$$

for all $x, y \in \mathbf{X}$ and for all $\lambda_1, \lambda_2 \in [0, 1]$, where $\lambda_1 + \lambda_2 = 1$. So according to the Definition 2.5, we know $\mathbf{F}$ is convex at $\overline{x}$. At the same time, by the Lemma 2.8 and the Definition 2.7, we have $\mathbf{F}$ is gH-directional differentiable for every $x \in \mathbf{X}$ and

$$\begin{aligned}
\mathbf{F}_{\mathscr{D}}(\overline{x})(h) &= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot (\mathbf{F}(\overline{x} + \lambda(x - \overline{x})) \ominus_{gH} \mathbf{F}(\overline{x})),\\
&= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot \{[(\overline{x} + \lambda h)^2 - 5, (\overline{x} + \lambda h)^2 - 2(\overline{x} + \lambda h)] \ominus_{gH} [\overline{x}^2 - 5, \overline{x}^2 - 2\overline{x}]\},\\
&= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \odot [\min\{2\lambda h\overline{x} + \lambda^2 h^2, 2\lambda h\overline{x} + \lambda^2 h^2 - 2\lambda h\},\\
&\quad \max\{2\lambda h\overline{x} + \lambda^2 h^2, 2\lambda h\overline{x} + \lambda^2 h^2 - 2\lambda h\}],\\
&= \lim_{\lambda \to 0^+} [\min\{2h\overline{x} + \lambda h^2, 2h\overline{x} + \lambda h^2 - 2\lambda h\}, \max\{2h\overline{x} + \lambda h^2, 2h\overline{x} + \lambda h^2 - 2\lambda h\}],\\
&= [\min\{2h\overline{x}, 2h\overline{x} - 2h\}, \max\{2h\overline{x}, 2h\overline{x} - 2h\}].
\end{aligned}$$

Hence

$$\mathbf{F}_{\mathscr{D}}(0)(h) = [\min\{-2h, 0\}, \max\{-2h, 0\}].$$

Therefore, there is no $h \in \mathbb{R}^n$ satisfying $\mathbf{F}_{\mathscr{D}}(0)(h) \ll \mathbf{0}$.

4. Conclusions

In this paper, we first propose a new interval number order relation based on the interval number order relation of fuzzy relations. And according to the definition of the gH-Gâteaux derivative and the gH-Fréchet derivative, the necessary conditions of interval-valued optimality problem are obtained. These optimality conditions make it easier to find the optimal solution point of the interval-valued function.

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