Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

IVI-octahedron ideals and bi-ideals in semigroups

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Received 14 February 2022; Revised 9 March 2022; Accepted 22 March 2022

Abstract. In this paper, we introduce the notions of IVI-octahedron ideals and bi-ideals, and obtain some of their properties. Also, we define an IVI-octahedron duo semigroup and give a characterization of a duo semigroup by it, and study some of its properties. Furthermore, we discuss some characterizations of a regular semigroup and a left [resp. right] regular semigroup by IVI-octahedron ideals and bi-ideals.

2020 AMS Classification: 03B52, 08A72, 20M12, 20M17

Keywords: IVI-octahedron ideal, IVI-octahedron bi-ideal, Level set, IVI-octahedron duo semigroup, Regular semigroup, Left [resp. right] regular semigroup.

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1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets as the generalization of crisp sets in order to solve the real world problems involving ambiguities and uncertainties. Rosenfeld [2] applied initially it to basic theory of groupoids and groups. After then, Das [3] defined a level subgroup and obtained a chracterization of a fuzzy group by its level subgroup. Liu [4] studied some properties of fuzzy invariant subgroups and fuzzy ideals. Mukherjee and Sen [5] investigated various properties of fuzzy ideals of a ring. In particular, Kuroki [6] dealt with properties of fuzzy ideals and bi-ideals in a semigroup. Also he discussed the characterizations of some special semigroups by fuzzy ideals and bi-ideals in [7]. Recently, Wattanatripop et al. [8] defined an almost bi-ideal in a semigroup and obtained some of its properties. Refer to [9, 10, 11, 12, 13] for further researches with respect to fuzzy ideals and fuzzy bi-ideals in semigroups.

In 1989, Biswas [14] applied the notion of intuitionistic fuzzy sets proposed by Atanassov [15] as a generalization of a fuzzy set to a group theory. After that time, Hur et al. [16] defined an intuitionistic fuzzy subgroupoid and an intuitionistic fuzzy ideal, and obtained their basic properties. Also, Hur et al. [17] studied various properties of intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings. Banerjee and Basnet [18] dealt with some properties of a group and an ideal based on intuitionistic fuzzy sets. In particular, Kim and Jun [19] introduced the concept of intuitionistic fuzzy ideals in a semigroup and investigated some of its properties. Kim and Lee [20] defined an intuitionistic fuzzy bi-ideal in a semigroup and had its various properties. Hur et al. [21] dealt with characterizations of some special semigroups by intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals. Refer to [22, 23, 24] for researches with respect to intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals.

In 1989, Atanassov and Gargov [25] proposed the concept of interval-valued intuitionistic fuzzy sets as a generalization of an intuitionistic fuzzy set. Yaqoob [26] applied it to ideals based on interval-valued intuitionistic fuzzy set in a regular LA-semigroup and obtained some of its properties. Abdullah et al. [27] defined an interval-valued (α, β)-intuitionistic fuzzy bi-ideal in a semigroup and investigated some of its properties. On the other hand, Krishnaswamy et al. [28] studied basic properties of interval-valued intuitionistic fuzzy bi-ideals in a ternary semiring. Also, Balasubramanian and Raja [29] dealt with properties of interval-valued intuitionistic Q-fuzzy k-ideals in a ternary semiring. Moreover, K. Arulmozhi1 et al. [30] introduced the concept of weak bi-ideals in a Γ near-ring and dealt with some of its properties.

Recently, Kim et al. [31] introduced the notion of IVI-octahedron sets and applied it to groupoid theory. The purpose of our study is not only to find the properties ideals and bi-ideals based on IVI-octahedron sets in semigroups, but also to obtain some characterizations of a regular semigroup by them. To accomplish this, our paper structured as follows: In Section 2, we list some definitions needed next sections. In Section 3, we define an IVI-octahedron ideal, and give characterizations of an IVI-octahedron ideal by the IVI-octahedron products and the level sets respectively (See Theorems 3.20 and 3.23 respectively). In Section 4, we define an IVI-octahedron bi-ideal, and obtain the characterizations of an IVI-octahedron bi-ideal by the IVI-octahedron products and the level sets respectively (See Theorems 4.9 and 4.15 respectively). In Section 5, we introduce the notion of IVI-octahedron duo semigroups and give a characterization of a duo semigroup by IVI-octahedron duo semigroups (See Theorem 5.6). Also, we deal with some relations between IVI-octahedron ideals and IVI-octahedron bi-ideals. In Section 6, We discuss with the characterizations of a regular semigroup by IVI-octahedron ideals and IVI-octahedron bi-ideals, and a characterization of a left [resp. right and completely] regular semigroup by IVI-octahedron left ideals [resp. IVI-octahedron right ideals and IVI-octahedron bi-ideals].

2. Preliminaries

In this section, we list some basic definitions needed in the next sections.

Let $I = [0, 1]$ and let X be a nonempty set. Then a mapping $A : X \to I$ is called a *fuzzy set* in X (See [1]). $\mathbf{0}$ and $\mathbf{1}$ denote the *fuzzy empty set* and the *fuzzy whole set* in X defined by: for each $x \in X$,

$$\mathbf{0}(x) = 0 \text{ and } \mathbf{1}(x) = 1.$$

Each member of a set $I \oplus I = \{(a^{\in}, a^{\notin}) : (a^{\in}, a^{\notin}) \in I \times I$ and $a^{\in} + a^{\notin} \leq 1\}$ is called an *intuitionistic fuzzy number* (See [32]). We denote intuitionistic fuzzy numbers $(a^{\in}, a^{\notin})$, $(b^{\in}, b^{\notin})$, $(c^{\in}, c^{\notin})$, etc. as $\bar{a}$, $\bar{b}$, $\bar{c}$, etc. In particular, $\bar{0} = (0, 1)$ and $\bar{1} = (1, 0)$. It is well-known (Theorem 2.1 in [32]) that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying DeMorgan's laws. For a nonempty set X, a mapping $\bar{A} = (A^{\in}, A^{\notin}) : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X (See [15]). $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

Let us denote the set of all IFSs in X as $IFS(X)$. Moreover, see [15] for the definitions of the inclusion, equality, intersection, union of two IFSs and the complement of an IFS, and operators $[\ ]\bar{A}$, $\diamond\bar{A}$ for each $\bar{A} \in IFS(X)$.

Let $[I] = \{\widetilde{a} = [a^{-},^{+}] \subset I : 0 \leq a^{-} \leq a^{+} \leq 1\}$ be the set of all closed subintervals of I. Then each member of $[I]$ are called *interval-valued fuzzy numbers* (See [33]). For a nonempty set X, a mapping $\widetilde{A} = [A^{-}, A^{+}] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVFS) in X (See [34]). $\widetilde{\mathbf{0}}$ and $\widetilde{\mathbf{1}}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X defined by: for each $x \in X$,

$$\widetilde{\mathbf{0}}(x) = [0, 0] \text{ and } \widetilde{\mathbf{1}}(x) = [1, 1].$$

We denote the set of all IVFSs in X as $IVFS(X)$. Furthermore, see [34] for the definitions of the inclusion, equality, intersection, union of two IVFSs and the complement of an IVFS.

Let $[I] \oplus [I] = \{\widetilde{\widetilde{a}} = (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) \in [I] \times [I] : a^{\in,+} + a^{\notin,+} \leq 1\}$. Then each member of $[I] \oplus [I]$ is called an *interval-valued intuitionistic fuzzy number* (briefly, IVIFN) (See [31]). In particular, $\widetilde{\widetilde{0}} = ([0, 0], [1, 1])$ and $\widetilde{\widetilde{1}} = ([1, 1], [0, 0])$. Also, see [31] for the definitions of the order, the equality, the inf and the sup of two IVIFNs. For a nonempty set X, a mapping $\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin}) : X \to [I] \oplus [I]$ is called an *interval-valued intuitionistic fuzzy set* (briefly, IVIS) in X (See [25]). $\widetilde{\widetilde{\mathbf{0}}}$ [resp. $\widetilde{\widetilde{\mathbf{1}}}$] denotes the *interval-valued intuitionistic fuzzy empty set* [resp. the *interval-valued intuitionistic fuzzy whole set*] in X defined by: for each $x \in X$,

$$\widetilde{\widetilde{\mathbf{0}}}(x) = ([0, 0], [1, 1]) \text{ [resp. } \widetilde{\widetilde{\mathbf{1}}}(x) = ([1,], [0, 0])].$$

We denote the set of all IVISs as $IVIFS(X)$.

Members of $([I] \oplus [I]) \times (I \oplus I) \times I$ are called *interval-valued intuitionistic fuzzy octahedron numbers* (briefly, IVI-octahedron numbers) and we write them as

$$\widetilde{\widetilde{\bar{a}}} = \left\langle \widetilde{\widetilde{a}}, \bar{a}, a \right\rangle, \ \widetilde{\widetilde{\bar{b}}} = \left\langle \widetilde{\widetilde{b}}, \bar{b}, b \right\rangle, \text{ etc. (See [31]).}$$

Moreover, see [31] for the definitions of the order, the equality, the inf and the sup of IVI-octahedron numbers.

Definition 2.1 ([31])**.** Let X be a nonempty set. Then a mapping $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle$: $X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ is called an *IVI-octahedron set* (briefly, IVIOS) in X. and let $\widetilde{\widetilde{A}} \in IVIFS(X),\ \bar{A} \in IFS(X),\ A \in I^X$. $\ddot{\mathbf{0}}$ and $\ddot{\mathbf{1}}$ denote the *IVI-octahedron empty set* and *IVI-octahedron whole set* in X defined by: for each $x \in X$,

$$\ddot{\mathbf{0}}(x) = \langle([0,0],[1,1],(0,1),0\rangle \text{ and } \ddot{\mathbf{1}}(x) = \langle([1,1],[0,0],(1,0),1\rangle .$$

We denote the set of all IVIOSs as $IVIOS(X)$.

It is clear that for each $A \in 2^X$, $\chi_{\mathcal{A}} = \left\langle \widetilde{\widetilde{\chi_A}}, \bar{\chi_A}, \chi_A \right\rangle \in IVIOS(X)$ and then $2^X \subset IVIOS(X)$, where $\widetilde{\widetilde{\chi_A}} = ([\chi_A, \chi_A], [\chi_{A^c}, \chi_{A^c}]) \in IVIFS(X)$, $\bar{\chi_A} = (\chi_A, \chi_{A^c}) \in IFS(X)$, 2^X denotes the set of all subsets of X and χ_A denotes the characteristic function of A.

Definition 2.2 ([31])**.** Let X be a nonempty set, let $\mathcal{A},\ \mathcal{B} \in IVIOS(X)$ and let $(\mathcal{A}_j)_{j\in J}$ be a family of IVIOSs in X. Then the *inclusion*, the *equality* between $\mathcal{A}$ and $\mathcal{B}$, the *union* and the *intersection* of $(\mathcal{A}_j)_{j\in J}$, the *complement* of $\mathcal{A}$, operators $[\ ]$ and $\diamond$ of $\mathcal{A}$ are defined as follows respectively:

(i) (The equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \widetilde{\widetilde{A}} = \widetilde{\widetilde{B}},\ \bar{A} = \bar{B},\ A = B$.

(ii) (The inclusion) $\mathcal{A} \subset \mathcal{B} \Leftrightarrow \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{B}},\ \bar{A} \subset \bar{B},\ A \leq B$.

(iii) (The union) $\bigcup_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \widetilde{\widetilde{A}}_j, \bigcup_{j\in J} \bar{A}_j, \bigcup_{j\in J} A_j \right\rangle$.

(iv) (The intersection) $\bigcap_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \widetilde{\widetilde{A}}_j, \bigcap_{j\in J} \bar{A}_j, \bigcap_{j\in J} A_j \right\rangle$.

(v) (The complement) $\mathcal{A}^c = \left\langle \widetilde{\widetilde{A}}^c, \bar{A}^c, A^c \right\rangle$.

(vi) $[\ ]\mathcal{A} = \left\langle [\ ]\widetilde{\widetilde{A}}, [\ ]\bar{A}, A \right\rangle$,

where $[\ ]\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, [A^{\notin,-}, 1 - A^{\in,+}])$ [resp. $[\ ]\bar{A} = (A^{\in}, 1 - A^{\in})$] (See [25] [resp. [15]]).

(vii) $\diamond\mathcal{A} = \left\langle \diamond\widetilde{\widetilde{A}}, \diamond\bar{A}, A \right\rangle$,

where $\diamond\widetilde{\widetilde{A}} = ([A^{\in,-}, 1 - A^{\notin,+}], \widetilde{A}^{\notin})$ [resp. $\diamond\bar{A} = (1 - A^{\notin}, A^{\notin})$] (See [25] [resp. [15]]).

3. IVI-octahedron subsemigroups and ideals

In this section, we introduce the concepts of IVI-octahedron subsemigroups [resp. left ideals, right ideals and ideals] of a semigroup S and deal with its characterizations respectively.

Let S be a semigroup and let $\varnothing \neq A \in 2^S$. Then

(i) A is called a *subsemigroup* of S, if $A^2 \subset A$,

(ii) A is called a *left ideal* [resp. *right ideal*] of S, if $SA \subset A$ [resp. $AS \subset A$],

(iii) A is called a *two-sided ideal* (briefly, *ideal*) of S, if it is both a left and a right ideal of S.

We will denote the set of all left ideals [resp. right ideals and ideals] of S as $LI(S)$ [resp. $RI(S)$ and $I(S)$].

Throughout this section and next section, unless otherwise noted, let us S denote a semigroup. From Proposition 6.13 in [31], we have the following definition.

Definition 3.1. Let $\ddot{\mathbf{0}} \neq \mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOS(S)$. Then $\mathcal{A}$ is called an *IVI-octahedron subsemigroup* (briefly, IVIOSG) of S, if it satisfies the following condition:

$$\text{for any } x,\ y \in X,\ \mathcal{A}(xy) \geq \mathcal{A}(x) \wedge \mathcal{A}(y), \quad \text{i.e.,}$$

(i) $A^{\in,-}(xy) \geq A^{\in,-}(x) \wedge A^{\in,-}(y),\ A^{\in,+}(xy) \geq A^{\in,+}(x) \wedge A^{\in,+}(y)$,
$A^{\notin,-}(xy) \leq A^{\notin,-}(x) \vee A^{\notin,-}(y),\ A^{\notin,+}(xy) \leq A^{\notin,+}(x) \vee A^{\notin,+}(y)$,
(ii) $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y)$,
(iii) $A(xy) \geq A(x) \wedge A(y)$.
We will denote the set of all IVIOSGs of S as $IVIOSG(S)$.

Remark 3.2. (1) Let $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle \in IVIOS(S)$ and let $IVIFSG(S)$ [resp. $IFSG(S)$, $FSG(S)$] denote the set of all interval-valued intuitionistic fuzzy [resp. intuitionistix fuzzy and fuzzy] subsemigroups of S (See [26] [resp. [16] and [2]] for the definition of an interval-valued intuiionistic fuzzy [resp. an intuitionistic fuzzy and a fuzzy] semigroup). Then from Definition 3.1, we can easily see that the following holds:

$\mathcal{A} \in IVIOSG(S)$ if and only if $\widetilde{\widetilde{A}} \in IVIFSG(S)$, $\bar{A} \in IFSG(S)$ and $A \in FSG(S)$.

Furthermore, if $\mathcal{A} \in IVIOSG(S)$, then $[\ ]\mathcal{A},\ \diamond\mathcal{A} \in IVIOSG(S)$.
(2) If $A \in FSG(S)$, then $\langle([A, A], [A^c, A^c]), (A, A^c), A\rangle \in IVIOSG(S)$.
(3) If $\bar{A} \in IFSG(S)$, then clearly, we have

$$\langle([A^{\in}, A^{\in}], [A^{\notin}, A^{\notin}]), \bar{A}, A\rangle,\ \left\langle([A^{\in}, A^{\in}], [A^{\notin}, A^{\notin}]), \bar{A}, A^{\notin^c}\right\rangle \in IVIOSG(S).$$

(4) If $\widetilde{\widetilde{A}} \in IVIFSG(S)$, then we get

$$\left\langle \widetilde{\widetilde{A}}, (A^{\in,-}, A^{\notin,-}), A^{\in,-}\right\rangle,\ \left\langle \widetilde{\widetilde{A}}, (A^{\in,+}, A^{\notin,+}), A^{\in,+}\right\rangle \in IVIOSG(S).$$

Example 3.3. Let $S = \{1, 2, 3\}$ be the semigroup with the following Cayley table:

$\cdot$	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

Table 3.1

Consider the mapping $\mathcal{A} : S \to ([I] \oplus [I]) \times (I \oplus I) \times I$ defined as follows:

$$\mathcal{A}(1) = \widetilde{\widetilde{\bar{a}}},\ \mathcal{A}(2) = \widetilde{\widetilde{\bar{b}}},\ \mathcal{A}(3) = \widetilde{\widetilde{\bar{c}}},$$

where $\widetilde{\widetilde{\bar{a}}}$, $\widetilde{\widetilde{\bar{a}}}$ and $\widetilde{\widetilde{\bar{a}}}$ are arbitrary IVI-octahedron numbers.
Then we can easily check that $\widetilde{\widetilde{A}} \in IVIFSG(S)$, $\bar{A} \in IFSG(S)$, $A \in FSG(S)$. Thus by Remark 3.2, $\mathcal{A} \in IVIOSG(S)$.

Theorem 3.4 (See Remark 6.21 [31]). *Let $\varnothing \neq A \in 2^S$. Then $\chi_{\mathcal{A}} \in IVIOSG(S)$ if and only if A is a subsemigroup of S.*

Proof. Straightforward. □

Definition 3.5 ([31]). Let $\ddot{\mathbf{0}} \neq \mathcal{A} \in IVIOS(S)$. Then $\mathcal{A}$ is called an:

(i) *IVI-octahedron left ideal* (briefly, IVIOLI) of S, if for any $x,\ y \in S$,

$$\mathcal{A}(xy) \geq \mathcal{A}(y), \text{ i.e.,}$$

$$\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(y),\ \widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(y),\ A^{\in}(xy) \geq A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(y),\ A(xy) \geq A(y),$$

(ii) *IVI-octahedron right ideal* (briefly, IVIORI) of S, if for any $x,\ y \in S$,

$$\mathcal{A}(xy) \geq \mathcal{A}(x),$$

$$\widetilde{A}^{\in}(xy) \geq \widetilde{A}^{\in}(x),\ \widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(x),\ A^{\in}(xy) \geq A^{\in}(x),\ A^{\notin}(xy) \leq A^{\notin}(x),\ A(xy) \geq A(x),$$

(iii) *IVI-octahedron ideal* (briefly, *i*-IVIOI) of S, if it is both an IVIOLI and an IVIORI of S.

In this case, we will denote the set of all IVIOIs [resp. IVIOLIs and IVIORIs] of S as $IVIOI(S)$ [resp. $IVIOLI(S)$ and $IVIORI(S)$].

For a semigroup S, let us denote the set of all fuzzy ideals [resp. left ideals and right ideals] (See [2]), the set of all intuitionistic fuzzy ideals [resp. intuitionistic fuzzy left and right ideals] (See [16]) and the set of all interval-valued intuitionistic fuzzy ideals [resp. interval-valued intuitionistic fuzzy left and right ideals] (See [26]) of S as $FI(S)$ [resp. $FLI(S)$ and $FRL(S)$], $IFI(S)$ [resp. $IFLI(S)$ and $IFRI(S)$] and $IVIFI(S)$ [resp. $IVIFLI(S)$ and $IVIFRI(S)$].

Remark 3.6. From Definition 3.5, we have the followings:

(1) $\mathcal{A} \in IVIOLI(S) \Longleftrightarrow \widetilde{\widetilde{A}} \in IVIFLI(S),\ \bar{A} \in IFLI(S),\ A \in FLI(S)$,

(2) $\mathcal{A} \in IVIORI(S) \Longleftrightarrow \widetilde{\widetilde{A}} \in IVIFRI(S),\ \bar{A} \in IFRI(S),\ A \in FRI(S)$,

(3) $\mathcal{A} \in IVIOI(S) \Longleftrightarrow \widetilde{\widetilde{A}} \in IVIFI(S),\ \bar{A} \in IFI(S),\ A \in FI(S)$.

Example 3.7. Let $\mathcal{A}$ be the IVI-octahedron subsemigroup of S given in Example 3.3. Then we can easily see that $\mathcal{A} \in IVIOLI(S)$. However, if $\mathcal{A}(1) \neq \mathcal{A}(2)$, $\mathcal{A}(1) \neq \mathcal{A}(3)$ or $\mathcal{A}(2) \neq \mathcal{A}(3)$, then clearly, $\mathcal{A} \notin IVIORI(S)$. Thus $\mathcal{A} \notin IVIOI(S)$. Moreover, if $\mathcal{A}(1) = \mathcal{A}(2) = \mathcal{A}(3)$, then $\mathcal{A} \in IVIOI(S)$.

Remark 3.8. (1) From Definitions 3.1 and 3.4, it follows that for each $\mathcal{A} \in IVIOI(S)$ [resp. $IVIOLI(S)$ and $IVIORI(S)$], $\mathcal{A} \in IVIOSG(S)$ but the converse does not hold in general (See Example 3.7).

(2) If $\mathcal{A} \in IVIOLI(S)$ [resp. $IVIORI(S)$ and $IVIOI(S)$], then $[\ ]\mathcal{A},\ \diamond\mathcal{A} \in IVIOLI(S)$ [resp. $IVIORI(S)$ and $IVIOI(S)$].

Theorem 3.9 (See Remark 6.26 [31]). *Let $\varnothing \neq A \in 2^S$. Then $\chi_{\mathcal{A}} \in IVIOLI_(S)$ [resp. $IVIORI(S)$ and $IVIOI(S)$] if and only if $A \in LI(S)$ [resp. $RI(S)$ and $I(S)$].*

Proof. Straightforward. □

Now we will list the product of two fuzzy [intuitionistic fuzzy, interval-valued intuitionistic fuzzy and IVI-octahedron] sets in a semigroup S.

Definition 3.10 (See [4]). Let $A,\ B \in I^S$. Then the *product* of A and B, denoted by $A \circ_F B$, is a fuzzy set in S defined as follows: for each $x \in S$,

$$(A \circ_F B)(x) = \begin{cases} \bigvee_{yz=x}[A(y) \wedge B(z)] & \text{if } yz = x \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.11 (See [16]). Let $\bar{A},\ \bar{B} \in IFS(S)$. Then the *product* of $\bar{A}$ and $\bar{B}$, denoted by $\bar{A} \circ_{IF} \bar{B}$, is an IFS in S defined as follows: for each $x \in S$,

$$(\bar{A} \circ_{IF} \bar{B})(x) = \begin{cases} (\bigvee_{yz=x}[A^{\in}(y) \wedge B^{\in}(z)], \bigwedge_{yz=x}[A^{\notin}(y) \vee B^{\notin}(z)] & \text{if } yz = x \\ (0,1) & \text{otherwise.} \end{cases}$$

Definition 3.12 (See [31]). Let $\widetilde{\widetilde{A}},\ \widetilde{\widetilde{B}} \in IVIS(S)$. Then the *product* of $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{A}}$, denoted by $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}$, is an IVIS in X defined as follows: for each $x \in S$,

$$(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}})(x) = \begin{cases} [\bigvee_{yz=x}[\widetilde{A}^{\in}(y) \wedge \widetilde{B}^{\in}(z)], \bigwedge_{yz=x}[\widetilde{A}^{\notin}(y) \vee \widetilde{B}^{\notin}(z)] & \text{if } yz = x \\ \widetilde{\widetilde{0}} & \text{otherwise,} \end{cases}$$

where $\widetilde{A}^{\in}(y) = [A^{\in,-}(y), A^{\in,+}(y)]$ and $\widetilde{A}^{\notin}(y) = [A^{\notin,-}(y), A^{\notin,+}(y)]$.

Definition 3.13 (See [31]). Let $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle,\ \mathcal{B} = \left\langle \widetilde{\widetilde{B}}, \bar{B}, B \right\rangle \in IVIOS(S)$. Then the *product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B}$, is an IVI-octahedron set in S defined as follows: for each $x \in S$,

$$(\mathcal{A} \circ \mathcal{B})(x) = \begin{cases} \bigvee_{yz=x}[\mathcal{A}(y) \wedge \mathcal{B}(z)] & \text{if } yz = x \\ \widetilde{\widetilde{\bar{0}}} & \text{otherwise.} \end{cases}$$

In fact, $\mathcal{A} \circ \mathcal{B} = \left\langle \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}}, \bar{A} \circ_{IF} \bar{B}, A \circ_F B \right\rangle$.

The following is an immediate consequence of Proposition 6.13 in [31].

Theorem 3.14 (See Proposition 6.13 [31]). *Let $\mathcal{A} \in IVIOS(S)$. Then*

$$\mathcal{A} \in IVIOSG(S) \text{ if and only if } \mathcal{A} \circ \mathcal{A} \subset \mathcal{A}.$$

Lemma 3.15. *Let $\widetilde{\widetilde{A}} \in IVIFS(S)$. Then*

$$\widetilde{\widetilde{A}} \in IVIFLI(S) \text{ if and only if } \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}.$$

Proof. Suppose $\widetilde{\widetilde{A}} \in IVIOLI(S)$ and let $a \in S$ such that $a = xy$ for some $x,\ y \in S$. Then we have

$$\begin{aligned}(\widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) &= \bigvee_{a=xy}[\widetilde{\mathbf{1}}^{\in}(x) \wedge \widetilde{A}^{\in}(y)] \\ &\leq \bigvee_{a=xy}[[1,1] \wedge \widetilde{A}^{\in}(xy)] \ [\text{Since } \widetilde{\widetilde{A}} \in IVIFLI(S)] \\ &= \bigvee_{a=xy}[[1,1] \wedge \widetilde{A}^{\in}(a)] \\ &= \widetilde{A}^{\in}(a).\end{aligned}$$

Similarly, we have $(\widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\notin}(a) \geq \widetilde{A}^{\notin}(a)$. Thus $\widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$.

Conversely, suppose the necessary condition holds, let $\widetilde{\widetilde{A}} \in IVIFS(S)$ and let $a \in S$ such that $a = xy$ for some $x,\ y \in S$. Then we get

$$\begin{aligned}\widetilde{A}^{\in}(xy) &= \widetilde{A}^{\in}(a)\\ &\geq (\widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) \text{ [Since } \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}]\\ &= \bigvee_{a=bc}[\widetilde{\mathbf{1}}^{\in}(b) \wedge \widetilde{A}^{\in}(c)]\\ &\geq \widetilde{\mathbf{1}}^{\in}(x) \wedge \widetilde{A}^{\in}(y) \text{ [Since } a = xy]\\ &= [1, 1] \wedge \widetilde{A}^{\in}(y)\\ &= \widetilde{A}^{\in}(y).\end{aligned}$$

Similarly, we get $\widetilde{A}^{\notin}(xy) \leq \widetilde{A}^{\notin}(y)$. Thus $\widetilde{\widetilde{A}} \in IVIFLI(S)$. □

From 3.15, 2.6 in [24] and 2.4 in [9], we get the following.

Theorem 3.16. *Let $\mathcal{A} \in IVIOS(S)$. Then*

$$\mathcal{A} \in IVIOLI(S) \text{ if and only if } \ddot{\mathbf{1}} \circ \mathcal{A} \subset \mathcal{A}.$$

The following is the dual of Lemma 3.15.

Lemma 3.17. *Let $\widetilde{\widetilde{A}} \in IVIFS(S)$. Then*

$$\widetilde{\widetilde{A}} \in IVIFRI(S) \text{ if and only if } \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \subset \widetilde{\widetilde{A}}.$$

From Lemmas 3.17, $2.7'$ in [24] and 2.5 in [9], we have the dual of Theorem 3.16.

Theorem 3.18. *Let $\mathcal{A} \in IVIOS(S)$. Then*

$$\mathcal{A} \in IVIORI(S) \text{ if and only if } \mathcal{A} \circ \ddot{\mathbf{1}} \subset \mathcal{A}.$$

The following is an immediate consequence of Lemmas 3.15 and 3.17.

Lemma 3.19. *Let $\widetilde{\widetilde{A}} \in IVIFS(S)$. Then*

$$\widetilde{\widetilde{A}} \in IVIFRI(S) \text{ if and only if } \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \text{ and } \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \subset \widetilde{\widetilde{A}}.$$

From Theorems 3.16 and 3.18, we get the following.

Theorem 3.20. *Let $\mathcal{A} \in IVIOS(S)$. Then*

$$\mathcal{A} \in IVIOI(S) \text{ if and only if } \ddot{\mathbf{1}} \circ \mathcal{A} \subset \mathcal{A} \text{ and } \mathcal{A} \circ \ddot{\mathbf{1}} \subset \mathcal{A}.$$

Definition 3.21 ([31])**.** Let X be a nonempty set, let $\widetilde{\widetilde{a}} \in ([I] \oplus [I]) \times (I \oplus I) \times I$ and let $\mathcal{A} =< \widetilde{\widetilde{A}}, \bar{A}, A >\in IVIOS(X)$. Then two subsets $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ and $[\mathcal{A}]^{*}_{\widetilde{\widetilde{a}}}$ of X are defined as follows:

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}} = \{x \in X : \widetilde{\widetilde{A}}(x) \geq \widetilde{\widetilde{a}},\ \bar{A}(x) \geq \bar{a},\ A(x) \geq a\}.$$

In this case, $[\mathcal{A}]_{\widetilde{\widetilde{a}}}$ is called an $\widetilde{\widetilde{a}}$*-level set* of $\mathcal{A}$.

From Remark 4.3 in [35], it is clear that

$$[\mathcal{A}]_{\widetilde{\widetilde{a}}} = [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \cap [\bar{A}]_{\bar{a}} \cap [A]_a,$$

where $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$, $[\bar{A}]_{\bar{a}}$ and $[A]_a$ denote the $\widetilde{\widetilde{a}}$-level set of $\widetilde{\widetilde{A}}$, the $\bar{a}$-level set of $\bar{A}$ and the a-level set of A (See [35], [16] and [3]).

Result 3.22 (Proposition 4.4 [35]). *Let X be a nonempty set, let $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$ be two IVI-octahedron numbers and let $\mathcal{A} \in IVIOS(X)$. If $\widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}}$, then $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{b}}} \subset [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$.*

Theorem 3.23. *Let $\mathcal{A} \in IVIOS(S)$. Then $\mathcal{A} \in IVIOSG(S)$* [resp. $IVIOI(S)$, $IVIOLI(S)$ and $IVIORI(S)$] *if and only if $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \in SG(S)$* [resp. $I(S)$, $LI(S)$ and $RI(S)$] *for each IVI-octahedron number $\widetilde{\widetilde{a}}$ such that $\widetilde{a} \neq \widetilde{\widetilde{0}}$, $\bar{a} \neq \bar{0}$ and $a \neq 0$, where $SG(S)$ denotes the set of all subsemigroups of S.*

Proof. The proof of the necessary condition is straightforward from 6.27 in [31].

Conversely, suppose the necessary condition holds. We prove the first part and the third part, and the remainder's proofs are omitted.

Suppose $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \in SG(S)$. Since $[\mathcal{A}]_{\widetilde{\widetilde{a}}} = [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \cap [\bar{A}]_{\bar{a}} \cap [A]_a$, $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$, $[\bar{A}]_{\bar{a}}$, $[A]_a \in SG(S)$. It is obvious that if $[\bar{A}]_{\bar{a}}$, then $\bar{A} \in IFSG(S)$ (See [23]). Then it is sufficient to show that if $[A]_a \in SG(S)$, then $A \in FSG(S)$ and if if $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in SG(S)$, then $\widetilde{\widetilde{A}} \in IVIFSG(S)$.

(i) Suppose $[A]_a \in SG(S)$ and for any $x,\ y \in S$, let $A(x) = a,\ A(y) = b$, where $a,\ b \in I$. Then clearly, $A(x) = a \geq a \wedge b,\ A(y) = b \geq a \wedge b$, where $a \wedge b \in I$. Thus $x,\ y \in [A]_{a \wedge b}$. Since $[A]_{a \wedge b} \in SG(S)$, $xy \in [A]_{a \wedge b}$. So $A(xy) \geq a \wedge b = A(x) \wedge A(y)$. Hence $A \in FSG(S)$.

(ii) Suppose $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in SG(S)$ and for any $x,\ y \in S$, let $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{a}},\ \widetilde{\widetilde{A}}(y) = \widetilde{\widetilde{b}}$, where $\widetilde{\widetilde{a}}$ and $\widetilde{\widetilde{b}}$ are interval-valued intuitionistic fuzzy numbers. Then we have

$$\widetilde{A}^{\in}(x) = \widetilde{a}^{\in} \geq \widetilde{a}^{\in} \wedge \widetilde{b}^{\in},\ \widetilde{A}^{\notin}(x) = \widetilde{a}^{\notin} \leq \widetilde{a}^{\notin} \vee \widetilde{b}^{\notin}$$

and

$$\widetilde{A}^{\in}(y) = \widetilde{b}^{\in} \geq \widetilde{a}^{\in} \wedge \widetilde{b}^{\in},\ \widetilde{A}^{\notin}(y) = \widetilde{b}^{\notin} \leq \widetilde{a}^{\notin} \vee \widetilde{b}^{\notin}.$$

Thus $x,\ y \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}}$. Since $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}} \in SG(S)$, $xy \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}}$. So we get

$$\widetilde{A}^{\in}(xy) \geq \widetilde{a}^{\in} \wedge \widetilde{b}^{\in} = \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y),\ \widetilde{A}^{\notin}(xy) \leq \widetilde{a}^{\notin} \vee \widetilde{b}^{\notin} = \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y).$$

Hence $\widetilde{\widetilde{A}} \in IVIFSG(S)$. Therefore by Remark 3.2 (1), $\mathcal{A} \in IVIOSG(S)$.

Suppose $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \in LI(S)$. It is well-known that if $[\bar{A}]_{\bar{a}}$, $[A]_a \in LI(S)$, then $\bar{A} \in IFLI(S)$, $A \in FLI(S)$ (See [23] and [11] respectively). Then it is sufficient to prove that $\widetilde{\widetilde{A}} \in IVIFLI(S)$. now suppose $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in LI(S)$ for each interval-valued intuitionistic fuzzy number $\widetilde{\widetilde{a}}$ and for each $y \in S$, let $\widetilde{\widetilde{A}} = \widetilde{\widetilde{a}}$. Then clearly, $y \in [\mathcal{A}]_{\widetilde{\widetilde{a}}}$. Let $x \in S$. Since $[\mathcal{A}]_{\widetilde{\widetilde{a}}} \in LI(S)$, $xy \in [\mathcal{A}]_{\widetilde{\widetilde{a}}}$. Then we have

$$\widetilde{A}^{\in}(xy) \geq \widetilde{a}^{\in} = \widetilde{A}^{\in}(y),\ \widetilde{A}^{\notin}(xy) \leq \widetilde{a}^{\notin} = \widetilde{A}^{\notin}(y).$$

Thus $\widetilde{\widetilde{A}} \in IVIFLI(S)$. So by by Remark 3.2 (1), $\mathcal{A} \in IVIOLI(S)$. □

4. IVI-octahedron bi-ideals

In this section, we define an IVI-octahedron bi-ideal and deal with some of its properties.

A subsemigroup A of a semigroup S is called a *bi-ideal* of S, if $ASA \subset A$. We will denote the set of all bi-ideals of S as $BI(S)$.

Definition 4.1 (See [13]). Let $A \in FSG(S)$. Then A is called a *fuzzy bi-ideal* (briefly, FBI) of S, if $A(xyz) \geq A(x) \wedge A(z)$ for any $x,\ y,\ z \in S$.

We will denote the set of all FBIs of S as $FBI(S)$.

Definition 4.2 ([23]). Let $\bar{A} \in IFSG(S)$. Then $\bar{A}$ is called an *intuitionistic fuzzy bi-ideal* (briefly, IFBI) of S, if it satisfies the following condition: for any $x,\ y,\ z \in S$,

$$\bar{A}(xyz) \geq A(x) \wedge \bar{A}(z),\ \text{i.e.,}\ A^{\in}(xyz) \geq A^{\in}(x) \wedge A^{\in}(z),\ A^{\notin}(xyz) \leq A^{\notin}(x) \vee A^{\notin}(z).$$

We will denote the set of all IFBIs of S as $IFBI(S)$.

Definition 4.3. Let $\widetilde{\widetilde{A}} \in IVIFSG(S)$. Then $\widetilde{\widetilde{A}}$ is called an *interval-valued intuitionistic fuzzy bi-ideal* (briefly, IVIFBI) of S, if it satisfies the following condition: for any $x,\ y,\ z \in S$,

$$\widetilde{\widetilde{A}}(xyz) \geq \widetilde{\widetilde{A}}(x) \wedge \widetilde{\widetilde{A}}(z),\ \text{i.e.,}\ \widetilde{A}^{\in}(xyz) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(z),\ \widetilde{A}^{\notin}(xyz) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(z).$$

We will denote the set of all IVIFBIs of S as $IVIFBI(S)$.

Definition 4.4. Let $\mathcal{A} \in IVIOSG(S)$. Then $\mathcal{A}$ is called an *IVI-octahedron bi-ideal* (briefly, IVIOBI) of S, if it satisfies the following condition: for any $x,\ y,\ z \in S$,

$$\mathcal{A}(xyz) \geq \mathcal{A}(x) \wedge \mathcal{A}(z),\ \text{i.e.,}$$

$$\widetilde{\widetilde{A}}(xyz) \geq \widetilde{\widetilde{A}}(x) \wedge \widetilde{\widetilde{A}}(z),\ \bar{A}(xyz) \geq \bar{A}(x) \wedge \bar{A}(z),\ A(xyz) \geq A(x) \wedge A(z).$$

We will denote the set of all IVIFBIs of S as $IVIFBI(S)$.

Remark 4.5. (1) From Definitions 4.1, 4.2, 4.3 and 4.4, it is obvious that for any $\mathcal{A} \in IVIOSG(S)$, $\mathcal{A} \in IVIOBI(S)$ if and only if $\widetilde{\widetilde{A}} \in IVIFBI(S)$, $\bar{A} \in IFBI(S)$ and $A \in FBI(S)$.

(2) If $A \in FBI(S)$, then $\langle([A, A], [A^c, A^c]), (A, A^c), A\rangle \in IVIOBI(S)$.

(3) If $\bar{A} \in IFBI(S)$, then we can easily see that

$$\left\langle([A^{\in}, A^{\in}], [A^{\notin}, A^{\notin}]), \bar{A}, A^{\in}\right\rangle,\ \left\langle([A^{\in}, A^{\in}], [A^{\notin}, A^{\notin}]), \bar{A}, A^{\notin^{c}}\right\rangle \in IVIOBI(S).$$

(4) If $\widetilde{\widetilde{A}} \in IVIFBI(S)$, then we can easily check that

$$\left\langle\widetilde{\widetilde{A}}, (A^{\in,-}, A^{\notin,-}), A^{\in,-}\right\rangle,\ \left\langle\widetilde{\widetilde{A}}, (A^{\in,+}, A^{\notin,+}), A^{\in,+}\right\rangle \in IVIOBI(S).$$

(5) If $\mathcal{A} \in IVIOBI(S)$, then $[\ ]\mathcal{A},\ \diamond\mathcal{A} \in IVIOBI(S)$.

Example 4.6. Let $S = \{1, 2, 3, 4\}$ be the semigroup with the following Cayley table:

Consider the mapping $\mathcal{A} : S \to ([I] \oplus [I]) \times (I \oplus I) \times I$ defined as follows:

$$\mathcal{A}(1) = \widetilde{\widetilde{\bar{a}}},\ \mathcal{A}(2) = \widetilde{\widetilde{\bar{b}}},\ \mathcal{A}(3) = \widetilde{\widetilde{\bar{c}}},\ \mathcal{A}(4) = \widetilde{\widetilde{\bar{d}}},$$

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2

Table 4.1

where $\widetilde{\widetilde{\bar{a}}}$, $\widetilde{\widetilde{\bar{b}}}$, $\widetilde{\widetilde{\bar{c}}}$, $\widetilde{\widetilde{\bar{d}}}$ are IVI-octahedron numbers such that $\widetilde{\widetilde{\bar{a}}} \geq \widetilde{\widetilde{\bar{b}}} \geq \widetilde{\widetilde{\bar{c}}}$ and $\widetilde{\widetilde{\bar{b}}} \geq \widetilde{\widetilde{\bar{d}}}$. Then clearly, $\mathcal{A} \in IVIOSG(S)$. Moreover, we can easily check that $\mathcal{A} \in IVIOBI(S)$ and thus $[\,]\mathcal{A},\ \diamond\mathcal{A} \in\in IVIOBI(S)$.

The following shows that the notion of an IVIOBI in a semigroup S is an one of a bi-ideal.

Theorem 4.7. *Let $\varnothing \neq A \in 2^S$. Then $A \in BI(S)$ if and only if $\chi_{\mathcal{A}} \in IVIOBI(S)$.*

Proof. It is well-known that $A \in BI(S)$ if and only if $\chi_A \in FBI(S)$ from Theorem 1 in [9] (Also, see Lemma 2.4 in [12]) and $A \in BI(S)$ if and only if $\bar{\chi_A} \in IFBI(S)$ from Proposition 2.5 in [23]. It is sufficient to prove that $A \in BI(S)$ if and only if $\widetilde{\widetilde{\chi_A}} \in IVIFBI(S)$.

Suppose $A \in BI(S)$. Then by Theorem 3.4, $\widetilde{\widetilde{\chi_A}} \in IVIFSG(S)$. Thus it is sufficient to show that for any $x,\ y,\ z \in S$,

$$\widetilde{\chi_A}^{\in}(xyz) = [\chi_A, \chi_A](xyz) \geq [\chi_A, \chi_A](x) \wedge [\chi_A, \chi_A](z), \tag{4.1}$$

$$\widetilde{\chi_A}^{\notin}(xyz) = [\chi_{A^c}, \chi_{A^c}](xyz) \leq [\chi_{A^c}, \chi_{A^c}](x) \vee [\chi_{A^c}, \chi_{A^c}](z). \tag{4.2}$$

Let $x,\ y,\ z \in S$. Then we have, $x,\ z \in A$ or $x \notin A$ or $z \notin A$.

Case (i) Suppose $x,\ z \in A$. Then clearly, $\chi_A(x) = \chi_A(z) = 1$ and $\chi_{A^c}(x) = \chi_{A^c}(z) = 0$. Since $A \in BI(S)$, $xyz \in ASA \subset A$. Thus we have

$$\chi_A(xyz) = 1 = \chi_A(x) \wedge \chi_A(z),\ \chi_{A^c}(xyz) = 0 = \chi_{A^c}(x) \wedge \chi_{A^c}(z).$$

Case (ii) Suppose $x \notin A$ or $z \notin A$. Then we get $\chi_A(x) = 0,\ \chi_{A^c}(x) = 1$ or $\chi_A(z) = 0,\ \chi_{A^c}(z) = 1$. Thus we have

$$\chi_A(xyz) \geq 0 = \chi_A(x) \wedge \chi_A(z),\ \chi_{A^c}(xyz) \geq 1 = \chi_{A^c}(x) \wedge \chi_{A^c}(z).$$

So in either cases, the inequalities (4.1) and (4.2) hold.

The proof of the converse is similar to one of Proposition 2.5 in [23]. This completes the proof. □

Lemma 4.8. *Let $\widetilde{\widetilde{A}} \in IVIFS(S)$. Then*

$$\widetilde{\widetilde{A}} \in IVIFBI(S) \text{ if and only if } \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}} \text{ and } \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}.$$

Proof. Suppose $\widetilde{\widetilde{A}} \in IVIFBI(S)$. Since $\widetilde{\widetilde{A}} \in IVIFSG(S)$, by Theorem 3.14, $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$. Let $a \in S$.

Suppose $(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})(a) = \widetilde{\widetilde{0}} = ([0,0],[1,1])$. Then clearly, we have

$$\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}.$$

Suppose $(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})(a) \neq \widetilde{\widetilde{0}}$. Then there are $x,\ y,\ p,\ q \in S$ such that

$$a = xy \text{ and } x = pq.$$

Thus we get

$$\begin{aligned}
(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) &= [(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}}) \circ_{IVI} \widetilde{\widetilde{A}}]^{\in}(a) \\
&= \bigvee_{a=xy}[(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(x) \wedge \widetilde{A}^{\in}(y)] \\
&= \bigvee_{a=xy}[(\bigvee_{x=pq}[\widetilde{A}^{\in}(p) \wedge \widetilde{\mathbf{1}}^{\in}(q)]) \wedge \widetilde{A}^{\in}(y)] \\
&= \bigvee_{a=xy}[(\bigvee_{x=pq}[\widetilde{A}^{\in}(p) \wedge [1,1]]) \wedge \widetilde{A}^{\in}(y)] \\
&= \bigvee_{a=xy}[\widetilde{A}^{\in}(p) \wedge \widetilde{A}^{\in}(y)] \\
&\leq \bigvee_{a=xy} \widetilde{A}^{\in}(pqy) \ [\text{Since } \widetilde{\widetilde{A}} \in IVIFSG(S)] \\
&= \widetilde{A}^{\in}(a),
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\notin}(a) &= [(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}}) \circ_{IVI} \widetilde{\widetilde{A}}]^{\notin}(a) \\
&= \bigwedge_{a=xy}[(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\notin}(x) \vee \widetilde{A}^{\notin}(y)] \\
&= \bigwedge_{a=xy}[(\bigwedge_{x=pq}[\widetilde{A}^{\notin}(p) \vee \widetilde{\mathbf{1}}^{\notin}(q)]) \vee \widetilde{A}^{\notin}(y)] \\
&= \bigwedge_{a=xy}[(\bigwedge_{x=pq}[\widetilde{A}^{\notin}(p) \vee [0,0]]) \vee \widetilde{A}^{\notin}(y)] \\
&= \bigwedge_{a=xy}[\widetilde{A}^{\notin}(p) \vee \widetilde{A}^{\notin}(y)] \\
&\geq \bigwedge_{a=xy} \widetilde{A}^{\notin}(pqy) \ [\text{Since } \widetilde{\widetilde{A}} \in IVIFSG(S)] \\
&= \widetilde{A}^{\notin}(a).
\end{aligned}$$

So $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$.

Conversely, suppose the necessary conditions hold. Since $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$, by Theorem 3.14, $\widetilde{\widetilde{A}} \in IVIFSG(S)$. Let $x,\ y,\ z \in S$ and let $a = xyz$. Then

$$\begin{aligned}
\widetilde{A}^{\in}(xyz) &= \widetilde{A}^{\in}(a) \\
&\geq (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) \ [\text{By the hypothesis}] \\
&= ((\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}}) \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) \\
&= \bigvee_{a=bc}[(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(b) \wedge \widetilde{A}^{\in}(c)] \\
&\geq (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(xy) \wedge \widetilde{A}^{\in}(z) \ [\text{Since } a = xyz] \\
&= (\bigvee_{xy=pq}[\widetilde{A}^{\in}(p) \wedge \widetilde{\mathbf{1}}(q)] \wedge \widetilde{A}^{\in}(z) \\
&\geq \widetilde{A}^{\in}(x) \wedge \widetilde{\mathbf{1}}(y) \wedge \widetilde{A}^{\in}(z) \\
&= \widetilde{A}^{\in}(x) \wedge [1,1] \wedge \widetilde{A}^{\in}(z) \\
&= \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(z).
\end{aligned}$$

Thus $\widetilde{A}^{\in}(xyz) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(z)$. Similarly, we have $\widetilde{A}^{\notin}(xyz) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(z)$. So $\widetilde{\widetilde{A}} \in IVIFBI(S)$. □

Theorem 4.9. *Let* $\mathcal{A} \in IVIOS(S)$. *Then*

$$\mathcal{A} \in IVIOBI(S) \text{ if and only if } \mathcal{A} \circ \mathcal{A} \subset \mathcal{A} \text{ and } \mathcal{A} \circ \ddot{\mathbf{1}} \circ \mathcal{A} \subset \mathcal{A}.$$

Proof. From Lemma 2.7 in [7], it is obvious that from lemma 2.7 the following holds:

(4.3) $\quad A \in FBI(S)$ if and only if $A \circ_F A \subset A$ and $\mathbf{1} \circ_F A \subset A$.

Also, from Lemma 4.8, we have

(4.4) $\quad \widetilde{\widetilde{A}} \in IVIFBI(S)$ if and only if $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$.

Then it is sufficient to show that the following hold:

(4.5) $\quad \bar{A} \in IFBI(S)$ if and only if $\bar{A} \circ_{IF} \bar{A} \subset \bar{A}$ and $\bar{A} \circ_{IF} \bar{\mathbf{1}} \circ_{IF} \bar{A} \subset \bar{A}$.

(4.5) can be proved similarly to Lemma 4.8. Thus from (4.3), (4.4), Lemma 4.8 and Remark 4.5 (1), the result holds. □

Lemma 4.10. *S is a group if and only if every IVIFBI of S is a constant mapping.*

Proof. Suppose S is a group with the identity e. Let $\widetilde{\widetilde{A}} \in IVIFBI(S)$ and let $a \in S$. Then we have

$$\begin{aligned}\widetilde{A}^{\in}(a) &= \widetilde{A}^{\in}(eae) \geq \widetilde{A}^{\in}(e) \wedge \widetilde{A}^{\in}(e) \text{ [Since } \widetilde{\widetilde{A}} \in IVIFBI(S)] \\ &= \widetilde{A}^{\in}(e) = \widetilde{A}^{\in}(ee) = \widetilde{A}^{\in}((aa^{-1})(a^{-1}a)) = \widetilde{A}^{\in}(a(a^{-1}a^{-1})a) \\ &\geq \widetilde{A}^{\in}(a) \wedge \widetilde{A}^{\in}(a) \text{ [Since } \widetilde{\widetilde{A}} \in IVIFBI(S)] \\ &= \widetilde{A}^{\in}(a).\end{aligned}$$

Similarly, we get $\widetilde{A}^{\notin}(a) = \widetilde{A}^{\notin}(e)$. Thus $\widetilde{\widetilde{A}}(a) = \widetilde{\widetilde{A}}(e)$. So $\widetilde{\widetilde{A}}$ is a constant mapping.

Conversely, suppose the necessary condition holds. Assume that S is not a group. Then it is well-known (84 pages in [36]) that S contains a proper bi-ideal A of S. Thus there is $x \in S$ such that $x \notin A$. Let $y \in A$ such that $y \neq x$. Since $A \in BI(S)$, by Theorem 4.7, $\widetilde{\widetilde{\chi_A}} \in IVIFBI(S)$. Then by the hypothesis, $\widetilde{\widetilde{\chi_A}}$ is a constant mapping. Thus we have

$$\widetilde{\widetilde{\chi_A}}(x) = \widetilde{\widetilde{\chi_A}}(y), \text{ i.e., } \widetilde{\chi_A}^{\in}(x) = \widetilde{\chi_A}^{\in}(y) \text{ and } \widetilde{\chi_A}^{\notin}(x) = \widetilde{\chi_A}^{\notin}(y).$$

Since $x \notin A$ and $y \in A$, we get

$$\widetilde{\chi_A}^{\in}(x) = \widetilde{0} = [0, 0] < [1, 1] = \widetilde{1} = \widetilde{\chi_A}^{\in}(y)$$

and

$$\widetilde{\chi_A}^{\notin}(x) = \widetilde{1} = [1, 1] > [0, 0] = \widetilde{0} = \widetilde{\chi_A}^{\notin}(y).$$

So $\widetilde{\widetilde{\chi_A}}(x) = \widetilde{\widetilde{0}} \neq \widetilde{\widetilde{1}} = \widetilde{\widetilde{\chi_A}}(y)$. This is a contradiction. Hence S is a group. □

From Remark 4.5 (1), Lemma 3.10, Proposition 2.6 in [23] and Theorem 2 in [9], we get the following.

Theorem 4.11. *S is a group if and only if every IVIOBI of S is a constant mapping.*

Lemma 4.12. *Every IVIFLI [resp. IVIFRI and IVIFI] of S is an IVIFBI of S.*

Proof. Let $\widetilde{\widetilde{A}} \in IVIFLI(S)$ and let $x,\ y,\ z \in S$. Then we get

$$\begin{aligned}\widetilde{A}^{\in}(xyz) &= \widetilde{A}^{\in}(((xy)z) \\ &\geq \widetilde{A}^{\in}(z) \text{ [Since } \widetilde{\widetilde{A}} \in IVIFLI(S)] \\ &\geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(z),\end{aligned}$$

$$\begin{aligned}\widetilde{A}^{\notin}(xyz) &= \widetilde{A}^{\notin}(((xy)z) \\ &\leq \widetilde{A}^{\notin}(z) \text{ [Since } \widetilde{\widetilde{A}} \in IVIFLI(S)]\end{aligned}$$

$$\leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(z).$$

Thus $\widetilde{\widetilde{A}} \in IVIFBI(S)$. Similarly, we can prove the remainders. □

From Lemma 4.12, Proposition 2.7 in [23] and Lemma 2.3 in [6], we have the following.

Proposition 4.13. *Every IVIOLI [resp. IVIORI and IVIOI] of S is an IVIOBI of S.*

Lemma 4.14. $\widetilde{\widetilde{A}} \in IVIS(S)$. *Then* $\widetilde{\widetilde{A}} \in IVIFBI(S)$ *if and only if* $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in BIS(S)$ *for each interval-valued intuitionistic fuzzy number* $\widetilde{\widetilde{a}}$.

Proof. Suppose $\widetilde{\widetilde{A}} \in IVIFBI(S)$ and let $\widetilde{\widetilde{a}}$ be any interval-valued intuitionistic fuzzy number. Then by the hypothesis and Theorem 3.23, $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in IVIFSG(S)$. Let $a \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} S [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Then there are $x,\ z \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$ and $y \in S$ such that $a = xyz$. Since $\widetilde{\widetilde{A}} \in IVIFBI(S)$, we have

$$\widetilde{A}^{\in}(a) \geq \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(z) \geq \widetilde{a}^{\in} \text{ and } \widetilde{A}^{\notin}(a) \leq \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(z) \leq \widetilde{a}^{\notin}.$$

Thus $a \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. So $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \subset [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}}$. Hence $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}}} \in BI(S)$.

Conversely, suppose the necessary condition holds. It is clear that $\widetilde{\widetilde{A}} \in IVIFSG(S)$. For any $x,\ z \in S$, let $\widetilde{\widetilde{A}}(x) = \widetilde{\widetilde{a}}$ and $\widetilde{\widetilde{A}}(z) = \widetilde{\widetilde{b}}$. Then by the process of the proof of the sufficient condition in Theorem 3.23, we get $x,\ z \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}}$. Let $y \in S$. Since $[\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}} \in BI(S)$, $xyz \in [\widetilde{\widetilde{A}}]_{\widetilde{\widetilde{a}} \wedge \widetilde{\widetilde{b}}}$. Then we have

$$\widetilde{A}^{\in}(xyz) \geq \widetilde{a}^{\in} \wedge \widetilde{a}^{\in} = \widetilde{A}^{\in}(x) \wedge \widetilde{A}^{\in}(y),$$
$$\widetilde{A}^{\notin}(xyz) \leq \widetilde{a}^{\notin} \vee \widetilde{a}^{\notin} = \widetilde{A}^{\notin}(x) \vee \widetilde{A}^{\notin}(y).$$

Thus $\widetilde{\widetilde{A}} \in IVIFBI(S)$. This completes the proof. □

From Lemma 4.14, Proposition 2.8 in [23] and Lemma 3.4 in [37], we obtain the following consequence.

Theorem 4.15. $\mathcal{A} \in IVIOS(S)$. *Then* $\mathcal{A} \in IVIOBI(S)$ *if and only if* $[\mathcal{A}]_{\widetilde{\widetilde{\widetilde{a}}}} \in BIS(S)$ *for each IVI-octahedron number* $\widetilde{\widetilde{\widetilde{a}}}$.

Lemma 4.16. $\widetilde{\widetilde{A}} \in IVIS(S)$ *and let* $\widetilde{\widetilde{B}} \in IVIFBI(S)$. *Then* $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}},\ \widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} \in IVIFBI(S)$.

Proof.
$$\begin{aligned}(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}) \circ_{IVI} (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}) &= \widetilde{\widetilde{A}} \circ_{IVI} [\widetilde{\widetilde{B}} \circ_{IVI} (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}})] \\ &\subset \widetilde{\widetilde{A}} \circ_{IVI} (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{B}}) \ [\text{Since} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{\mathbf{1}}}] \\ &\subset \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}. \ [\text{By Lemma 4.8}]\end{aligned}$$

Then by Theorem 3.14 and Remark 3.2 (1), $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}} \in IVIFSG(S)$. On the other hand,

$$(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}) \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}) = \widetilde{\widetilde{A}} \circ_{IVI} [\widetilde{\widetilde{B}} \circ_{IVI} (\widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}}) \circ_{IVI} \widetilde{\widetilde{B}}]$$

$$\subset \widetilde{\widetilde{A}} \circ_{IVI} (\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{B}})$$
$$[\text{Since } \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} = \widetilde{\widetilde{\mathbf{1}}}]$$
$$\subset \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}}. \text{ [By Lemma 4.8]}$$

Thus by Lemma 4.8, $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{B}} \in IVIBI(S)$. It can be proved a similar way that $\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{A}} \in IVIBI(S)$. □

From Lemma 4.16, Proposition 2.4 in [21] and Lemma 2.8 in [7], we get the following.

Proposition 4.17. *$\mathcal{A} \in IVIOS(S)$ and let $\mathcal{B} \in IVIOBI(S)$. Then $\mathcal{A} \circ \mathcal{A}$, $\mathcal{B} \circ \mathcal{A} \in IVIOBI(S)$.*

5. Duo semigroups

In this section, we define an IVI-octahedron duo and obtain some of its properties.

A semigroup S is said to be *left* [resp. *right*] *duo* (briefly, LD [resp. RD]), if every left [resp. right] ideal of S is an idealof S. A semigroup S is said to be *duo* (briefly, D], if it is both left and right duo (See [36]). A semigroup S is said to be *fuzzy left* [resp. *right*] *duo* (briefly, FLD [resp. FRD]), if every fuzzy left [resp. right] ideal of S is a fuzzy ideal of S and S is said to be *fuzzy duo* (briefly, FD), if it is both fuzzy left and fuzzy right duo (See [6]). A semigroup S is said to be *intuitionistic fuzzy left* [resp. *right*] *duo* (briefly, IFLD [resp. IFTD]), if every intuitionistic fuzzy left [resp. right] ideal of S is an intuitionistic fuzzy ideal of S and S is said to be *intuitionistic fuzzy duo* (briefly, IFD), if it is both intuitionistic fuzzy left and intuitionistic fuzzy right duo (See [23]). A semigroup S is said to be *regular*, if for each $a \in S$, there is $x \in S$ such that $a = axa$.

Now we have the similar definitions.

Definition 5.1. A semigroup S is said to be:

(i) *interval-valued intuitionistic fuzzy left duo* (briefly, IVIFLD), if every IVIFLI of S is an IVIFI of S,

(ii) *interval-valued intuitionistic fuzzy right duo* (briefly, IVIFRD), if every IVIFRI of S is an IVIFI of S,

(iii) *interval-valued intuitionistic fuzzy duo* (briefly, IVIFD), if it is both IVIFLD and IVIFRD.

Definition 5.2. A semigroup S is said to be:

(i) *IVI-octahedron left duo* (briefly, IVIOLD), if every IVIOLI of S is an IVIOI of S,

(ii) *IVI-octahedron right duo* (briefly, IVIORD), if every IVIORI of S is an IVIOI of S,

(iii) *IVI-octahedron duo* (briefly, IVIOD), if it is both IVIOLD and IVIORD.

Lemma 5.3. *Let S be a regular semigroup. Then S is LD if and only if S is IVIFLD.*

Proof. Suppose S is LD. Let $\widetilde{A} \in IVIFLI(S)$ and let $a,\ b \in S$. Since the left ideal Sa is an ideal of S and S is regular, we get

$$ab \in (aSa)b \subset (Sa)S \subset Sa.$$

Then there is $x \in S$ such that $ab = xa$. Since $\widetilde{\widetilde{A}} \in IVIFLI(S)$, we get

$$\widetilde{A}^{\in}(ab) = \widetilde{A}^{\in}(xa) \geq \widetilde{A}^{\in}(a) \text{ and } \widetilde{A}^{\notin}(ab) = \widetilde{A}^{\notin}(xa) \leq \widetilde{A}^{\notin}(a).$$

Thus $\widetilde{\widetilde{A}} \in IVIFRI(S)$. So $\widetilde{\widetilde{A}} \in IVIFI(S)$. Hence S is IVIFLD.

Conversely, suppose S is IVIFLD and let $A \in LI(S)$. Then by Remark 3.6 (1) and Theorem 3.9, $\widetilde{\widetilde{\chi_A}} \in IVIFLI(S)$. Thus by the hypothesis, $\widetilde{\widetilde{\chi_A}} \in IVIFI(S)$. Since $A \neq \varnothing$, by Remark 3.6 (3) and Theorem 3.9, $A \in I(S)$. So S is LD. □

From Lemma 5.3, Proposition 3.1 in [23] and Theorem 3.1 in [6], we have the following consequence.

Theorem 5.4. *Let S be a regular semigroup. Then S is LD if and only if S is IVIOLD.*

The following is the dual of Lemma 5.3.

Lemma 5.5. *Let S be a regular semigroup. Then S is RD if and only if S is IVIFRD.*

From Lemma 5.5, Proposition $3.1'$ in [23] and Theorem 3.2 in [6], we have the following consequence.

Theorem 5.6. *Let S be a regular semigroup. Then S is RD if and only if S is IVIORD.*

The following is an immediate consequence of Theorems 5.4 and 5.6.

Corollary 5.7. *Let S be a regular semigroup. Then S is D if and only if S is IVIOD.*

Lemma 5.8. *Let S be a regular semigroup. Then every bi-ideal of S is a right ideal of S if and only if every IVIFBI of S is an IVIFRI of S.*

Proof. Suppose every bi-ideal of S is a right ideal of S. Let $\widetilde{\widetilde{A}} \in IVIFBI(S)$ and let $a,\ b \in S$. Then clearly, $aSa \in BI(S)$. Thus by the hypothesis, $aSa \in RI(S)$. Since S is regular, we have

$$ab \in (aSa)S \subset aSa.$$

So there is $x \in S$ such that $ab = axa$. Since $\widetilde{\widetilde{A}} \in IVIFBI(S)$, we get

$$\widetilde{A}^{\in}(ab) = \widetilde{A}^{\in}(axa) \geq \widetilde{A}^{\in}(a) \wedge \widetilde{A}^{\in}(a) = \widetilde{A}^{\in}(a).$$

Similarly, we have $\widetilde{A}^{\notin}(ab) \leq \widetilde{A}^{\notin}(a)$. Hence $\widetilde{\widetilde{A}} \in IVIFRI(S)$.

Conversely, suppose the necessary condition holds and let $A \in BI(S)$. Then by Remark 4.5 (1) and Theorem 4.6, $\widetilde{\widetilde{\chi_A}} \in IVIFBI(S)$. Thus by the hypothesis, $\widetilde{\widetilde{\chi_A}} \in IVIFRI(S)$. Since $A \neq \varnothing$, by Remark 3.6 (2) and Theorem 3.9, $A \in RI(S)$. So the sufficient condition holds. □

From Lemma 5.8, Proposition 3.3 in [23] and Theorem 3.4 in [6], we obtain the following.

Theorem 5.9. *Let S be a regular semigroup. Then every bi-ideal of S is a right ideal of S if and only if every IVIOBI of S is an IVIORI of S.*

The followings are the duals of Lemma 5.8 and Theorem 5.9 respectively.

Lemma 5.10. *Let S be a regular semigroup. Then every bi-ideal of S is a left ideal of S if and only if every IVIFBI of S is an IVIFLI of S.*

Theorem 5.11. *Let S be a regular semigroup. Then every bi-ideal of S is a right ideal of S if and only if every IVIOBI of S is an IVIOLI of S.*

The following is an immediate consequence of Theorems 5.9 and 5.11.

Theorem 5.12. *Let S be a regular semigroup. Then every bi-ideal of S is an ideal of S if and only if every IVIOBI of S is an IVIOI of S.*

Corollary 5.13. *Let S be a regular duo semigroup. Then $\mathcal{A} \in IVIORI(S)$ for each $\mathcal{A} \in IVIOBI(S)$.*

Proof. Let $\mathcal{A} \in IVIOBI(S)$. It is well-known that every bi-ideal of a regular left duo semigroup is a right ideal of it (See [38], Theorem 30). Then from this and Theorem 5.9, $\mathcal{A} \in IVIORI(S)$. □

A semigroup is called a *semilattice of groups* ([36]), if it is the set-theoretical union of a set of mutually disjoint subgroups G_α $(\alpha \in \Gamma)$, i.e., $S = \bigcup_{\alpha\in\Gamma} G_\alpha$ such that for any $\alpha,\ \beta \in \Gamma$, $G_\alpha G_\beta \subset G_\gamma$ and $G_\beta G_\alpha \subset G_\gamma$ for some $\gamma \in \Gamma$.

Corollary 5.14. *Let S be semigroup which is a semilattice of groups. Then $\mathcal{A} \in IVIOI(S)$ for each $\mathcal{A} \in IVIOBI(S)$.*

Proof. Let $\mathcal{A} \in IVIOBI(S)$. It is well-known that every bi-ideal of such semigroup S is an ideal of S (See [39], Theorem 4). Then from this and Theorem 5.12, $\mathcal{A} \in IVIOI(S)$. □

Let us $L[a]$ [resp. $J[a]$] denote the principal left ideal [resp. ideal] of a semigroup S generated by $a \in S$, i.e.,

$$L[a] = \{a\} \cup Sa,$$
$$J[a] = \{a\} \cup Sa \cup aS \cup SaS.$$

It is well-known ([36], Lemma 2.13) that if S is a regular semigroup, then $L[a] = Sa$ for each $a \in S$.

A semigroup S is said to be *right* [resp. *left*] *zero*, if $xy = y$ [resp. $xy = x$] for any $x,\ y \in S$. Then we get the following.

Lemma 5.15. *Let S be a regular semigroup and let E_S be the set of all idempotent elements of S. Then E_S forms a left zero subsemigroup of S if and only if for each $\widetilde{\widetilde{A}} \in IVIFLI(S)$, $\widetilde{\widetilde{A}}(e) = \widetilde{\widetilde{A}}(f)$ for any $e,\ f \in E_S$.*

Proof. Suppose E_S forms a left zero subsemigroup of S. Let $\widetilde{\widetilde{A}} \in IVIFLI(S)$ and let $e,\ f \in E_S$. Then by the hypothesis, $ef = e$ and $fe = f$. Since $\widetilde{\widetilde{A}} \in IVIFLI(S)$, we have

$$\widetilde{A}^{\in}(e) = \widetilde{A}^{\in}(ef) \geq \widetilde{A}^{\in}(f) = \widetilde{A}^{\in}(fe) \geq \widetilde{A}^{\in}(e)$$

and

$$\widetilde{A}^{\notin}(e) = \widetilde{A}^{\notin}(ef) \leq \widetilde{A}^{\notin}(f) = \widetilde{A}^{\notin}(fe) \leq \widetilde{A}^{\notin}(e).$$

Thus $\widetilde{\widetilde{A}}(e) = \widetilde{\widetilde{A}}(f)$.

Conversely, suppose the necessary condition holds. Since S is regular, $E_S \neq \varnothing$. Let $e,\ f \in E_S$. Then by Remark 3.6 (1) and Theorem 3.9, it is clear that

$$\widetilde{\widetilde{\chi_{L(f)}}} \in IVIFLI(S).$$

Thus we have

$$\widetilde{\chi_{L(f)}}^{\in}(e) = \widetilde{\chi_{L(f)}}^{\in}(f) = [1,1] \text{ and } \widetilde{\chi_{L(f)}}^{\notin}(e) = \widetilde{\chi_{L(f)}}^{\notin}(f) = [0,0].$$

So $e \in L(f) = Sf$. Hence there is $x \in S$ such that $e = xf = xff = ef$. Therefore E_S is a left zero semigroup. □

Corollary 5.16. *Let S be an idempotent semigroup. Then E_S is left zero if and only if for each $\widetilde{\widetilde{A}} \in IVIFLI(S)$, $\widetilde{\widetilde{A}}(e) = \widetilde{\widetilde{A}}(f)$ for any $e,\ f \in E_S$.*

From Lemma 5.15, Proposition 3.5 in [23] and Theorem 3.9 in [6], we get the following.

Theorem 5.17. *Let S be a regular semigroup. Then E_S forms a left zero subsemigroup of S if and only if for each $\mathcal{A} \in IVIOLI(S)$, $\mathcal{A}(e) = \mathcal{A}(f)$ for any $e,\ f \in E_S$.*

The following is an immediate consequence of Corollaries 3.16, 3.5 in [23] and 3.10 in [6].

Corollary 5.18. *Let S be an idempotent semigroup. Then E_S is left zero if and only if for each $\mathcal{A} \in IVIOLI(S)$, $\mathcal{A}(e) = \mathcal{A}(f)$ for any $e,\ f \in E_S$.*

The following is the dual of Theorem 5.17.

Theorem 5.19. *Let S be a regular semigroup. Then E_S forms a right zero subsemigroup of S if and only if for each $\mathcal{A} \in IVIORI(S)$, $\mathcal{A}(e) = \mathcal{A}(f)$ for any $e,\ f \in E_S$.*

The following is the dual of Corollary 5.18.

Corollary 5.20. *Let S be an idempotent semigroup. Then E_S is right zero if and only if for each $\mathcal{A} \in IVIORI(S)$, $\mathcal{A}(e) = \mathcal{A}(f)$ for any $e,\ f \in E_S$.*

Lemma 5.21. *Let S be a regular semigroup. Then S is a group if and only if for each $\widetilde{\widetilde{A}} \in IVIFBI(S)$, $\widetilde{\widetilde{A}}(e) = \widetilde{\widetilde{A}}(f)$ for any $e,\ f \in E_S$.*

Proof. Suppose S is a group and let $\widetilde{\widetilde{A}} \in IVIFBI(S)$. Then by Lemma 4.10, $\widetilde{\widetilde{A}}$ is a constant mapping. Thus $\widetilde{\widetilde{A}}(e) = \widetilde{\widetilde{A}}(f)$ for any $e,\ f \in E_S$.

Conversely, suppose the necessary condition holds and let $e,\ f \in E_S$. Let $B[x]$ denote the principal bi-ideal of S generated by $x \in S$, i.e., $B[x] = \{x\} \cup \{x^2\} \cup xSx$ (See p. 84 in [36]). Furthermore, if S is regular, then $B[x] = xSx$. Since $B[x]$ is bi-ideal of S, by Theorem 4.7, $\widetilde{\widetilde{\chi_{B[f]}}} \in IVIFBI(S)$. Since $f \in B[f]$, we get

$$\widetilde{\chi_{B[f]}}^{\in}(e) = \widetilde{\chi_{B[f]}}^{\in}(f) = [1,1] \text{ and } \widetilde{\chi_{B[f]}}^{\notin}(e) = \widetilde{\chi_{B[f]}}^{\notin}(f) = [0,0].$$

Then $e \in B[f] = fSf$. Thus by the process of the proof Theorem 3.14 in [6], $e = f$. Since S is regular, $E_S \neq \varnothing$ and S contains exactly one idempotent. So from p. 33 (Ex. 4) in [36], it is obvious that S is a group. This completes the proof. □

From Lemma 5.21, Proposition 3.6 in [23] and Theorem 3.14 in [6], we have the following.

Theorem 5.22. *Let S be a regular semigroup. Then S is a group if and only if for each $\mathcal{A} \in IVIOBI(S)$, $\mathcal{A}(e) = \mathcal{A}(f)$ for any $e,\ f \in E_S$.*

6. Regular semigroups

In this section, we deal with some characterizations of a regular semigroup by IVI-octahedron ideals and bi-ideals. It is well-known ([38], Theorem 2.6) that a semigroup S is regular if and only if $B = BSB$ for each $B \in BI(S)$. Also we give a characterization of a left [resp. right and completely] regular semigroup by IVIOLIs [resp. IVIORIs and IVIOBIs]. First of all, we will give a characterization of a regular semigroup by IVIFBIs.

Lemma 6.1. *Let S be a semigroup. Then S is regular if and only if $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}}$ for each $\widetilde{\widetilde{A}} \in IVIFBI(S)$.*

Proof. Suppose S is regular. Let $\widetilde{\widetilde{A}} \in IVIFBI(S)$ and let $a \in S$. Since S is regular, there is $x \in S$ such that $a = axa$. Then we get

$$\begin{aligned}(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\in}(a) &= \bigvee_{a=xy}[(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(x) \wedge \widetilde{A}^{\in}(y)] \\ &\geq (\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(ax) \wedge \widetilde{A}^{\in}(a) \text{ [Since } a = axa] \\ &= (\bigvee_{ax=pq}[\widetilde{A}^{\in}(p) \wedge \widetilde{\mathbf{1}}^{\in}(q)]) \wedge \widetilde{A}^{\in}(a) \\ &\geq (\widetilde{A}^{\in}(a) \wedge \widetilde{\mathbf{1}}^{\in}(x)) \wedge \widetilde{A}^{\in}(a) \\ &= (\widetilde{A}^{\in}(a) \wedge [1,1]) \wedge \widetilde{A}^{\in}(a) \\ &= \widetilde{A}^{\in}(a).\end{aligned}$$

Similarly, we have $(\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}})^{\notin}(a) \leq \widetilde{A}^{\notin}(a)$. Thus $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \supset \widetilde{\widetilde{A}}$. Since $\widetilde{\widetilde{A}} \in IVIFBI(S)$, by Lemma 4.7, $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}}$. So $\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} = \widetilde{\widetilde{A}}$.

Conversely, suppose the necessary condition holds. Let $A \in BI(S)$ and let $a \in S$. By Remark 4.5 (1) and Theorem 4.7, $\widetilde{\widetilde{\chi_A}} \in IVIFBI(S)$. By the hypothesis, we have

$$\begin{aligned}\bigvee_{a=yz}[(\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(y) \wedge \widetilde{\chi_A}^{\in}(y)] &= [(\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}}) \circ_{IVI} \widetilde{\widetilde{\chi_A}}]^{\in}(a) \\ &= \widetilde{\chi_A}^{\in}(a) \\ &= [1,1]. \text{ [By the hypothesis]}\end{aligned}$$

Similarly, we get $\bigwedge_{a=yz}[(\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\notin}(y) \vee \widetilde{\chi_A}^{\notin}(y)] = [0,0]$. Then there are $b,\ c \in S$ with $a = bc$ such that

$$(\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})(b) = ([1,1],[0,0]) \text{ and } \widetilde{\widetilde{\chi_A}}(c) = ([1,1],[0,0]).$$

Since $(\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})(b) = ([1,1],[0,0])$, we have

$$\bigvee_{b=pq}[\widetilde{\chi_A}^{\in}(p) \wedge \widetilde{\mathbf{1}}^{\in}(q)] = (\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\in}(b) = [1,1],$$

$$\bigwedge_{b=pq}[\widetilde{\chi_A}^{\notin}(p) \vee \widetilde{\mathbf{1}}^{\notin}(q)] = (\widetilde{\widetilde{\chi_A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}})^{\notin}(b) = [0,0].$$

Thus there are $d,\ e \in S$ with $b = de$ such that

$$\widetilde{\widetilde{\chi_A}}(d) = ([1,1],[0,0]) \text{ and } \widetilde{\widetilde{\mathbf{1}}}(e) = ([1,1],[0,0]).$$

So $d \in A$, $e \in S$, $c \in A$ and $a = bc = (de)c \in ASA$, i.e., $A \subset ASA$. Since $A \in BI(S)$, $ASA \subset A$. Hence $A = ASA$. Therefore S is regular. □

From Lemma 6.1, Theorem 3.1 in [21] and Theorem 3.1 in [7], we have a characterization of a regular semigroup by IVIOBIs.

Theorem 6.2. *Let S be a semigroup. Then S is regular if and only if $\mathcal{A} = \mathcal{A} \circ \ddot{\mathbf{1}} \circ \mathcal{A}$ for each $\mathcal{A} \in IVIOBI(S)$.*

Theorem 6.3. *Let S be a regular semigroup and let $\mathcal{A} \in IVIOS(S)$. Then $\mathcal{A} \in IVIOBI(S)$ if and only if there are $\mathcal{B} \in IVIORI(S)$ and $\mathcal{C} \in IVIOLI(S)$ such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$.*

Proof. Suppose $\mathcal{A} \in IVIOBI(S)$. Then we have

$$\begin{aligned}\mathcal{A} &= \mathcal{A} \circ \ddot{\mathbf{1}} \circ \mathcal{A} \text{ [By Theorem 6.2]}\\ &= \mathcal{A} \circ \ddot{\mathbf{1}} \circ (\mathcal{A} \circ \ddot{\mathbf{1}} \circ \mathcal{A}) \text{ [By Theorem 6.2]}\\ &= [\mathcal{A} \circ (\ddot{\mathbf{1}} \circ \mathcal{A}] \circ (\ddot{\mathbf{1}} \circ \mathcal{A})\\ &\subset (\mathcal{A} \circ \ddot{\mathbf{1}}) \circ (\ddot{\mathbf{1}} \circ \mathcal{A})\\ &= \mathcal{A} \circ (\ddot{\mathbf{1}} \circ \ddot{\mathbf{1}}) \circ \mathcal{A}\\ &\subset \mathcal{A} \circ \ddot{\mathbf{1}} \circ \circ \mathcal{A}\\ &\subset \mathcal{A}. \text{ [By Theorem 4.9]}\end{aligned}$$

Thus we have

$$\mathcal{A} = (\mathcal{A} \circ \ddot{\mathbf{1}}) \circ (\ddot{\mathbf{1}} \circ \mathcal{A}). \tag{6.1}$$

On the other hand, we get

$$(\mathcal{A} \circ \ddot{\mathbf{1}}) \circ \ddot{\mathbf{1}} = \mathcal{A} \circ (\ddot{\mathbf{1}} \circ \ddot{\mathbf{1}}) \subset \mathcal{A} \circ \ddot{\mathbf{1}}.$$

So by Theorem 3.19, $\mathcal{A} \circ \ddot{\mathbf{1}} \in IVIORI(S)$. Similarly, we can see that $\ddot{\mathbf{1}} \circ \mathcal{A} \in IVIOLI(S)$. Hence the necessary condition holds.

Conversely, suppose that there are $\mathcal{B} \in IVIORI(S)$ and $\mathcal{C} \in IVIOLI(S)$ such that $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$. Then by Proposition 4.13, $\mathcal{B},\ \mathcal{C} \in IVIOBI(S)$. Thus by Proposition 4.17, $\mathcal{B} \circ \mathcal{C} \in IVIOBI(S)$. So $\mathcal{A} \in IVIOBI(S)$. □

Result 6.4 (Theorem 5, [40]; Theorem 41, [38])**.** *Let S be a semigroup. Then S is regular if and only if $B \cap J = BJB$ for each $B \in BI(S)$ and each $J \in I(S)$.*

Lemma 6.5. *Let S be a semigroup. Then S is regular if and only if*

$$\widetilde{\widetilde{B}} \cap \widetilde{\widetilde{J}} = \widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{J}} \circ_{IVI} \widetilde{\widetilde{B}} \text{ for each } \widetilde{\widetilde{B}} \in IVIFBI(S) \text{ and each } \widetilde{\widetilde{J}} \in IVIFI(S). \tag{6.2}$$

Proof. Suppose S is regular and let $\widetilde{\widetilde{B}} \in IVIFBI(S)$ and $\widetilde{\widetilde{J}} \in IVIFI(S)$. Since $\widetilde{\widetilde{B}} \in IVIFBI(S)$, by Lemma 4.8, we get

$$\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{J}} \circ_{IVI} \widetilde{\widetilde{B}} \subset \widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{B}} \subset \widetilde{\widetilde{B}}.$$

Since $\widetilde{\widetilde{J}} \in IVIFI(S)$, by Theorem 3.19, we have

$$\widetilde{\widetilde{B}} \circ_{IVI} \widetilde{\widetilde{J}} \circ_{IVI} \widetilde{\widetilde{B}} \subset \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{J}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \subset \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{J}} \subset \widetilde{\widetilde{J}}.$$

Then we have

$$\tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}} \subset \tilde{\tilde{B}} \cap \tilde{\tilde{J}}. \tag{6.3}$$

In order to show that the converse inclusion holds, let $a \in S$. Since S is regular, there is $x \in S$ such that $a = axa(= axaxa)$. Since $\tilde{\tilde{J}} \in IVIFI(S)$, $\tilde{\tilde{J}}(xax) \geq \tilde{\tilde{J}}(ax) \geq \tilde{\tilde{J}}(a)$. Then we get

$$\begin{aligned}
(\tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}})^{\in}(a) &= \bigvee_{a=yz}[\widetilde{B}^{\in}(y) \wedge (\tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}})^{\in}(z)] \\
&\geq \widetilde{B}^{\in}(a) \wedge (\tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}})^{\in}(xaxa) \text{ [Since } a = axaxa] \\
&= \widetilde{B}^{\in}(a) \wedge (\bigvee_{xaxa=pq}[\widetilde{J}^{\in}(p) \wedge \widetilde{B}^{\in}(q)]) \\
&\geq \widetilde{B}^{\in}(a) \wedge (\widetilde{J}^{\in}(xax) \wedge \widetilde{B}^{\in}(a)) \\
&\geq \widetilde{B}^{\in}(a) \wedge (\widetilde{J}^{\in}(a) \wedge \widetilde{B}^{\in}(a)) \\
&= \widetilde{B}^{\in}(a) \wedge \wedge \widetilde{B}^{\in}(a)) \\
&= (\tilde{\tilde{B}} \cap \tilde{\tilde{J}})^{\in}(a).
\end{aligned}$$

Similarly, we have $(\tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}})^{\notin}(a) \leq (\tilde{\tilde{B}} \cap \tilde{\tilde{J}})^{\notin}(a)$. Thus we get

$$\tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}} \supset \tilde{\tilde{B}} \cap \tilde{\tilde{J}}. \tag{6.4}$$

So by (6.3) and (6.4), $\tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{J}} \circ_{IVI} \tilde{\tilde{B}} = \tilde{\tilde{B}} \cap \tilde{\tilde{J}}$.

Conversely, suppose the condition (6.2) holds and let $\tilde{\tilde{B}} \in IVIFBI(S)$. Since $\tilde{\tilde{\mathbf{1}}} \in IVIFI(S)$, we have

$$\tilde{\tilde{B}} = \tilde{\tilde{B}} \cap \tilde{\tilde{\mathbf{1}}} = \tilde{\tilde{B}} \circ_{IVI} \tilde{\tilde{\mathbf{1}}} \circ_{IVI} \tilde{\tilde{B}}.$$

Then by Lemma 6.1, S is regular. This completes the proof. □

From Lemma 6.5, Theorem 3.3 in [21] and Theorem 3.4 in [7], we obtain a characterization of a regular semigroup by an IVIOBI and an IVIOI as a generalization of Result 6.4.

Theorem 6.6. *Let S be a semigroup. Then S is regular if and only if*

$$\mathcal{B} \cap \mathcal{J} = \mathcal{B} \circ \mathcal{J} \circ \mathcal{B} \text{ for each } \mathcal{B} \in IVIOBI(S) \text{ and each } \mathcal{J} \in IVIOI(S). \tag{6.5}$$

The following characterization of a regular semigroup is due to Theorem 1 of Iséki [41].

Result 6.7. *Let S be a semigroup. Then S is regular if and only if*

$$RL = R \cap L \text{ for each } R \in RI(S) \text{ and each } L \in LI(S). \tag{6.6}$$

Lemma 6.8. *Let S be a semigroup. Then S is regular if and only if*

$$\tilde{\tilde{R}} \circ_{IVI} \tilde{\tilde{L}} = \tilde{\tilde{R}} \cap \tilde{\tilde{L}} \text{ for each } \tilde{\tilde{R}} \in IVIFRI(S) \text{ and each } \tilde{\tilde{L}} \in IVIFLI(S). \tag{6.7}$$

Proof. Suppose S is regular and let $\tilde{\tilde{R}} \in IVIFRI(S)$, $\tilde{\tilde{L}} \in IVIFLI(S)$. Then by Lemmas 3.17 and 3.15, we get

$$\tilde{\tilde{R}} \circ_{IVI} \tilde{\tilde{L}} \subset \tilde{\tilde{R}} \circ_{IVI} \tilde{\tilde{\mathbf{1}}} \subset \tilde{\tilde{R}} \text{ and } \tilde{\tilde{R}} \circ_{IVI} \tilde{\tilde{L}} \subset \tilde{\tilde{\mathbf{1}}} \circ_{IVI} \tilde{\tilde{L}} \subset \tilde{\tilde{L}}.$$

Thus we have

$$\widetilde{\widetilde{R}} \cap \widetilde{\widetilde{L}} \subset \widetilde{\widetilde{R}} \circ_{IVI} \widetilde{\widetilde{L}}. \tag{6.8}$$

Now let $a \in S$. Since S is regular, there is $x \in S$ such that $a = axa$. Then we have

$$\begin{aligned}(\widetilde{\widetilde{R}} \circ_{IVI} \widetilde{\widetilde{L}})^{\in}(a) &= \bigvee_{a=yz}[\widetilde{R}^{\in}(y) \wedge \widetilde{L}^{\in}(z)] \\ &\geq \widetilde{R}^{\in}(ax) \wedge \widetilde{L}^{\in}(a) \text{ [Since } a = axa] \\ &\geq \widetilde{R}^{\in}(a) \wedge \widetilde{L}^{\in}(a) \text{ [Since } \widetilde{\widetilde{R}} \in IVIFRI(S)] \\ &= (\widetilde{\widetilde{R}} \cap \widetilde{\widetilde{L}})^{\in}(a).\end{aligned}$$

Similarly, we get $(\widetilde{\widetilde{R}} \circ_{IVI} \widetilde{\widetilde{L}})^{\notin}(a) \leq (\widetilde{\widetilde{R}} \cap \widetilde{\widetilde{L}})^{\notin}(a)$. Thus we have

$$\widetilde{\widetilde{R}} \cap \widetilde{\widetilde{L}} \supset \widetilde{\widetilde{R}} \circ_{IVI} \widetilde{\widetilde{L}}. \tag{6.9}$$

So by (6.8) and (6.9), the condition (6.7) holds.

Conversely, suppose the condition (6.7) holds and let $R \in RI(S)$, $L \in LI(S)$. Then by Remark 3.6 and Theorem 3.9, $\widetilde{\widetilde{\chi_R}} \in IVIFRI(S)$ and $\widetilde{\widetilde{\chi_L}} \in IVIFLI(S)$. Thus by the hypothesis, we have

$$\widetilde{\widetilde{\chi_R}} \circ_{IVI} \widetilde{\widetilde{\chi_L}} = \widetilde{\widetilde{\chi_R}} \cap \widetilde{\widetilde{\chi_L}}. \tag{6.10}$$

In order to prove that $R \cap L \subset RL$, let $a \in R \cap L$. Then we get

$$\begin{aligned}\bigvee_{a=yz}[\widetilde{\chi_R}^{\in}(y) \wedge \widetilde{\chi_L}^{\in}(z)] &= (\widetilde{\widetilde{\chi_R}} \circ_{IVI} \widetilde{\widetilde{\chi_L}})^{\in}(a) \\ &= (\widetilde{\widetilde{\chi_R}} \cap \widetilde{\widetilde{\chi_L}})^{\in}(a) \\ &= \widetilde{\chi_R}^{\in}(a) \wedge \widetilde{\chi_L}^{\in}(a) \\ &= [1,1] \wedge [1,1] \\ &= [1,1].\end{aligned}$$

Similarly, we have $\bigwedge_{a=yz}[\widetilde{\chi_R}^{\notin}(y) \vee \widetilde{\chi_L}^{\notin}(z)] = [0,0]$. This implies that there are $b,\ c \in S$ with $a = bc$ such that

$$\widetilde{\widetilde{\chi_R}}(b) = ([1,1],[0,0]) \text{ and } \widetilde{\widetilde{\chi_L}}(c) = ([1,1],[0,0]).$$

Thus $b \in R$ and $c \in L$, i.e., $a = bc \in RL$. So $R \cap L \subset RL$. It is obvious that $RL \subset R \cap L$. Hence $R \cap L = RL$, i.e., the condition (6.6) holds. Therefore by Result 6.7, S is regular. This completes the proof. □

From Lemma 6.8, Theorem 3.4 in [21] and Theorem 3.6 in [7], We obtain another characterization of a regular semigroup by an IVIORI and an IVIOLI as a generalization of Result 6.7.

Theorem 6.9. *Let S be a semigroup. Then S is regular if and only if*

$$\mathcal{R} \circ \mathcal{L} = \mathcal{R} \cap \mathcal{L} \text{ for each } \mathcal{R} \in IVIORI(S) \text{ and each } \mathcal{L} \in IVIOLI(S). \tag{6.11}$$

Lemma 6.10. *Every IVIFI of a regular semigroup S is idempotent, i.e., $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}}$ for each $\widetilde{\widetilde{A}} \in IVIFI(S)$.*

Proof. Let S be a regular semigroup and let $\widetilde{\widetilde{A}} \in IVIFI(S)$. Then by Remark 3.6 (3) and Theorem 3.19, we have

$$\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \subset \widetilde{\widetilde{A}}$$

and

$$\widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \subset \widetilde{\widetilde{A}}.$$

Thus by Lemma 4.8, $\widetilde{\widetilde{A}} \in IVIFBI(S)$. Since S is regular, by Lemma 6.1, we get

$$\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{\mathbf{1}}} \circ_{IVI} \widetilde{\widetilde{A}} \subset \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}}.$$

So $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}} \circ_{IVI} \widetilde{\widetilde{A}}$. Hence $\widetilde{\widetilde{A}}$ is idempotent. □

From Lemma 6.10, Proposition 3.5 in [21] and Theorem 3.7 in [7], we get the following.

Proposition 6.11. *Every IVIOI of a regular semigroup S is idempotent, i.e., $\mathcal{A} = \mathcal{A} \circ \mathcal{A}$ for each $\mathcal{A} \in IVIOI(S)$.*

Now we deal with a characterization of a left [resp. right and completely] regular semigroup by IVIOLIs [resp. IVIORIs and IVIOBIs].

A semigroup S is said to be *left* [resp. *right*] *regular*, if for each $a \in S$, there is $x \in S$ such that $a = xa^2$ [resp. $a = a^2x$].

A semigroup S is said to be *completely regular*, if for each $a \in S$, there is $x \in S$ such that $a = axa$ and $ax = xa$.

For characterization of a left [resp. right] regular semigroup, see Theorem 4.2 in [36]. Also it is well-known ([36], Theorem 4.3) that S is completely regular if and only if it is left and right regular.

Lemma 6.12. *Let S be a semigroup. Then S is left regular if and only if for each $\widetilde{\widetilde{L}} \in IVIFLI(S)$, $\widetilde{\widetilde{L}}(a) = \widetilde{\widetilde{L}}(a^2)$ for each $a \in S$.*

Proof. Suppose S is left regular. Let $\widetilde{\widetilde{L}} \in IVIFLI(S)$ and let $a \in S$. Then by the hypothesis, there is $x \in S$ such that $a = xa^2$. Since $\widetilde{\widetilde{L}} \in IVIFLI(S)$, we have

$$\widetilde{L}^{\in}(a) = \widetilde{L}^{\in}(xa^2) \geq \widetilde{L}^{\in}(a^2) \geq \widetilde{L}^{\in}(a).$$

Similarly, we get $\widetilde{L}^{\notin}(a) \leq \widetilde{L}^{\notin}(a^2) \leq \widetilde{L}^{\notin}(a)$. Thus $\widetilde{\widetilde{L}}(a) = \widetilde{\widetilde{L}}(a^2)$.

Conversely, suppose the necessary condition holds and let $a \in S$. Then by Remark 3.6 (1) and Theorem 3.9, $\widetilde{\widetilde{\chi_{L[a^2]}}} \in IVIFLI(S)$. Since $a^2 \in L[a^2]$,

$$\widetilde{\chi_{L[a^2]}}^{\in}(a) = \widetilde{\chi_{L[a^2]}}^{\in}(a^2) = [1,1] \text{ and } \widetilde{\chi_{L[a^2]}}^{\notin}(a) = \widetilde{\chi_{L[a^2]}}^{\notin}(a^2) = [0,0].$$

Thus $a \in L[a^2] = \{a^2\} \cup Sa^2$, i.e., there is $x \in S$ such that $a = xa^2$. So S is left regular. □

From Lemma 6.12, Proposition 5.1 in [23] and Theorem 5.1 in [6], we give a characterization of a left regular semigroup by IVIOLIs as a generalization of Theorem 4.2 in [36].

Theorem 6.13. *Let S be a semigroup. Then S is left regular if and only if for each $\mathcal{L} \in IVIOLI(S)$, $\mathcal{L}(a) = \mathcal{L}(a^2)$ for each $a \in S$.*

The following is the dual of Lemma 6.12.

Lemma 6.14. *Let S be a semigroup. Then S is right regular if and only if for each $\tilde{\tilde{R}} \in IVIFRI(S)$, $\tilde{\tilde{R}}(a) = \tilde{\tilde{R}}(a^2)$ for each $a \in S$.*

From Lemma 6.14, Proposition $5.1'$ in [23] and Theorem 5.2 in [6], we give a characterization of a right regular semigroup by IVIOLIs as a generalization of Theorem 4.2 in [36].

Theorem 6.15. *Let S be a semigroup. Then S is right regular if and only if for each $\mathcal{R} \in IVIORI(S)$, $\mathcal{R}(a) = \mathcal{R}(a^2)$ for each $a \in S$.*

Result 6.16 (p. 105, [42])**.** *Let S be a semigroup. Then the followings are equivalent:*
(1) *S is completely regular.*
(2) *S is a union of groups.*
(3) *$a \in a^2Sa^2$ for each $a \in S$.*

Lemma 6.17. *Let S be a semigroup. Then the followings are equivalent:*
(1) *S is completely regular.*
(2) *For each $\tilde{\tilde{B}} \in IVIFBI(S)$, $\tilde{\tilde{B}}(a) = \tilde{\tilde{B}}(a^2)$ for each $a \in S$.*
(3) *For each $\tilde{\tilde{L}} \in IVIFLI(S)$ and each $\tilde{\tilde{R}} \in IVIFRI(S)$,*

$$\tilde{\tilde{L}}(a) = \tilde{\tilde{L}}(a^2) \text{ and } \tilde{\tilde{R}}(a) = \tilde{\tilde{R}}(a^2) \text{ for each } a \in S.$$

Proof. It is obvious that (1) $\Longleftrightarrow$ (3) by Lemmas 6.12 and 6.14. It is sufficient to prove that (1) $\Longleftrightarrow$ (2).

Suppose the condition (1) holds. Let $\tilde{\tilde{B}} \in IVIFBI(S)$ and let $a \in S$. Then by Result 6.16, there is $x \in S$ such that $a = a^2xa^2$. Since $\tilde{\tilde{B}} \in IVIFBI(S)$, we have

$$\begin{aligned}\tilde{B}^{\in}(a) &= \tilde{B}^{\in}(a^2xa^2) \geq \tilde{B}^{\in}(a^2) \wedge \tilde{B}^{\in}(a^2)\\ &= \tilde{B}^{\in}(a^2) \geq \tilde{B}^{\in}(a) \wedge \tilde{B}^{\in}(a)\\ &= \tilde{B}^{\in}(a).\end{aligned}$$

Similarly, $\tilde{B}^{\notin}(a) \leq \tilde{B}^{\notin}(a^2) \leq \tilde{B}^{\notin}(a)$. Thus $\tilde{\tilde{B}}(a) = \tilde{\tilde{B}}(a^2)$.

Conversely, suppose the condition (2) holds. For each $x \in S$, let $B[x]$ denote the principal bi-ideal of S generated by x, i.e.,

$$B[x] = \{x\} \cup \{x^2\} \cup xSx.$$

Let $a \in S$. Then by Remark 4.5 (1) and Theorem 4.7, $\widetilde{\widetilde{\chi_{B[a^2]}}} \in IVIFBI(S)$. Since $a^2 \in B[a^2]$, we get

$$\widetilde{\chi_{B[a^2]}}^{\in}(a) = \widetilde{\chi_{B[a^2]}}^{\in}(a^2) = [1,1] \text{ and } \widetilde{\chi_{B[a^2]}}^{\notin}(a) = \widetilde{\chi_{B[a^2]}}^{\notin}(a^2) = [0,0].$$

Thus $a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2$. So by Result 6.16, S is completely regular. □

From Lemma 6.17, Proposition 5.2 in [23] and Theorem 5.3 in [6], we give a characterization of a right regular semigroup by IVIOLIs as a generalization of Result 6.16.

Theorem 6.18. *Let S be a semigroup. Then the followings are equivalent:*
(1) *S is completely regular.*

(2) *For each* $\mathcal{B} \in IVIOBI(S)$, $\mathcal{B}(a) = \mathcal{B}(a^2)$ *for each* $a \in S$.
(3) *For each* $\mathcal{L} \in IVIOLI(S)$ *and each* $\mathcal{R} \in IVIORI(S)$,

$$\mathcal{L}(a) = \mathcal{L}(a^2) \text{ and } \mathcal{R}(a) = \mathcal{R}(a^2) \text{ for each } a \in S.$$

7. Conclusions

We introduced the notions of IVI-octahedron ideals and bi-ideals in a semigroup, and IVI-octahwedron duo semigroups and studied some of their properties. Moreover, we discussed some characterizations of a regular semigroup and a left [resp. right] regular semigroup by IVIOIs and IVIOBIs.

In the future, we expect that one applies IVI-octahedron sets to BCI/BCK-algebras, topologies, category theory and decision-making problems, etc. Furthermore, we will try to study group structures and (semi)ring structures based on IVI-octahedron sets.

Acknowledgements. The authors wish to thank the anonymous reviewers for valuable suggestions.

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