

Lukasiewicz fuzzy subalgebras in BCK -algebras and BCI -algebras

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ABSTRACT. The idea of Lukasiewicz t -norm is used to construct the concept of Lukasiewicz fuzzy sets based on a given fuzzy set. The Lukasiewicz fuzzy sets are applied to BCK -algebras and BCI -algebras. Moreover, the notion of Lukasiewicz fuzzy subalgebra is introduced and its various properties are investigated. Three types of subsets so called \in -set, q -set and O -set are constructed, and the conditions under which they can be subalgebras are explored.

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1. INTRODUCTION

A fuzzy concept, which is introduced by Zadeh [1], is understood as a concept which is “to an extent applicable” in a situation. That means the concept has gradations of significance or unsharp (variable) boundaries of application. Prior to the emergence of the fuzzy set, the very idea of inferring as an unclear concept faced considerable resistance from the elite in the academic world. They did not want to endorse the use of imprecise concepts in research or argumentation. Yet although people might not be aware of it, the use of fuzzy concepts has risen gigantically in all walks of life from the 1970s onward. That is mainly due to advances in electronic engineering, fuzzy mathematics and digital computer programming. The new technology allows very complex inferences about “variations on a theme” to be anticipated and fixed in a program. Lukasiewicz logic, which is the logic of the Lukasiewicz t -norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Lukasiewicz as a three-valued logic. BCK/BCI -algebras originally defined by Iséki and Tanaka in [2] to generalize the set difference in

set theory. In 1999, Jun et al. [3] studied fuzzy subalgebras and fuzzy ideals in *BCK*-algebras. Also, Jun [4] studied fuzzy subalgebras with thresholds in *BCK/BCI*-algebras.

In this paper, using the idea of Łukasiewicz *t*-norm, we construct the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and apply it to *BCK*-algebras and *BCI*-algebras. We define the concepts of (strong) Łukasiewicz fuzzy subalgebras, and investigate several properties. We provide conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy subalgebra. We explore the conditions under which Łukasiewicz fuzzy subalgebra becomes strong. We discuss characterizations of Łukasiewicz fuzzy subalgebras. We construct a three kind of subsets so called \in -set, q -set and O -set, and we find the conditions under which they can be subalgebras.

2. PRELIMINARIES

A *BCK/BCI*-algebra is an important class of logical algebras introduced by Iséki (See [5] and [2]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [6, 7] for further information regarding *BCK*-algebras and *BCI*-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

- (I_1) $(\forall a, b, c \in X) (((a * b) * (a * c)) * (c * b) = 0)$,
- (I_2) $(\forall a, b \in X) ((a * (a * b)) * b = 0)$,
- (I_3) $(\forall a \in X) (a * a = 0)$,
- (I_4) $(\forall a, b \in X) (a * b = 0, b * a = 0 \Rightarrow a = b)$,

then we say that X is a *BCI*-algebra. If a *BCI*-algebra X satisfies the following identity:

$$(K) (\forall a \in X) (0 * a = 0),$$

then X is called a *BCK*-algebra.

A *BCI*-algebra X is said to be *p-semisimple* (See [6]), if $0 * (0 * a) = a$ for all $a \in X$.

The order relation “ \leq ” in a *BCK/BCI*-algebra X is defined as follows:

$$(2.1) \quad (\forall a, b \in X) (a \leq b \Leftrightarrow a * b = 0).$$

Every *BCK/BCI*-algebra X satisfies the following conditions (See [6, 7]):

$$(2.2) \quad (\forall a \in X) (a * 0 = a),$$

$$(2.3) \quad (\forall a, b, c \in X) (a \leq b \Rightarrow a * c \leq b * c, c * b \leq c * a),$$

$$(2.4) \quad (\forall a, b, c \in X) ((a * b) * c = (a * c) * b).$$

Every *BCI*-algebra X satisfies (See [6]):

$$(2.5) \quad (\forall a, b \in X) (a * (a * (a * b)) = a * b),$$

$$(2.6) \quad (\forall a, b \in X) (0 * (a * b) = (0 * a) * (0 * b)).$$

A subset A of a *BCK/BCI*-algebra X is called a *subalgebra* of X (See [6, 7]), if it satisfies:

$$(2.7) \quad (\forall a, b \in A) (a * b \in A),$$

A fuzzy set f in a set X of the form

$$f(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set f in a set X , we say that a fuzzy point $[a/t]$ is

- (i) *contained* in f , denoted by $[a/t] \in f$ ([8]), if $f(a) \geq t$,
- (ii) *quasi-coincident* with f , denoted by $[a/t] q f$ ([8]), if $f(a) + t > 1$.

A fuzzy set f in a *BCK/BCI*-algebra X is called a *fuzzy subalgebra* of X , if it satisfies:

$$(2.8) \quad (\forall a, b \in X)(f(a * b) \geq \min\{f(a), f(b)\}).$$

3. LUKASIEWICZ FUZZY SUBALGEBRAS

Definition 3.1. Let f be a fuzzy set in a set X and let $\varepsilon \in [0, 1]$. A function

$$L_f^\varepsilon : X \rightarrow [0, 1], \quad x \mapsto \max\{0, f(x) + \varepsilon - 1\}$$

is called an ε -*Lukasiewicz fuzzy set* of f in X .

Let L_f^ε be an ε -Lukasiewicz fuzzy set of a fuzzy set f in X . If $\varepsilon = 1$, then $L_f^\varepsilon(x) = \max\{0, f(x) + 1 - 1\} = \max\{0, f(x)\} = f(x)$ for all $x \in X$. This shows that if $\varepsilon = 1$, then the ε -Lukasiewicz fuzzy set of a fuzzy set f in X is the classical fuzzy set f itself in X . If $\varepsilon = 0$, then $L_f^\varepsilon(x) = \max\{0, f(x) + 0 - 1\} = \max\{0, f(x) - 1\} = 0$ for all $x \in X$, that is, if $\varepsilon = 0$, then the ε -Lukasiewicz fuzzy set is the zero fuzzy set. Therefore, in handling the ε -Lukasiewicz fuzzy set, the value of ε can always be considered to be in $(0, 1)$.

Let f be a fuzzy set in a set X and $\varepsilon \in (0, 1)$. If $f(x) + \varepsilon \leq 1$ for all $x \in X$, then the ε -Lukasiewicz fuzzy set L_f^ε of f in X is the 0-constant function, that is, $L_f^\varepsilon(x) = 0$ for all $x \in X$. Therefore, in order for the ε -Lukasiewicz fuzzy set to have a meaningful form, a fuzzy set f in X and $\varepsilon \in (0, 1)$ must be set to satisfy the following condition:

$$(3.1) \quad (\exists x \in X)(f(x) + \varepsilon > 1).$$

Proposition 3.2. *If f is a fuzzy set in a set X and $\varepsilon \in (0, 1)$, then its ε -Lukasiewicz fuzzy set L_f^ε satisfies:*

$$(3.2) \quad (\forall x, y \in X)(f(x) \geq f(y) \Rightarrow L_f^\varepsilon(x) \geq L_f^\varepsilon(y)),$$

$$(3.3) \quad (\forall x \in X)([x/\varepsilon] q f \Rightarrow L_f^\varepsilon(x) = f(x) + \varepsilon - 1),$$

$$(3.4) \quad (\forall x \in X)(\forall \delta \in (0, 1))(\varepsilon \geq \delta \Rightarrow L_f^\varepsilon(x) \geq L_f^\delta(x)).$$

Proof. Straightforward. □

Proposition 3.3. *If f and g are fuzzy sets in a set X , then*

$$(3.5) \quad (\forall \varepsilon \in (0, 1)) (L_{f \cap g}^\varepsilon = L_f^\varepsilon \cap L_g^\varepsilon).$$

Proof. For every $x \in X$, we have

$$\begin{aligned} L_{f \cap g}^\varepsilon(x) &= \max\{0, (f \cap g)(x) + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x), g(x)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x) + \varepsilon - 1, g(x) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, g(x) + \varepsilon - 1\}\} \\ &= \min\{L_f^\varepsilon(x), L_g^\varepsilon(x)\} = (L_f^\varepsilon \cap L_g^\varepsilon)(x) \end{aligned}$$

which proves (3.5). □

In what follows, let X be a *BCK*-algebra or a *BCI*-algebra, and ε is an element of $(0, 1)$ unless otherwise specified.

Definition 3.4. Let f be a fuzzy set in X . Then its ε -Lukasiewicz fuzzy set L_f^ε in X is called an ε -Lukasiewicz fuzzy subalgebra of X if it satisfies:

$$(3.6) \quad [x/t_a] \in L_f^\varepsilon, [y/t_b] \in L_f^\varepsilon \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in L_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Example 3.5. Consider a *BCK*-algebra $X = \{0, a_1, a_2, a_3, a_4\}$ (See [7]) with a binary operation “*” given by Table 1.

TABLE 1. Cayley table for the binary operation “*”

*	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	a_1	0	a_1	0	0
a_2	a_2	a_2	0	0	0
a_3	a_3	a_3	a_3	0	0
a_4	a_4	a_3	a_4	a_1	0

Define a fuzzy set f in X as follows:

$$f : X \rightarrow [0, 1], x \mapsto \begin{cases} 0.76 & \text{if } x = 0, \\ 0.69 & \text{if } x = a_1, \\ 0.63 & \text{if } x = a_2, \\ 0.57 & \text{if } x = a_3, \\ 0.42 & \text{if } x = a_4. \end{cases}$$

Given $\varepsilon := 0.55$, the ε -Lukasiewicz fuzzy set L_f^ε of f in X is given as follows:

$$L_f^\varepsilon : X \rightarrow [0, 1], x \mapsto \begin{cases} 0.31 & \text{if } x = 0, \\ 0.24 & \text{if } x = a_1, \\ 0.18 & \text{if } x = a_2, \\ 0.12 & \text{if } x = a_3, \\ 0 & \text{if } x = a_4. \end{cases}$$

It is routine to verify that L_f^ε is an ε -Lukasiewicz fuzzy subalgebra of X .

Theorem 3.6. *If f is a fuzzy subalgebra of X , then its ε -Lukasiewicz fuzzy set L_f^ε in X is an ε -Lukasiewicz fuzzy subalgebra of X .*

Proof. Assume that f is a fuzzy subalgebra of X . Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_f^\varepsilon$ and $[y/t_b] \in L_f^\varepsilon$. Then $L_f^\varepsilon(x) \geq t_a$ and $L_f^\varepsilon(y) \geq t_b$. Thus

$$\begin{aligned} L_f^\varepsilon(x * y) &= \max\{0, f(x * y) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{f(x), f(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\ &= \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}. \end{aligned}$$

So $[(x * y)/\min\{t_a, t_b\}] \in L_f^\varepsilon$. Hence L_f^ε is an ε -Łukasiewicz fuzzy subalgebra of X . \square

The following example shows that the converse of Theorem 3.6 may not be true.

Example 3.7. Consider a *BCI*-algebra $X = \{0, a_1, a_2, a_3, a_4\}$ (See [6]) with a binary operation “ $*$ ” given by Table 2.

TABLE 2. Cayley table for the binary operation “ $*$ ”

$*$	0	a_1	a_2	a_3	a_4
0	0	0	a_2	a_3	a_4
a_1	a_1	0	a_2	a_3	a_4
a_2	a_2	a_2	0	a_4	a_3
a_3	a_3	a_3	a_4	0	a_2
a_4	a_4	a_4	a_3	a_2	0

Define a fuzzy set f in X as follows:

$$f : X \rightarrow [0, 1], x \mapsto \begin{cases} 0.72 & \text{if } x = 0, \\ 0.68 & \text{if } x = a_1, \\ 0.61 & \text{if } x = a_2, \\ 0.57 & \text{if } x = a_3, \\ 0.39 & \text{if } x = a_4. \end{cases}$$

Given $\varepsilon := 0.42$, the ε -Łukasiewicz fuzzy set L_f^ε of f in X is given as follows:

$$L_f^\varepsilon : X \rightarrow [0, 1], x \mapsto \begin{cases} 0.14 & \text{if } x = 0, \\ 0.10 & \text{if } x = a_1, \\ 0.03 & \text{if } x = a_2, \\ 0 & \text{if } x = a_3, \\ 0 & \text{if } x = a_4. \end{cases}$$

It is routine to verify that L_f^ε is an ε -Łukasiewicz fuzzy subalgebra of X . But f is not a fuzzy subalgebra of X because of

$$f(a_2 * a_3) = f(a_4) = 0.39 \not\geq 0.57 = \min\{f(a_2), f(a_3)\}.$$

We consider a characterization of ε -Łukasiewicz fuzzy subalgebra.

Theorem 3.8. *Let f be a fuzzy set in X . Then its ε -Lukasiewicz fuzzy set L_f^ε in X is an ε -Lukasiewicz fuzzy subalgebra of X if and only if it satisfies:*

$$(3.7) \quad (\forall x, y \in X)(L_f^\varepsilon(x * y) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\}).$$

Proof. Assume that L_f^ε is an ε -Lukasiewicz fuzzy subalgebra of X . Let $x, y \in X$. It is clear that $[x/L_f^\varepsilon(x)] \in L_f^\varepsilon$ and $[y/L_f^\varepsilon(y)] \in L_f^\varepsilon$. Then

$$[(x * y)/\min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\}] \in L_f^\varepsilon$$

by (3.6), which implies that $L_f^\varepsilon(x * y) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\}$.

Conversely, suppose that L_f^ε satisfies the condition (3.7). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_f^\varepsilon$ and $[y/t_b] \in L_f^\varepsilon$. Then $L_f^\varepsilon(x) \geq t_a$ and $L_f^\varepsilon(y) \geq t_b$, which implies from (3.7) that

$$L_f^\varepsilon(x * y) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}.$$

Thus $[(x * y)_{\min\{t_a, t_b\}}] \in L_f^\varepsilon$. So L_f^ε is an ε -Lukasiewicz fuzzy subalgebra of X . \square

Proposition 3.9. *If f is a fuzzy subalgebra of X , then its ε -Lukasiewicz fuzzy set L_f^ε satisfies:*

$$(3.8) \quad (\forall x \in X)(L_f^\varepsilon(0) \geq L_f^\varepsilon(x)).$$

Proof. If f is a fuzzy subalgebra of X , then $f(0) = f(x * x) \geq \min\{f(x), f(x)\} = f(x)$ for all $x \in X$. It follows from (3.2) that $L_f^\varepsilon(0) \geq L_f^\varepsilon(x)$ for all $x \in X$. \square

Proposition 3.10. *If f is a fuzzy subalgebra of X , then its ε -Lukasiewicz fuzzy set L_f^ε satisfies:*

$$(3.9) \quad (\forall x, y \in X)(L_f^\varepsilon(x) = L_f^\varepsilon(0) \Leftrightarrow L_f^\varepsilon(x * y) \geq L_f^\varepsilon(y)).$$

Proof. Assume that $L_f^\varepsilon(x) = L_f^\varepsilon(0)$ for all $x \in X$. Then

$$L_f^\varepsilon(x * y) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} = \min\{L_f^\varepsilon(0), L_f^\varepsilon(y)\} = L_f^\varepsilon(y)$$

for all $x, y \in X$ by the combination of Theorem 3.6 and Proposition 3.9.

Conversely, suppose that $L_f^\varepsilon(x * y) \geq L_f^\varepsilon(y)$ for all $x, y \in X$. Using (2.2) induces $L_f^\varepsilon(x) = L_f^\varepsilon(x * 0) \geq L_f^\varepsilon(0)$. The combination of this and Proposition 3.9 leads to $L_f^\varepsilon(x) = L_f^\varepsilon(0)$ for all $x \in X$. \square

Proposition 3.11. *If f is a fuzzy subalgebra of a BCI-algebra X , then its ε -Lukasiewicz fuzzy set L_f^ε satisfies:*

$$(3.10) \quad (\forall x \in X)(L_f^\varepsilon(0 * x) \geq L_f^\varepsilon(x)).$$

Proof. If f is a fuzzy subalgebra of a BCI-algebra X , then

$$f(0 * x) \geq \min\{f(0), f(x)\} = f(x)$$

for all $x \in X$. It follows from (3.2) that $L_f^\varepsilon(0 * x) \geq L_f^\varepsilon(x)$ for all $x \in X$. \square

Proposition 3.12. *If f is a fuzzy subalgebra of a BCI-algebra X , then its ε -Lukasiewicz fuzzy set L_f^ε satisfies:*

$$(3.11) \quad [x/t_a] \in L_f^\varepsilon, [y/t_b] \in L_f^\varepsilon \Rightarrow [(x * (0 * y))/\min\{t_a, t_b\}] \in L_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_f^\varepsilon$ and $[y/t_b] \in L_f^\varepsilon$. Then $L_f^\varepsilon(x) \geq t_a$ and $L_f^\varepsilon(y) \geq t_b$. Thus

$$\begin{aligned} L_f^\varepsilon(x * (0 * y)) &= \max\{0, f(x * (0 * y)) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{f(x), f(0 * y)\} + \varepsilon - 1\} \\ &\geq \max\{0, \min\{f(x), \min\{f(0), f(y)\}\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x), f(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\ &= \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \\ &\geq \min\{t_a, t_b\}. \end{aligned}$$

So $[(x * (0 * y))/\min\{t_a, t_b\}] \in L_f^\varepsilon$. □

We provide conditions for a Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy subalgebra.

Theorem 3.13. *Let f be a fuzzy set in X . If ε -Łukasiewicz fuzzy set L_f^ε of f in X satisfies:*

$$(3.12) \quad [y/t_b] \in L_f^\varepsilon, [z/t_c] \in L_f^\varepsilon \Rightarrow [(x * y)/\min\{t_b, t_c\}] \in L_f^\varepsilon$$

for all $t_b, t_c \in (0, 1]$ and $x, y, z \in X$ with $z \leq x$, then L_f^ε is an ε -Łukasiewicz fuzzy subalgebra of X .

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_f^\varepsilon$ and $[y/t_b] \in L_f^\varepsilon$. Since $x \leq x$ for all $x \in X$, it follows from (3.12) that $[(x * y)/\min\{t_a, t_b\}] \in L_f^\varepsilon$. Hence L_f^ε is an ε -Łukasiewicz fuzzy subalgebra of X . □

Proposition 3.14. *Let f be a fuzzy set in a BCI-algebra X . Then every ε -Łukasiewicz fuzzy subalgebra L_f^ε of X satisfies:*

$$(3.13) \quad [x/t_a] \in L_f^\varepsilon, [y/t_b] \in L_f^\varepsilon \Rightarrow [(x * (0 * y))/\min\{t_a, t_b\}] \in L_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_f^\varepsilon$ and $[y/t_b] \in L_f^\varepsilon$. Then $L_f^\varepsilon(x) \geq t_a$ and $L_f^\varepsilon(y) \geq t_b$. It follows from Theorem 3.8 and Proposition 3.9 that

$$\begin{aligned} L_f^\varepsilon(x * (0 * y)) &\geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(0 * y)\} \\ &\geq \min\{L_f^\varepsilon(x), \min\{L_f^\varepsilon(0), L_f^\varepsilon(y)\}\} \\ &= \{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}, \end{aligned}$$

i.e., $[(x * (0 * y))/\min\{t_a, t_b\}] \in L_f^\varepsilon$. □

Given a fuzzy set f in a BCI-algebra X , its ε -Łukasiewicz fuzzy subalgebra L_f^ε does not satisfy the following equality:

$$(3.14) \quad (\forall x \in X)(L_f^\varepsilon(0 * x) = L_f^\varepsilon(x)).$$

In fact, the $\varepsilon(= 0.42)$ -Łukasiewicz fuzzy subalgebra L_f^ε in Example 3.7 does not satisfy (3.14) since $L_f^\varepsilon(0 * a_1) = L_f^\varepsilon(0) = 0.14 \neq 0.10 = L_f^\varepsilon(a_1)$.

Let f be a fuzzy set in a BCI-algebra X . If an ε -Łukasiewicz fuzzy subalgebra L_f^ε of X satisfies the condition (3.14), we say it is *strong*.

We provide a condition for a Łukasiewicz fuzzy subalgebra to be strong.

Theorem 3.15. *Let f be a fuzzy set in a BCI-algebra X . If X is p -semisimple, then every ε -Łukasiewicz fuzzy subalgebra L_f^ε of X is strong.*

Proof. Assume that X is a p -semisimple BCI-algebra and let L_f^ε be an ε -Łukasiewicz fuzzy subalgebra of X . Then it satisfies the condition (3.13) (See Proposition 3.14). In the same way as the proof of Theorem 3.8, it can be revealed that (3.13) is equivalent to the following.

$$(3.15) \quad (\forall x, y \in X)(L_f^\varepsilon(x * (0 * y)) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\}).$$

Using (2.5), Theorem 3.8, Proposition 3.9 and the p -semisimplicity of X , we have

$$\begin{aligned} L_f^\varepsilon(x) &= L_f^\varepsilon(0 * (0 * x)) \geq \min\{L_f^\varepsilon(0), L_f^\varepsilon(0 * x)\} \\ &= L_f^\varepsilon(0 * x) = L_f^\varepsilon(0 * (0 * (0 * x))) \\ &\geq \min\{L_f^\varepsilon(0), L_f^\varepsilon(0 * (0 * x))\} = L_f^\varepsilon(0 * (0 * x)) \\ &\geq L_f^\varepsilon(x). \end{aligned}$$

Thus $L_f^\varepsilon(x) = L_f^\varepsilon(0 * x)$ for all $x \in X$. So L_f^ε is a strong ε -Łukasiewicz fuzzy subalgebra of X . \square

Corollary 3.16. *Let X be a BCI-algebra which satisfies any one of the following conditions:*

$$(3.16) \quad X = \{0 * x \mid x \in X\},$$

$$(3.17) \quad (\forall x, y \in X)(x * (0 * y) = y * (0 * x)),$$

$$(3.18) \quad (\forall x \in X)(0 * x = 0 \Rightarrow x = 0),$$

$$(3.19) \quad (\forall x, y \in X)(0 * (y * x) = x * y),$$

$$(3.20) \quad (\forall x, y, z \in X)(z * x = z * y \Rightarrow x = y),$$

$$(3.21) \quad (\forall x, y, z \in X)((x * y) * (x * z) = z * y).$$

If f is a fuzzy subalgebra of X , then L_f^ε is a strong ε -Łukasiewicz fuzzy subalgebra of X .

Proof. It is strateforward by Theorems 3.6 and 3.15 because the p -semisimplicity of BCI algebra is equivalent to each of conditions (3.16)–(3.21) (See [9]). \square

Let f be a fuzzy set in X . For an ε -Łukasiewicz fuzzy set L_f^ε of f in X and $t \in (0, 1]$, consider the sets

$$(L_f^\varepsilon, t)_\varepsilon := \{x \in X \mid [x/t] \in L_f^\varepsilon\},$$

$$(L_f^\varepsilon, t)_q := \{x \in X \mid [x/t] q L_f^\varepsilon\},$$

which are called the ε -set and q -set, respectively, of L_f^ε (with value t).

We explore the conditions under which the ε -set and q -set of Łukasiewicz fuzzy set can be subalgebras.

Theorem 3.17. Let L_f^ε be an ε -Lukasiewicz fuzzy set of a fuzzy set f in X . Then the \in -set $(L_f^\varepsilon, t)_\in$ of L_f^ε with value $t \in (0.5, 1]$ is a subalgebra of X if and only if the following assertion is valid.

$$(3.22) \quad (\forall x, y \in X) (\min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \leq \max\{L_f^\varepsilon(x * y), 0.5\}).$$

Proof. Assume that the \in -set $(L_f^\varepsilon, t)_\in$ of L_f^ε with value $t \in (0.5, 1]$ is a subalgebra of X . If the condition (3.22) is not valid, then there exist $a, b \in X$ such that

$$\min\{L_f^\varepsilon(a), L_f^\varepsilon(b)\} > \max\{L_f^\varepsilon(a * b), 0.5\}.$$

If we take $s := \min\{L_f^\varepsilon(a), L_f^\varepsilon(b)\}$, then $s \in (0.5, 1]$ and $[a/s], [b/s] \in L_f^\varepsilon$, i.e., $a, b \in (L_f^\varepsilon, s)_\in$. Since $(L_f^\varepsilon, s)_\in$ is a subalgebra of X , we have $a * b \in (L_f^\varepsilon, s)_\in$. But $[(a * b)/s] \notin L_f^\varepsilon$ implies $a * b \notin (L_f^\varepsilon, s)_\in$, a contradiction. Thus we have

$$\min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \leq \max\{L_f^\varepsilon(x * y), 0.5\} \text{ for all } x, y \in X.$$

Conversely, suppose that L_f^ε satisfies (3.22). Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $x \in (L_f^\varepsilon, t)_\in$ and $y \in (L_f^\varepsilon, t)_\in$. Then $L_f^\varepsilon(x) \geq t$ and $L_f^\varepsilon(y) \geq t$, which imply from (3.22) that

$$0.5 < t \leq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \leq \max\{L_f^\varepsilon(x * y), 0.5\}.$$

Thus $[(x * y)/t] \in L_f^\varepsilon$, i.e., $x * y \in (L_f^\varepsilon, t)_\in$. So $(L_f^\varepsilon, t)_\in$ is a subalgebra of X for $t \in (0.5, 1]$. \square

Theorem 3.18. Let L_f^ε be an ε -Lukasiewicz fuzzy set of a fuzzy set f in X . If f is a fuzzy subalgebra of X , then the q -set $(L_f^\varepsilon, t)_q$ of L_f^ε with value $t \in (0, 1]$ is a subalgebra of X .

Proof. Let $t \in (0, 1]$ and $x, y \in (L_f^\varepsilon, t)_q$. Then $[x/t] q L_f^\varepsilon$ and $[y/t] q L_f^\varepsilon$, i.e., $L_f^\varepsilon(x) + t > 1$ and $L_f^\varepsilon(y) + t > 1$. It follows from Theorems 3.6 and 3.8 that

$$L_f^\varepsilon(x * y) + t \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} + t = \min\{L_f^\varepsilon(x) + t, L_f^\varepsilon(y) + t\} > 1.$$

Thus $[(x * y)/t] q L_f^\varepsilon$. So $x * y \in (L_f^\varepsilon, t)_q$. Hence $(L_f^\varepsilon, t)_q$ is a subalgebra of X . \square

Theorem 3.19. Let f be a fuzzy set in X . For an ε -Lukasiewicz fuzzy set L_f^ε of f in X , if the q -set $(L_f^\varepsilon, t)_q$ is a subalgebra of X , then L_f^ε satisfies:

$$(3.23) \quad [x/t_a] q L_f^\varepsilon, [y/t_b] q L_f^\varepsilon \Rightarrow [(x * y)/\max\{t_a, t_b\}] \in L_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 0.5]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $[x/t_a] q L_f^\varepsilon$ and $[y/t_b] q L_f^\varepsilon$. Then $x \in (L_f^\varepsilon, t_a)_q \subseteq (L_f^\varepsilon, \max\{t_a, t_b\})_q$ and $y \in (L_f^\varepsilon, t_b)_q \subseteq (L_f^\varepsilon, \max\{t_a, t_b\})_q$. Thus $x * y \in (L_f^\varepsilon, \max\{t_a, t_b\})_q$. Since $\max\{t_a, t_b\} \leq 0.5$,

$$L_f^\varepsilon(x * y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}.$$

So $[(x * y)/\max\{t_a, t_b\}] \in L_f^\varepsilon$. \square

Let f be a fuzzy set in X . For an ε -Lukasiewicz fuzzy set L_f^ε of f in X , consider a set:

$$(3.24) \quad O(L_f^\varepsilon) := \{x \in X \mid L_f^\varepsilon(x) > 0\}$$

which is called an O -set of L_f^ε . It is observed that

$$O(L_f^\varepsilon) = \{x \in X \mid f(x) + \varepsilon - 1 > 0\}.$$

Theorem 3.20. *Let L_f^ε be an ε -Lukasiewicz fuzzy set of a fuzzy set f in X . If f is a fuzzy subalgebra of X , then the O -set $O(L_f^\varepsilon)$ of L_f^ε is a subalgebra of X .*

Proof. Let $x, y \in O(L_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. If f is a fuzzy subalgebra of X , then L_f^ε is an ε -Lukasiewicz fuzzy subalgebra of X (See Theorem 3.6). It follows from Theorem 3.8 that

$$L_f^\varepsilon(x * y) \geq \min\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} = \min\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\} > 0$$

Thus $x * y \in O(L_f^\varepsilon)$. So $O(L_f^\varepsilon)$ is a subalgebra of X . \square

Theorem 3.21. *Let f be a fuzzy set in X . If an ε -Lukasiewicz fuzzy set L_f^ε of f in X satisfies:*

$$(3.25) \quad [x/t_a] \in L_f^\varepsilon, [y/t_b] \in L_f^\varepsilon \Rightarrow [(x * y)/\max\{t_a, t_b\}] q L_f^\varepsilon$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the O -set $O(L_f^\varepsilon)$ of L_f^ε is a subalgebra of X .

Proof. Assume that L_f^ε satisfies the condition (3.25) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in O(L_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. Since $[x/L_f^\varepsilon(x)] \in L_f^\varepsilon$ and $[y/L_f^\varepsilon(y)] \in L_f^\varepsilon$, it follows from (3.25) that

$$(3.26) \quad [(x * y)/\max\{L_f^\varepsilon(x), L_f^\varepsilon(y)\}] q L_f^\varepsilon.$$

If $x * y \notin O(L_f^\varepsilon)$, then $L_f^\varepsilon(x * y) = 0$. Thus we get

$$\begin{aligned} L_f^\varepsilon(x * y) + \max\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} &= \max\{L_f^\varepsilon(x), L_f^\varepsilon(y)\} \\ &= \max\{\max\{0, f(x) + \varepsilon - 1\}, \max\{0, f(y) + \varepsilon - 1\}\} \\ &= \max\{f(x) + \varepsilon - 1, f(y) + \varepsilon - 1\} \\ &= \max\{f(x), f(y)\} + \varepsilon - 1 \\ &\leq 1 + \varepsilon - 1 = \varepsilon \leq 1, \end{aligned}$$

which shows that (3.26) is not valid. This is a contradiction. So $x * y \in O(L_f^\varepsilon)$. Hence $O(L_f^\varepsilon)$ is a subalgebra of X . \square

Theorem 3.22. *Let f be a fuzzy set in X . If an ε -Lukasiewicz fuzzy set L_f^ε of f in X satisfies the condition (3.23) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the O -set $O(L_f^\varepsilon)$ of L_f^ε is a subalgebra of X .*

Proof. Let $x, y \in O(L_f^\varepsilon)$. Then $f(x) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. Hence

$$L_f^\varepsilon(x) + 1 = \max\{0, f(x) + \varepsilon - 1\} + 1 = f(x) + \varepsilon - 1 + 1 = f(x) + \varepsilon > 1$$

and

$$L_f^\varepsilon(y) + 1 = \max\{0, f(y) + \varepsilon - 1\} + 1 = f(y) + \varepsilon - 1 + 1 = f(y) + \varepsilon > 1,$$

i.e., $[x/1] q L_f^\varepsilon$ and $[y/1] q L_f^\varepsilon$. It follows from (3.23) that

$$(3.27) \quad [(x * y)/1] = [(x * y)/\max\{1, 1\}] \in L_f^\varepsilon.$$

If $x * y \notin O(L_f^\varepsilon)$, then $L_f^\varepsilon(x * y) = 0 < 1$ and so (3.27) is not valid. This is a contradiction. Thus $x * y \in O(L_f^\varepsilon)$. So $O(L_f^\varepsilon)$ is a subalgebra of X . \square

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