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IVI-octahedron subgroups and level subgroups

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Abstract. In this paper, we introduce the concept of IVI-octahedron subgroups and investigate some of its properties. In particular, we give the characterization for an IVIOG of a cyclic group. Next, we define a level subgroup of an IVIOG and obtain some of its properties. Furthermore, we give the characterization for an IVIOG of a finite cyclic group by using its chain of subgroups.

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1. Introduction

In 2020, as a tool to solve the problems involving ambiguities and uncertainties in the real world, Kim et al. [1] proposed the notion of IVI-octahedron sets combined with interval-valued intuitionistic fuzzy sets (Atanassov and Gargov [2]), intuitionistic fuzzy sets (Atanassov [3]) and fuzzy sets (Zadeh [4]) and they studied its basic algebraic properties, and applied it to groupoids.

First of all, we would like to examine research trends on group theory based on fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets through literature. In 1971, Rosenfeld [5] defined a fuzzy groupoid and a fuzzy subgroup based on fuzzy sets, and studied their various properties. Das [6] introduced the concept of level subgroups of a fuzzy subgroup and discussed some of its properties. After then, many researchers [7, 8, 9, 10, 11, 12, 13, 14, 15] investigated group structures via fuzzy sets. In 1997, Biswas [16] defined an intuitionistic fuzzy subgroup and obtained some of its properties. After that time, Hur et al. [17] introduced the notion of intuitionisric fuzzy groupoids and dealt with its various properties. Banerjee and Basnet [18] studied some properties of intuitionistic fuzzy subrings and ideals. Moreover, Ahn et al. [19] defined a level subgroup of an intuitionistic fuzzy subgroup and discussed with the relationships between level subgroups and intuitionistic fuzzy subgroups (See [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] for further researches of group structures based on intuitionistix fuzzy sets). In 2009, Hedayati [32] dealt with substructures in semigroups based on interval-valued intuitionistic fuzzy sets. Aygünoğlu et al. [33] defined an interval-valued intuitionistic fuzzy subgroup and an interval-valued intuitionistic fuzzy normal subgroup in terms of a double t-norm, and dealt with their various properties. Yaqoob [34] discussed with interval-valued intuitionistic fuzzy ideals in a regular LA-semigroup. Furthermore, Vetrivel and Murugadas [35] introduced the concept of bi-ideals in Γ near-rings via interval-valued intuitionistic fuzzy sets and investigated some of its properties.

Our research's aim is to study group structures based on IVI-octahedron sets. In order to accomplish it, this paper is organized as follows: In Section 2, we list basic definitions related to intuitionistic fuzzy sets, interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets octahedron sets and IVI-octahedron sets. In Section 3, we define an IVI-octahedron subgroup of a group by using interval-valued intuitionistic fuzzy subgroups, intuitionistic fuzzy subgroups and fuzzy subgroups, and obtained some of its properties. In particular, we give the necessary and sufficient condition that an IVI-octahedron set of a cyclic group G_p of a prime order p is an IVI-octahedron subgroup (See Theorem 3.24). Moreover, we introduce the notion of IVI-octahedron normal subgroups and discuss with some of its properties. In Section 4, we define an level subgroup of an IVI-octahedron subgroup and give the necessary and sufficient condition that two level subgroups are equal (See Theorem 4.10). Further more, we give the the necessary and sufficient condition that two IVI-octahedron subgroups of a finite group are equal (See Theorem 4.19). Also, we give the necessary and sufficient condition that an IVI-octahedron set in a finite cyclic group is an IVI-octahedron subgroup (See Theorem 4.23).

2. Preliminaries

In this section, we list some basic definitions needed in the next sections.

For a set X, let I^X denotes the set of all fuzzy sets in X and members of I^X will write $A,\ B,\ C$, etc., where $I = [0, 1]$. In particular, 0 and 1 denote the fuzzy empty set and the fuzzy whole set in X respectively (See [4]).

Each member of a set $I \oplus I = \{(a^\in, a^\notin) : (a^\in, a^\notin) \in I \times I \text{ and } a^\in + a^\notin \leq 1\}$ is called an *intuitionistic fuzzy number*, and $(0, 1)$ and $(1, 0)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively (See [36]). We will denote intuitionistic fuzzy numbers $(a^\in, a^\notin)$, $(b^\in, b^\notin)$, $(c^\in, c^\notin)$, etc. as $\bar{a},\ \bar{b},\ \bar{c}$, etc. It is well-known (Theorem 2.1 in [36]) that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying DeMorgan's laws.

Definition 2.1 ([3])**.** For a nonempty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X, where for each $x \in X$, $\bar{A}(x) = (A^\in(x), A^\notin(x))$, and $A^\in(x)$ and $A^\notin(x)$ represent the degree of membership and the degree of nonmembership of an element x to A respectively. Let us denote the set of all IFSs in X as $IFS(X)$ and for each $\bar{A} \in IFSX$, we write $\bar{A} = (A^\in, A^\notin)$. In particular, $\bar{0}$ and $\bar{1}$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by respectively: for each $x \in X$,

$$\bar{0}(x) = \bar{0} \text{ and } \bar{1}(x) = \bar{1}.$$

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval-valued fuzzy numbers* and denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$ (See [37]).

Definition 2.2 ([38, 39])**.** For a nonempty set X, a mapping $\widetilde{A} : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X. Let us denote the set of all IVFSs in X as $IVS(X)$. For each $\widetilde{A} \in IVS(X)$ and $x \in X$, $\widetilde{A}(x) = [A^-(x), A^+(x)]$ is called the *degree of membership of an element x to $\widetilde{A}$*, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and an *upper fuzzy set* in X respectively. For each $\widetilde{A} \in IVS(X)$, we write $\widetilde{A} = [A^-, A^+]$. In particular, $\widetilde{0}$ [resp. $\widetilde{1}$] denotes the interval-valued fuzzy empty set [resp. the interval-valued fuzzy whole set] in X and is defined as follows:

$$\widetilde{0}(x) = \mathbf{0} \text{ [resp. } \widetilde{1}(x) = \mathbf{1}] \text{ for each } x \in X.$$

Let $[I] \oplus [I] = \{(\widetilde{a}^{\in}, \widetilde{a}^{\notin}) : (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) \in [I] \times [I] \text{ and } a^{\in,+} + a^{\notin,+} \leq 1\}$, where

$$\widetilde{a}^{\in} = [a^{\in,-}, a^{\in,+}],\ \widetilde{a}^{\notin} = [a^{\notin,-}, a^{\notin,+}] \in [I].$$

Each member of $[I] \oplus [I]$ is called an *interval-valued intuitionistic fuzzy number* (See [1]). In particular, we write as $\widetilde{\widetilde{0}} = (\mathbf{0}, \mathbf{1})$ and $\widetilde{\widetilde{1}} = (\mathbf{1}, \mathbf{0})$. We will interval-valued intuitionistic fuzzy numbers $(\widetilde{a}^{\in}, \widetilde{a}^{\notin})$, $(\widetilde{b}^{\in}, \widetilde{b}^{\notin})$, $(\widetilde{c}^{\in}, \widetilde{c}^{\notin})$, etc. write as $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$, $\widetilde{\widetilde{c}}$, etc.

We define relations $\leq$ and $=$ on $[I] \oplus [I]$ as follows: for any $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}} \in [I] \oplus [I]$,

$$\widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}} \Longleftrightarrow a^{\in,-} \leq b^{\in,-},\ a^{\in,+} \leq b^{\in,+} \text{ and } a^{\notin,-} \geq b^{\notin,-},\ a^{\notin,+} \geq b^{\notin,+},$$

$$\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}} \text{ and } \widetilde{\widetilde{a}} \geq \widetilde{\widetilde{b}}.$$

For each $(\widetilde{\widetilde{a}}_j)_{j \in J} \subset [I] \oplus [I]$, its inf and sup, denoted by $\bigwedge_{j \in J} \widetilde{\widetilde{a}}_j$ and $\bigvee_{j \in J} \widetilde{\widetilde{a}}_j$, are defined as follows:

$$\bigwedge_{j \in J} \widetilde{\widetilde{a}}_j = ([\bigwedge_{j \in J} a_j^{\in,-}, \bigwedge_{j \in J} a_j^{\in,+}], [\bigvee_{j \in J} a_j^{\notin,-}, \bigvee_{j \in J} a_j^{\notin,+}]),$$

$$\bigvee_{j \in J} \widetilde{\widetilde{a}}_j = ([\bigvee j \in J a_j^{\in,-}, \bigvee_{j \in J} a_j^{\in,+}], [\bigwedge_{j \in J} a_j^{\notin,-}, \bigwedge_{j \in J} a_j^{\notin,+}]).$$

Definition 2.3 ([2])**.** Let X be a nonempty set. Then a mapping $\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin}) : X \to [I] \oplus [I]$ is called the *interval-valued intuitionistic fuzzy set* (briefly, IVIS) in X, where for each $x \in X$, $\widetilde{A}^{\in} = [A^{\in,-}(x), A^{\in,+}(x)]$, $\widetilde{A}^{\notin} = [A^{\notin,-}(x), A^{\notin,+}(x)]$ and $A^{\in,+}(x) + A^{\notin,+}(x) \leq 1$.

In particular, $\widetilde{\widetilde{\mathbf{0}}}$ [resp. $\widetilde{\widetilde{\mathbf{1}}}$] is called an *interval-valued intuitionistic fuzzy empty set* (resp. *interval-valued intuitionistic fuzzy whole set*) in X. We denote the set of all IVISs as $IVIS(X)$.

The relations $(\subset, =)$, operations $(\cup, \cap, {}^c)$ and operators $([\ ], \diamond)$ on $IVIS(X)$ are defined as follows.

Definition 2.4 ([2])**.** Let $\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin})$, $\widetilde{\widetilde{B}} = (\widetilde{B}^{\in}, \widetilde{B}^{\notin}) \in IVIS(X)$ and let $(\widetilde{\widetilde{A}}_j)_{j\in J} = ((\widetilde{A}^{\in}_j, \widetilde{A}^{\notin}_j))_{j\in J} \subset IVIS(X)$. Then

(i) $\widetilde{\widetilde{A}} \subset \widetilde{\widetilde{B}} \Longleftrightarrow (\forall x \in X)(A^{\in,-}(x) \leq B^{\in,-}(x),\ A^{\in,+}(x) \leq B^{\in,+}(x)$ and $A^{\notin,-}(x) \geq B^{\notin,-}(x),\ A^{\notin,+}(x) \geq B^{\notin,+}(x))$,

(ii) $\widetilde{\widetilde{A}} = \widetilde{\widetilde{B}} \Longleftrightarrow \widetilde{A} \subset \widetilde{B}$ and $\widetilde{B} \subset \widetilde{A}$,

(iii) $\widetilde{\widetilde{A}}^c(x) = (\widetilde{A}^{\notin}(x), \widetilde{A}^{\in}(x))$ for each $x \in X$,

(iv) $(\widetilde{\widetilde{A}} \cup \widetilde{\widetilde{B}})(x) = ([A^{\in,-}(x) \vee B^{\in,-}(x), A^{\in,+}(x) \vee B^{\in,+}(x)],$ $[A^{\notin,-}(x) \wedge B^{\notin,-}(x), A^{\notin,+}(x) \wedge B^{\notin,+}(x)])$ for each $x \in X$,

(v) $(\widetilde{\widetilde{A}} \cap \widetilde{\widetilde{B}})(x) = ([A^{\in,-}(x) \wedge B^{\in,-}(x), A^{\in,+}(x) \wedge B^{\in,+}(x)],$ $[A^{\notin,-}(x) \vee B^{\notin,-}(x), A^{\notin,+}(x) \vee B^{\notin,+}(x)])$ for each $x \in X$,

(vi) $(\bigcup_{j\in J} \widetilde{\widetilde{A}}_j)(x) = ([\bigvee_{j\in J} A^{\in,-}_j(x), \bigvee_{j\in J} A^{\in,+}_j(x)],$ $[\bigwedge_{j\in J} A^{\notin,-}_j(x), \bigwedge_{j\in J} A^{\notin,+}_j(x)])$ for each $x \in X$,

(vii) $(\bigcap_{j\in J} \widetilde{\widetilde{A}}_j)(x) = ([\bigwedge_{j\in J} A^{\in,-}_j(x), \bigwedge_{j\in J} A^{\in,+}_j(x)],$ $[\bigvee_{j\in J} A^{\notin,-}_j(x), \bigvee_{j\in J} A^{\notin,+}_j(x)])$ for each $x \in X$,

(viii) $[\ ]\widetilde{\widetilde{A}}(x) = (\widetilde{A}^{\in}(x), [(A^{\notin,-}(x), 1 - A^{\in,+}(x)])$ for each $x \in X$,

(ix) $\diamond\widetilde{\widetilde{A}}(x) = ([A^{\in,-}(x), 1 - A^{\notin,+}(x)], \widetilde{A}^{\notin}(x))$ for each $x \in X$.

Definition 2.5 ([37])**.** Let X be a nonempty set and let $\widetilde{A} = [A^-, A^+] \in IVS(X)$, $\bar{A} = (A^{\in}, A^{\notin}) \in IFS(X)$, $A \in I^X$. Then the triple $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle$ is called an *octahedron set* in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping. In particular, $\dot{0}$ [resp. $\dot{1}$] is called the *octahedron empty set* [resp. *octahedron whole set*] in X and is defined by $\dot{0}(x) = \langle \mathbf{0}, \bar{0}, 0\rangle$ [resp. $\dot{1}(x) = \langle \mathbf{1}, \bar{1}, 1\rangle$ for each $x \in X$. We denote the set of all octahedron sets as $\mathcal{O}(X)$.

Members of $([I] \oplus [I]) \times (I \oplus I) \times I$ are called *interval-valued intuitionistic fuzzy octahedron numbers* (briefly, IVI-octahedron numbers) (See [1]) and we write them as

$$\widetilde{\widetilde{\widetilde{a}}} = \left\langle \widetilde{\widetilde{a}}, \bar{a}, a \right\rangle,\ \widetilde{\widetilde{\widetilde{b}}} = \left\langle \widetilde{\widetilde{b}}, \bar{b}, b \right\rangle,\ \text{etc},$$

where $\widetilde{\widetilde{a}} = (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) = ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}])$, $\bar{a} = (a^{\in}, a^{\notin})$. In particular, $\left\langle \widetilde{\widetilde{0}}, \bar{0}, 0 \right\rangle$ and $\left\langle \widetilde{\widetilde{1}}, \bar{1}, 1 \right\rangle$ as $\widetilde{\widetilde{\widetilde{0}}}$ and $\widetilde{\widetilde{\widetilde{1}}}$ respectively.

We define relations $\leq$ and $=$ on $([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for any $\widetilde{\widetilde{\widetilde{a}}}$, $\widetilde{\widetilde{\widetilde{b}}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$,

$$\widetilde{\widetilde{\widetilde{a}}} \leq \widetilde{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \leq \bar{b},\ a \leq b, \quad \widetilde{\widetilde{\widetilde{a}}} = \widetilde{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{\widetilde{a}}} \leq \widetilde{\widetilde{\widetilde{b}}},\ \widetilde{\widetilde{\widetilde{b}}} \leq \widetilde{\widetilde{\widetilde{a}}}.$$

For any $(\widetilde{\widetilde{\widetilde{a}}}_j)_{j\in J} \subset ([I] \oplus [I]) \times (I \oplus I) \times I$, its inf $\bigwedge_{j\in J} \widetilde{\widetilde{\widetilde{a}}}_j$ and sup $\bigvee_{j\in J} \widetilde{\widetilde{\widetilde{a}}}_j$ are defined as follows:

$$\bigwedge_{j\in J} \widetilde{\widetilde{\widetilde{a}}}_j = \left\langle \bigwedge_{j\in J} \widetilde{\widetilde{a}}_j, \bigwedge_{j\in J} \bar{a}_j, \bigwedge_{j\in J} a_j \right\rangle, \quad \bigvee_{j\in J} \widetilde{\widetilde{\widetilde{a}}}_j = \left\langle \bigvee_{j\in J} \widetilde{\widetilde{a}}_j, \bigvee_{j\in J} \bar{a}_j, \bigvee_{j\in J} a_j \right\rangle.$$

Definition 2.6 ([1])**.** Let X be a nonempty set. Then a triple $\mathcal{A} =< \widetilde{\widetilde{A}}, A, \lambda >$ is called an *interval-valued intuitionistic fuzzy octahedron set* (briefly, IVI-octahedron set) in X, where $\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin}) = ([A^{\in,-}, A^{\in,+}], [A^{\notin,-}, A^{\notin,+}]) \in ([I] \oplus [I])^X$, $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$, $\lambda \in I^X$. In fact, $\mathcal{A} : X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ is a mapping. In particular, $\ddot{\mathbf{0}} =< \widetilde{\widetilde{0}}, \bar{0}, 0 >$ [resp. $\ddot{\mathbf{1}} =< \widetilde{\widetilde{1}}, \bar{1}, 1 >$] is called the *IVI-octahedron empty set* [resp. the *IVI-octahedron whole set*] in X and is defined by $\ddot{\mathbf{0}}(x) =< \widetilde{\widetilde{0}}, \bar{0}, 0 >$ [resp. $\ddot{\mathbf{1}}(x) =< \widetilde{\widetilde{1}}, \bar{1}, 1 >$] for each $x \in X$. We denote the set of all IVI-octahedron sets as $IVIOS(X)$.

It is obvious that for each $A \in 2^X$,

$$\chi_{\mathcal{A}} = \langle ([\chi_A, \chi_A], [\chi_{A^c}, \chi_{A^c}]), (\chi_A, \chi_{A^c}), \chi_A \rangle \in IVIOS(X),$$

where 2^X [resp. χ_A] denotes the power set of X [resp. the characteristic function of A].

Remark 2.7. (1) Let $A \in I^X$. Then clearly, $\langle [A, A], A \rangle \in C(X)$.

(2) Let $\mathbf{A} = \left\langle \widetilde{A}, A \right\rangle \in C(X)$. Then we can easily see that

$$\left\langle \widetilde{A}, (A^-, 1 - A^+), A \right\rangle \in \mathcal{O}(X).$$

(3) Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle \in \mathcal{O}(X)$. Then we can easily check that

$$\langle ([A^-, A^+], [1 - A^+, 1 - A^+], \bar{A}, A \rangle \in IVIO(X).$$

Thus from (2) and (3), we can consider an IVI-octahedron set in a set X as the generalization of both a cubic set and an octahedron set in X.

From orders of IVI-octahedron numbers, we can define the following.

Definition 2.8 ([1])**.** Let X be a nonempty set and let $\mathcal{A} =< \widetilde{\widetilde{A}}, \bar{A}, A >$, $\mathcal{B} = \widetilde{\widetilde{B}}, \bar{B}, B > \in IVIOS(X)$.

(i) We say that $\mathcal{A}$ *is equal to* $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B}$, if $\widetilde{\widetilde{A}} = \widetilde{\widetilde{B}}$, $\bar{A} = \bar{B}$, $A = B$,

(ii) We say that $\mathcal{A}$ *is a subset of* $\mathcal{B}$, denoted by $\mathcal{A} \subset \mathcal{B}$, if $\widetilde{\widetilde{A}} \subset \widetilde{\widetilde{B}}$, $\bar{A} \subset \bar{B}$, $A \subset B$.

Definition 2.9 ([1])**.** Let X be a nonempty set and let $(\mathcal{A}_j)_{j \in J} = (< \widetilde{\widetilde{A}}_j, \bar{A}_j, A_j >)_{j \in J}$ be a family of IVI-octahedron sets in X. Then the *union* $\bigcup_{j \in J} \mathcal{A}_j$ and the *intersection* $\bigcap_{j \in J} \mathcal{A}_j$ of $(\mathcal{A}_j)_{j \in J}$, are IVI-octahedron sets in X defined as follows respectively:

$$\bigcup_{j \in J} \mathcal{A}_j =< \bigcup_{j \in J} \widetilde{\widetilde{A}}_j, \bigcup_{j \in J} \bar{A}_j, \bigcup_{j \in J} A_j >,$$

$$\bigcap_{j \in J} \mathcal{A}_j =< \bigcap_{j \in J} \widetilde{\widetilde{A}}_j, \bigcap_{j \in J} \bar{A}_j, \bigcap_{j \in J} A_j > .$$

Definition 2.10. Let X be a nonempty set and let $\mathcal{A} =< \widetilde{\widetilde{A}}, \bar{A}, A > \in IVIOS(X)$. Then the *complement* $\mathcal{A}^c$, operators $[\,]$ and $\diamond$ of $\mathcal{A}$ are defined as follows respectively: for each $x \in X$,

(i) $\mathcal{A}^c =< \widetilde{\widetilde{A}}^c, \bar{A}^c, A^c >$,
(ii) $[\ ]\mathcal{A} =< [\ ]\widetilde{\widetilde{A}}, [\ ]\bar{A}, A >$,
(iii) $\diamond\mathcal{A} =< \diamond\widetilde{\widetilde{A}}, \diamond\bar{A}, A >$.

3. IVI-octahedron subgroups

In this section, we introduce the concepts of an IVI-octahedron subgroup and an IVI-octahedron normal subgroup, and study some of their properties. Unless stated otherwise in this section and next sections, G denotes a group with the identity e.

Definition 3.1 ([5])**.** Let $A \in I^X$. Then A is called a *fuzzy subgroup* (briefly, FG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,
(i) $A(xy) \geq A(x) \wedge A(y)$,
(ii) $A(x^{-1}) \geq A(x)$.
The set of all fuzzy subgroups of G is denoted by $FG(G)$.

Definition 3.2 ([16, 20])**.** Let $\bar{A} \in IFS(G)$. Then $\bar{A}$ is called an *intuitionistic fuzzy subgroup* (briefly, IFG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,
(i) $\bar{A}(xy) \geq \bar{A}(x) \wedge \bar{A}(y)$, i.e., $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y)$, $A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y)$,
(ii) $\bar{A}(x^{-1}) \geq \bar{A}(x)$, i.e., $A^{\in}(x^{-1}) \geq A^{\in}(x)$, $A^{\notin}(x^{-1}) \leq A^{\notin}(x)$.
The set of IFGs of G is denoted by $IFG(G)$.

Definition 3.3 ([40, 41])**.** Let $\widetilde{A} \in IVS(G)$. Then $\widetilde{A}$ is called an *interval-valued fuzzy subgroup* (briefly, IVFG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,
(i) $\widetilde{A}(xy) \geq \widetilde{A}(x) \wedge \widetilde{A}(y)$, i.e., $A^{-}(xy) \geq A^{-}(x) \wedge A^{-}(y)$, $A^{+}(xy) \geq A^{+}(x) \wedge A^{+}(y)$,
(ii) $\widetilde{A}(x^{-1}) \geq \widetilde{A}(x)$, i.e., $A^{-}(x^{-1}) \geq A^{-}(x)$, $A^{+}(x^{-1}) \geq A^{+}(x)$.
The set of IVFGs of G is denoted by $IVG(G)$.

Definition 3.4 ([33])**.** Let $\widetilde{\widetilde{A}} \in IVIS(G)$. Then $\widetilde{\widetilde{A}}$ is called an *interval-valued intuitionistic fuzzy subgroup* (briefly, IVIFG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,
(i) $\widetilde{\widetilde{A}}(xy) \geq \widetilde{\widetilde{A}}(x) \wedge \widetilde{\widetilde{A}}(y)$, i.e., $\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y)$, $\widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y)$,
(ii) $\widetilde{\widetilde{A}}(x^{-1}) \geq \widetilde{\widetilde{A}}(x)$, i.e., $\widetilde{A}^{\in}(x^{-1}) \geq \widetilde{A}^{\in}(x)$, $\widetilde{A}^{\notin}(x^{-1}) \leq \widetilde{A}^{\notin}(x)$.
The set of IVIFGs of G is denoted by $IVIG(G)$.

Definition 3.5 ([42])**.** Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A \right\rangle \in \mathcal{O}(G)$. Then $\mathcal{A}$ is called an *octahedron subgroup* (briefly, OG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,
(i) $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y)$, i.e., $\widetilde{A}(xy) \geq \widetilde{A}(x) \wedge \widetilde{A}(y)$, $\bar{A}(xy) \geq \bar{A}(x) \wedge \bar{A}(y)$, $A(xy) \geq A(x) \wedge A(y)$,
(ii) $\mathcal{A}(x^{-1}) \geq \mathcal{A}(x)$, i.e., $\widetilde{A}(x^{-1}) \geq \widetilde{A}(x)$, $\bar{A}(x^{-1}) \geq \bar{A}(x)$, $A(x^{-1}) \geq A(x)$.
The set of OGs of G is denoted by $OG(G)$.

Definition 3.6. Let $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOS(G)$. Then $\mathcal{A}$ is called an *IVI-octahedron subgroup* (briefly, IVIOG) of G, if it satisfies the following conditions: for any $x,\ y \in G$,

(i) $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y)$, i.e., $\widetilde{\widetilde{A}}(xy) \geq \widetilde{\widetilde{A}}(x) \wedge \widetilde{\widetilde{A}}(y)$, $\bar{A}(xy) \geq \bar{A}(x) \wedge \bar{A}(y)$, $A(xy) \geq A(x) \wedge A(y)$,

(ii) $\mathcal{A}(x^{-1}) \geq \mathcal{A}(x)$, i.e., $\widetilde{\widetilde{A}}(x^{-1}) \geq \widetilde{\widetilde{A}}(x)$, $\bar{A}(x^{-1}) \geq \bar{A}(x)$, $A(x^{-1}) \geq A(x)$.
The set of IVIOGs of G is denoted by $IVIOG(G)$.

The following is an immediate consequence of Definitions 3.1, 3.2, 3.4 and 3.6

Theorem 3.7. *Let $\mathcal{A} \in IVIOS(G)$. Then $\mathcal{A} \in IVIOG(G)$ if and only if $\widetilde{\widetilde{A}} \in IVIG(G)$, $\bar{A} \in IFG(G)$ and $A \in FG(G)$.*

Remark 3.8. (1) If $A \in FG(G)$, then we can easily check that

$$(A, A^c) \in IFG(G),\ [A, A] \in IVG(G),\ ([A, A], [A^c, A^c]) \in IVIG(G),$$
$$\langle [A, A], (A, A^c), A\rangle \in OG(G),\ \langle ([A, A], [A^c, A^c]), (A, A^c), A\rangle \in IVIOG(G).$$

(2) If $\widetilde{A} \in IVG(G)$, then we can easily see that

$$A^-,\ A^+ \in FG(G),\ (A^-, 1 - A^+) \in IFG(G),\ (\widetilde{A}, [1 - A^+, 1 - A^+]) \in IVIG(G),$$
$$\left\langle \widetilde{A}, (A^-, 1 - A^+),\ A^-\right\rangle,\ \left\langle \widetilde{A}, (A^-, 1 - A^+),\ A^+\right\rangle \in OG(G),$$
$$\left\langle (\widetilde{A}, [1 - A^+, 1 - A^+]), (A^-, 1 - A^+),\ A^-\right\rangle \in IVIOG(G),$$
$$\left\langle (\widetilde{A}, [1 - A^+, 1 - A^+]), (A^-, 1 - A^+),\ A^+\right\rangle \in IVIOG(G).$$

(3) If $\bar{A} \in IFG(G)$, then we can easily show that

$$A^\in,\ 1 - A^\notin \in FG(G),\ [A^\in, 1 - A^\notin] \in IVG(G),\ ([A^\in, 1 - A^\notin], [A^\notin, 1 - A^\in) \in IVIG(G),$$
$$\langle [A^\in, 1 - A^\notin], \bar{A}, A^\in\rangle,\ \langle [A^\in, 1 - A^\notin], \bar{A}, 1 - A^\notin\rangle \in OG(G),$$
$$\langle ([A^\in, 1 - A^\notin], [A^\notin, 1 - A^\in), \bar{A}, A^\in\rangle \in IVIOG(G),$$
$$\langle ([A^\in, 1 - A^\notin], [A^\notin, 1 - A^\in), \bar{A}, 1 - A^\notin\rangle \in IVIOG(G).$$

(4) If $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A\right\rangle \in OG(G)$, then we have

$$A^-,\ A^+,\ A^\in,\ 1 - A^\notin,\ A \in FG(G),$$
$$(A^-, 1 - A^+),\ \bar{A},\ (A, A^c) \in IFG(G),$$
$$\widetilde{A},\ [A^\in, 1 - A^\notin],\ [\lambda, \lambda] \in IVG(G),$$
$$(\widetilde{A}, 1 - \widetilde{A}),\ ([A^\in, 1 - A^\notin], [A^\notin, 1 - A^\in]),\ ([A, A], [A^c, A^c]) \in IVIG(G),$$
$$\left\langle (\widetilde{A}, [1 - A^+, 1 - A^+]), \bar{A}, \lambda\right\rangle \in IVIOG(G).$$

(5) If $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A\right\rangle \in IVIOG(G)$, then we get

$$A^{\in,-},\ A^{\in,+},\ 1 - A^{\notin,-},\ 1 - A^{\notin,+},\ A^\in,\ 1 - A^\notin,\ A \in FG(G),$$
$$\widetilde{A}^\in,\ 1 - \widetilde{A}^\notin,\ [A^\in, 1 - A^\notin],\ [A, A] \in IVG(G),$$
$$(A^{\in,-}, 1 - A^{\notin,-}),\ (1 - A^{\notin,+}, 1 - A^{\notin,-}), \bar{A}, (A, A^c) \in IFG(G),$$
$$\widetilde{\widetilde{A}},\ ([A^\in, 1 - A^\notin], [A^\notin, 1 - A^\in]),\ ([A, A], [A^c, A^c]) \in IVIG(G),$$
$$\left\langle \widetilde{A}^\in, \bar{A}, A\right\rangle,\ \left\langle 1 - \widetilde{A}^\notin, \bar{A}, A\right\rangle \in OG(G).$$

(6) If $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOG(G)$, then we can easily see that

$$[\,]\mathcal{A},\ \left\langle [\,]\widetilde{\widetilde{A}}, \bar{A}, A \right\rangle,\ \left\langle \widetilde{\widetilde{A}}, [\,]\bar{A}, A \right\rangle,\ \diamond\mathcal{A},\ \left\langle \diamond\widetilde{\widetilde{A}}, \bar{A}, A \right\rangle,\ \left\langle \widetilde{\widetilde{A}}, \diamond\bar{A}, A \right\rangle \in IVIOG(G).$$

(7) If $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOG(G)$, then we can easily check that

$$\mathcal{IO}_{\widetilde{\widetilde{A}}},\ \mathcal{IO}_A \in IVIOG(G),$$

where $\mathcal{IO}_{\widetilde{\widetilde{A}}}$ [resp. $\mathcal{IO}_A$] is the IVI-octahedron set in induced by $\widetilde{\widetilde{A}}$ [resp. A] (See Example 3.2 (3) [resp. (4)] in [1]).

Example 3.9. (1) Let $V = \{e, x, y, z\}$ be the Klein's four group, where $x^2 = y^2 = z^2 = e$ and $xy = yx = z$. Consider the IVI-octahedron $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle$ in G defined by:

$$\widetilde{\widetilde{A}}(e) = ([0.3, 0.9], [0.1, 0.7]),\ \widetilde{\widetilde{A}}(x) = ([0.1, 0.7], [0.3, 0.9]),$$
$$\widetilde{\widetilde{A}}(y) = ([0.1, 0.9], [0.1, 0.9]),\ \widetilde{\widetilde{A}}(z) = ([0.3, 0.7], [0.3, 0.7]),$$
$$\bar{A}(e) = (0.8, 0.1),\ \bar{A}(x) = (0.6, 0.3),\ \bar{A}(y) = (0.5, 0.4),\ \bar{A}(z) = (0.7, 0.2),$$
$$A(e) = 0.9,\ A(x) = 0.7,\ A(y) = 0.6,\ A(z) = 0.8.$$

Then we can easily check that $\widetilde{\widetilde{A}} \in IVIG(G)$, $\bar{A} \in IFG(G)$ and $A \in FG(G)$. Thus $\mathcal{A} \in IVIOG(G)$.

(2) Consider the additive group $(\mathbb{Z}, +)$. We define three mappings $\widetilde{\widetilde{A}} : \mathbb{Z} \to [I] \oplus [I]$, $\bar{A} : \mathbb{Z} \to I \oplus I$ and $A : \mathbb{Z} \to I$ respectively as follows: for each $0 \neq n \in \mathbb{Z}$,

$$\widetilde{\widetilde{A}}(0) = ([1, 1], [0, 0]),\ A(0) = (1, 0),\ \lambda(0) = 1, \tag{3.1}$$

$$\widetilde{\widetilde{A}}(n) = \begin{cases} \left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right) & \text{if } n \text{ is odd} \\ \left(\left[\frac{1}{3}, \frac{4}{5}\right], \left[\frac{1}{5}, \frac{2}{3}\right]\right) & \text{if } n \text{ is even,} \end{cases} \tag{3.2}$$

$$\bar{A}(n) = \begin{cases} \left(\frac{1}{2}, \frac{1}{3}\right) & \text{if } n \text{ is odd} \\ \left(\frac{2}{3}, \frac{1}{5}\right) & \text{if } n \text{ is even,} \end{cases} \tag{3.3}$$

$$A(n) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{3}{5} & \text{if } n \text{ is even.} \end{cases} \tag{3.4}$$

Then we can easily see that $\widetilde{\widetilde{A}} \in IVG(\mathbb{Z})$, $\bar{A} \in IFG(\mathbb{Z})$ and $A \in FG(\mathbb{Z})$. Thus $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, A, \lambda \right\rangle \in IVIOG(G)$.

Proposition 3.10. *Let $\mathcal{A} \in IVIOG(G)$. Then for each $x \in G$,*

(1) *$\mathcal{A}(e) \geq \mathcal{A}(x)$, i.e., $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{A}}(x)$, $\bar{A}(e) \geq \bar{A}(x)$ and $A(e) \geq A(x)$,*
(2) *$\mathcal{A}(x^{-1}) = \mathcal{A}(x)$.*

Proof. From Theorem 3.7, it is obvious that $\widetilde{\widetilde{A}} \in IVIG(G)$, $\bar{A} \in IFG(G)$ and $A \in F(G)$.

(1) By Propositions 2.8 in [20] and 5.4 in [5],

$$\bar{A}(e) \geq \bar{A}(x),\ A(e) \geq A(x) \text{for each } x \in G.$$

Then it is sufficient to show that $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{A}}(x)$, i.e.,

$$\widetilde{A}^{\in}(e) \geq \widetilde{A}^{\in}(x) \text{ and } \widetilde{A}^{\notin}(e) \leq \widetilde{A}^{\notin}(x).$$

Let $x \in G$. Then we have

$$\begin{aligned}\widetilde{A}^{\in}(e) &= \widetilde{A}^{\in}(xx^{-1})\\ &\geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(x^{-1}) \text{ [By Definition 3.6 (i)]}\\ &= \widetilde{A}^{\in}(x), \text{ [By Definition 3.6 (ii)]}\\ \widetilde{A}^{\notin}(e) &= \widetilde{A}^{\notin}(xx^{-1})\\ &\leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(x^{-1})\\ &= \widetilde{A}^{\notin}(x).\end{aligned}$$

Thus $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{A}}(x)$. So $\mathcal{A}(e) \geq \mathcal{A}(x)$.

(2) From Propositions 2.6 in [20] and 5.4 in [5],

$$\bar{A}(x^{-1}) = \bar{A}(x),\ A(x^{-1}) = A(x) \text{for each } x \in G.$$

Then it is sufficient to prove that $\widetilde{\widetilde{A}}(x^{-1}) = \widetilde{\widetilde{A}}(x)$, i.e.,

$$\widetilde{A}^{\in}(x^{-1}) = \widetilde{A}^{\in}(x) \text{ and } \widetilde{A}^{\notin}(x^{-1}) = \widetilde{A}^{\notin}(x).$$

Let $x \in G$. Then we have

$$\begin{aligned}\widetilde{A}^{\in}(x) &= \widetilde{A}^{\in}((x^{-1})^{-1})\\ &\geq \widetilde{A}^{\in}(x^{-1}) \text{ [By Definition 3.6 (ii)]}\\ &= \widetilde{A}^{\in}(x), \text{ [By Definition 3.6 (ii)]}\\ \widetilde{A}^{\notin}(x) &= \widetilde{A}^{\notin}((x^{-1})^{-1})\\ &\leq \widetilde{A}^{\notin}(x^{-1})\\ &= \widetilde{A}^{\notin}(x).\end{aligned}$$

Thus $\widetilde{\widetilde{A}}((x^{-1})^{-1}) = \widetilde{\widetilde{A}}(x)$. So $\mathcal{A}((x^{-1})^{-1}) = \mathcal{A}(x)$. □

Theorem 3.11. *Let $H \in 2^G$. Then H is a subgroup of G if and only if $\chi_{\mathcal{H}} = \langle([\chi_H, \chi_H], [\chi_{H^c}, \chi_{H^c}]), (\chi_H, \chi_{H^c}), \chi_H\rangle \in IVIOG(G)$.*

Proof. From Propositions 2.2 in [20] and 5.1 in [5], it is clear that $(\chi_H, \chi_{H^c}) \in IFG(G)$ and $\chi_H \in FG(G)$. Then it is sufficient to prove that $([\chi_H, \chi_H], [\chi_{H^c}, \chi_{H^c}]) \in IVIG(G)$. Let $x,\ y \in G$. Since $(\chi_H, \chi_{H^c}) \in IFG(G)$, we can easily check that

$$[\chi_H, \chi_H](x, y) \geq [\chi_H, \chi_H](x) \wedge [\chi_H, \chi_H](y),$$

$$[\chi_{H^c}, \chi_{H^c}](x, y) \leq [\chi_{H^c}, \chi_{H^c}](x) \vee [\chi_{H^c}, \chi_{H^c}](y),$$

$$[\chi_H, \chi_H](x^{-1}) \geq [\chi_H, \chi_H](x),\ [\chi_{H^c}, \chi_{H^c}](x^{-1}) \leq [\chi_{H^c}, \chi_{H^c}](x).$$

Thus $([\chi_H, \chi_H], [\chi_{H^c}, \chi_{H^c}]) \in IVIG(G)$. So by Theorem 3.7, $\chi_{\mathcal{H}} \in IVIOG(G)$. □

Proposition 3.12. *If $(\mathcal{A}_j)_{j\in J} = (\left\langle \widetilde{\widetilde{A}}_j, \bar{A}_j, A_j \right\rangle)_{j\in J} \subset IVIOG(G)$, then $\bigcap_{j\in J} \mathcal{A}_j \in IVIOG(G)$, where J denotes an index set.*

Proof. Let $\mathcal{A} = \bigcap_{j\in J}\mathcal{A}_j = \left\langle \bigcap_{j\in J}\widetilde{\widetilde{A}}_j, \bigcap_{j\in J}\bar{A}_j, \bigcap_{j\in J}A_j \right\rangle$. Then clearly, $\bar{A} = \bigcap_{j\in J}\bar{A}_j \in IFG(G)$ by Proposition 2.3 in [20] and $A = \bigcap_{j\in J}A_j \in FG(G)$ by Proposition 5.2 in [20]. Thus it is sufficient to show that $\widetilde{\widetilde{A}} = \bigcap_{j\in J}\widetilde{\widetilde{A}}_j \in IVIG(G)$. Let $x,\ y \in G$. Then we have

$$\begin{aligned}
\widetilde{A}^{\in}(x,y) &= (\textstyle\bigcap_{j\in J}\widetilde{A}^{\in}_j)(x,y)\\
&= \textstyle\bigwedge_{j\in J}\widetilde{A}^{\in}_j(x,y)\\
&\geq \textstyle\bigwedge_{j\in J}[\widetilde{A}^{\in}_j(x)\wedge \widetilde{A}^{\in}_j(y)] \text{ [By Definition 3.6 (i)]}\\
&= (\textstyle\bigwedge_{j\in J}\widetilde{A}^{\in}_j(x))\wedge(\bigwedge_{j\in J}\widetilde{A}^{\in}_j(y))\\
&= (\widetilde{A}^{\in})(x)\wedge(\widetilde{A}^{\in})(y)\\
&= \widetilde{A}^{\in}(x)\wedge\widetilde{A}^{\in}(y),\\
\widetilde{A}^{\in}(x^{-1}) &= (\textstyle\bigcap_{j\in J}\widetilde{A}^{\in}_j)(x^{-1})\\
&= \textstyle\bigwedge_{j\in J}\widetilde{A}^{\in}_j(x^{-1})\\
&\geq \textstyle\bigwedge_{j\in J}\widetilde{A}^{\in}_j(x) \text{ [By Definition 3.6 (ii)]}\\
&= (\textstyle\bigcap_{j\in J}\widetilde{A}^{\in}_j)(x)\\
&= A^{\in}(x).
\end{aligned}$$

Similarly, we get $A^{\notin}(x,y) \leq A^{\notin}(x)\vee A^{\notin}(y)$ and $A^{\notin}(x^{-1}) \leq A^{\notin}(x)$. Thus $\widetilde{\widetilde{A}} \in IVIG(G)$. So by Theorem 3.7, $\mathcal{A} = \bigcap_{j\in J}\mathcal{A}_j \in IVIOG(G)$. □

The following is an immediate consequence of Proposition 3.12.

Corollary 3.13. *Let $\mathcal{A} \in IVIOG(G)$ and let*

$$(\mathcal{A}) = \left\langle (\widetilde{\widetilde{A}}), (\bar{A}), (A) \right\rangle = \bigcap\{\mathcal{B} = \left\langle \widetilde{\widetilde{B}}, \bar{B}, B \right\rangle \in IVIOG(G) : \mathcal{A} \subset \mathcal{B}\}.$$

Then $(\mathcal{A}) \in IVIOG(G)$.

In this case, $(\mathcal{A})$ is called the *IVI-octahedron subgroup of G generated by* $\mathcal{A}$.

The following is an immediate result of Theorem 3.11 and Corollary 3.13.

Corollary 3.14. *For each $H \in 2^G$, let*

$$\chi_{(\mathcal{H})} = \langle([\chi_{(H)}, \chi_{(H)}], [\chi_{(H^c)}, \chi_{(H^c)}]), (\chi_{(H)}, \chi_{(H^c)}), \chi_{(H)}\rangle.$$

Then $(\chi_{\mathcal{H}}) = \chi_{(\mathcal{H})}$.

We obtain the characterization of an IVI-octahedron subgroup of G.

Theorem 3.15. *Let $\mathcal{A} \in IVIOS(G)$. Then $\mathcal{A} \in IVIOG(G)$ if and only if $\mathcal{A}(xy^{-1}) \geq \mathcal{A}(x)\wedge\mathcal{A}(y)$ for any $x,\ y \in G$.*

Proof. The proof is straightforward. □

Corollary 3.16. *If $\mathcal{A} \in IVIOG(G)$, then $G_{\mathcal{A}} = \{x \in G : \mathcal{A}(x) = \mathcal{A}(e)\}$ is a subgroup of G.*

In this case, $G_{\mathcal{A}}$ is called the *subgroup of G induced by $\mathcal{A}$*.

Proof. Let $G_{\widetilde{\widetilde{A}}} = \{x \in G : \widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{A}}(e)\}$, $G_{\bar{A}} = \{x \in G : \bar{A}(x) = \bar{A}(e)\}$ and $G_A = \{x \in G : A(x) = A(e)\}$. Then clearly, $G_{\mathcal{A}} = G_{\widetilde{\widetilde{A}}} \cap G_{\bar{A}} \cap G_A$. It is clear that

$G_{\bar{A}}$ is a subgroup of G by Proposition 2.7 in [20] and G_{A} is a subgroup of G by Corollary of Proposition 5.4 in [5]. In order to show that $G_{\tilde{\tilde{A}}}$ is a subgroup of G, let $x,\ y \in G_{\tilde{\tilde{A}}}$. Then

$$\begin{aligned} \widetilde{A}^{\in}(xy^{-1}) &\geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \text{ [By Theorem 3.15]} \\ &= \widetilde{A}^{\in}(e). \text{ [Since } x,\ y \in G_{\tilde{\tilde{A}}}] \end{aligned}$$

Thus by Proposition 3.10 (1), $\widetilde{A}^{\in}(xy^{-1}) = \widetilde{A}^{\in}(e)$. Similarly, $\widetilde{A}^{\notin}(xy^{-1}) = \widetilde{A}^{\notin}(e)$. So $\widetilde{\widetilde{A}}(xy^{-1}) = \widetilde{\widetilde{A}}(e)$. Hence $xy^{-1} \in G_{\tilde{\tilde{A}}}$. Therefore $G_{\mathcal{A}}$ is a subgroup of G. □

Proposition 3.17. *Let $\mathcal{A} \in IVIOG(G)$. If $\mathcal{A}(xy^{-1}) = \mathcal{A}(e)$ for any $x,\ y \in G$, then $\mathcal{A}(x) = \mathcal{A}(y)$.*

Proof. Suppose $\mathcal{A}(xy^{-1}) = \mathcal{A}(e)$ for any $x,\ y \in G$. Then clearly, $\widetilde{\widetilde{A}}(xy^{-1}) = \widetilde{\widetilde{A}}(e)$, $\bar{A}(xy^{-1}) = \bar{A}(e)$ and $A(xy^{-1}) = A(e)$. Since $\bar{A} \in IFG(G)$ and $A \in FG(G)$, it is obvious that $\bar{A}(x) = \bar{A}(y)$ by Proposition 2.8 in [20] and $A(x) = A(y)$ by Proposition 5.4 in [5]. Thus it is sufficient to prove that $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{A}}(y)$. Let $x,\ y \in G$. Then we have

$$\begin{aligned} \widetilde{A}^{\in}(x) &= \widetilde{A}^{\in}((xy^{-1})y) \\ &\geq \widetilde{A}^{\in}(xy^{-1}) \wedge \widetilde{A}^{\in}(y) \text{ [Since } \widetilde{\widetilde{A}} \in IVIG(G)] \\ &= \widetilde{A}^{\in}(e) \wedge \widetilde{A}^{\in}(y) \text{ [By the hypothesis]} \\ &= \widetilde{A}^{\in}(y), \text{ [Since } \widetilde{A}^{\in}(e) \geq \widetilde{A}^{\in}(y) \text{ by Proposition 3.10 (1)]} \end{aligned}$$

$$\begin{aligned} \widetilde{A}^{\in}(y) &= \widetilde{A}^{\in}((yx^{-1})x) \\ &\geq \widetilde{A}^{\in}(yx^{-1}) \wedge \widetilde{A}^{\in}(x) \\ &= \widetilde{A}^{\in}(xy^{-1}) \wedge \widetilde{A}^{\in}(x) \text{ [By Proposition 3.10 (2)]} \\ &= \widetilde{A}^{\in}(e) \wedge \widetilde{A}^{\in}(x) \\ &= \widetilde{A}^{\in}(x). \end{aligned}$$

Thus $\widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(y)$. Similarly, we can see that $\widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(y)$. So $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{A}}(y)$. Hence $\mathcal{A}(x) = \mathcal{A}(y)$. □

Corollary 3.18. *Let $\mathcal{A} \in IVIOG(G)$. If $G_{\mathcal{A}}$ is a normal subgroup of G, then $\mathcal{A}$ is constant on each coset of $G_{\mathcal{A}}$.*

Proof. Suppose $G_{\mathcal{A}}$ is a normal subgroup of G. Then we can easily see that $G_{\tilde{\tilde{A}}}$, $G_{\bar{A}}$ and G_{A} are normal subgroups of G respectively. Thus from Corollary 2.8-1 in [20] and the first Corollary of Proposition 5.4 in [5], $\bar{A}$ is constant on each coset of $G_{\bar{A}}$ and A is constant on each coset of G_{A}. So it is sufficient show that $G_{\tilde{\tilde{A}}}$ is constant on each coset of $G_{\tilde{\tilde{A}}}$.

Let $a \in G$ and let $x \in aG_{\tilde{\tilde{A}}}$. Then there is $x^{'} \in G_{\tilde{\tilde{A}}}$ such that $x = ax^{'}$. Since $G_{\tilde{\tilde{A}}}$ is normal subgroup of G and $x^{'} \in G_{\tilde{\tilde{A}}}$, $xa^{-1} = ax^{'}a^{-1} \in G_{\tilde{\tilde{A}}}$. Thus $G_{\tilde{\tilde{A}}}(xa^{-1}) = G_{\tilde{\tilde{A}}}(e)$. By Proposition 3.17, $G_{\tilde{\tilde{A}}}(x) = G_{\tilde{\tilde{A}}}(a)$. So $\widetilde{\widetilde{A}}$ is constant on $aG_{\tilde{\tilde{A}}}$ for each $a \in G$. Hence $\mathcal{A}$ is constant on each coset of $G_{\mathcal{A}}$. □

Let H be a subgroup of G. Then the number of right [resp. left] cosets of H in G is called the *index of H in G* and denoted by $[G : H]$. If G is a finite group, then there can be only a finite number of distinct right [resp. left] cosets of H, i.e., $[G : H]$ is finite. If G is an infinite group, then $[G : H]$ may be either finite or infinite.

Definition 3.19. Let X be a nonempty set and let $\widetilde{\widetilde{A}} \in IVIS(X)$. Then we say that $\widetilde{\widetilde{A}}$ *has the sup-property*, if for each $T \in 2^X$, there is $t_0 \in T$ such that

$$\widetilde{\widetilde{A}}(t_0) = (\bigcup_{t\in T} \widetilde{\widetilde{A}})(t) = \bigvee_{t\in T} \widetilde{\widetilde{A}}(t), \text{ i.e.,}$$

$$\widetilde{A}^{\in}(t_0) = \bigvee_{t\in T} \widetilde{A}^{\in}(t),\ \widetilde{A}^{\notin}(t_0) = \bigwedge_{t\in T} \widetilde{A}^{\notin}(t).$$

Definition 3.20 ([1])**.** Let X be a nonempty set and let $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIO(X)$. Then we say that $\mathcal{A}$ *has the sup-property*, if for each $T \in 2^X$, there is $t_0 \in T$ such that $\mathcal{A}(t_0) = (\bigcup_{t\in T} \mathcal{A})(t) = \left\langle \bigvee_{t\in T} \widetilde{\widetilde{A}}(t), \bigvee_{t\in T} \bar{A}(t), \bigvee_{t\in T} A(t) \right\rangle$.

It is obvious that $\mathcal{A} \in IVIO(X)$ has the sup-property if and only if $\widetilde{\widetilde{A}}$, $\bar{A}$ and A have the sup-property.

Corollary 3.21. *Let $\mathcal{A} \in IVIOG(G)$ and let $G_{\mathcal{A}}$ be a normal subgroup of G. If $G_{\mathcal{A}}$ has a finite index, then $\mathcal{A}$ has the sup-property.*

Proof. Suppose $G_{\mathcal{A}}$ has a finite index. Then clearly, $G_{\widetilde{\widetilde{A}}}$, $G_{\bar{A}}$ and G_A have finite indices respectively. From Corollary 2.8-2 in [20] and the second Corollary of Proposition 5.4 in [5], $\bar{A}$ and A have the sup-property respectively. Thus it is sufficient to prove that $\widetilde{\widetilde{A}}$ has the sup-property.

Let $T \in 2^G$ and let $G/G_{\widetilde{\widetilde{A}}}$ denote the set of all right cosets of $G_{\widetilde{\widetilde{A}}}$. Since $G_{\widetilde{\widetilde{A}}}$ has a finite index, let $[G, G_{\widetilde{\widetilde{A}}}] = n$, say $G/G_{\widetilde{\widetilde{A}}} = \{a_1 G_{\widetilde{\widetilde{A}}}, \cdots, a_n G_{\widetilde{\widetilde{A}}}\}$, where $a_i \in G$ $(i = 1,\ 2,\ ,\cdots,\ n)$ and $a_i G_{\widetilde{\widetilde{A}}} \cap a_j G_{\widetilde{\widetilde{A}}} = \varnothing$ for any $i \neq j$. Let $t \in T$. Since $G = \bigcup G/G_{\widetilde{\widetilde{A}}} = \bigcup_{i=1}^n a_i G_{\widetilde{\widetilde{A}}}$ and $T \in 2^G$, $t \in \bigcup_{i=1}^n a_i G_{\widetilde{\widetilde{A}}}$. Then there is $i \in \{1, \cdots, n\}$ such that $t \in a_i G_{\widetilde{\widetilde{A}}}$. Since $G_{\widetilde{\widetilde{A}}}$ is normal, by Corollary 3.18, $\widetilde{\widetilde{A}}(t) = \widetilde{\widetilde{A}}(a_i)$ on $a_i G_{\widetilde{\widetilde{A}}}$, say $\widetilde{\widetilde{A}}(t) = \widetilde{\widetilde{a}}_i$, i.e., $\widetilde{A}^{\in}(t) = \widetilde{a}_i{}^{\in}$ and $\widetilde{A}^{\notin}(t) = \widetilde{a}_i{}^{\notin}$. Thus there is $t_0 \in T$ such that $\widetilde{A}^{\in}(t_0) = \bigvee_{t\in T} \widetilde{A}^{\in}(t) = \bigvee_{i=1}^n \widetilde{a}_i{}^{\in}$ and $\widetilde{A}^{\notin}(t_0) = \bigwedge_{t\in T} \widetilde{A}^{\notin}(t) = \bigwedge_{i=1}^n \widetilde{a}_i{}^{\notin}$. So $\widetilde{\widetilde{A}}$ has the sup-property. Hence $\mathcal{A}$ has the sup-property. □

Definition 3.22 ([1])**.** Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} =< \widetilde{\widetilde{A}}, \bar{A}, A >\in IVIOS(X)$, $\mathcal{B} =< \widetilde{\widetilde{B}}, \bar{B}, B >\in IVIOS(Y)$.

(i) The *preimage of $\mathcal{B}$ under f*, denoted by $f^{-1}(\mathcal{B})$, is an IVI-octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) = \left\langle f^{-1}(\widetilde{\widetilde{B}})(x), f^{-1}(\bar{B})(x), f^{-1}(B)(x) \right\rangle,$$

where $\quad f^{-1}(\widetilde{\widetilde{B}})(x) = (\widetilde{\widetilde{B}} \circ f)(x) = ((\widetilde{B}^{\in} \circ f)(x), (\widetilde{B}^{\notin} \circ f)(x)),$

$$f^{-1}(\bar{B})(x) = ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)),\ f^{-1}(B)(x) = (B \circ f)(x).$$

In fact, $f^{-1}(\widetilde{\widetilde{B}})$ [resp. $f^{-1}(\bar{B})$ and $f^{-1}(B)$] is the preimage of an IVIS $\widetilde{\widetilde{B}}$ [resp. an IFS $\bar{B}$ and a fuzzy set B] under f (See [43] [resp. [44] and [4]]).

(ii) The *image of $\mathcal{A}$ under f*, denoted by $f(\mathcal{A})$, is an IVI-octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\mathcal{A})(y) = \left\langle f(\widetilde{\widetilde{A}})(y), f(\bar{A})(y), f(A)(y) \right\rangle = \left\langle (f(\widetilde{A}^{\in})(y), f(\widetilde{A}^{\notin})(y)), f(\bar{A})(y), f(A)(y) \right\rangle,$$

where

$$f(\widetilde{A}^{\in})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^{\in,-}(x), \bigvee_{x \in f^{-1}(y)} A^{\in,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ [0,0] & \text{otherwise,} \end{cases}$$

$$f(\widetilde{A}^{\notin})(y) = \begin{cases} [\bigwedge_{x \in f^{-1}(y)} A^{\notin,-}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin,+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ [1,1] & \text{otherwise,} \end{cases}$$

$$f(\bar{A})(y) = \begin{cases} (\bigvee_{x \in f^{-1}(y)} A^{\in}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ (0,1) & \text{otherwise,} \end{cases}$$

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $f(\widetilde{\widetilde{A}})$ [resp. $f(\bar{A})$ and $f(A)$] is the image of an IVIS $\widetilde{\widetilde{A}}$ [resp. an IFS $\bar{A}$ and a fuzzy set A] under f (See [43] [resp. [44] and [4]]).

Proposition 3.23. *Let $f : G \to G'$ be a group homomorphism, let $\mathcal{A} \in IVIOG(G)$ and let $\mathcal{B} \in IVIOG(G')$. Then the followings hold:*
(1) *if $\mathcal{A}$ has the sup-property, then $f(\mathcal{A}) \in IVIOG(G')$,*
(2) *$f^{-1}(\mathcal{B}) \in IVIOG(G)$.*

Proof. (1) Suppose $\mathcal{A}$ has the sup-property. Then clearly, $\widetilde{\widetilde{A}}$, $\bar{A}$ and A have the sup-property respectively. By Proposition 2.13 (1) in [20], $f(\bar{A}) \in IFG(G)$. From Proposition 4.2 in [5], $f(A)(xy) \geq f(A)(x) \wedge f(A)(y)$ for any $x,\ y \in f(\lambda)$. Moreover, we can easily see that $f(A)(y^{-1}) \geq f(A)(y)$ for each $y \in f(A)$. Thus $f(A) \in FG(G)$. So it is sufficient to show that $f(\widetilde{\widetilde{A}}) \in IVIG(G)$.

Let $y,\ y' \in f(G)$. Then $\varnothing \neq f^{-1}(y) \subset G$ and $\varnothing \neq f^{-1}(y') \subset G$. Since $\widetilde{\widetilde{A}}$ has the sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0' \in f^{-1}(y')$ such that

$$\widetilde{A}^{\in}(x_0) = \bigvee_{t \in f^{-1}(y)} \widetilde{A}^{\in}(t),\ \widetilde{A}^{\notin}(x_0) = \bigvee_{t \in f^{-1}(y)} \widetilde{A}^{\notin}(t)$$

and

$$\widetilde{A}^{\in}(x_0') = \bigvee_{t' \in f^{-1}(y')} \widetilde{A}^{\in}(t'),\ \widetilde{A}^{\notin}(x_0') = \bigvee_{t' \in f^{-1}(y')} \widetilde{A}^{\notin}(t').$$

Thus we have

$$\begin{aligned} f(\widetilde{A}^{\in})(yy') &= \textstyle\bigvee_{x \in f^{-1}(yy')} \widetilde{A}^{\in}(x) \text{ [By Definition 3.22 (ii)]} \\ &\geq \widetilde{A}^{\in}(x_0x_0') \text{ [Since } x_0x_0' \in f^{-1}(yy')] \\ &\geq \widetilde{A}^{\in}(x_0) \wedge \widetilde{A}^{\in}(x_0') \text{ [Since } \widetilde{\widetilde{A}} \in IVIG(G)] \end{aligned}$$

$= (\bigvee_{t\in f^{-1}(y)} \widetilde{A}^{\in}(t)) \wedge (\bigvee_{t'\in f^{-1}(y')} \widetilde{A}^{\in}(t'))$
$= f(\widetilde{A}^{\in})(y) \wedge f(\widetilde{A}^{\in})(y')$.

Similarly, we get $f(\widetilde{A}^{\notin})(yy') \leq f(\widetilde{A}^{\notin})(y) \vee f(\widetilde{A}^{\notin})(y')$. On the other hand,

$f(\widetilde{A}^{\in})(y^{-1}) = \bigvee_{t\in f^{-1}(y^{-1})} \widetilde{A}^{\in}(t)$
$\geq \widetilde{A}^{\in}(x_0^{-1})$ [Since $x_0^{-1} \in f^{-1}(y^{-1})$]
$\geq \widetilde{A}^{\in}(x_0)$ [Since $\widetilde{\widetilde{A}} \in IVIG(G)$]
$= f(\widetilde{A}^{\in})(y)$.

Similarly, we have $f(\widetilde{A}^{\notin})(y^{-1}) \leq f(\widetilde{A}^{\notin})(y)$. So $f(\widetilde{\widetilde{A}}) \in IVIG(G)$. Hence $f(\mathcal{A}) \in IVIOG(G')$.

(2) Since $\mathcal{B} \in IVIOG(G')$, it is clear that $\widetilde{\widetilde{B}} \in IVIG(G')$, $B \in IFG(G')$ and $\mu \in FG(G')$. From Proposition 2.13 (2) in [20], $f^{-1}(B) \in IFG(G)$. By Proposition 4.1, $f^{-1}(\mu)(xy) \geq f^{-1}(\mu)(x) \wedge f^{-1}(\mu)(y)$ for any $x,\ y \in G$. It can easily see that $f^{-1}(\mu)(x^{-1}) \geq f^{-1}(\mu)(x)$ for each $x \in G$. Then $f^{-1}(\mu) \in FG(G)$. Thus it is enough to prove that $f^{-1}(\widetilde{\widetilde{B}}) \in IVIG(G)$.

Let $x,\ y \in G$. Then we have

$f^{-1}(\widetilde{B}^{\in})(xy) = \widetilde{B}^{\in}(f(xy))$
$= \widetilde{B}^{-in}(f(x)f(y))$ [Since f is a group homomorphism]
$\geq \widetilde{B}^{\in}(f(x)) \wedge \widetilde{B}^{\in}(f(y))$
$= f^{-1}(\widetilde{B}^{\in})(x) \wedge f^{-1}(\widetilde{B}^{\in})(y)$.

Similarly, we get $f^{-1}(\widetilde{B}^{\in})(xy) \leq f^{-1}(\widetilde{B}^{\in})(x) \vee f^{-1}(\widetilde{B}^{\in})(y)$. On the other hand,

$f^{-1}(\widetilde{B}^{\in})(x^{-1}) = \widetilde{B}^{\in}(f(x^{-1}))$
$= \widetilde{B}^{\in}(f(x)^{-1})$ [Since f is a group homomorphism]
$\geq \widetilde{B}^{\in}(f(x))$ [Since $\widetilde{\widetilde{B}} \in IVIG(G')$]
$= f^{-1}(\widetilde{B}^{\in})(x)$.

Similarly, we have $f^{-1}(\widetilde{B}^{\notin})(x^{-1}) \leq f^{-1}(\widetilde{B}^{\notin})(x)$. Thus $f^{-1}(\widetilde{\widetilde{B}}) \in IVIG(G)$. So $f^{-1}(\mathcal{B}) \in IVIOG(G)$. □

Theorem 3.24. *Let G_p be the cyclic group of a prime order p. Then $\mathcal{A} \in IVIOG(G_p)$ if and only if for each $0 \neq x \in G_p$, $\mathcal{A}(x) = \mathcal{A}(1) \leq \mathcal{A}(0)$, i.e.,*

$$\widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(1) \leq \widetilde{A}^{\in}(0),\ \widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(1) \geq \widetilde{A}^{\notin}(0), \tag{3.5}$$

$$A^{\in}(x) = A^{\in}(1) \leq A^{\in}(0),\ A^{\notin}(x) = A^{\notin}(1) \leq A^{\notin}(0),\ A(x) = A(1) \leq A(0). \tag{3.6}$$

Proof. It is obvious that $\mathcal{A} \in IVIOG(G_p)$ if and only if $\widetilde{\widetilde{A}} \in IVIG(G_p)$, $\bar{A} \in IFG(G_p)$ and $A \in FG(G_p)$. Moreover, from Propositions 2.14 in [20] and 5.10 in [5], $\bar{A} \in IFG(G_p)$ and $A \in FG(G_p)$ if and only if (3.6) holds. Then it is sufficient to prove that $\widetilde{\widetilde{A}} \in IVIG(G_p)$ if and only if (3.5) holds.

(⇒): Suppose $\widetilde{\widetilde{A}} \in IVIG(G_p)$ and let $0 \neq x \in G_p$. Since $\widetilde{\widetilde{A}} \in IVIG(G_p)$, we have for each $y \in G_p$,

$$\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \text{ and } \widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y).$$

Since G_p is the cyclic group of a prime order p, $G_p = \{0, 1, 2, \cdots, p-1\}$. Since x is the sum of 1's and 1 is the sum of x's, we get

$$\widetilde{A}^{\in}(x) \geq \widetilde{A}^{\in}(1) \geq \widetilde{A}^{\in}(x) \text{ and } \widetilde{A}^{\notin}(x) \leq \widetilde{A}^{\notin}(1) \leq \widetilde{A}^{\notin}(x).$$

Thus $\widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(1)$ and $\widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(1)$. Since 0 is the identity element of G_p, $\widetilde{A}^{\in}(x) \leq \widetilde{A}^{\in}(0)$ and $\widetilde{A}^{\notin}(x) \geq \widetilde{A}^{\notin}(0)$. So (3.5) holds.

($\Leftarrow$): Suppose the conditions (3.5) holds and let $x,\ y \in G_p$. Then we have the following four cases:

(i) $x \neq 0,\ y \neq 0$ and $x = y$, (ii) $x \neq 0,\ y = 0$,
(iii) $x = 0,\ y \neq 0$, (iv) $x \neq 0,\ y \neq 0$ and $x \neq y$.

Case (i): Suppose $x \neq 0,\ y \neq 0$ and $x = y$. Then by the hypothesis, we get

$$\widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(y) = \widetilde{A}^{\in}(1) \leq \widetilde{A}^{\in}(0) \text{ and } \widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(y) = \widetilde{A}^{\notin}(1) \geq \widetilde{A}^{\notin}(0).$$

Thus we have

$$\widetilde{A}^{\in}(x-y) = \widetilde{A}^{\in}(0) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \text{ and } \widetilde{A}^{\notin}(x-y) = \widetilde{A}^{\notin}(0) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y).$$

Case (ii): Suppose $x \neq 0,\ y = 0$. Since $x - y \neq 0$, by the hypothesis,

$$\widetilde{A}^{\in}(x-y) = \widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(1) \leq \widetilde{A}^{\in}(0) = \widetilde{A}^{\in}(y)$$

and

$$\widetilde{A}^{\notin}(x-y) = \widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(1) \geq \widetilde{A}^{\notin}(0) = \widetilde{A}^{\notin}(y).$$

Thus $\widetilde{A}^{\in}(x-y) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y)$ and $\widetilde{A}^{\notin}(x-y) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y)$.

Case (iii): The proof is similar to Case (ii).

Case (iv): Suppose $x \neq 0,\ y \neq 0$ and $x \neq y$. Since $x - y \neq 0$, by the hypothesis,

$$\widetilde{A}^{\in}(x-y) = \widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(y) = \widetilde{A}^{\in}(1) \leq \widetilde{A}^{\in}(0)$$

and

$$\widetilde{A}^{\notin}(x-y) = \widetilde{A}^{\notin}(x) = \widetilde{A}^{\notin}(y) = \widetilde{A}^{\notin}(1) \geq \widetilde{A}^{\notin}(0).$$

Then $\widetilde{A}^{\in}(x-y) \geq \widetilde{A}^{\in} \wedge \widetilde{A}^{\in}(y)$ and $\widetilde{A}^{\notin}(x-y) \leq \widetilde{A}^{\notin} \vee \widetilde{A}^{\notin}(y)$. Thus in all cases, we get the following inequalities:

$$\widetilde{A}^{\in}(x-y) \geq \widetilde{A}^{\in} \wedge \widetilde{A}^{\in}(y) \text{ and } \widetilde{A}^{\notin}(x-y) \leq \widetilde{A}^{\notin} \vee \widetilde{A}^{\notin}(y).$$

So $\widetilde{\widetilde{A}} \in IVIG(G_p)$. Hence by Theorem 3.15, $\mathcal{A} \in IVIOG(G_p)$. □

Definition 3.25. Let $\mathcal{A} \in IVIOG(G)$. Then $\mathcal{A}$ is called an *IVI-octahedron normal subgroup* (in briefly, IVIONG) of G, if $\mathcal{A}(xy) = \mathcal{A}(yx)$ for any $x,\ y \in G$.

It is obvious that if G is an abelian, then every IVIOG of G is an IVIONG of G.

We will denote the set of all IVIONGs [resp. IVINGs, IFNGs and FNGs] of G by $IVIONG(G)$ [resp. $IVING(G)$, $IFNG(G)$ and $FNG(G)$] (See [33], [20] and [7] for definitions of an IVING, an IFNG and a FNG).

The following is an immediate consequence of Theorem 3.7 and Definition 3.25.

Theorem 3.26. *Let* $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOS(G)$. *Then* $\mathcal{A} \in IVIONG(G)$ *if and only if* $\widetilde{\widetilde{A}} \in IVING(G)$, $\bar{A} \in IFNG(G)$ *and* $A \in FNG(G)$.

Example 3.27. Consider the general linear group of degree n, $GL(n,\mathbb{R})$ and let I_n be the unit matrix. Define $\mathcal{A} : GL(n,\mathbb{R}) \to ([I]\oplus[I])\times(I\oplus I)\times I$ as follows: for each $I_n \neq M \in GL(n,\mathbb{R})$,

$$\mathcal{A}(I_n) = \langle([1,1],(0,0]),(1,0),1\rangle\,, \tag{3.7}$$

$$\widetilde{\widetilde{A}}(M) = \begin{cases} \left(\left[\frac{2}{3},\frac{4}{5}\right],\left[\frac{1}{3},\frac{1}{2}\right]\right) & \text{if } M \text{ is not a triangular matrix} \\ \left(\left[\frac{1}{2},\frac{2}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right) & \text{if } M \text{ is a triangular matrix,} \end{cases} \tag{3.8}$$

$$\bar{A}(M) = \begin{cases} \left(\frac{2}{3},\frac{1}{5}\right) & \text{if } M \text{ is not a triangular matrix} \\ \left(\frac{1}{2},\frac{1}{3}\right) & \text{if } M \text{ is a triangular marix,} \end{cases} \tag{3.9}$$

$$A(M) = \begin{cases} \frac{1}{2} & \text{if } M \text{ is not a triangular matrix} \\ \frac{3}{5} & \text{if } M \text{ is a triangular matrix.} \end{cases} \tag{3.10}$$

Then we can easily check that $\widetilde{\widetilde{A}} \in IVING(GL(n,\mathbb{R}))$, $\bar{A} \in IFNG(GL(n,\mathbb{R}))$ and $A \in FNG(GL(n,\mathbb{R}))$. Thus by Theorem 3.26, $\mathcal{A} \in IVIONG(G)$.

Definition 3.28 ([7])**.** Let $(X,\cdot)$ be a groupoid and let $A,\ B \in I^X$. Then the product of A and B, denoted by $A \circ_F B$, is a fuzzy set in X defined as follows: for each $x \in X$,

$$(A \circ_F B)(x) = \begin{cases} \bigvee_{yz=x,\ y,\ z\in X}[A(y)\wedge B(z)] & \text{if } yz = x \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that if $A \in FG(G)$, then $A \circ_F A \subset A$.

Definition 3.29 ([17])**.** Let $(X,\cdot)$ be a groupoid and let $\bar{A},\ \bar{B} \in IFS(X)$. Then the product of $\bar{A}$ and $\bar{B}$, denoted by $A \circ_{IF} B$, is an IF set in X defined as follows: for each $x \in X$,

$$(\bar{A}\circ_{IF}\bar{B})(x) = \begin{cases} (\bigvee_{yz=x,\ y,\ z\in X}[A^{\in}(y)\wedge B^{\in}(z)], \bigwedge_{yz=x,\ y,\ z\in X}[A^{\notin}(y)\wedge B^{\notin}(z)] & \text{if } yz = x \\ (0,1) & \text{otherwise.} \end{cases}$$

We can easily see that if $\bar{A} \in IFG(G)$, then $\bar{A} \circ_{IF} \bar{A} \subset \bar{A}$.

Definition 3.30 ([1])**.** Let $(X,\cdot)$ be a groupoid and let $\widetilde{\widetilde{A}},\ \widetilde{\widetilde{B}} \in IVIS(X)$. Then the product of $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$, denoted by $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}$, is an IVI set in X defined as follows: for each $x \in X$,

$$(\widetilde{\widetilde{A}}\circ_{IVI}\widetilde{\widetilde{B}})(x) = \begin{cases} [\bigvee_{yz=x,\ y,\ z\in X}[\widetilde{A}^{\in}(y)\wedge \widetilde{B}^{\in}(z)], \bigwedge_{yz=x,\ y,\ z\in X}[\widetilde{A}^{\notin}(y)\wedge \widetilde{B}^{\notin}(z)] & \text{if } yz = x \\ ([0,0],[1,1]) & \text{otherwise.} \end{cases}$$

It can be easily seen that if $A \in IFG(G)$, then $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$.

Definition 3.31 ([1]). Let $(X, \cdot)$ be a groupoid and let $\mathcal{A},\ \mathcal{B} \in IVIO(X)$. Then the *product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B}$, is an IVI-octahedron set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \circ \mathcal{B})(x) = \begin{cases} \bigvee_{yz=x,\ y,\ z \in X}[\mathcal{A}(y) \wedge \mathcal{B}(z)] & \text{if } yz = x \\ \langle([0,0][1,1]),(0,1),0\rangle & \text{otherwise.} \end{cases}$$

From Definitions 3.28, 3.29, 3.30 and 3.31, it is clear that

$$\mathcal{A} \circ \mathcal{B} = \left\langle \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}, \bar{A} \circ_{IF} \bar{B}, A \circ_F B \right\rangle.$$

Then it is clear that if $\mathcal{A} \in IVIOG(G)$, then $\mathcal{A} \circ \mathcal{A} \subset \mathcal{A}$.

Lemma 3.32. *Let* $\widetilde{\widetilde{A}} \in IVING(G)$.

(1) $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}} = \widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}}$ *for each* $\widetilde{\widetilde{B}} \in IVIS(G)$,

(2) *if* $\widetilde{\widetilde{B}} \in IVIG(G)$, *then* $\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} \in IVIG(G)$.

Proof. (1) The proof is easy from Definition 3.30.

(2)
$$\begin{aligned}(\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}}) \circ_{IVI} (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}}) &= \widetilde{\widetilde{B}} \circ_{IVI} (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}) \circ_{IVI} \widetilde{\widetilde{A}} \\ &= \widetilde{\widetilde{B}} \circ_{IVI} (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}}) \circ_{IVI} \widetilde{\widetilde{A}} \text{ [By (1)]} \\ &= (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{B}}) \circ (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}}) \\ &\subset \widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}}. \text{ [Since } \widetilde{\widetilde{A}}, \widetilde{\widetilde{B}} \in IVIG(G)]\end{aligned}$$

Then $(\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})(xy) \geq (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})(x) \wedge (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})(y)$ for any $x,\ y \in G$.

Now let $x \in G$. Then we have

$$\begin{aligned}(\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(x^{-1}) &= \bigvee_{yz=x^{-1}}[\widetilde{B}^{\in}(y) \wedge \widetilde{A}^{\in}(z)] \\ &= \bigvee_{z^{-1}y^{-1}=x}[\widetilde{B}^{\in}((y^{-1})^{-1}) \wedge \widetilde{A}^{\in}((z^{-1})^{-1})] \\ &\geq \bigvee_{z^{-1}y^{-1}=x}[\widetilde{B}^{\in}(y^{-1}) \wedge \widetilde{A}^{\in}(z^{-1})] \\ &= \bigvee_{z^{-1}y^{-1}=x}[\widetilde{A}^{\in}(z^{-1}) \wedge \widetilde{B}^{\in}(y^{-1}) \\ &= (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}})^{\in}(x) \\ &= (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(x).\end{aligned}$$

Similarly, we get $(\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})^{\notin}(x^{-1}) \leq (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}})^{\notin}(x)$. Thus $\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} \in IVIG(G)$. □

Proposition 3.33. $\mathcal{A} \in IVIOG(G)$.

(1) $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$ *for each* $\mathcal{B} \in IVIOS(G)$,

(2) *if* $\mathcal{B} \in IVIOG(G)$, *then* $\mathcal{B} \circ \mathcal{A} \in IVIOG(G)$.

Proof. (1) From Lemma 3.32 (1), Propositions 3.2 in [20] and 2.1 (i) in [7], we have

$$\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} = \widetilde{\widetilde{A}} \circ \widetilde{\widetilde{B}},\ \ \bar{B} \circ_{IF} \bar{A} = \bar{A} \circ_{IF} \bar{B} \text{ and } B \circ_F A = A \circ_F B.$$

Then by Definition 3.31, $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$.

(2) From Lemma 3.32 (2), Propositions 3.3 in [20] and Proposition 2.1 (ii) in [7], it is obvious that $\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} \in IVIG(G)$, $\bar{B} \circ_{IF} \bar{A} \in IFG(G)$ and $B \circ_F A \in FG(G)$. Then by Definitions 3.6 and 3.31, $\mathcal{B} \circ \mathcal{A} \in IVIOG(G)$. □

Lemma 3.34. *If* $\widetilde{\widetilde{A}} \in IVING(G)$, *then* $G_{\widetilde{\widetilde{A}}}$ *is a normal subgroup of* G.

Proof. From Corollary 3.16, it is obvious that $G_{\tilde{\tilde{A}}}$ is a subgroup of$G$ and $G_{\tilde{\tilde{A}}} \neq \varnothing$. Let $x \in G_{\tilde{\tilde{A}}}$ and let $y \in G$. Then we get

$$\widetilde{A}^{\in}(yxy^{-1}) = \widetilde{A}^{\in}((yx)y^{-1}) = \widetilde{A}^{\in}(y^{-1}((yx)) = \widetilde{A}^{\in}(x) = \widetilde{A}^{\in}(e).$$

Similarly, $\widetilde{A}^{\notin}(yxy^{-1}) = \widetilde{A}^{\notin}(e)$. Thus $yxy^{-1} \in G_{\tilde{\tilde{A}}}$. So $G_{\tilde{\tilde{A}}}$ is a normal subgroup of G. □

From Lemma 3.24, Propositions 3.5 in [20] and 2.2 in [7], we have the following.

Proposition 3.35. *If $\mathcal{A} \in IVIONG(G)$, then $G_{\mathcal{A}}$ is a normal subgroup of G.*

Remark 3.36. If H is a normal subgroup of G, it is clear that $\chi_{\mathcal{H}} \in IVIONG(G)$ and $G_{\chi_{\mathcal{H}}} = H$ (See Theorem 3.11).

Definition 3.37. Let $\mathcal{A} \in IVIONG(G)$. Then the quotient group $G/G_{\mathcal{A}}$ is called an *IVI- octahedron quotient subgroup* (in briefly, IVIOQG) of G *with respect to* $\mathcal{A}$.

Definition 3.38. Let $\tilde{\tilde{A}} \in IVING(G)$. Then the quotient group $G/G_{\tilde{\tilde{A}}}$ is called an *interval-valued intuitionistic fuzzy quotient subgroup* (in briefly, IVIFQG) of G *with respect to* $\tilde{\tilde{A}}$.

Lemma 3.39. *Let $\tilde{\tilde{A}} \in IVING(G)$ and let $\tilde{\tilde{B}} \in IVIS(G)$. Then $\pi^{-1}(\pi(\tilde{\tilde{B}})) = G_{\tilde{\tilde{A}}} \circ_{IVI} \tilde{\tilde{B}}$, where $\pi : G \to G/G_{\tilde{\tilde{A}}}$ is the natural projection.*

Proof. Let $x \in G$. Then we get

$$\begin{aligned}\pi^{-1}(\pi(\widetilde{B}^{\in}))(x) &= \pi(\widetilde{B}^{\in})(\pi(x)) \\ &= \bigvee\nolimits_{\pi(y)=\pi(x)} \widetilde{B}^{\in}(y) \\ &= \bigvee\nolimits_{xy^{-1}\in G_{\tilde{\tilde{A}}}} \widetilde{B}^{\in}(y) \\ &= \bigvee\nolimits_{zy=x}[G_{\tilde{\tilde{A}}}(z) \wedge \widetilde{B}^{\in}(y)] \\ &= (G_{\tilde{\tilde{A}}} \circ_{IVI} \tilde{\tilde{B}})^{\in}(x).\end{aligned}$$

Thus $\pi^{-1}(\pi(\widetilde{B}^{\in}))(x) = (G_{\tilde{\tilde{A}}} \circ_{IVI} \tilde{\tilde{B}})^{\in}(x)$. Similarly, we have

$$\pi^{-1}(\pi(\widetilde{B}^{\notin}))(x) = (G_{\tilde{\tilde{A}}} \circ_{IVI} \tilde{\tilde{B}})^{\notin}(x).$$

So $\pi^{-1}(\pi(\tilde{\tilde{B}})) = G_{\tilde{\tilde{A}}} \circ_{IVI} \tilde{\tilde{B}}$. □

From Lemma 3.39, Propositions 3.7 in [20] and 2.3 in [7], we have the following

Proposition 3.40. *Let $\mathcal{A} \in IVIONG(G)$ and let $\mathcal{B} \in IVIOS(G)$. Then $\varphi^{-1}(\varphi(\mathcal{B})) = G_{\mathcal{A}} \circ \mathcal{B}$, where $\varphi : G \to G/G_{\mathcal{A}}$ is the natural projection.*

4. Level subgroups

In this section, we define the level subgroup of an IVI-octahedron subgroup and find some of its properties.

Definition 4.1 ([1]). Let X be a nonempty set, let $\widetilde{\widetilde{a}}$ be an IVI-octahedron number and let $\mathcal{A} =< \widetilde{\widetilde{A}}, \bar{A}, A >\in IVIO(X)$. Then the $\widetilde{\widetilde{a}}$*-level set* of $\mathcal{A}$, denoted by $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$, is a subset of X, is defined by:

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}} = \{x \in X : \widetilde{\widetilde{A}}(x) \geq \widetilde{\widetilde{a}},\ \bar{A}(x) \geq \bar{a},\ A(x) \geq a\}.$$

Definition 4.2. Let X be a nonempty set, let $\widetilde{\widetilde{A}} \in IVIS(X)$ and let $\widetilde{\widetilde{a}}$ be an interval-valued intuitionistic fuzzy number. Then the $\widetilde{\widetilde{a}}$*-level set* of $\widetilde{\widetilde{A}}$, denoted by $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$, is a subset of X defined as follows:

$$[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} = \{x \in X : \widetilde{\widetilde{A}}(x) \geq \widetilde{\widetilde{a}}\} = \{x \in X : \widetilde{A}^{\in}(x) \geq \widetilde{a}^{\in}, \widetilde{A}^{\notin}(x) \leq \widetilde{a}^{\notin}\}.$$

Remark 4.3. Let X be a nonempty set, let $\widetilde{\widetilde{a}}$ be an IVI-octahedron number and let $\mathcal{A} \in IVIO(X)$. Then we have

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}} = [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \cap [\bar{A}]_{\bar{a}} \cap [A]_a,$$

where $[\bar{A}]_{\bar{a}}$ and $[A]_a$ denote the $\bar{a}$-level set of $\bar{A}$ and the a-level set of A respectively (See [17] and [6]).

The following is an immediate consequence of Definition 4.1

Proposition 4.4. *Let X be a nonempty set, let $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$ be two IVI-octahedron numbers and let $\mathcal{A} \in IVIO(X)$. If $\widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}}$, then $[\mathcal{A}]_{\widetilde{\widetilde{b}}} \subset [\mathcal{A}]_{\widetilde{\widetilde{a}}}$.*

Lemma 4.5. *Let $\widetilde{\widetilde{A}} \in IVIG(G)$ and let $\widetilde{\widetilde{a}}$ be an interval-valued intuitionistic fuzzy number with $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{a}}$. Then $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$ is a subgroup of G.*

Proof. It is clear that $e \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$, i.e., $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \neq \varnothing$. Let $x,\ y \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Then we get

$$\widetilde{A}^{\in}(x) \geq \widetilde{a}^{\in},\ \widetilde{A}^{\in}(y) \geq \widetilde{a}^{\in} \text{ and } \widetilde{A}^{\notin}(x) \leq \widetilde{a}^{\notin},\ \widetilde{A}^{\notin}(y) \geq \widetilde{a}^{\notin}.$$

Since $\widetilde{\widetilde{A}} \in IVIG(G)$, we have

$$\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \geq \widetilde{a}^{\in} \text{ and } \widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y) \leq \widetilde{a}^{\notin}.$$

Thus $\widetilde{A}^{\in}(xy) \geq \widetilde{a}^{\in}$ and $\widetilde{A}^{\notin}(xy) \leq \widetilde{a}^{\notin}$, i.e., $xy \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Moreover, we get

$$\widetilde{A}^{\in}(x^{-1}) \geq \widetilde{A}^{\in}(x) \geq \widetilde{a}^{\in} \text{ and } \widetilde{A}^{\notin}(x^{-1}) \leq \widetilde{A}^{\notin}(x) \leq \widetilde{a}^{\notin}.$$

So $\widetilde{A}^{\in}(x^{-1}) \geq \widetilde{a}^{\in}$ and $\widetilde{A}^{\notin}(x^{-1}) \leq \widetilde{a}^{\notin}$, i.e., $x^{-1} \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Hence $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$ is a subgroup of G. □

Proposition 4.6. *Let $\mathcal{A} \in IVIOG(G)$ and let $\widetilde{\widetilde{a}}$ be an IVI-octahedron number with $\mathcal{A}(e) \geq \widetilde{\widetilde{a}}$. Then $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ is a subgroup of G.*

In this case, $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ is called an $\widetilde{\widetilde{a}}$*-level subgroup of* $\mathcal{A}$.

Proof. From Lemma 4.5, Proposition 2.18 in [20] and Theorem 2.1 in [6], $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$, $[\bar{A}]_{\bar{a}}$ and $[A]_a$ are subgroups of $\widetilde{\widetilde{A}}$, $\bar{A}$ and A respectively. Then by Remark 4.3, $[\mathcal{A}]_{\widetilde{\widetilde{\bar{a}}}}$ is a subgroup of G. □

The following is the converse of Lemma 4.5.

Lemma 4.7. *Let $\widetilde{\widetilde{A}} \in IVIS(G)$. If $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$ is a subgroup of G for each interval-valued intuitionistic fuzzy number $\widetilde{\widetilde{a}}$ with $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{a}}$, then $\widetilde{\widetilde{A}} \in IVIG(G)$.*

Proof. For any $x,\ y \in G$, let $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{b}}$ and let $\widetilde{\widetilde{A}}(y) = \widetilde{\widetilde{c}}$, where $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{b}}$ and $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{c}}$. Then clearly, $x \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}}$ and $y \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{c}}}$. Suppose $\widetilde{\widetilde{b}} < \widetilde{\widetilde{c}}$, i.e., $\widetilde{b}^{\in} < \widetilde{c}^{\in}$ and $\widetilde{b}^{\notin} > \widetilde{c}^{\notin}$. Then $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{c}}} \subset [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}}$. Thus $y \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}}$. Since $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}}$ is a subgroup of G, $xy \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}}$, i.e., $\widetilde{A}^{\in}(xy) \geq \widetilde{b}^{\in}$ and $\widetilde{A}^{\notin}(xy) \leq \widetilde{b}^{\notin}$. So we get

$$\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \text{ and } \widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y).$$

Now for each $x \in G$, let $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{a}}$, where $\widetilde{\widetilde{A}}(e) \geq \widetilde{\widetilde{a}}$. Then clearly, $x \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Since $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$ is a subgroup of G, $x^{-1} \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Thus $\widetilde{\widetilde{A}}(x^{-1}) \geq \widetilde{\widetilde{a}}$, i.e., $\widetilde{A}^{\in}(x^{-1}) \geq \widetilde{a}^{\in}$ and $\widetilde{A}^{\notin}(x^{-1}) \leq \widetilde{a}^{\notin}$. So $\widetilde{A}^{\in}(x^{-1}) \geq \widetilde{A}^{\in}(x)$ and $\widetilde{A}^{\notin}(x^{-1}) \leq \widetilde{A}^{\notin}(x)$. Hence $\widetilde{\widetilde{A}} \in IVIG(G)$. □

The following is the converse of Proposition 4.6.

Proposition 4.8. *Let $\mathcal{A} \in IVIOS(G)$. If $[\mathcal{A}]_{\widetilde{\widetilde{\bar{a}}}}$ is a subgroup of G for each IVI-octahedron number $\widetilde{\widetilde{\bar{a}}}$ with $\mathcal{A}(e) \geq \widetilde{\widetilde{\bar{a}}}$, then $\mathcal{A} \in IVIOG(G)$.*

Proof. From Lemma 4.7, Proposition 2.19 in [20] and Theorem 2.2 in [6], $\widetilde{\widetilde{A}} \in IVIG(G)$, $\bar{A} \in IFG(G)$ and $A \in FG(G)$. Then by Theorem 3.7, $\mathcal{A} \in IVIOG(G)$. □

Let G be a finite group. Then the number of subgroups of G is finite. But the number of level subgroups of an IVIOG $\mathcal{A}$ appears to be infinite.

Example 4.9. Consider the Klein four-group $V = \{e, x, y, z\}$ given in Example 3.9 (1). Then clearly, the number of subgroups of V is finite. Now we define $\mathcal{A} : V \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows:

$$\mathcal{A}(e) = \widetilde{\widetilde{\bar{a}}}_0,\ \mathcal{A}(x) = \widetilde{\widetilde{\bar{a}}}_1,\ \mathcal{A}(y) = \widetilde{\widetilde{\bar{a}}}_2,\ \mathcal{A}(z) = \widetilde{\widetilde{\bar{a}}}_3,$$

where $\widetilde{\widetilde{\bar{a}}}_0 \geq \widetilde{\widetilde{\bar{a}}}_i$ $(i = 1,\ 2,\ 3)$ and $\widetilde{\widetilde{\bar{a}}}_3 \geq \widetilde{\widetilde{\bar{a}}}_1 \wedge \widetilde{\widetilde{\bar{a}}}_2$.
Then we can easily check that $\mathcal{A} \in IVIOG(V)$. Consider the family

$$\mathcal{P} = \{[\mathcal{A}]_{\widetilde{\widetilde{\bar{a}}}} : \widetilde{\widetilde{\bar{a}}}_i \text{ is an } IVI\text{-octahedron number with } \widetilde{\widetilde{\bar{a}}}_i \leq \mathcal{A}(e)\}.$$

Then by Proposition 4.6, $\mathcal{P}$ is a family of level subgroups of $\mathcal{A}$, i.e., subgroups of V. Moreover, $\mathcal{P}$ is infinite. However, we can see that all members of $\mathcal{P}$ are not distinct.

Theorem 4.10. *Let $\mathcal{A} \in IVIOG(G)$ and let $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$ and $[\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$ (with $\tilde{\tilde{\bar{a}}} < \tilde{\tilde{\bar{b}}}$) be two level subgroups of G. Then $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} = [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$ if and only if there is no $x \in G$ such that $\tilde{\tilde{\bar{a}}} < \mathcal{A}(x) < \tilde{\tilde{\bar{b}}}$.*

Proof. Suppose $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} = [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$. Assume that there is $x \in G$ such that $\tilde{\tilde{\bar{a}}} < \mathcal{A}(x) < \tilde{\tilde{\bar{b}}}$. Then $\tilde{\tilde{A}}(x) > \tilde{\tilde{a}}$, $\bar{A}(x) > \bar{a}, A(x) > a$ and $\tilde{\tilde{A}}(x) < \tilde{\tilde{b}}$, $\bar{A}(x) < \bar{b}, A(x) < b$. Thus $x \in [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$ but $x \notin [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$. So by Proposition 4.4, $[\mathcal{A}]_{\tilde{\tilde{\bar{b}}}} \subsetneq \not\supseteq [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$. This contradicts the hypothesis.

Conversely, suppose the necessary condition holds. Since $\tilde{\tilde{\bar{a}}} < \tilde{\tilde{\bar{b}}}$, $[\mathcal{A}]_{\tilde{\tilde{\bar{b}}}} \subset [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$ by Proposition 4.4. Let $x \in [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$. Then $\mathcal{A}(x) \geq \tilde{\tilde{\bar{a}}}$. By the hypothesis, $\mathcal{A}(x) \geq \tilde{\tilde{\bar{a}}}$. Thus $x \in [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$. So $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} \subset [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$. Hence $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} = [\mathcal{A}]_{\tilde{\tilde{\bar{b}}}}$. □

Corollary 4.11. *Let G be a finite group of order n and let $\mathcal{A} \in IVIOG(G)$. Let $\mathrm{Im}\mathcal{A} = \{\tilde{\tilde{\bar{a}}}_i : \mathcal{A}(x) = \tilde{\tilde{\bar{a}}}_i \text{ for some } x \in G\}$. Then $\{[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_i}\}$ is the set of the only level subgroups of $\mathcal{A}$.*

Proof. From Proposition 4.6, it is obvious that $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_i}$ is a subgroup of $\mathcal{A}$. Let $\tilde{\tilde{\bar{a}}}$ be an IVI-octahedron number such that $\tilde{\tilde{\bar{a}}} \notin \mathrm{Im}\mathcal{A}$.

Case (i): Suppose $\tilde{\tilde{\bar{a}}}_i < \tilde{\tilde{\bar{a}}} < \tilde{\tilde{\bar{a}}}_j$, where $\tilde{\tilde{\bar{a}}}_i,\ \tilde{\tilde{\bar{a}}}_j \in \mathrm{Im}\mathcal{A}$. Then by Theorem 4.10, $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_i} = [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} = [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_j}$.

Case (ii): Suppose $\tilde{\tilde{\bar{a}}} < \tilde{\tilde{\bar{a}}}_r$, where $\tilde{\tilde{\bar{a}}}_r$ is the least element in $\mathrm{Im}\mathcal{A}$. Then by Theorem 4.10, $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_r} = G = [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$.

Case (iii): Suppose $\tilde{\tilde{\bar{a}}}_0 < \tilde{\tilde{\bar{a}}}$, where $\tilde{\tilde{\bar{a}}}_r$ is the greatest element in $\mathrm{Im}\mathcal{A}$. Then by Theorem 4.10, $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}} = \{e\} = [\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_0}$.

Thus in all cases, for any IVI-octahedron number $\tilde{\tilde{\bar{a}}}$, the $\tilde{\tilde{\bar{a}}}$-level subgroup $[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}}$ of $\mathcal{A}$ is one of $\{[\mathcal{A}]_{\tilde{\tilde{\bar{a}}}_i}\}$, where $\tilde{\tilde{\bar{a}}}_i \in \mathrm{Im}\mathcal{A}$. So the result holds. □

Proposition 4.12. *Let H be a subgroup of G. Then H can be realized as a level subgroup of some $\mathcal{A} \in VIOG(G)$.*

Proof. We define a mapping $\mathcal{A} : G \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for each $x \in G$,

$$\mathcal{A}(x) = \begin{cases} \tilde{\tilde{\bar{a}}} & \text{if } x \in H \\ \langle([0,0],[1,1]),(0,1),0\rangle & \text{if } x \notin H, \end{cases} \tag{4.1}$$

where $\tilde{\tilde{\bar{a}}}$ is an IVI-octahedron number. Then we can easily show that $\mathcal{A} \in IVIOG(G)$. Let $x,\ y \in G$.

Case (i): Suppose $x,\ y \in H$. Then clearly, $xy \in H$. Thus $\mathcal{A}(xy) = \mathcal{A}(x) = \mathcal{A}(y)$. So $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y)$. On the other hand, Since H is a subgroup of G and $x \in H$, $x^{-1} \in H$. Then $\mathcal{A}(x^{-1}) = \tilde{\tilde{\bar{a}}}$. Thus $\mathcal{A}(x^{-1}) \geq \mathcal{A}(x)$.

Case (ii): Suppose $x \in H$ but $y \notin H$. Then clearly, $xy \notin H$. Thus $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}$, $\mathcal{A}(y) = \mathcal{A}(xy) = \langle([0,0],[1,1],(0,1),0\rangle$. So $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y)$. Also we get $\mathcal{A}(x^{-1}) \geq \mathcal{A}(x)$.

Case (iii): Suppose $x \notin H$ but $y \in H$. Then the proof is similar to Case (ii).

Case (iv): Suppose $x,\ y \notin H$. Then either $xy \in H$ or $xy \notin H$. In any case, we have $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y)$ and $\mathcal{A}(x^{-1}) \geq \mathcal{A}(x)$. Thus in all cases, $\mathcal{A} \in IVIOG(G)$. □

We obtain the following generalization of Proposition 4.12.

Proposition 4.13. *Let the following be any chain of subgroups of G:*

$$G_0 \subset G_1 \subset \cdots \subset G_r = G. \tag{4.2}$$

Then there is $\mathcal{A} \in IVIOG(G)$ whose level subgroups are precisely the members of (4.2).

Proof. Consider the following sequence of IVI-octahedron numbers:

$$\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \cdots > \tilde{\tilde{\tilde{a}}}_r. \tag{4.3}$$

We define a mapping $\mathcal{A} : G \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for each $x \in G$,

$$\mathcal{A}(G_0) = \tilde{\tilde{\tilde{a}}}_0 \text{ and } \mathcal{A}(\hat{G}_i) = \tilde{\tilde{\tilde{a}}}_i, \tag{4.4}$$

where $\hat{G}_i = G_i \setminus G_{i-1}$ for $i = 1,\ 2,\ \cdots,\ r$. From the definition of $\mathcal{A}$, it is clear that $\mathcal{A} \in IVIOS(G)$. Let $x,\ y \in G$.

Case (i): Suppose $x,\ y \in \hat{G}_i$. Then clearly, $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(y)$. Since G_i is a subgroup of G, $xy \in G_i$. Thus either $xy \in G_i$ or $xy \in G_{i-1}$. So in any case,

$$\mathcal{A}(xy) \geq \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x) \wedge \mathcal{A}(y).$$

Since G_i is a subgroup of G and $x \in G_i$, $x^{-1} \in G_i$. Hence we have

$$\mathcal{A}(x^{-1}) \geq \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x).$$

Case (ii): Suppose $x \in \hat{G}_i,\ y \in \hat{G}_j$ and $i > j$. Then $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}_i$ and $\mathcal{A}(y) = \tilde{\tilde{\tilde{a}}}_j$. Since $G_j \subset G_i$ and G_i is a subgroup of G, $xy \in G_i$. Thus we get

$$\mathcal{A}(xy) = \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x) \wedge \mathcal{A}(y).$$

Since G_i is a subgroup of G and $x \in G_i$, $x^{-1} \in G_i$. So we have

$$\mathcal{A}(x^{-1}) \geq \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x).$$

Hence in either cases, $\mathcal{A} \in IVIOG(G)$.

From (4.4), $\text{Im}\mathcal{A} = \{\tilde{\tilde{\tilde{a}}}_0, \tilde{\tilde{\tilde{a}}}_1, \cdots, \tilde{\tilde{\tilde{a}}}_r\}$. Then by (4.3), the level subgroups of $\mathcal{A}$ are given as the following chain of subgroups of G:

$$[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} \subset \cdots \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = G.$$

Now we claim that $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} = G_i,\ 0 < i \leq r$. It is obvious that $G_i \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}$. Let $x \in [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}$. Then $\mathcal{A}(x) \geq \tilde{\tilde{\tilde{a}}}_i$ and $\mathcal{A}(x) < \tilde{\tilde{\tilde{a}}}_j$ for $j > i$. Thus $\mathcal{A}(x) \in \{\tilde{\tilde{\tilde{a}}}_1, \cdots, \tilde{\tilde{\tilde{a}}}_i\}$, i.e., $x \in G_k$ for some $k \leq i$. By (4.2), $G_k \subset G_i$. Thus $x \in G_i$. So $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} \subset G_i$. Hence $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} =\subset G_i$, $0 \leq i \leq r$. This completes the proof. □

As a consequence of Proposition 4.13, we can see that the level subgroups of an IVIOG $\mathcal{A}$ form a chain. Since $\mathcal{A}(x) \leq \mathcal{A}(e)$ for each $x \in G$, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0}$ is the smallest level subgroup of $\mathcal{A}$, where $\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0$. Thus we get the following chain:

$$(e) = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} \subset \cdots \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = G, \tag{4.5}$$

where $\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \cdots > \tilde{\tilde{\tilde{a}}}_r$. We will denote the chain (4.5) of level subgroups of $\mathcal{A}$ by $C(\mathcal{A})$. In general, as all subgroups of G do not form a chain, it follows that all subgroups of G are not level subgroups of a given IVIOG. So it is an interesting problem to find an IVIOG $\mathcal{A}$ of G which accommodates as many subgroups of G as possible in $C(\mathcal{A})$.

Proposition 4.14. *Let G be a finite group such that $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_r}$, where the G_{p_i} are prime cyclic groups of order p_i. Then there is $\mathcal{A} \in IVIOG(G)$ such that $C(\mathcal{A})$ is a maximal chain of length $r + 1$.*

Proof. We show by the induction on r. Suppose $r = 1$. Then clearly, $G = G_{p_1}$. By Theorem 3.24, there is $\mathcal{A} \in IVIOG(G)$ such that $\mathcal{A}(x) \leq \mathcal{A}(e)$ for each $e \neq x \in G$, where $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}_1$, $\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0$ and $\tilde{\tilde{\tilde{a}}}_1 \leq \tilde{\tilde{\tilde{a}}}_0$. Thus $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} = (e)$ and $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} = G$. So $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1}$ is a maximal chain of length 2. Hence the theorem is true for $r = 1$.

Now let $r > 1$ and suppose the theorem is true for all integers $\leq r - 1$. Let $H = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_{r-1}}$. Then $G = H \times G_{p_r}$. Let us define a mapping

$$\mathcal{A} = \left\langle \tilde{\tilde{A}}, \bar{A}, A \right\rangle : G \rightarrow ([I] \oplus [I]) \times (I \oplus I) \times I \text{ as follows: for each} x \in G,$$

$$\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0,\ \mathcal{A}(\hat{G_{p_1}}) = \tilde{\tilde{\tilde{a}}}_1,\ \mathcal{A}(G_{p_1} \hat{\times} G_{p_2}) = \tilde{\tilde{\tilde{a}}}_2,\ \cdots,\ \mathcal{A}(H \hat{\times} G_{p_r}) = \tilde{\tilde{\tilde{a}}}_r,$$

where $\tilde{\tilde{\tilde{a}}}_0 \geq \tilde{\tilde{\tilde{a}}}_1 \geq \tilde{\tilde{\tilde{a}}}_2 \geq \cdots \geq \tilde{\tilde{\tilde{a}}}_r$ and $\hat{G_{p_1}} = G_{p_1} \setminus (e)$, $G_{p_1} \hat{\times} G_{p_2} = G_{p_1} \times G_{p_2} \setminus G_{p_1}$, and so on. Then clearly, $\mathcal{A} \in IVIO(G)$. we will prove that $\mathcal{A} \in IVIOG(G)$. Let $x,\ y \in G$.

Case (i): Suppose $x,\ y \in H$. Then clearly, $xy \in H$. Thus by the induction, we can easily see that the followings hold:

$$\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y),\ \mathcal{A}(x^{-1}) \geq \mathcal{A}(x).$$

Case (ii): Suppose $x \in H$, $y \in G \setminus H$. Then $xy \notin H$. Thus $\mathcal{A}(xy) = \tilde{\tilde{\tilde{a}}}_r$, $\mathcal{A}(x) \geq \widetilde{\widetilde{\widetilde{a_{r-1}}}}$ and $\mathcal{A}(y) = \tilde{\tilde{\tilde{a}}}_r$. So we have

$$\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y),\ \mathcal{A}(x^{-1}) \geq \mathcal{A}(x).$$

Case (iii): Suppose $x \in G \setminus H$, $y \in H$. Then we get the same results by the similar proof of Case (ii).

Case (iv): Suppose $x,\ y \in G \setminus H$. Then we can easily show that the followings hold:

$$\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y),\ \mathcal{A}(x^{-1}) \geq \mathcal{A}(x).$$

Thus in all cases, $\mathcal{A} \in IVIOG(G)$. Furthermore, we have

$$[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} = (e),\ [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} = G_{p_1},\ [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_2} = G_{p_1} \times G_{p_2},\ \cdots,\ [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = H \times G_{p_r}.$$

So $C(\mathcal{A}) = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} \subset \cdots \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r}$ is maximal and is of length $r + 1$. □

Remark 4.15. In the same way, we can obtain an $\mathcal{A} \in IVIOG(G)$ with the maximal $C(\mathcal{A})$ in the following cases:
(i) G is a cyclic p-group,
(ii) G is the direct product of cyclic p-groups,
(iii) G is a finite abelian group.
As we adopt the same technique of Proposition 4.14 in proving these cases, we will omit the proofs.

From the following example, we can see that two IVIOGs of G may have an identical family of level subgroups but the IVIOGs may not be equal.

Example 4.16. Consider the Klein four-group $V = \{e, x, y, z\}$ given in Example 3.9 (1). Let $\tilde{\tilde{a}}_i$ be an IVI-octahedron number such that $\tilde{\tilde{a}}_0 > \tilde{\tilde{a}}_1 > \tilde{\tilde{a}}_2$ for $i = 0,\ 1,\ 2$. We define a mapping $\mathcal{A} : V \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for each $t \in V$,

$$\mathcal{A}(e) = \tilde{\tilde{a}}_0,\ \mathcal{A}(x) = \tilde{\tilde{a}}_1,\ \mathcal{A}(y) = \mathcal{A}(z) = \tilde{\tilde{a}}_2.$$

Then it is obvious that $\mathcal{A} \in IVIOG(V)$ and $\mathrm{Im}\mathcal{A} = \{\tilde{\tilde{a}}_0, \tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2\}$. Moreover, the level subgroups of $\mathcal{A}$ are

$$[\mathcal{A}]_{\tilde{\tilde{a}}_0} = \{e\},\ [\mathcal{A}]_{\tilde{\tilde{a}}_1} = \{e, x\},\ [\mathcal{A}]_{\tilde{\tilde{a}}_2} = V.$$

Now let $\tilde{\tilde{b}}_i$ $(i = 0,\ 1,\ 2)$ be an IVI-octahedron number such that

$$\tilde{\tilde{b}}_0 \geq \tilde{\tilde{b}}_1 \geq \tilde{\tilde{b}}_2 \text{ and } \{\tilde{\tilde{a}}_0, \tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2\} \cap \{\tilde{\tilde{b}}_0, \tilde{\tilde{b}}_1, \tilde{\tilde{b}}_2\} = \varnothing.$$

We define a mapping $\mathcal{B} : V \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows: for each $t \in V$,

$$\mathcal{A}(e) = \tilde{\tilde{b}}_0,\ \mathcal{A}(x) = \tilde{\tilde{b}}_1,\ \mathcal{A}(y) = \tilde{\tilde{b}}_2,\ \mathcal{A}(c) = \tilde{\tilde{b}}_2.$$

Then we can easily see that $\mathcal{B} \in IVUOG(V)$ and the level subgroups of $\mathcal{B}$ are

$$[\mathcal{A}]_{\tilde{\tilde{b}}_0} = \{e\},\ [\mathcal{A}]_{\tilde{\tilde{b}}_1} = \{e, x\},\ [\mathcal{A}]_{\tilde{\tilde{b}}_2} = V.$$

Thus $\mathcal{A}$ and $\mathcal{B}$ have the same family of level subgroups but $\mathcal{A} \neq \mathcal{B}$.

The following is an immediate consequence of Definition 4.2.

Lemma 4.17. *Let G be a finite group and let $\mathcal{A} \in IVIOG(G)$. If $\tilde{\tilde{a}},\ \tilde{\tilde{b}} \in \mathrm{Im}\mathcal{A}$ such that $[\mathcal{A}]_{\tilde{\tilde{a}}} = [\mathcal{A}]_{\tilde{\tilde{b}}}$, then $\tilde{\tilde{a}} = \tilde{\tilde{b}}$.*

Proposition 4.18. *Let G be a finite group and let $\mathcal{A},\ \mathcal{B} \in IVIOG(G)$ with the identical family of level subgroups. Let $\tilde{\tilde{a}}_i$ and $\tilde{\tilde{b}}_j$ be IVI-octahedron numbers $(i = 1,\ 2,\ \cdots,\ r;\ j = 1,\ 2,\ \cdots,\ k)$ such that $\tilde{\tilde{a}}_0 > \mathcal{A} > \cdots > \tilde{\tilde{a}}_r$ and $\tilde{\tilde{b}}_0 > \tilde{\tilde{b}}_1 > \cdots > \tilde{\tilde{b}}_k$. If $\mathrm{Im}\mathcal{A} = \{\tilde{\tilde{a}}_0, \tilde{\tilde{a}}_1, \cdots, \tilde{\tilde{a}}_r\}$ and $\mathrm{Im}\mathcal{B} = \{\tilde{\tilde{b}}_0, \tilde{\tilde{1}}_1, \cdots, \tilde{\tilde{b}}_r\}$, then we have*
(1) *$r = k$,*
(2) *$[\mathcal{A}]_{\tilde{\tilde{a}}_i} = [\mathcal{B}]_{\tilde{\tilde{b}}_i}$ for $0 \leq i \leq r$,*
(3) *if $x \in G$ such that $\mathcal{A}(x) = \tilde{\tilde{a}}_i$, then $\mathcal{B}(x) = \tilde{\tilde{b}}_i$ for $0 \leq i \leq r$.*

Proof. (1) By Corollary 4.11, the only level subgroups of $\mathcal{A}$ and $\mathcal{B}$ are two families $\{[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}\}$ and $\{[\mathcal{B}]_{\tilde{\tilde{\tilde{a}}}_k}\}$. Then by the hypothesis, $r = k$.

(2) By (1) and Corollary 4.11, there are two chains of level subgroups:

$$[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} \subset \cdots \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = G$$

and

$$[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_0} \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_1} \subset \cdots \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_r} = G.$$

Suppose $\tilde{\tilde{\tilde{a}}}_i,\ \tilde{\tilde{\tilde{a}}}_j \in \mathrm{Im}\mathcal{A}$ such that $\tilde{\tilde{\tilde{a}}}_i > \tilde{\tilde{\tilde{a}}}_j$. Then we get

$$[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_j}. \tag{4.6}$$

Suppose $\tilde{\tilde{\tilde{b}}}_i,\ \tilde{\tilde{\tilde{b}}}_j \in \mathrm{Im}\mathcal{B}$ such that $\tilde{\tilde{\tilde{b}}}_i > \tilde{\tilde{\tilde{b}}}_j$. Then we have

$$[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i} \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}. \tag{4.7}$$

Since $\{[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}\} = \{[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i}\}$, it is obvious that $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_0}$. By the hypothesis, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$ for some $j > 0$. Assume that $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$ for some $j > 1$. Again we get $[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_1} = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}$ for some $\tilde{\tilde{\tilde{a}}}_i > \tilde{\tilde{\tilde{a}}}_1$. Then clearly, $\tilde{\tilde{\tilde{a}}}_i = \tilde{\tilde{\tilde{a}}}_1$. Thus by (4.6), we have

$$[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_1} \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}. \tag{4.8}$$

Also by (4.7), we get

$$[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j} = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}. \tag{4.9}$$

However, (4.8) and (4.9) contradict one another as the inclusions are both proper. So $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_1}$. The remainder's proofs follow by the induction on i by using arguments exactly on the same as above. Hence the result holds.

(3) Let $x \in G$ such that $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}_i$ and $\mathcal{B}(x) = \tilde{\tilde{\tilde{b}}}_j$. Then by (2), $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i}$. Since $x \in [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}$, $x \in [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i}$. Thus $\mathcal{B}(x) = \tilde{\tilde{\tilde{b}}}_j \geq \tilde{\tilde{\tilde{b}}}_i$. By (4.7), $[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j} \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i}$. By (2), $[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j} = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_j}$. Since $x \in [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$, $x \in [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_j}$. So $\mathcal{A}(x) = \tilde{\tilde{\tilde{a}}}_i \geq \tilde{\tilde{\tilde{a}}}_j$. By (4.6), $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_j}$. By (2), $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i}$ and $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_j} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$. Consequently, $[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i} \subset [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$. Hence $[\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_i} = [\mathcal{B}]_{\tilde{\tilde{\tilde{b}}}_j}$. Therefore by Lemma 4.17, $\tilde{\tilde{\tilde{b}}}_i = \tilde{\tilde{\tilde{b}}}_j$. □

Theorem 4.19. *Let G be a finite group and let $\mathcal{A}$, $\mathcal{B} \in IVIOG(G)$ with the identical family of level subgroups. Then $\mathcal{A} = \mathcal{B}$ if and only if* $\mathrm{Im}\mathcal{A} = \mathrm{Im}\mathcal{B}$.

Proof. The proof of the necessary condition is easy. Conversely, suppose $\mathrm{Im}\mathcal{A} = \mathrm{Im}\mathcal{B}$. Let $\mathrm{Im}\mathcal{A} = \{\tilde{\tilde{\tilde{a}}}_0, \cdots, \tilde{\tilde{\tilde{a}}}_r\}$ and let $\mathrm{Im}\mathcal{B} = \{\tilde{\tilde{\tilde{b}}}_0, \cdots, \tilde{\tilde{\tilde{r}}}_r\}$ such that

$$\tilde{\tilde{\tilde{a}}}_0 > \cdots > \tilde{\tilde{\tilde{a}}}_r \text{ and } \tilde{\tilde{\tilde{b}}}_0 > \cdots > \tilde{\tilde{\tilde{b}}}_r.$$

Since $\tilde{\tilde{\tilde{b}}}_0 \in \mathrm{Im}\mathcal{B}$, by the hypothesis, there is $k_0 \in \{0, 1, \cdots, r\}$ such that $\tilde{\tilde{\tilde{b}}}_0 = \widetilde{\widetilde{\widetilde{a_{k_0}}}}$. Suppose $\widetilde{\widetilde{\widetilde{a_{k_0}}}} \neq \tilde{\tilde{\tilde{a}}}_0$. Then $\widetilde{\widetilde{\widetilde{a_{k_0}}}} < \tilde{\tilde{\tilde{a}}}_1$. Since $\tilde{\tilde{\tilde{b}}}_1 \in \mathrm{Im}\mathcal{B}$, by the hypothesis, there is $k_1 \in \{0, 1, \cdots, r\}$ such that $\tilde{\tilde{\tilde{b}}}_1 = \widetilde{\widetilde{\widetilde{a_{k_1}}}}$. Since $\tilde{\tilde{\tilde{b}}}_0 > \tilde{\tilde{\tilde{b}}}_1$, $\widetilde{\widetilde{\widetilde{a_{k_0}}}} > \widetilde{\widetilde{\widetilde{a_{k_1}}}}$. By proceeding in this way, we get

$$\widetilde{\widetilde{\widetilde{a_{k_0}}}} > \widetilde{\widetilde{\widetilde{a_{k_1}}}} > \cdots > \widetilde{\widetilde{\widetilde{a_{k_r}}}},$$

where $\tilde{\tilde{\tilde{b}}}_0 = \widetilde{\widetilde{\widetilde{a_{k_0}}}}$ and $\widetilde{\widetilde{\widetilde{a_{k_0}}}} > \tilde{\tilde{\tilde{a}}}_0$. Thus they contradict the fact that $\mathrm{Im}\mathcal{A} = \mathrm{Im}\mathcal{B}$. So $\tilde{\tilde{\tilde{a}}}_0 = \tilde{\tilde{\tilde{b}}}_0$. Arguing this manner, we have

$$\tilde{\tilde{\tilde{a}}}_i = \tilde{\tilde{\tilde{b}}}_i,\ 0 \leq i \leq r. \tag{4.10}$$

Now let $x_0,\ x_1,\ \cdots,\ x_r$ be distinct elements of G such that

$$\mathcal{A}(x_i) = \tilde{\tilde{\tilde{a}}}_i,\ 0 \leq i \leq r.$$

Then by Proposition 4.18, $\mathcal{B}(x_i) = \tilde{\tilde{\tilde{b}}}_i,\ 0 \leq i \leq r$. Thus by (4.10), $\mathcal{A}(x_i) = \mathcal{B}(x_i)$ for each $x_i \in G$. So $\mathcal{A} = \mathcal{B}$. This completes the proof. □

Proposition 4.20. *Let G be a cyclic p-group of order p^n, where p is a prime. Let $\mathcal{A} \in IVIOG(G)$, let $x,\ y \in G$ and let $O(x)$ denote the order of x. If $O(x) > O(y)$, then $\mathcal{A}(y) \geq \mathcal{A}(x)$. Furthermore, if $O(x) = O(y)$, then $\mathcal{A}(x) = \mathcal{A}(y)$.*

Proof. We prove by the induction on n. Suppose $n = 1$. Then clearly, $O(G) = p$. Thus the theorem is true by Proposition 3.10. Let $n > 1$ and suppose the theorem is true for all integers $\leq n - 1$. Let H be a subgroup of order p^{n-1} and let $x,\ y \in G$.

Case (i): Suppose $x,\ y \in H$. Then by the induction, the result holds.

Case (ii): Suppose $x \notin H$ and $y \in H$. Then clearly, $O(x) = p^n$ and $O(y) = p^r$, where $r \leq n - 1$. Thus x is a generator of G and there is an integer m such that $y = x^m$. So we have

$$\mathcal{A}(y) = \mathcal{A}(x) \wedge \mathcal{A}(x) \wedge \cdots \wedge \mathcal{A}(x) \text{ (m times)} \geq \mathcal{A}(x).$$

Case (iii): Suppose $x \in H$ and $y \notin H$. The the proof is similar to Case (ii).

Case (iv): Suppose $x \notin H$ and $y \notin H$. Then clearly, $O(x) = O(y) = p^n$. Thus x and y are generators of G. So there are integers m and s such that $y = x^m$ and $x = y^s$. Hence we get

$$\mathcal{A}(x) \geq \mathcal{A}(y) \wedge \mathcal{A}(y) \wedge \cdots \wedge \mathcal{A}(y) \text{ (s times)} \geq \mathcal{A}(y)$$

and

$$\mathcal{A}(y) \geq \mathcal{A}(x) \wedge \mathcal{A}(x) \wedge \cdots \wedge \mathcal{A}(x) \text{ (m times)} \geq \mathcal{A}(x).$$

Therefore $\mathcal{A}(x) = \mathcal{A}(y)$. □

The following is an example which Proposition 4.20 does not hold in general.

Example 4.21. Consider the Klein's four group V given in Example 3.9 and $\mathcal{A} \in IVIOG(V)$ given in Example 4.16. Then clearly, $O(x) = O(y)$ but $\mathcal{A}(x) \neq \mathcal{A}(y)$.

The following is an example which for an IVIOG $\mathcal{A}$ of a cyclic group G, $O(x) \neq O(y)$ but $\mathcal{A}(x) = \mathcal{A}(y)$.

Example 4.22. Let $G = (x)$ be a cyclic group of order 6. We define the mapping $\mathcal{A} : G \to ([I] \oplus [I]) \times (I \oplus I) \times I$ as follows:

$$\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0,\ \mathcal{A}(x) = \mathcal{A}(x^3) = \mathcal{A}(x^5) = \tilde{\tilde{\tilde{a}}}_1,\ \mathcal{A}(x^2) = \tilde{\tilde{\tilde{a}}}_2,$$

where $\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \tilde{\tilde{\tilde{a}}}_2$. Then clearly, $\mathcal{A} \in IVIOG(G)$ and $O(x^3) \neq O(x)$. But $\mathcal{A}(x) = \mathcal{A}(x^3)$.

Now we give the characterization of all IVIOGs of a finite cyclic group.

Theorem 4.23. *Let G be a finite cyclic group and let $\mathcal{A} \in IVIOS(G)$. Then $\mathcal{A} \in IVIOG(G)$ if and only if there is a maximal chain of subgroups*

$$(e) = G_0 \subset G_1 \subset \cdots \subset G_r = G$$

such that for any $\tilde{\tilde{\tilde{a}}}_0,\ \tilde{\tilde{\tilde{a}}}_1,\ \cdots,\ \tilde{\tilde{\tilde{a}}}_r \in \mathrm{Im}\mathcal{A}$ with $\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \cdots > \tilde{\tilde{\tilde{a}}}_r$,

$$\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0,\ \mathcal{A}(\hat{G}_1) = \tilde{\tilde{\tilde{a}}}_1,\ \cdots,\ \mathcal{A}(\hat{G}_r) = \tilde{\tilde{\tilde{a}}}_r,$$

where $\hat{G}_i = G_i \setminus G_{i-1}$ for $i = 1,\ 2,\ \cdots,\ r$.

Proof. Suppose $\mathcal{A} \in IVIOG(G)$. Then by Corollary 4.11, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0}$, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1}$, $\cdots$, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r}$ are the only level subgroups of $\mathcal{A}$, where $\{\tilde{\tilde{\tilde{a}}}_0, \tilde{\tilde{\tilde{a}}}_1, \cdots, \tilde{\tilde{\tilde{a}}}_r\} = \mathrm{Im}\mathcal{A}$ and $\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \cdots > \tilde{\tilde{\tilde{a}}}_r$. Furthermore, the level subgroups form a chain

$$C(\mathcal{A}) = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_2} \subset \cdots \subset [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r}.$$

Thus clearly, $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} = (e)$ and $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = G$. Assume that $C(\mathcal{A})$ is not maximal. Then we redefine $C(\mathcal{A})$ introducing subgroups of G. Let us call the redefined chain as

$$G_0 \subset G_1 \subset \cdots \subset G_s,$$

where $G_0 = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0} = (e)$ and $G_s = [\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_r} = G$. Then for each G_i between $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_0}(= G_0)$ and $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_1}(= G_j$ for some $j)$, $\mathcal{A}(\hat{G}_i) = \tilde{\tilde{\tilde{a}}}_1$. Similarly, we get for each G_k between $[\mathcal{A}]_{\tilde{\tilde{\tilde{a}}}_i}$ and $[\mathcal{A}]_{\widetilde{\widetilde{\widetilde{a_{i+1}}}}}$, $\mathcal{A}(\hat{G}_k) = \widetilde{\widetilde{\widetilde{a_{i+1}}}}$ and $\mathcal{A}(\hat{G}_s) = \tilde{\tilde{\tilde{a}}}_r$. Thus we have

$$\mathcal{A}(G_0) = \tilde{\tilde{\tilde{a}}}_0,\ \mathcal{A}(\hat{G}_1) \cdots = \mathcal{A}(\hat{G}_i) = \tilde{\tilde{\tilde{a}}}_1,\ \mathcal{A}(\widehat{G_{i+1}}) \cdots = \mathcal{A}(\hat{G}_k) = \widetilde{\widetilde{\widetilde{a_{i+1}}}},\ \mathcal{A}(\hat{G}_s) = \tilde{\tilde{\tilde{a}}}_r,$$

where $\hat{G}_1 = G_1 \setminus G_0,\ \hat{G}_2 = G_2 \setminus G_1,\ \cdots,\ \hat{G}_s = G_s \setminus G_{s-1}$ and $\tilde{\tilde{\tilde{a}}}_0 > \tilde{\tilde{\tilde{a}}}_1 > \cdots > \tilde{\tilde{\tilde{a}}}_r$. So the result holds.

Conversely, suppose the necessary condition holds and we define the mapping

$$\mathcal{A} : G \to ([I] \oplus [I]) \times (I \oplus I) \times I \text{ as follows:}$$

$$\mathcal{A}(e) = \tilde{\tilde{\tilde{a}}}_0,\ \mathcal{A}(\hat{G}_1) = \tilde{\tilde{\tilde{a}}}_1,\ \cdots,\ \mathcal{A}(\hat{G}_r) = \tilde{\tilde{\tilde{a}}}_r.$$

Then clearly, $\mathcal{A} \in IVIOG(G)$. Let $x,\ y \in G$.

Case (i): Suppose $x,\ y \in G_i$ but $x,\ y \notin G_{i-1}$. Then $\mathcal{A}(x) = \mathcal{A}(y) = \tilde{\tilde{\tilde{a}}}_i$ and $xy \in G_i$ or $xy \in G_{i-1}$. Thus $\mathcal{A}(xy) \geq \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x) \wedge \mathcal{A}(y)$. Since $x^{-1} \in G_i$, we have $\mathcal{A}(x^{-1}) \geq \tilde{\tilde{\tilde{a}}}_i = \mathcal{A}(x)$.

Case (ii): Suppose $x \in G_i$, $x \notin G_{i-1}$ and $y \in G_j$, $x \notin G_{j-1}$, where $i > j$. Then $\mathcal{A}(x) = \tilde{\tilde{a}}_i$ and $\mathcal{A}(y) = \tilde{\tilde{a}}_j$. Thus $\mathcal{A}(xy) \geq \tilde{\tilde{a}}_i = \mathcal{A}(x) \wedge \mathcal{A}(y)$. Moreover, $\mathcal{A}(x^{-1}) \geq \tilde{\tilde{a}}_i = \mathcal{A}(x)$. So in either cases, $\mathcal{A} \in IVIOG(G)$. □

The following is an immediate consequence of Theorem 4.23.

Corollary 4.24. *Let G be a cyclic p-group of order p^r and let $\mathcal{A} \in IVIOS(G)$. Then $\mathcal{A} \in IVIOG(G)$ if and only if for each $x \in G$ with $O(x) = p^i$, $\mathcal{A}(x) = \tilde{\tilde{a}}_i$, where $i = 0,\ 1,\ \cdots,\ r$ and $\tilde{\tilde{a}}_0 > \tilde{\tilde{a}}_1 > \cdots > \tilde{\tilde{a}}_r$.*

Remark 4.25. Corollary 4.24 can be seen by Proposition 4.20.

5. Conclusions

We introduced the notions of IVI-octahedron subgroups and IVI-octahedron normal subgroups of a group and obtained some of their properties, and gave some examples. In particular, we gave a sufficient condition that the image of an IVI-octahedron subgroup under a group homomorphism is an IVI-octahedron subgroup. Also, we obtained the characterization that an IVI-octahedron set in a cyclic group of a prime order is an IVI-octahedron subgroup. Furthermore, we defined the level subgroup of an IVI-octahedron subgroup and studied some relationships between them.

In the future, we expect that one applies IVI-octahedron sets to ring structures, BCI/BCK-algebraic structures, topologies, topological group structures, category theories and decision-making problems, etc. Also, we will propose a new concept combined with a neutrosophic set and an IVI-octahedron set, and research for its properties.

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