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On intuitionistic fuzzy multiplication modules

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Abstract. In this paper, we introduce and investigate the intuitionistic fuzzy multiplication modules over a commutative ring with non-zero identity. The basic properties of the intuitionistic fuzzy prime submodules of an intuitionistic fuzzy multiplication modules are characterized.

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1. Introduction

The idea of investigating a mathematical structure via its representation in simpler structure is commonly used and often successful. The representation theory of multiplication modules over a commutative ring has developed greatly in the recent years. Among the most interesting modules are multiplication modules because, for example, they are top module (an R-module M equipped with Zariski topology is called top module, see [1]). Let R be a commutative ring and M an R-module. Then M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M) = \{r \in R : rM \subseteq N\}$. The literature on multiplication ideals and modules are quite extensive, for example, see [2, 3, 4, 5, 6, 7]. In particular [2], [4] and [6] contain a number of characterizations of multiplication modules.

Research on the theory of intuitionistic fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, neural networks and planning. Consequently, intuitionistic fuzzy set theory has emerged as a potential area of interdisciplinary research and intuitionistic fuzzy module theory is of recent interest.

In the last few years a considerable amount of work has been done on intuitionistic fuzzy modules. Atanassov in [8, 9, 10] introduced the notion of an intuitionistic fuzzy subset A of a non-empty set X as an ordered function (μ_A, ν_A) from X to $[0,1] \times [0,1]$. In [11], Biswas considered the intuitionistic fuzzification of algebraic structures. Hur et al. [12] introduced and examined the notion of an intuitionistic fuzzy ideal of a ring. Since then several authors have obtained interesting results on intuitionistic fuzzy ideals of a ring and intuitionistic fuzzy modules (See [13, 14, 15, 16, 17, 18, 19, 20, 21]. Also, See [14] for a comprehensive survey of the literature of these developments. Hence the study of the intuitionistic fuzzy multiplication modules theory is worthy of study.

In the present paper, we introduce and study the intuitionistic fuzzy multiplication modules over a commutative ring with non-zero identity. There are many basic open questions concerning the intuitionistic fuzzy module theory. The most essential one among them is to know whether or not an intuitionistic fuzzy P-module is a P-module and vice versa. We give a condition giving an affirmative answer to these questions. Our main purpose is to establish a connection between the intuitionistic fuzzy multiplication modules (resp. the intuitionistic fuzzy Noetherian modules) and the multiplication modules (resp. the Noetherian modules) over a commutative ring. In Section 3, we introduce the intuitionistic fuzzy multiplication modules and make an intensive study of this notion. It is shown that, in Theorem 3.11, an R-module M is a multiplication module if and only if M is an intuitionistic fuzzy multiplication module (so it is top module). Also, we introduce intuitionistic fuzzy Noetherian modules and show that every intuitionistic fuzzy Noetherian module is a Noetherian module (Theorem 3.8), but the converse is not true. In Section 4, we introduce the notion of intuitionistic fuzzy radical of an intuitionistic fuzzy submodule of an intuitionistic fuzzy module over a commutative ring. Finally, in Theorem 4.13, we formulate the intuitionistic fuzzy radical of intuitionistic fuzzy submodules of an intuitionistic fuzzy multiplication module.

2. Preliminaries

Throughout this paper R is a commutative ring with non-zero identity, M is an unitary R-module. Let θ denote the zero element of M. In order to make this paper easier to follow, we recall in this section various notions from intuitionistic fuzzy commutative algebra theory which will be used in the sequel.

Given a nonempty set X, an intuitionistic fuzzy subset A is an ordered function (μ_A, ν_A) from X to $[0,1] \times [0,1]$. We denote by $IFS(X)$ the set of all intuitionistic fuzzy subsets of X. For $A, B \in IFS(X)$ we write $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. By an intuitionistic fuzzy point (IFP) $x_{(p,q)}$ of X, $x \in X$, $p, q \in (0,1]$ such that $p + q \le 1$ we mean $x_{(p,q)} \in IFS(X)$ is defined by

$$x_{(p,q)}(y) = \begin{cases} (p,q), & \text{if } y = x \\ (0,1), & \text{if otherwise.} \end{cases}$$

The intuitionistic fuzzy characterisitic function of X with respect to a subset Y is denoted by χ_Y and is defined as:

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases}; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $\chi_X(x) = (1, 0), \forall x \in X$ is a constant intuitionistic fuzzy point in X and it is the maximal element of $IFS(X)$.

Definition 2.1 ([12, 22]). Let $A \in IFS(R)$. Then A is called an *intuitionistic fuzzy ideal* (briefly, IFI) of R, if it satisfies the following conditions: for all $x,\ y \in R$,
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$,
(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(iv)$\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 2.2 ([22]). Let $A \in IFS(M)$. Then A is called an *intuitionistic fuzzy module* (briefly, IFM) of M, if it satisfies the following conditions: for all $x,\ y \in M$,
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(rx) \geq \mu_A(x)$,
(iii) $\mu_A(\theta) = 1$,
(iv) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(v) $\nu_A(rx) \leq \nu_A(x)$,
(vi) $\nu_A(\theta) = 0$.

It can be easily verified that both $\chi_{\{\theta\}}$ and χ_M are intuitionistic fuzzy modules of M and these are called trivial intuitionistic fuzzy modules of M. Any intuitionistic fuzzy modules of M other than these are called proper intuitionistic fuzzy modules of M (See [14]). Let $IFM(M)$ denote the set of all intuitionistic fuzzy R-modules of M and let $IFI(R)$ denote the set of all intuitionistic fuzzy ideals of R. We note that when $R = M$, then $A \in IFM(M)$ if and only if $\mu_A(\theta) = 1,\ \nu_A(\theta) = 0$ and $A \in IFI(R)$.

Let $A \in IFS(M)$ and $p,\ q \in [0, 1]$ with $p + q \leq 1$. Then the set

$$A_{(p,q)} = \{x \in M : \mu_A(x) \geq p \text{ and } \nu_A(x) \leq q\}$$

is called the (p, q)-*cut subset of* M *with respect to* A. In particular, we denote $A_{(\mu_A(\theta),\nu_A(\theta))}$ by A_*. Of course, $A_* = \{x \in M : \mu_A(x) = \mu_A(\theta) \text{ and } \nu_A(x) = \nu_A(\theta)\}$. The *support* of an IFS A, is denoted by A^*, is a subset of M defined as:

$$A^* = \{x \in M : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}.$$

Note that if $A = \chi_M$, then $A^* = M$, but converse need not be true. The following theorem gives a relation between an intuitionistic fuzzy modules on M and submodules of M. It is a very practical method to construct an intuitionistic fuzzy module on M.

Theorem 2.3 ([18]). *Let $A \in IFS(M)$. Then A is an intuitionistic fuzzy module if and only if for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ such that $A_{(\alpha,\beta)}$ is an R-submodules of M. In particular, A_* is an R-submodules of M.*

Definition 2.4 ([13])**.** For a non-constant $C \in IFI(R)$, C is called an *intuitionistic fuzzy prime ideal* of R, if for any $x_{(p,q)}, y_{(s,t)} \in IFP(R)$, whenever $x_{(p,q)}y_{(s,t)} \subseteq C$ implies that either $x_{(p,q)} \subseteq C$ or $y_{(s,t)} \subseteq C$.

The set of intuitionistic fuzzy prime ideals of R is denoted by $IF-Spec(R)$.

Definition 2.5 ([19])**.** Let $A,\ B \in IFM(M)$. Then A is called an *intuitionistic fuzzy submodule* of B, if $A \subseteq B$. In particular, if $B = \chi_M$, then we say A is an *intuitionistic fuzzy submodule* of M.

Definition 2.6 ([19])**.** A non-constant intuitionistic fuzzy submodule A of B is said to be *prime*, if for $C \in IFI(R)$ and $D \in IFM(M)$ such that $C \cdot D \subseteq A$, either $D \subseteq A$ or $C \subseteq (A : B)$.

In particular, when $B = \chi_M$, we say that A is called an *intuitionistic fuzzy prime submodule* of M, if for $C \in IFI(R)$ and $D \in IFM(M)$ such that $C \cdot D \subseteq A$, then either $D \subseteq A$ or $C \subseteq (A : \chi_M)$.

The set of intuitionistic fuzzy prime submodules of M is denoted by $IF-Spec(M)$.

Theorem 2.7 ([19])**.** *Let A be an intuitionistic fuzzy prime submodule of B. If $A_{(\alpha,\beta)} \neq B_{(\alpha,\beta)}, \alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, then $A_{(\alpha,\beta)}$ is a prime submodule of $B_{(\alpha,\beta)}$.*

Corollary 2.8. *Let A be an intuitionistic fuzzy prime submodule of M. Then*

$$A_* = \{x \in M : \mu_A(x) = \mu_A(\theta) \text{ and } \nu_A(x) = \nu_A(\theta)\}$$

is a prime submodule of M.

Proof. Clear from Theorem 2.7 as $A_{(\alpha,\beta)} = A_*$, when $\alpha = \mu_A(\theta)$ and $\beta = \nu_A(\theta)$ and $B = \chi_M$. □

Definition 2.9. Let $C \in IFI(R)$ and $B \in IFM(M)$. Then the *product* of C and D, denoted by $C \cdot B$, is an intuitionistic fuzzy subset of M defined as follows: for all $x \in M$,

$$\mu_{C \cdot B}(x) = \begin{cases} Sup[\mu_C(r) \wedge \mu_B(y)] & \text{if } x = ry, r \in R, y \in M \\ 0, & \text{if } x \text{ is not expressible as } x = ry, \end{cases}$$

$$\nu_{C \cdot B}(x) = \begin{cases} Inf[\nu_C(r) \vee \nu_B(y)] & \text{if } x = ry, r \in R, y \in M \\ 1, & \text{if } x \text{ is not expressible as } x = ry, \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively.

Definition 2.10 ([22])**.** For $A, B \in IFS(M)$ and $C \in IFS(R)$, define the *residual quotient* $(A : B)$ and $(A : C)$ as follows:

$$(A : B) = \bigcup\{D : D \in IFS(R) \text{ such that } D \cdot B \subseteq A\}$$

and

$$(A : C) = \bigcup\{E : E \in IFS(M) \text{ such that } C \cdot E \subseteq A\}.$$

Then it is well-known ([22], Theorem 3.6) that if $A, B \in IFM(M)$ and $C \in IFI(R)$, then $(A : B) \in IFI(R)$ and $(A : C) \in IFM(M)$.

Theorem 2.11 ([22])**.** *For* $A,\ B \in IFS(M)$ *and* $C \in IFS(R)$*. Then we have*
(1) $(A : B) \cdot B \subseteq A$,
(2) $C \cdot (A : C) \subseteq A$,
(3) $C \cdot B \subseteq A \Leftrightarrow C \subseteq (A : B) \Leftrightarrow B \subseteq (A : C)$.

Definition 2.12 ([20])**.** Let A and B be two intuitionistic fuzzy modules of an R-module M. Then

$$(A : B) = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in (0, 1] \text{ with } \alpha + \beta \leq 1, r_{(\alpha,\beta)} \cdot B \subseteq A\}$$

is called an *intuitionistic fuzzy ideal* in R.

Definition 2.13 ([20])**.** Let M be an R-module M and $A \in IFS(M)$. Then the *annihilator* of A, denoted by $ann(A)$, is an intuitionistic fuzzy subset of M defined as:

$$ann(A) = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in (0, 1] \text{ with } \alpha + \beta \leq 1, r_{(\alpha,\beta)}A \subseteq \chi_{\{\theta\}}\}.$$

Note that for any $r \in R$, we have $\mu_{ann(A)}(r) = Sup\{\alpha : \alpha \in (0, 1], r_{(\alpha,\beta)}A \subseteq \chi_{\{\theta\}}\}$ and $\nu_{ann(A)}(r) = Inf\{\beta : \beta \in (0, 1], r_{(\alpha,\beta)}A \subseteq \chi_{\{\theta\}}\}$. In other words, $ann(A) = (\chi_{\{\theta\}} : A)$.

Definition 2.14 ([20])**.** An intuitionistic fuzzy module A of an R-module M is said to be *faithful*, if $ann(A) = \chi_{\{0\}}$.

Definition 2.15. Let $A \in IFS(M)$. Then $\langle A\rangle = \bigcap\{B : B \supseteq A, B \in IFM(M)\}$ denote the intuitionistic fuzzy submodule of M generated by A.

Proposition 2.16. *Let A be a proper intuitionistic fuzzy submodule of an R-module M, then A is an intuitionistic fuzzy prime submodule of M if and only if $(A : B) = (A : \chi_M)$ for all intuitionistic fuzzy submodule B of M containing A properly.*

Proof. Let A be a proper intuitionistic fuzzy prime submodule of M and let B be an intuitionistic fuzzy submodule B of M containing A properly. It is obvious that $(A : \chi_M) \subseteq (A : B)$. For other inclusion, assume that $A \subsetneqq B$. Then there exist at least one $x \in M$ such that $x_{(p,q)} \in B$ but $x_{(p,q)} \notin A$, where $p, q \in [0, 1]$ such that $p + q \leq 1$. Let $r_{(s,t)} \in IFP(R)$ such that $r_{(s,t)} \in (A : B)$. Then $r_{(s,t)}B \subseteq A$. In particular, $r_{(s,t)}x_{(p,q)} \subseteq A$. As A is an intuitionistic fuzzy prime submodule of M and $x_{(p,q)} \notin A$ implies that $r_{(s,t)} \in (A : \chi_M)$. Thus $(A : B) \subseteq (A : \chi_M)$. So $(A : B) = (A : \chi_M)$.

Conversely, suppose $r_{(s,t)}x_{(p,q)} \in A$ for some $r_{(s,t)} \in IFP(R)$ and $x_{(p,q)} \in IFP(M)$. Assume that $x_{(p,q)} \notin A$. We only have to show that $r_{(s,t)} \in (A : \chi_M)$.

It is clear that $(A : \chi_M) = (A : A + \langle x_{(p,q)}\rangle)$, but $r_{(s,t)} \in (A : A + \langle x_{(p,q)}\rangle)$. Then $r_{(s,t)} \in (A : \chi_M)$. Thus A is an intuitionistic fuzzy prime submodule of M. □

3. Intuitionistic fuzzy multiplication modules

In this section we list some basic properties concerning intuitionistic fuzzy multiplication modules over a commutative ring. We begin with the key definition of this paper.

Definition 3.1. An R-module M is called an intuitionistic fuzzy multiplication module if and only if for each intuitionistic fuzzy submodule A of M: there exists a intuitionistic fuzzy ideal C of R with $C(0_R) = (1, 0)$ such that $A = C \cdot \chi_M$. One can easily show that if $A = C \cdot \chi_M$ then $A = (A : \chi_M) \cdot \chi_M$.

From this onwards, we write the product $C \cdot D$ by CD. Note that above intuitionistic fuzzy ideal C of R is called a presentation intuitionistic fuzzy ideal of an intuitionistic fuzzy submodule A (or for short, a presentation of A). It is clear that such a presentation is not unique, that is, there exist some intuitionistic fuzzy ideals C and D of R such that $C\chi_M = D\chi_M$ even if $C \neq D$.

Definition 3.2. Let A and B be intuitionistic fuzzy submodules of an intuitionistic fuzzy multiplication module M, where $A = C\chi_M$ and $B = D\chi_M$ for some intuitionistic fuzzy ideals C, D of a ring R. The *product* of A and B is denoted by AB and is defined by $AB = CD\chi_M$. Clearly AB is an intuitionistic fuzzy submodule of M and contained in $A \cap B$.

Since a presentation of an intuitionistic fuzzy module is not unique we must prove that the product is well defined, is the product and is independent of presentations of two intuitionistic fuzzy submodules.

Proposition 3.3. *Let $A = C\chi_M$ and $B = D\chi_M$ be intuitionistic fuzzy submodules of an intuitionistic fuzzy multiplication module M where C and D are intuitionistic fuzzy ideals of R. Then the product of A and B is independent of the presentation of A and B.*

Proof. Let $A = C\chi_M = C^{'}\chi_M$ and $B = D\chi_M = D^{'}\chi_M$ where $C^{'}$ and $D^{'}$ are intuitionistic fuzzy ideals of R. Then we have

$$\begin{aligned}(CD)\chi_M &= C(D\chi_M) = C(D^{'}\chi_M)\\ &= (CD^{'})\chi_M = (D^{'}C)\chi_M\\ &= D^{'}(C\chi_M) = D^{'}(C^{'}\chi_M)\\ &= (D^{'}C^{'})\chi_M = (C^{'}D^{'})\chi_M.\end{aligned}$$

Thus the product of A and B is well defined. □

In [6], Proposition 1.1, it was proved that an R-module M is a multiplication module if and only if for each m in M there exists an ideal I of R such that $Rm = IM$. Now, we have the following proposition for intuitionistic fuzzy multiplication R-modules.

Proposition 3.4. *An R-module M is an intuitionistic fuzzy multiplication module if and only if for each $x \in M$ and $p, q \in (0, 1]$ such that $p + q \leq 1$, there exists an intuitionistic fuzzy ideal C of R with $C(0_R) = (1, 0)$ such that $< x_{(p,q)} >= C\chi_M$.*

Proof. The necessity is clear. Conversely, suppose for each $x \in M$ and $p, q \in (0, 1]$ such that $p + q \leq 1$, there exists $C \in IFI(R)$ with $\mu_C(0) = 1$ and $\nu_C(0) = 0$ such that $\langle x_{(p,q)} \rangle = C\chi_M$. Let $A \in IFM(M)$ and $x \in M$. It is clear that there exists $p, q \in (0, 1]$ such that $p+q \leq 1$, $\mu_A(x) = p$ and $\nu_A(x) = q$. Then $x_{(p,q)} \in A$ and thus $\langle x_{(p,q)} \rangle \subseteq A$ by definition. By assumption, there exists $C_x \in IFI(R)$ with $\mu_{C_x}(0) = 1$ and $\nu_{C_x}(0) = 0$ such that $\langle x_{(p,q)} \rangle = C_x\chi_M$. Thus $C_x\chi_M \subseteq A$. By Theorem 2.11,

$C_x \subseteq (A : \chi_M)$. It follows that $\mu_{C_x}(r) \leq \mu_{(A:\chi_M)}(r)$ and $\nu_{C_x}(r) \geq \nu_{(A:\chi_M)}(r)$ for each $r \in R$.

Now let $C = \bigcup\{C_x : x \in M\}$. Then $\mu_C(r) = \vee\{\mu_{C_x}(r) : x \in M\}$ and $\nu_C(r) = \wedge\{\nu_{C_x}(r) : x \in M\}$ for each $r \in R$ and $\mu_C(0) = 1$ and $\nu_C(0) = 0$. Thus $\mu_C(r) \leq \mu_{(A:\chi_M)}(r)$ and $\nu_C(r) \geq \nu_{(A:\chi_M)}(r)$ for every $r \in R$. So $C \subseteq (A : \chi_M)$, i.e., $C\chi_M \subseteq A$ by Theorem 2.11. For the other inclusion, assume that $m \in M$. Then there exists $p,\ q \in (0, 1]$ such that $p+q \leq 1$, $\mu_A(m) = p$ and $\nu_A(m) = q$, and hence $m_{(p,q)} \in A$. It follows that $< m_{(p,q)} >= C_{m_{(p,q)}}\chi_M \subseteq C\chi_M$. On the other hand, $\mu_{<m_{(p,q)}>}(m) = p$ and $\nu_{<m_{(p,q)}>}(m) = q$. Thus $\mu_A(m) = \mu_{\langle m_{(p,q)}\rangle}(m) \leq \mu_{C\chi_M}(m)$ and $\nu_A(m) = \nu_{\langle m_{(p,q)}\rangle}(m) \geq \nu_{C\chi_M}(m)$. Thus, $A \subseteq C\chi_M$. So we have the equality. □

Lemma 3.5. *Let M be an R-module and $A \in IFM(M)$. Then $(A : \chi_M) = \bigcup\{r_{(s,t)} : s, t \in [0, 1]$ such that $s + t \leq 1, r \in R \cap (A_{(s,t)} : M)\}$.*

Proof. By Theorem 2.12 in [19], we have

$$(A : \chi_M) = \bigcup\{r_{(s,t)} : r \in R, s, t \in [0, 1], s + t \leq 1, \text{ such that } r_{(s,t)}\chi_M \subseteq A\}.$$

On the other hand, for each $y \in M$, by definition we have

$$\mu_{r_{(s,t)}\chi_M}(y) = \begin{cases} s, & \text{if } y = rm, r \in R, m \in M \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{r_{(s,t)}\chi_M}(y) = \begin{cases} t, & \text{if } y = rm, r \in R, m \in M \\ 1, & \text{otherwise.} \end{cases}$$

Then we get

$$\begin{aligned}(A : \chi_M) &= \bigcup\{r_{(s,t)} : r \in R \text{ such that } \mu_A(rm) \geq s; \nu_A(rm) \leq t \text{ for each } m \in M\} \\ &= \bigcup\{r_{(s,t)} : r \in R \text{ such that } rm \in A_{(s,t)} \text{ for each } m \in M\} \\ &= \bigcup\{r_{(s,t)} : r \in R \text{ such that } rM \subseteq A_{(s,t)}\} \\ &= \bigcup\{r_{(s,t)} : r \in R \text{ such that } r \in (A_{(s,t)} : M)\} \\ &= \bigcup\{r_{(s,t)} : s, t \in [0, 1], s + t \leq 1 \text{ such that } r \in R \cap (A_{(s,t)} : M)\}.\end{aligned}$$
□

Let us now define a basic concept and new properties of them over commutative rings.

Definition 3.6. An R-module M is called an *intuitionistic fuzzy Noetherian module*, if every ascending chain of intuitionistic fuzzy submodules in M is stationary.

Theorem 3.7. *Let R be an intuitionistic fuzzy Noetherian ring and M be an intuitionistic fuzzy multiplication module. Then M is an intuitionistic fuzzy Noetherian module.*

Proof. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$ be an ascending chain of intuitionistic fuzzy submodules of M. Then $(A_1 : \chi_M) \subseteq (A_2 : \chi_M) \subseteq (A_3 : \chi_M) \subseteq ....$ is an ascending chain of intuitionistic fuzzy ideals of R. By assumption, there is a positive integer t such that $(A_t : \chi_M) = (A_{t+s} : \chi_M)$ for every positive integer s. Thus $A_t = (A_t : \chi_M)\chi_M = (A_{t+s} : \chi_M)\chi_M = A_{t+s}$ for every positive integer s. So the chain is stationary. □

Theorem 3.8. *If M is an intuitionistic fuzzy Noetherian module, then M is a Noetherian module.*

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq ....$ be an ascending chain of submodules of M. For each positive integer i, we define IFSMs A_i by

$$\mu_{A_i}(x) = \begin{cases} 1, & \text{if } x \in N_i \\ 0, & \text{otherwise} \end{cases}, \nu_{A_i}(x) = \begin{cases} 0, & \text{if } x \in N_i \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $A_i = \chi_{N_i}$. Then $A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$ be an ascending chain of intuitionistic fuzzy submodules of M. Thus there exist a positive integer n such that $A_n = A_{n+k}$ for every positive integer k. Now we show that $N_n = N_{n+k}$ for all k.

Let k be a positive integer and $x \in N_{n+k}$. So $\mu_{A_n}(x) = \mu_{A_{n+k}}(x) = 1$ and $\nu_{A_n}(x) = \nu_{A_{n+k}}(x) = 0$. Then $x \in N_n$. Thus $N_n \subseteq N_{n+k} \subseteq N_n$ and so we have the equality. So M is a Noetherian R-module. □

Example 3.9. Let $M = R$ denote the field of real numbers with usual addition and multiplication. So M is a Noetherian R-module. We define IFSs A_n of M by

$$\mu_{A_n}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 1 - 1/n, & \text{otherwise} \end{cases}, \nu_{A_n}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1/n, & \text{otherwise,} \end{cases}$$

for all positive integer n. It is easy to verify that each A_n is an intuitionistic fuzzy submodule of M. Also $A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$ is an infinite strictly ascending chain of intuitionistic fuzzy submodules of M, and so M is not an intuitionistic fuzzy Noetherian R-module.

Remark 3.10. The above example shows that converse of the theorem is not true. However we have the following theorem for the multiplication modules

Theorem 3.11. *Let M be an R-module. Then M is a multiplication module if and only if M is an intuitionistic fuzzy multiplication module.*

Proof. Let M be a multiplication module, and let $A \in IFM(M)$. Since the inclusion $(A : \chi_M)\chi_M \subseteq A$ is clear, we will prove the reverse inclusion. Let $x \in M$ and $\mu_A(x) = p$ and $\nu_A(x) = q$ for some $p, q \in [0, 1]$ with $p+q \leq 1$. It suffices to show that $p \leq \mu_{(A:\chi_M)\chi_M}(x)$ and $q \geq \nu_{(A:\chi_M)\chi_M}(x)$. By assumption, $A_{(p,q)} = (A_{(p,q)} : M)M$. Since $\mu_A(x) = p$ and $\nu_A(x) = q$. Then $x \in A_{(p,q)}$. Thus $x = \sum_{i=1}^{n} r_i x_i$, where $x_i \in M$ and $r_i \in (A_{(p,q)} : M)$ $(i = 1, 2, ..., n)$. It follows from Lemma 3.5 that $(r_i)_{(p,q)} \in (A : \chi_M)$ for each i.

On the other hand, for each i $(x_i)_{(p,q)} \in \chi_M$. So $(r_i x_i)_{(p,q)} = (r_i)_{(p,q)}(x_i)_{(p,q)} \in (A : \chi_M)\chi_M$. Then $(r_i x_i)_{(p,q)} \in (A : \chi_M)\chi_M$ $(1 \leq i \leq n)$. It follows that $\mu_{(A:\chi_M)\chi_M}(r_i x_i) \geq p$ and $\nu_{(A:\chi_M)\chi_M}(r_i x_i) \leq q$ for each i $(1 \leq i \leq n)$. Moreover, we have

$$\mu_{(A:\chi_M)\chi_M}(x) = \mu_{(A:\chi_M)\chi_M}(\sum_{i=1}^{n} r_i x_i) \geq \wedge_{i=1}^{n} \mu_{(A:\chi_M)\chi_M}(r_i x_i) \geq p.$$

Thus $\mu_{(A:\chi_M)\chi_M}(x) \geq p$. Similarly, we can show that $\nu_{(A:\chi_M)\chi_M}(x) \leq q$. So $\mu_A(x) = p \leq \mu_{(A:\chi_M)\chi_M}(x)$ and $\nu_A(x) = q \geq \nu_{(A:\chi_M)\chi_M}(x)$ for every $x \in M$. Hence we have $A \subseteq (A : \chi_M)\chi_M$. Therefore $A = (A : \chi_M)\chi_M$, i.e., M is an intuitionistic fuzzy multiplication module.

Conversely, assume that M is an intuitionistic fuzzy multiplication module and let N be a proper submodule of M. It suffices to show that $N \subseteq (N : M)M$. We define an IFS A of M by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{otherwise} \end{cases}, \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ 1, & \text{otherwise.} \end{cases}$$

One can easily see that $A = \chi_N$ and $A \in IFM(M)$. Also $A_{(p,q)} = N$ for each $p,\ q \in (0, 1]$ with $p + q \leq 1$. Since M is an intuitionistic fuzzy multiplication module, $A = (A : \chi_M)\chi_M$. Let $m \in N$. Then $\mu_A(m) = \mu_{(A:\chi_M)\chi_M}(m) = 1$ and $\nu_A(m) = \nu_{(A:\chi_M)\chi_M}(m) = 0$. But

$$\begin{aligned} \mu_{(A:\chi_M)\chi_M}(m) &= \vee\{\mu_{(A:\chi_M)}(s) \wedge \chi_M(x) : sx = m \text{ for some } s \in R, x \in M\} \\ &= \vee\{\mu_{(A:\chi_M)}(s) : m \in sM, s \in R\}. \end{aligned}$$

Similarly, we can show that $\nu_{(A:\chi_M)\chi_M}(m) = \wedge\{\nu_{(A:\chi_M)}(s) : m \in sM, s \in R\}$.

On the other hand, we have $\mu_{(A:\chi_M)}(s) = \vee\{r_{(p,q)}(s) : r \in (N : M)\}$ and $\nu_{(A:\chi_M)}(s) = \wedge\{r_{(p,q)}(s) : r \in (N : M)\}$ and the fact that $A_{(p,q)} = N$ for each $p, q \in (0, 1]$ with $p + q \leq 1$. If there is no $1 \neq r \in R$ such that $m \in rM$, then $\mu_{(A:\chi_M)}(1) = \vee\{r_{(p,q)}(1) : r \in (N : M)\} = 0$ and $\nu_{(A:\chi_M)}(1) = \wedge\{r_{(p,q)}(1) : r \in (N : M)\} = 1$. Which is a contradiction. Thus there exists $r^{'} \in R$ such that $m \in r^{'}M$. Set $S = \{r \in (N : M) : m \in rM\}$. If $S = \varnothing$, then for each $t \in R$ with $m \in tM$, we have $t \notin (N : M)$. Thus

$$\mu_{(A:\chi_M)}(t) = \vee\{r_{(p,q)}(t) : r \in (N : M)\} = 0$$

and

$$\nu_{(A:\chi_M)}(t) = \wedge\{r_{(p,q)}(t) : r \in (N : M)\} = 1.$$

So we get

$$\mu_{(A:\chi_M)\chi_M}(m) = \vee\{\mu_{(A:\chi_M)}(s) : m \in sM, s \in R\} = 0$$

and

$$\nu_{(A:\chi_M)\chi_M}(m) = \wedge\{\nu_{(A:\chi_M)}(t) : m \in tM, t \in R\} = 1.$$

A contradiction. Hence we may assume that $S \neq \varnothing$. Then there exists $r \in R$ such that $m \in rM$ and $r \in (N : M)$. Thus $N \subseteq (N : M)M$. So M is a multiplication R-module. □

Let M be an R-module. A submodule N of M is said to be *prime*, if $N \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in N$, then $rM \subseteq N$ or $m \in N$. Let $Spec(M)$ denote the collection of prime submodules of M. We define $V(N)$ to be the set of all prime submodules of M containing N (So $V(M) = \varnothing$ and $V(0) = Spec(M)$). If $\mathcal{C}^*(M)$ denotes the collection of all subsets $V(N)$ of $Spec(M)$, then $\mathcal{C}^*(M)$ contains the empty set and $Spec(M)$, and $\mathcal{C}^*(M)$ is closed under intersections. We shall say that M is a module with Zariski topology, or a top module for short, if $\mathcal{C}^*(M)$ is closed under finite unions, i.e., for any submodules N and J of M there exists a submodule K of M such that $V(N) \cup V(J) = V(K)$, for in this case $\mathcal{C}^*(M)$ satisfies the axioms for the closed subsets of a topological space [23]. We recall that by ([23], Theorem 1) every multiplication module is a top module. Now by Theorem 3.11, we obtained the following theorem.

Theorem 3.12. *Every intuitionistic fuzzy multiplication R-module is top module.*

4. Radical of an intuitionistic fuzzy submodule

In [24], the notion of intuitionistic fuzzy radical of an intuitionistic fuzzy ideal and its properties are given. We generalized this definition to any intuitionistic fuzzy submodule of an intuitionistic fuzzy module over a commutative ring.

Definition 4.1. Let M be an R-module and $A \in IFM(M)$. Let $P(A)$ be the family of all intuitionistic fuzzy prime submodules of M containing A. The *intuitionistic fuzzy radical* of A, denoted by $IFrad_{\chi_M}(A)$, is defined by

$$IFrad_{\chi_M}(A) = \bigcap\{B : B \in P(A)\}.$$

We first have the following lemma.

Lemma 4.2. *Let M be an R-module and $A \in IFM(M)$. If A is an intuitionistic fuzzy submodule of M, then $rad_M(A_*) \subseteq (IFrad_{\chi_M}(A))_*$.*

Proof. If $P(A) = \varnothing$, then $IFrad_{\chi_M}(A) = \chi_M$. Thus $(IFrad_{\chi_M}(A))_* = M$. So we may assume that $P(A) \neq \varnothing$. Let $B \in P(A)$. Then B_* is a prime submodule of M by Corollary 2.8 and $A_* \subseteq B_*$. Thus $rad_M(A_*) \subseteq \bigcap\{B_* : B \in P(A)\}$. Let $m \in rad_M(A_*) \Rightarrow m \in \bigcap\{B_* : B \in P(A)\}$. Then $m \in B_*$ for all $B \in P(A)$. Thus $\mu_B(m) = 1$ and $\nu_B(m) = 0$ for every $B \in P(A)$. So we have

$$\mu_{IFrad_{\chi_M}(A)}(m) = 1 \text{ and } \nu_{IFrad_{\chi_M}(A)}(m) = 0.$$

It follows that $m \in (IFrad_{\chi_M}(A))_*$, so $rad_M(A_*) \subseteq (IFrad_{\chi_M}(A))_*$. □

In view of above Lemma 4.2 and ([19], Theorem 3.10), we have the following theorem,

Theorem 4.3. *Let A be a non-constant intuitionistic fuzzy submodule of M, and let $\wedge\{\mu_A(x) : x \notin rad_M(A_*)\} = s$ and $\vee\{\nu_A(x) : x \notin rad_M(A_*)\} = t$. If $s, t \in (0, 1)$ such that $s + t < 1$, then $rad_M(A_*) = (IFrad_{\chi_M}(A))_*$.*

Proof. By Lemma 4.2, it suffices to show that $rad_M(A_*) \subseteq (IFrad_{\chi_M}(A))_*$. Let P be a prime submodule of M containing A_*. Then we define the IFS Q on M as follow:

$$\mu_Q(x) = \begin{cases} 1, & \text{if } x \in P \\ s, & \text{otherwise} \end{cases} \quad , \nu_Q(x) = \begin{cases} 0, & \text{if } x \in P \\ t, & \text{otherwise.} \end{cases}$$

Clearly, Q is an intuitionistic fuzzy prime submodule of M by Theorem 3.10 in [19]. Now we show that $A \subseteq Q$. If $x \in P$, then we get

$$\mu_A(x) \le \mu_Q(x) = 1 \text{ and } \nu_A(x) \ge \nu_Q(x) = 0.$$

Also when $x \notin P$, then $\mu_Q(x) = s$ and $\nu_Q(x) = t$. Since by hypothesis we have $\mu_A(x) \le s$ and $\nu_A(x) \ge t$. Then we must have $\mu_A(x) \le s = \mu_Q(x)$ and $\nu_A(x) \ge t = \nu_Q(x)$. Thus we have $A \subseteq Q$. It follows that Q is an intuitionistic fuzzy prime submodule of M containing A and also $Q_* = P$. Now let $x \in (IFrad_{\chi_M}(A))_*$. Then $\mu_{IFrad_{\chi_M}(A)}(x) = 1$ and $\nu_{IFrad_{\chi_M}(A)}(x) = 0$. Thus $\mu_Q(x) = 1$ and $\nu_Q(x) = 0$ and $x \in Q_* = P$ So $x \in rad_M(A_*)$. Hence $(IFrad_{\chi_M}(A))_* \subseteq rad_M(A_*)$. Therefore we have the equality. □

Definition 4.4 ([19]). Let M be an R-module and $A \in IFM(M)$. Then A is called an *intuitionistic fuzzy maximal submodule* of M, if for any intuitionistic fuzzy submodule B of M, if $A \subseteq B$, then either $A_* = B_*$ or $B = \chi_M$.

In [6], Theorem 2.5, it was proved that every proper submodule of a non-zero multiplication R-module is contained in a maximal submodule of M. Now we have the following theorem for intuitionistic fuzzy multiplication R-modules.

Theorem 4.5. *Let M be a non-zero intuitionistic fuzzy multiplication R-module. Then every intuitionistic fuzzy submodule $A \neq \chi_M$ of M is contained in an intuitionistic fuzzy maximal submodule of M.*

Proof. Let A be a non-constant intuitionistic fuzzy submodule of M. So $A_* \neq M$ and there exists a maximal submodule N of M such that $A_* \subseteq N$ by ([6], Theorem 2.5) and Theorem 3.12. Let $s = \vee\{\mu_A(x) : x \in M\}$ and $t = \wedge\{\nu_A(x) : x \in M\}$. We define an IFS B of M defined by

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in N \\ s, & \text{otherwise} \end{cases} ,\nu_B(x) = \begin{cases} 0, & \text{if } x \in N \\ t, & \text{otherwise.} \end{cases}$$

One can easily see that B is an intuitionistic fuzzy submodule of M and $A \subseteq B$. Now we show that B is an intuitionistic fuzzy maximal submodule of M. Let $B \subseteq D$ and $D \in IFM(M)$. Then $N = B_* \subseteq D_*$. Thus either $N = D_*$ or $D_* = M$, since N is a maximal submodule of M. So $B_* = N = D_*$ or $D = \chi_M$. □

Definition 4.6. Let M be an R-module. We define the *intuitionistic fuzzy jacobson radical* of M, denoted by $IF_{Jac}(\chi_M)$, to be the intersection of all the intuitionistic fuzzy maximal submodules of M, if such exist and χ_M, otherwise.

Proposition 4.7. *Let A be a non-constant intuitionistic fuzzy submodule of an intuitionistic fuzzy multiplication module M such that $\chi_M = A + IF_{Jac}(\chi_M)$. Then $\chi_M = A$.*

Proof. If $\chi_M \neq A$, then A is contained in an intuitionistic fuzzy maximal submodule B of M. Thus $\chi_M = A + IF_{Jac}(\chi_M) \subseteq B$, which is a contradiction. □

Here we have the following proposition that is a generalization of Lemma 2.10 in [6] for intuitionistic fuzzy multiplication R-modules.

Proposition 4.8. *Assume that M be an intuitionistic fuzzy faithful multiplication R-module and let C be an intuitionistic fuzzy prime ideal of R, $r,\ s,\ p,\ q \in [0,1]$ such that $r+s \leq 1$ and $p+q \leq 1$ and $r_{(s,t)}x_{(p,q)} \in C\chi_M$ for some $r \in R$ and $x \in M$. Then $r_{(s,t)} \in C$ or $x_{(p,q)} \in C\chi_M$.*

Proof. Since C is an intuitionistic fuzzy prime ideal of R, by Proposition 3.5 in [12], C_* is prime ideal of R. Moreover, C can be defined as follow:

$$\mu_C(r) = \begin{cases} 1, & \text{if } r \in C_* \\ \alpha, & \text{otherwise} \end{cases} ,\nu_C(r) = \begin{cases} 0, & \text{if } r \in C_* \\ \beta, & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$. Since $r_{(s,t)}x_{(p,q)} = (rx)_{(s\wedge p, t\vee q)} \in C\chi_M$, $\mu_{C\chi_M}(rx) \geq s \wedge p$ and $\nu_{C\chi_M}(rx) \leq t \wedge q$. But

$$\begin{aligned}\mu_{C\chi_M}(rx) &= \vee\{\mu_C(a) \wedge \mu_{\chi_M}(y) : a \in R, y \in M, rx = ay\} \\ &= \vee\{\mu_C(a) : a \in R, rx \in aM\}.\end{aligned}$$

Similarly, we have $\nu_{C\chi_M}(rx) = \wedge\{\nu_C(a) : a \in R, rx \in aM\}$. Set $K = \{a \in C_* : rx \in aM\}$. First suppose that $K = \varnothing$. Then there is no $a \in C_*$ such that $rx \in aM$. Thus $\mu_{C\chi_M}(rx) = \alpha \geq s \wedge p$ and $\nu_{C\chi_M}(rx) = \beta \leq t \vee q$. This implies that either $\alpha \geq s$ or $\alpha \geq p$ and $\beta \leq t$ or $\beta \leq q$. We split the proof into two cases:

Case(i). When $\alpha \geq s$ and $\beta \leq t$. Since $C\chi_M(R) = \{(1,0), (\alpha, \beta)\}$, i.e., $\mu_{C\chi_M}(r) \in \{1, \alpha\}$. Then $\mu_C(r) \geq s$. Also, $\nu_{C\chi_M}(r) \in \{0, \beta\}$. Thus $\nu_C(r) \leq t$. So $r_{(s,t)} \in C$.

Case(ii). When $\alpha \geq p$ and $\beta \leq q$. Similarly, we have

$$\mu_{C\chi_M}(x) = \vee\{\mu_C(a') : a' \in R, x \in a'M\} \text{ and } \nu_{C\chi_M}(x) = \wedge\{\nu_C(a') : a' \in R, x \in a'M\}.$$

Then $\mu_{C\chi_M}(x) \in \{1, \alpha\}$ and $\nu_{C\chi_M}(x) \in \{0, \beta\}$. Thus we get

$$\mu_{C\chi_M}(x) \geq p \text{ and } \nu_{C\chi_M}(x) \leq q.$$

This implies $x_{(p,q)} \in C\chi_M$. So we may assume that $K \neq \varnothing$. Then there exist $b \in C_*$ such that $rx \in bM$. Thus we have

$$\mu_{C\chi_M}(rx) = \vee\{\mu_C(a) : a \in R, rx \in aM\} = 1,$$

$$\nu_{C\chi_M}(rx) = \wedge\{\nu_C(a) : a \in R, rx \in aM\} = 0,$$

$$rx \in bM \subseteq C_*M.$$

It follows from ([6], Lemma 2.10) and Theorem 3.12 that $r \in C_*M$ or $x \in C_*M$.

If $r \in C_*M$, then $\mu_{C_*} = 1 \geq s$ and $\nu_{C_*} = 0 \leq t$. Thus $r_{(s,t)} \in C$.

If $x \in C_*M$, then $x = \sum_{i=1}^{n} r_i x_i$ for some $r_i \in C_*$ and $x_i \in M$ for $i = 1, 2, ..., n$. Thus we get

$$\mu_{C\chi_M}(rx) = \mu_{C\chi_M}(\sum_{i=1}^{n} r_i x_i) \geq \wedge_{i=1}^{n} \mu_{C\chi_M}(r_i x_i) \geq 1 \geq p.$$

Similarly, $\nu_{C\chi_M}(rx) \leq 0 \leq q$ implies that $x_{(p,q)} \in C\chi_M$. So the proof is complete. □

Proposition 4.9. *Assume that M is a faithful intuitionistic fuzzy multiplication R-module and let C be an intuitionistic fuzzy ideal of R. If E is an intuitionistic fuzzy ideal of R such that $E\chi_M \subseteq C\chi_M$ and $C\chi_M \neq \chi_M$, then $E \subseteq C$. In particular, $(C\chi_M : \chi_M) = C$.*

Proof. Let $r \in R$ and $\mu_E(r) = \alpha, \nu_E(r) = \beta$ for some $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$. Then $r_{(\alpha,\beta)} \in E$ and there exist $m \in M$ such that $\mu_{C\chi_M}(m) < 1$ and $\nu_{C\chi_M}(m) > 0$, since $C\chi_M \neq \chi_M$. Thus $m_{(1,0)} \notin C\chi_M$. So $r_{(\alpha,\beta)}m_{(1,0)} \in E\chi_M \subseteq C\chi_M$. By Proposition 4.8, $r_{(\alpha,\beta)} \in C$. Hence $\mu_C(r) \geq \alpha = \mu_E(r)$ and $\nu_C(r) \leq \beta = \nu_E(r)$. Therefore $E \subseteq C$. The particular statement is clear. □

The Proposition 4.9 can be restated thus: If M be a faithful intuitionistic fuzzy multiplication module and C be an intuitionistic fuzzy prime ideal of R such that $\chi_M \neq C\chi_M$, then $C\chi_M$ is an intuitionistic fuzzy prime submodule of M. Thus in view of Propositions 4.8, 4.9 and Corollary 3.17 in [19], we have the following theorem

Theorem 4.10. *The following statements are equivalent for a non-constant IFSM A of an intuitionistic fuzzy multiplication module M:*

(1) *A is an intuitionistic fuzzy prime submodule of M,*
(2) *$(A : \chi_M)$ is an intuitionistic fuzzy prime ideal of R,*
(3) *$A = C\chi_M$ for some intuitionistic fuzzy prime ideal C of R.*

Proposition 4.11. *Let M be an R-module. If A is an IFSM of M, then $\sqrt{(A : \chi_M)}\chi_M \subseteq IFrad_{\chi_M}(A)$.*

Proof. If $IFrad_{\chi_M}(A) = \chi_M$, then the result is clear. Otherwise, if $B \in P(A)$, then we have $(A : \chi_M) \subseteq (B : \chi_M)$. By Corollary 3.17 in [19], $\sqrt{(A : \chi_M)} \subseteq (B : \chi_M)$. Thus $\sqrt{(A : \chi_M)}\chi_M \subseteq (B : \chi_M)\chi_M \subseteq B$. Since B is an arbitrary intuitionistic fuzzy prime submodule of M containing A, we have $\sqrt{(A : \chi_M)}\chi_M \subseteq IFrad_{\chi_M}(A)$. □

Lemma 4.12. *Let A, B be IFSMs of an R-module M. If E is an intuitionistic fuzzy ideal of R, then*

(1) *$(A_* : M) \subseteq (A : \chi_M)_*$,*
(2) *$E_*A_* \subseteq (EA)_*$.*

Proof. (1) Let $r \in (A_* : M)$. Then $rM \subseteq A_* \subseteq A_{(s,t)}$ for every $s,\ t \in [0,1]$ such that $s + t \leq 1$. Thus we have

$$\begin{aligned}\mu_{(A:\chi_M)}(r) &= \vee\{a_{(s,t)}(r) : a \in R \text{ such that } a \in (A_{(s,t)} : \chi_M)\}\\ &\geq \vee\{r_{(s,t)}(r) = s, s \in [0,1]\}\\ &= 1.\end{aligned}$$

Similarly, $\nu_{(A:\chi_M)}(r) \leq 0$ implies that $r \in (A : \chi_M)_*$. So $(A_* : M) \subseteq (A : \chi_M)_*$.

(2) Let $m \in E_*A_*$. Then $m = \sum_{i=1}^{n} r_i m_i$ for some $r_i \in E_*$ and $x_i \in A_*$, $i = 1, 2, ..., n$. Thus we get

$$\begin{aligned}\mu_{EA}(m) &= \mu_{EA}(\sum_{i=1}^{n} r_i m_i)\\ &\geq \wedge_{i=1}^{n}\mu_{EA}(r_i m_i)\\ &\geq \wedge_{i=1}^{n} 1 = 1.\end{aligned}$$

Similarly, we can show that $\nu_{EA}(m) \leq 0$. This implies that $m \in (EA)_*$. So we have $E_*A_* \subseteq (EA)_*$. □

Now, we have the following theorem that is a generalization of Theorem 2.12 in [6].

Theorem 4.13. *Let M be a faithful intuitionistic fuzzy multiplication R-module. If A is an IFSM of M, then $IFrad_{\chi_M}(A) = \sqrt{(A : \chi_M)}\chi_M$.*

Proof. By Proposition 4.11, it suffices to show that $IFrad_{\chi_M}(A) \subseteq \sqrt{(A:\chi_M)}\chi_M$. Since M is an intuitionistic fuzzy multiplication module, we must have

$$IFrad_{\chi_M}(A) = (IFrad_{\chi_M}(A) : \chi_M)\chi_M.$$

Now it is enough to show that $(IFrad_{\chi_M}(A) : \chi_M) \subseteq \sqrt{(A:\chi_M)}$. Let C be an intuitionistic fuzzy prime ideal of R containing $(A : \chi_M)$. By Proposition 3.5 in [12], C_* is a prime ideal of R. By Lemma 4.12, we have $(0 : M) \subseteq (A_* : M) \subseteq (A : \chi_M)_* \subseteq C_*$. By Corollary 2.11 in [6], C_*M is a prime submodule of M. Then $C_*M \neq M$. By Lemma 4.12, $C\chi_M \neq \chi_M$. By Theorem 4.9, $C\chi_M$ is an intuitionistic fuzzy prime submodule of M. Thus $A = (A : \chi_M)\chi_M \subseteq C\chi_M$ and $IFrad_{\chi_M}(A) \subseteq C\chi_M$. So $(IFrad_{\chi_M}(A) : \chi_M)\chi_M \subseteq C\chi_M$. It then follows from Proposition 4.10 that $(IFrad_{\chi_M}(A) : \chi_M) \subseteq C \subseteq \sqrt{(A:\chi_M)}$. Hence we have the equality. □

Conclusion

Letting $\mathcal{C}^*(M) = \{V(C\chi_M) : C \in IFI(R))\}$. Then in [21] it has been shown that $\mathcal{C}^*(M)$ induces a topology which is called Zariski topology if and only if M is a top module. By following these we define intuitionistic fuzzy multiplication R-modules and we show that every intuitionistic fuzzy multiplication R-module is a top module. Also we find a connection between the intuitionistic fuzzy multiplication R-modules and the multiplication R-modules.

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