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Triangular shaped type-2 fuzzy numbers:

Application to type-2 fuzzy differential equations

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Abstract. It is generally difficult to compute the level cut sets of type-2 fuzzy sets, so type-2 fuzzy sets that are more naturally represented and whose level cut sets can be computed relatively easily are desired. Then, we introduce a new type-2 fuzzy number called Triangular Shaped Type-2 Fuzzy Number in this paper. Moreover, we study calculus for type-2 fuzzy-number-valued functions and solve the problem of a fuzzy differential equation whose initial value is this type-2 fuzzy number.

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Keywords: Type-1 / type-2 fuzzy number, Type-1 / type-2 fuzzy-number-valued function, Type-1 / type-2 H-derivative, Type-1 / type-2 fuzzy differential equation, Type-1 / type-2 fuzzy initial value problem.

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1. Introduction

The concept of type-2 fuzzy sets is introduced by Zadeh [1] in 1975. This event was exactly 10 years after he [2] introduced the concept of (type-1) fuzzy sets. Also, the concept of fuzzy numbers, a special fuzzy set, was considered. For more information on type-1 fuzzy numbers, see Appendix A.

Type-2 fuzzy sets are fuzzy sets with ambiguous membership grades. Generally speaking, membership functions of a fuzzy set are subjectively determined by the observer, so it is rather natural that membership grades contain ambiguity. We can say, in other words, the type-2 fuzzy set includes not only the uncertainty of the data, but also the membership function which indicates the uncertainty. Then, we consider the membership function of a type-2 fuzzy set by characterizing it *like* a two-variable function. In type-2 fuzzy theory, we have to consider two level cut sets, i.e., (α, β)-cut sets, where α and β are the level on data and the observer, respectively. In 2014, Mazandarani and Najariyan [3] gave the way to represent the (α, β)-cut set of a special type-2 fuzzy number called "triangular perfect quasi-type-2 fuzzy number" on the study of type-2 fuzzy differential equation theory. Recently, Bayeğ et al. [4] conducted a study on this triangular perfect quasi-type-2 fuzzy number. However, it is not easy to compute the (α, β)-cut set of a type-2 fuzzy set and there are several research papers on it (e.g., [5, 6]). Related to these studies, we give the way to represent the (α, β)-cut set of more general type-2 fuzzy numbers. We study the application of them to type-2 fuzzy differential equation theory in this paper.

As such, type-2 fuzzy set theory is applying to fuzzy differential equation theory (See, e.g., [3, 4, 7]). The concept of fuzzy derivatives was proposed by Chang and Zadeh [8]. In addition, fuzzy derivatives using the extension principle were proposed by Dubois and Prade [9], and some other concepts related to fuzzy derivatives were discussed by Puri and Ralescu [10]. Fuzzy initial value problems have been researched since Kaleva [11], and Bede and Gal [12], and several attempts have been proposed to define the differentiability of fuzzy functions. Among them, Hukuhara differentiability and strongly generalized differentiability [12, 10] have attracted particular attention. This concept has been applied to the theory of differential equations, and hence fuzzy differential equation theory has been established. These studies can also be found in other references, such as [13, 14, 15, 16, 17]. To discuss the differentiability of fuzzy functions, the concept of topology needs to be introduced for them. This is summarized in detail in, e.g., [18, 17].

For the sake of simplicity, "type-n fuzzy differential equations" and "type-n fuzzy initial value problems" are often abbreviated as "TnFDEs" and "TnFIVPs", respectively in this paper. (It should be added that [19, 20] are also available for the studies on the approximation by $[0, 1]$-valued solutions of (probabilistic) differential equations.)

For example, suppose that we have a highly experienced expert and an inexperienced student measure the temperature of a certain substance and ask them to indicate the membership function of temperature (More specifically, see Section 5 of [3]). Then, there is a possibility that different membership functions will be expressed. In such a case, by applying the type-2 fuzzy concept, we can express the appearance of measurements of a highly experienced expert and an inexperienced student in a "coherent" form. To be more specific, the appearance of measurements of a highly experienced expert can be expressed as the principal set (Definition 2.6) of the type-2 fuzzy set and that of an inexperienced student expressed as vertical slices (Definition 2.3) of the type-2 fuzzy set. The footprint (Definition 2.6) of the type-2 fuzzy set represents the appearance of the measurement of the student with the worst measurement accuracy. Moreover, type-2 fuzzy sets are useful if the exact form of the membership function is not known, or if the grade of the membership function itself is ambiguous or inaccurate. Since there are so many problems where the exact form of the membership function cannot be determined, type-2 fuzzy sets are suitable for dealing with high levels of uncertainty that involve more complicated calculations. Also, the parameters and variables appearing in differential equations in real problems are usually very imprecise, but we may well be able to model them by type-2 fuzzy theory and hence T2FDEs.

This paper is organized as follows: Section 2 gives some basic knowledge of type-2 fuzzy sets or numbers, which is necessary for this paper. To discuss type-2 fuzzy sets, basic knowledge of type-1 fuzzy sets is required, which are summarized in an appendix at the end of this paper (Appendix A). So, refer to it as needed. Section 3 gives the definition of a new type-2 fuzzy number which we will call a "triangular shaped type-2 fuzzy number". Moreover, we will define the operations for it and compute the (α, β)-cut set of it. We also derive some theorems on type-2 fuzzy differentiation. Section 4 gives how to solve a concrete problem on a T2FDE. For this purpose, we will define the "type-2 fuzzy solution" as the solution of the T2FDE. Section 5 gives the conclusion of this paper.

Throughout this paper, we propose some new notations (e.g., $\dagger, \ddagger$) that seem better because type-2 fuzzy theory tends to be complicated in notations.

2. Preparation

We prepare definitions of terms, notations and known results on type-2 fuzzy theory required in this paper.

For convenience, we use a notation such as "the fuzzy set $A : X \to [0, 1]$" and basically write $A(x)$ for the membership grade of A. "Crisp" means "non-fuzzy".

2.1. Type-2 fuzzy sets.

We first introduce type-2 fuzzy sets in our way. $E^1(X)$ denotes the type-1 fuzzy space on X (See Appendix A.1). The definitions that appear below (Definitions 2.1-2.7) are listed primarily with reference to [3, 21].

Definition 2.1. $\widetilde{A}$ is a *type-2 fuzzy set* on X, if the membership function of $\widetilde{A}$ is defined by

$$\mu_{\widetilde{A}} : I \times J_x \to [0, 1] \quad (\ (x, u) \mapsto \mu_{\widetilde{A}}(x, u)\ ), \tag{2.1}$$

where $I \subset X$ is the universe for the primary variable $x \in X$ and $J_x \subset [0, 1]$ is the interval determined for each $x \in I$. $\widetilde{A}$ is also denoted by

$$\widetilde{A} = \{(x, u; \mu_{\widetilde{A}}(x, u)) : x \in I,\ u \in J_x\} = \int_{x \in I} \int_{u \in J_x} \mu_{\widetilde{A}}(x, u)/(x, u). \tag{2.2}$$

In (2.2), "$\int$" means a continuous union for sets, not an integral. As well, "/" means a marker, not a division. If $\widetilde{A}$ is discrete, we rewrite "$\int$" to "$\sum$".

Definition 2.2. Let $\widetilde{A}$ be a type-2 fuzzy set with $\mu_{\widetilde{A}} : I \times J_x \to [0, 1]$. Then I and J_x for each $x \in I$ are called the *primary domain* and *secondary domain* of $\widetilde{A}$ respectively.

As with type-1, type-2 fuzzy numbers are special type-2 fuzzy sets. In order to discuss a type-2 fuzzy number we think is ideal, we would like to propose a new type-2 fuzzy number whose bottom (called "footprint" later) is a parallelogram.

Definition 2.3. Let $\widetilde{A}$ be a type-2 fuzzy set with $\mu_{\widetilde{A}} : I \times J_x \to [0, 1]$. By fixing $x \in I$ arbitrarily, one type-1 fuzzy set appears. We call it the *vertical slice* of $\widetilde{A}$. Then its membership function is denoted by

$$\nu^x_{\widetilde{A}} : J_x \to [0, 1]$$

and called the *secondary membership function* of $\widetilde{A}$ at x. This $\nu^x_{\widetilde{A}}$ characterizes $\widetilde{A}$:

$$\widetilde{A} = \int_{x\in X} \left(\int_{u\in J_x} \nu^x_{\widetilde{A}}(u)/u \right) \Big/ x,$$

and then $\nu^x_{\widetilde{A}}(u)$ is called the *secondary membership grade* of $\widetilde{A}$.

Remark 2.4. Vertical slices and secondary membership functions are often treated equally, and hence "secondary membership functions" are also called "vertical slices" in this paper.

There are two kinds of cutting with respect to β and what is important is what to cut with respect to β.

Definition 2.5 (Cutting for vertical slices)**.** Let $\widetilde{A}$ be a type-2 fuzzy set on X. Then

$$S_{\widetilde{A}}(x|\beta) := \begin{cases} \{u \in J_x : \nu^x_{\widetilde{A}}(u) \geq \beta\} & \beta \in (0,1], \\ \mathrm{cl}(\{u \in J_x : \nu^x_{\widetilde{A}}(u) > 0\}) & \beta = 0 \end{cases}$$

is called the β-*cut set* of the vertical slice of $\widetilde{A}$.

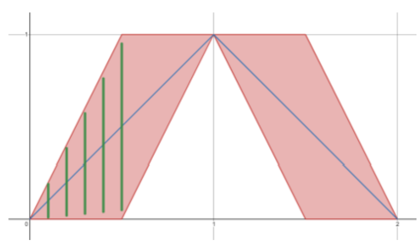

FIGURE 1. 0-cut set of the vertical slice

Definition 2.6 (Cutting for type-2 fuzzy sets)**.** Let $\widetilde{A}$ be a type-2 fuzzy set on X. Then for $\beta \in [0,1]$,

$$\widetilde{A}_\beta := \bigcup_{x\in I} S_{\widetilde{A}}(x|\beta)$$

is called the β-*plane* of $\widetilde{A}$. In particular, $\widetilde{A}_1$ and $\widetilde{A}_0$ are called the *principal* (or *principle*) *set* and *footprint* (*set*) of $\widetilde{A}$ respectively. If the β-plane of $\widetilde{A}$ is the interval-valued fuzzy set, we denote

$$\widetilde{A}_\beta = \left\langle \underline{A_\beta},\ \overline{A_\beta} \right\rangle \tag{2.3}$$

for convenience, where two type-1 fuzzy sets $\underline{A_\beta}$ and $\overline{A_\beta}$ are called the *lower membership function* (briefly, LMF) and *upper membership function* (briefly, UMF) on $\widetilde{A}$ respectively.

Definition 2.7. Let $\widetilde{A}$ be a type-2 fuzzy set on X. Then, the coupling, which becomes the crisp set, of α-cut sets of $\underline{A_\beta}$ and $\overline{A_\beta}$ is called the (α, β)-*cut set* of $\widetilde{A}$:

$$[\widetilde{A}]^\alpha_\beta := \left\langle [\underline{A_\beta}]_\alpha,\ [\overline{A_\beta}]_\alpha \right\rangle \tag{2.4}$$

for any $\alpha \in [0, 1]$.

As with type-1, it is sufficient to discuss β-planes or (α, β)-cut sets of type-2 fuzzy sets. This is because the level cut sets make up the original type-2 fuzzy set $\widetilde{A}$:

Proposition 2.8 (Hamrawi, [5]). *For any type-2 fuzzy set $\widetilde{A}$ on X, one has*

$$\widetilde{A} = \bigcup_{\beta \in [0,1]} \beta \bigcup_{\alpha \in [0,1]} \alpha \widetilde{A}^\alpha_\beta,$$

where $\alpha \widetilde{A}^\alpha_\beta : X \to \{0, \alpha\}$ is a type-1 fuzzy set.

Example 2.9. Let $\widetilde{A}$ be a type-2 fuzzy set on $\mathbb{R}$. The principal set $\widetilde{A}_1$ is given by

$$\widetilde{A}_1(x) = \max\{1 - |x - 1|,\ 0\}$$

and the secondary membership function $\nu^x_{\widetilde{A}}$ is given by

$$\nu^x_{\widetilde{A}}(u) = \max\{1 - 4|u - \widetilde{A}_1(x)|,\ 0\}, \quad u \in [0, 1].$$

The graph of $\nu^x_{\widetilde{A}}$ can be seen in Figure 2. Let us compute the concrete β-plane of

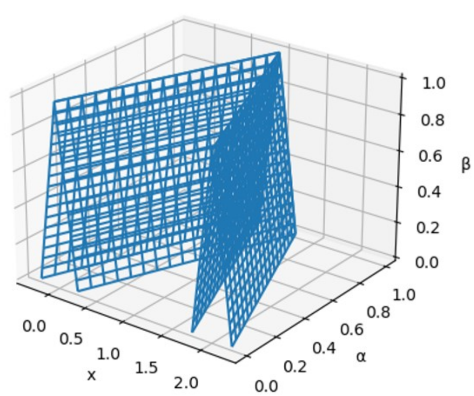

FIGURE 2. Graph of $\nu^x_{\widetilde{A}}$

$\widetilde{A}$. Set $\beta = 0.5$. Then, we have

$$\underline{A_\beta} = \max\left\{\frac{7}{8} - |x - 1|,\ 0\right\},$$

$$\overline{A_\beta} = \min\left\{\max\left\{\frac{9}{8} - |x - 1|,\ 0\right\},\ 1\right\}.$$

Their α-cut sets for $(\alpha, \beta) = (0.3, 0.5)$ is obtained as

$$[\underline{A_\beta}]_\alpha = \left[\frac{17}{40}, \frac{63}{40}\right] = [0.425,\ 1.575],$$
$$[\overline{A_\beta}]_\alpha = \left[\frac{7}{40}, \frac{73}{40}\right] = [0.175,\ 1.825].$$

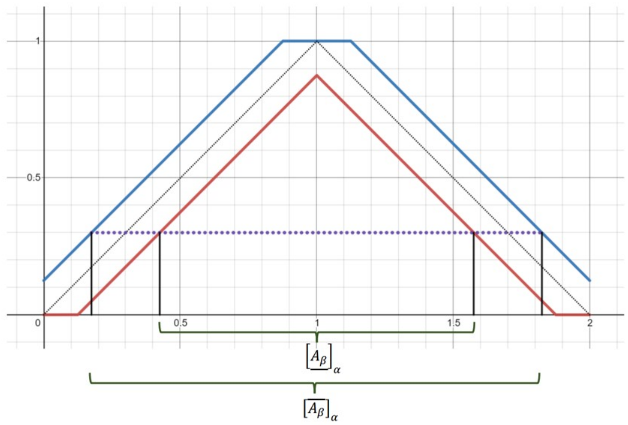

FIGURE 3. LMF (red line) and UMF (blue line) of the type-2 fuzzy set whose core is 1 and their α-cut sets

2.2. **Operations for type-2 fuzzy sets.** In order to solve T2FDEs, for example, one must define the equality and (arithmetic) operations for type-2 fuzzy sets. We list their definitions with reference to [3] below.

Definition 2.10 (Equality for type-2 fuzzy sets)**.** Let $\widetilde{A}$ and $\widetilde{B}$ be type-2 fuzzy sets on X. Then $\widetilde{A} = \widetilde{B}$ if and only if $S_{\widetilde{A}}(x|\beta) = S_{\widetilde{B}}(x|\beta)$ for all $x \in I$ and $\beta \in [0, 1]$.

Definition 2.11 (Linearity for type-2 fuzzy sets)**.** Let $\widetilde{A}$ and $\widetilde{B}$ be type-2 fuzzy sets on X and let $k \in \mathbb{R}$. The *sum* $\widetilde{A} + \widetilde{B}$ of $\widetilde{A}$ and $\widetilde{B}$ is defined by

$$[\widetilde{A} + \widetilde{B}]^\alpha_\beta := \left\langle [\underline{A_\beta} + \underline{B_\beta}]_\alpha,\ [\overline{A_\beta} + \overline{B_\beta}]_\alpha \right\rangle = \left\langle [\underline{A_\beta}]_\alpha + [\underline{B_\beta}]_\alpha,\ [\overline{A_\beta}]_\alpha + [\overline{B_\beta}]_\alpha \right\rangle.$$

Moreover, the *scalar multiple* $k\widetilde{A}$ of $\widetilde{A}$ is defined by

$$[k\widetilde{A}]^\alpha_\beta := \left\langle [k\underline{A_\beta}]_\alpha,\ [k\overline{A_\beta}]_\alpha \right\rangle = \left\langle k[\underline{A_\beta}]_\alpha,\ k[\overline{A_\beta}]_\alpha \right\rangle.$$

Example 2.12. We consider only the example about the sum operation of type-2 fuzzy sets. Let $\widetilde{2}$ and $\widetilde{3}$ be a type-2 fuzzy set on $\mathbb{R}$. Their principal sets $\widetilde{2}_1, \widetilde{3}_1$ and the secondary membership functions $\nu^x_{\widetilde{2}}, \nu^x_{\widetilde{3}}$ are given by

$$\nu^x_{\widetilde{2}}(u) = \max\{1 - 4|u - \widetilde{2}_1(x)|,\ 0\},$$
$$\nu^x_{\widetilde{3}}(u) = \max\{1 - 4|u - \widetilde{3}_1(x)|,\ 0\}$$

where

$$\widetilde{2}_1(x) = \max\{1 - |x-2|,\ 0\},$$
$$\widetilde{3}_1(x) = \max\{1 - |x-2|,\ 0\}.$$

On the other hand, since

$$\underline{2_\beta} = \max\left\{\frac{7+\beta}{8} - |x-2|,\ 0\right\},$$
$$\overline{2_\beta} = \min\left\{\max\left\{\frac{9-\beta}{8} - |x-2|,\ 0\right\},\ 1\right\},$$
$$\underline{3_\beta} = \max\left\{\frac{7+\beta}{8} - |x-3|,\ 0\right\},$$
$$\overline{3_\beta} = \min\left\{\max\left\{\frac{9-\beta}{8} - |x-3|,\ 0\right\},\ 1\right\},$$

we obtain their α-cut sets as

$$[\underline{2_\beta}]_\alpha = \left[\frac{9-\beta}{8} + \alpha,\ \frac{23+\beta}{8} - \alpha\right],$$
$$[\underline{3_\beta}]_\alpha = \left[\frac{17-\beta}{8} + \alpha,\ \frac{31+\beta}{8} - \alpha\right]$$

for $\alpha \in [0, 7/8]$ and

$$[\overline{2_\beta}]_\alpha = \left[\frac{7-\beta}{8} + \alpha,\ \frac{25+\beta}{8} - \alpha\right],$$
$$[\overline{3_\beta}]_\alpha = \left[\frac{15-\beta}{8} + \alpha,\ \frac{33+\beta}{8} - \alpha\right]$$

for $\alpha \in [0, 1]$. Then from these, we have

$$[\widetilde{2} + \widetilde{3}]^\alpha_\beta = \left\langle [\underline{2_\beta}]_\alpha + [\underline{3_\beta}]_\alpha,\ [\overline{2_\beta}]_\alpha + [\overline{3_\beta}]_\alpha \right\rangle$$
$$= \left\langle \left[\frac{13-\beta}{4} + 2\alpha,\ \frac{27+\beta}{4} - 2\alpha\right],\ \left[\frac{11-\beta}{4} + 2\alpha,\ \frac{29+\beta}{4} - 2\alpha\right] \right\rangle.$$

Thus by this result, we can calculate the LMF and UMF of $\widetilde{2} + \widetilde{3}$ as follows:

$$\underline{(\widetilde{2} + \widetilde{3})_\beta} = \max\left\{\frac{7+\beta}{8} - \frac{1}{2}|x-5|,\ 0\right\},$$
$$\overline{(\widetilde{2} + \widetilde{3})_\beta} = \min\left\{\max\left\{\frac{9+\beta}{8} - \frac{1}{2}|x-5|,\ 0\right\},\ 1\right\}.$$

So we have the secondary membership function

$$\nu^x_{\widetilde{2}+\widetilde{3}}(u) = \max\{1 - 4|u - (\widetilde{2} + \widetilde{3})_1(x)|,\ 0\}$$

where

$$(\widetilde{2} + \widetilde{3})_1(x) = \max\left\{1 - \frac{1}{2}|x-5|,\ 0\right\}$$

is the principal set.

We consider the type-2 version of H-differences introduced in [3].

Definition 2.13. Let $\widetilde{A}$ and $\widetilde{B}$ be type-2 fuzzy sets on X. If there exists some type-2 fuzzy set $\widetilde{C}$ such that

$$\widetilde{A} = \widetilde{B} + \widetilde{C},$$

then we call $\widetilde{C}$ the *T2-H-difference* of $\widetilde{A}$ and $\widetilde{B}$, and is denoted by $\widetilde{A} - \widetilde{B}$.

Proposition 2.14 ([3])**.** *Let $\widetilde{A}$ and $\widetilde{B}$ be type-2 fuzzy sets on X. Then,*

$$[\widetilde{A} - \widetilde{B}]_\beta = \left\langle \underline{A_\beta} - \underline{B_\beta},\ \overline{A_\beta} - \overline{B_\beta} \right\rangle$$

where the differences appearing in the right-hand side are T1-H-differences.

We adopt the following distance to set the type-2 fuzzy topology.

Definition 2.15 ([25])**.** Let $\widetilde{A}$ and $\widetilde{B}$ be type-2 fuzzy sets on X. A *Hung–Yang distance* between $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$d_{\mathrm{HY}}(\widetilde{A}, \widetilde{B}) := \int_I H_{\mathrm{f}}(\nu^x_{\widetilde{A}}, \nu^x_{\widetilde{B}})\, dx, \tag{2.5}$$

where

$$H_{\mathrm{f}}(\nu^x_{\widetilde{A}}, \nu^x_{\widetilde{B}}) := \frac{\int_0^1 \beta d_{\mathrm{H}}(S_{\widetilde{A}}(x|\beta), S_{\widetilde{B}}(x|\beta))\, d\beta}{\int_0^1 \beta\, d\beta} = 2\int_0^1 \beta d_{\mathrm{H}}(S_{\widetilde{A}}(x|\beta), S_{\widetilde{B}}(x|\beta))\, d\beta$$

and the integrals are in the sense of Riemann. We denote by $E^2(X)$ the space of type-2 fuzzy numbers on X equipped with d_{HY}-topology.

Remark 2.16. See Theorem 2.3 of [3] to make sure $E^2(X)$ is a crisp metric space, i.e., d_{HY} satisfies the metric axiom.

2.3. **Triangular shaped type-2 fuzzy numbers and their level cut sets.** We introduce a type-2 version of triangular shaped type-1 fuzzy numbers. It is important to define the principal set as a symmetric triangular type-1 fuzzy number. The type-2 fuzzy number is the fuzzy number whose membership grade is type-1. So, we decide to treat membership grades of 1 and 0 not as "about 1" and "about 0" but as crisp "1" and "0" without fuzziness, and to treat a grade of 0.5 as having the highest degree of fuzziness. Based on this thought, we propose a new type-2 fuzzy number as follows.

Definition 2.17. Let $\widetilde{A}$ be a type-2 fuzzy set on $\mathbb{R}$ and $a \in \mathbb{R}$ its core. Then, $\widetilde{A}$ is called a *triangular shaped type-2 fuzzy number* (shortly, TST2FN) on $\mathbb{R}$, if the principal set $\widetilde{A}_1$ is given by

$$\widetilde{A}_1(x) := \max\{1 - |x - a|,\ 0\} \tag{2.6}$$

and the *secondary membership function* $\nu^x_{\widetilde{A}}$ at $x \in I$ is given by

$$\nu^x_{\widetilde{A}}(t) := \begin{cases} 1 - \dfrac{|t - \widetilde{A}_1(x)|}{\min\{\widetilde{A}_1(x), 1 - \widetilde{A}_1(x)\}} & if\ \widetilde{A}_1(x) \in (0,1), \\ \left\{ \begin{array}{ll} 1 & if\ t = \widetilde{A}_1(x) \\ 0 & if\ t \neq \widetilde{A}_1(x) \end{array} \right\} & if\ \widetilde{A}_1(x) \in \{0,1\}. \end{cases} \tag{2.7}$$

We denote by $T^2(\mathbb{R})$ the space of TST2FNs on $\mathbb{R}$ and note that $T^2(\mathbb{R})$ is equipped with d_{HY}.

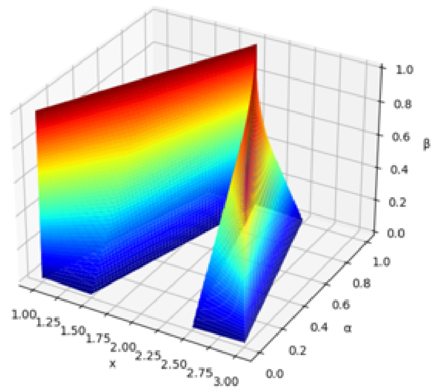

FIGURE 4. Type-2 fuzzy set for "about 2"

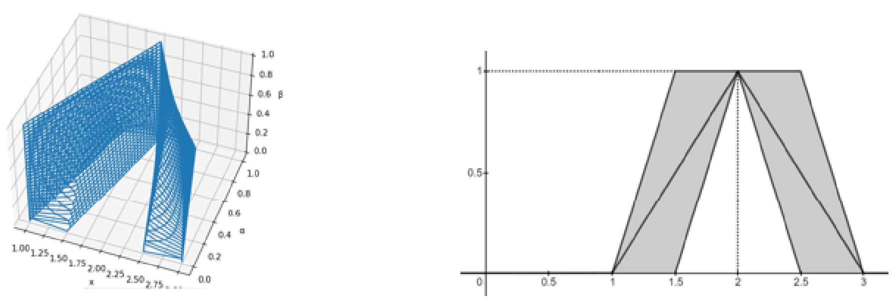

FIGURE 5. Footprint set of "about 2" in the sense of type-2

Remark 2.18. For any $\widetilde{A} \in T^2(\mathbb{R})$, the left and right sides of its footprint are congruently parallelograms with a width length of 0.5 (See Figure 5). Moreover, the principal set of $\widetilde{A}$ consists of diagonals of the two parallelograms (See Figure 5-6).

TST2FNs are relatively easy to compute their level cut sets. Let us find the (α, β)-cut set of a TST2FN. For convenience, consider $\widetilde{A} \in T^2(\mathbb{R})$ whose core is $a \in \mathbb{R}$. The footprint of $\widetilde{A}$ is the area consisting of two parallelograms (See the red area in Figure 7)

$$\min\{y,\ y - 1 + 2|x - a|,\ -y + 2 - 2|x - a|,\ -y + 1\} \geq 0.$$

Moreover, the principal set of $\widetilde{A}$ is

$$\widetilde{A}_1 = \max\{1 - |x - a|,\ 0\}$$

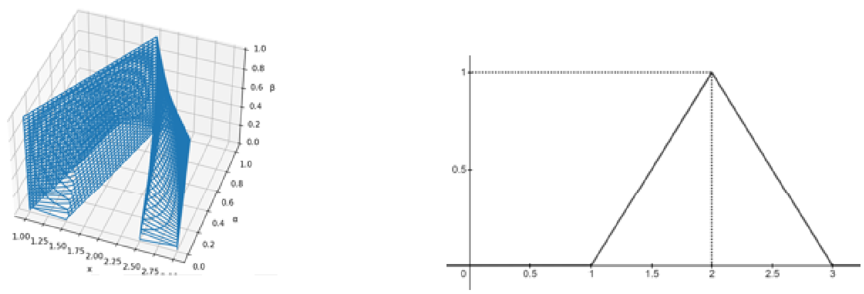

FIGURE 6. Principal set of "about 2" in the sense of type-2

within $a-1 \leq x \leq a+1$ (See the blue line in Figure 7). For any $\beta \in [0,1]$, the β-plane $\widetilde{A}_\beta$ is given as the continuous set of all $(x,u) \in I \times J_x$ such that

$$\begin{aligned} &\min\{\,(2-\beta)-|(2-\beta)(x-a)|-y,\ 1-\beta|x-a|-y,\\ &\qquad y-\beta+\beta|x-a|,\ -1+|(2-\beta)(x-a)|+y\,\} \geq 0. \end{aligned} \tag{2.8}$$

As we can see from Figure 8, this is the area consisting of two parallelograms (blue area). By (2.8), we can compute the (α,β)-cut set of $\widetilde{A}$ (See Figure 9). That is, it can be summarized as follows.

Theorem 2.19. *Let $\widetilde{A} \in T^2(\mathbb{R})$ have core $a \in \mathbb{R}$ and let $\alpha,\ \beta \in [0,1]$.*

(1) *If $0 < \alpha \leq \beta/2$, then one has*

$$[\widetilde{A}]^\alpha_\beta = \left\langle \left[a-\frac{2-\alpha-\beta}{2-\beta},\ a-\frac{\beta-\alpha}{\beta}\right],\ \left[a+\frac{\beta-\alpha}{\beta},\ a+\frac{2-\alpha-\beta}{2-\beta}\right] \right\rangle. \tag{2.9}$$

(2) *If $\beta/2 < \alpha \leq 1-\beta/2$, then one has*

$$[\widetilde{A}]^\alpha_\beta = \left\langle \left[a-\frac{2-\alpha-\beta}{2-\beta},\ a-\frac{1-\alpha}{2-\beta}\right],\ \left[a+\frac{1-\alpha}{2-\beta},\ a+\frac{2-\alpha-\beta}{2-\beta}\right] \right\rangle. \tag{2.10}$$

(3) *If $1-\beta/2 < \alpha \leq 1$, then one has*

$$[\widetilde{A}]^\alpha_\beta = \left\langle \left[a-\frac{1-\alpha}{\beta},\ a-\frac{1-\alpha}{2-\beta}\right],\ \left[a+\frac{1-\alpha}{2-\beta},\ a+\frac{1-\alpha}{\beta}\right] \right\rangle. \tag{2.11}$$

Concrete examples of TST2FNs will be also seen in Section 4.

2.4. **Type-2 fuzzy-number-valued functions.** Let $I \subset \mathbb{R}$. We consider the crisp function $f: I \to \mathbb{R}$. Zadeh's extension principle derives the type-2 fuzzy function $\widetilde{F}: E^2(I) \to E^2(\mathbb{R})$ via f in the same way as type-1. In what follows, we treat the case of $E^2(I) = I$ and call $\widetilde{F}: I \to E^2(\mathbb{R})$ the *type-2 fuzzy-number-valued function* (shortly, T2FNV function). The β-plane and (α,β)-cut set of $\widetilde{F}: I \to E^2(\mathbb{R})$ are represented by

$$\left[\widetilde{F}(x)\right]_\beta = \left\langle \underline{F_\beta}(x),\ \overline{F_\beta}(x) \right\rangle$$

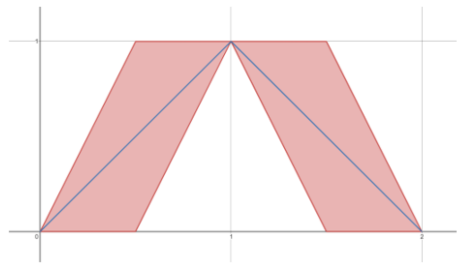

FIGURE 7. Footprint and principal set of $\widetilde{A}$ whose core is 1

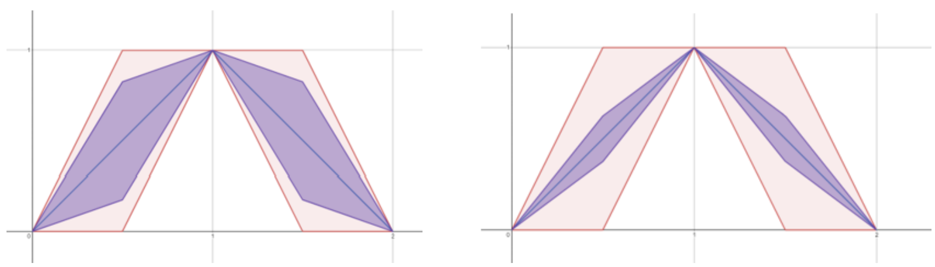

FIGURE 8. β-plane if $\beta \in (0, 0.5]$ and β-plane if $\beta \in (0.5, 1)$

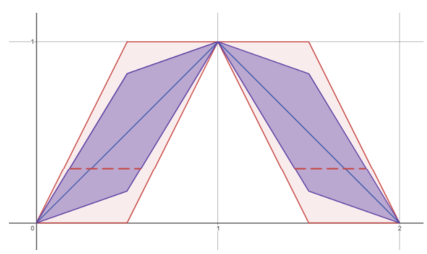

FIGURE 9. α-cut set of the β-plane

and

$$\left[\widetilde{F}(x)\right]_\beta^\alpha = \left\langle [\underline{F_\beta}(x)]_\alpha,\ [\overline{F_\beta}(x)]_\alpha \right\rangle,$$

respectively. Here $\underline{F_\beta}, \overline{F_\beta} : I \to E^1(\mathbb{R})$ are type-1 fuzzy-number-valued functions and we denote

$$\left[\underline{F_\beta}(x)\right]_\alpha := \left[\underline{F_\beta}_-(x;\alpha),\ \underline{F_\beta}_+(x;\alpha)\right],$$
$$\left[\overline{F_\beta}(x)\right]_\alpha := \left[\overline{F_\beta}_-(x;\alpha),\ \overline{F_\beta}_+(x;\alpha)\right]$$

for all $x \in I$ and any $\alpha, \beta \in [0,1]$. For convenience, we also denote crisp function values $\underline{F_\beta}_\pm(x;\alpha)$ and $\overline{F_\beta}_\pm(x;\alpha)$ by $\underline{F}_{\alpha,\beta,\pm}(x)$ and $\overline{F}_{\alpha,\beta,\pm}(x)$, respectively.

We use daggers $\dagger, \ddagger$ introduced in Appendix A.2 as symbols for type-2 fuzzy derivatives. The following definition was introduced in [3] with reference to the case of type-1.

Definition 2.20. Let $\widetilde{F} : I \to E^2(\mathbb{R})$ and $h > 0$ be a crisp number.

- $\widetilde{F}$ is *(1)-T2-differentiable at some* $x_0 \in I$ if and only if there exist $\widetilde{F}(x_0 + h) - \widetilde{F}(x_0)$ and $\widetilde{F}(x_0) - \widetilde{F}(x_0 - h)$ satisfying that the fuzzy limit

$$\widetilde{F}^\dagger(x_0) := \lim_{h\downarrow 0} \frac{\widetilde{F}(x_0+h) - \widetilde{F}(x_0)}{h} = \lim_{h\downarrow 0} \frac{\widetilde{F}(x_0) - \widetilde{F}(x_0-h)}{h} \tag{2.12}$$

  exists.
- $\widetilde{F}$ is *(2)-T2-differentiable at some* $x_0 \in I$ if and only if there exist $\widetilde{F}(x_0) - \widetilde{F}(x_0 + h)$ and $\widetilde{F}(x_0 - h) - \widetilde{F}(x_0)$ satisfying that the fuzzy limit

$$\widetilde{F}^\ddagger(x_0) := \lim_{h\uparrow 0} \frac{\widetilde{F}(x_0) - \widetilde{F}(x_0+h)}{-h} = \lim_{h\uparrow 0} \frac{\widetilde{F}(x_0-h) - \widetilde{F}(x_0)}{-h} \tag{2.13}$$

  exists.

Here the above differences and limits are due to the meaning of T2-Hukuhara and d_{HY}, respectively. Moreover, if $\widetilde{F}$ is T2-differentiable in both senses at any $x \in I$, then $\widetilde{F}^\dagger$ and $\widetilde{F}^\ddagger$ is called the *(1)-T2-derivative* and *(2)-T2-derivative of* $\widetilde{F}$ respectively.

Remark 2.21. (1) Like Remark A.11, we shall ignore T2-derivatives in the third and fourth forms. The limits of both the forms become crisp numbers as with type-1. This thing has already been mentioned in Note 4.1 of [3].

(2) As with type-1, T2-derivatives of order n, $n \geq 2$, are obtained by applying first-order T2-derivatives to (2.12) and (2.13).

Theorem 2.22 ([3])**.** *Let $\widetilde{F} : I \to E^2(\mathbb{R})$ be T2-differentiable on I. Then, the parametric forms of its T2-derivatives are given by*

(1) *the (1)-form:*

$$\begin{aligned}\left[\widetilde{F}^\dagger(x)\right]_\beta^\alpha &= \left\langle \left[\underline{F_\beta}^\dagger(x)\right]_\alpha, \left[\overline{F_\beta}^\dagger(x)\right]_\alpha \right\rangle \\ &= \left\langle \left[\underline{F}'_{\alpha,\beta,-}(x), \underline{F}'_{\alpha,\beta,+}(x)\right], \left[\overline{F}'_{\alpha,\beta,-}(x), \overline{F}'_{\alpha,\beta,+}(x)\right] \right\rangle,\end{aligned}$$

(2) *the (2)-form:*

$$\begin{aligned}\left[\widetilde{F}^\ddagger(x)\right]_\beta^\alpha &= \left\langle \left[\underline{F_\beta}^\ddagger(x)\right]_\alpha, \left[\overline{F_\beta}^\ddagger(x)\right]_\alpha \right\rangle \\ &= \left\langle \left[\underline{F}'_{\alpha,\beta,+}(x), \underline{F}'_{\alpha,\beta,-}(x)\right], \left[\overline{F}'_{\alpha,\beta,+}(x), \overline{F}'_{\alpha,\beta,-}(x)\right] \right\rangle.\end{aligned}$$

3. Theorems on Type-2 fuzzy differentiation

We study basic properties on type-2 fuzzy differentiation in this section.

3.1. **Continuity of T2FNV functions.** We begin by defining the continuity of T2FNV Functions.

Definition 3.1. Let $\widetilde{F} : I \to E^2(\mathbb{R})$ and $h > 0$ be a crisp number. $\widetilde{F}$ is *T2-continuous* on I, if it follows in d_{HY} that

$$\lim_{h\downarrow 0}\{\widetilde{F}(x+h) - \widetilde{F}(x)\} = \lim_{h\downarrow 0}\{\widetilde{F}(x) - \widetilde{F}(x-h)\} = 0 \tag{3.1}$$

for any $x \in I$. We denote by $C(I; E^2(\mathbb{R}))$ the space of such T2FNV functions.

Proposition 3.2. *If $\widetilde{F} : I \to E^2(\mathbb{R})$ is T2-differentiable on I, then $\widetilde{F}$ is T2-continuous on I.*

Proof. Since $\widetilde{F}$ is T2-differentiable on I from the assumption, the limit

$$\lim_{h\downarrow 0}\frac{\widetilde{F}(x+h) - \widetilde{F}(x)}{h} = \lim_{h\downarrow 0}\frac{\widetilde{F}(x) - \widetilde{F}(x-h)}{h}$$

exists. By expressing, for any $x \in I$,

$$\widetilde{F}(x+h) - \widetilde{F}(x) = \frac{\widetilde{F}(x+h) - \widetilde{F}(x)}{h}h \tag{3.2}$$

$$\widetilde{F}(x) - \widetilde{F}(x-h) = \frac{\widetilde{F}(x) - \widetilde{F}(x-h)}{h}h \tag{3.3}$$

and by letting $h \downarrow 0$ in both sides of these, we have (3.1). □

Remark 3.3. Expressions (3.2) and (3.3) are not always possible if h is fuzzy, since it is *not generally* the case that $u/u = 1$ for any $u \in E^1(\mathbb{R})$. The above proof was made because h is crisp.

3.2. **Differentiation of four operation rules of T2FNV functions.**

Theorem 3.4. *Let $\widetilde{F}, \widetilde{G} : I \to E^2(\mathbb{R})$ be T2-differentiable on I. Then, $\widetilde{F} + \widetilde{G} : I \to E^2(\mathbb{R})$ is T2-differentiable on I and*

$$(\widetilde{F} \pm \widetilde{G})^{\dagger}(x) = \widetilde{F}^{\dagger}(x) \pm \widetilde{G}^{\dagger}(x), \tag{3.4}$$

$$(\widetilde{F} \pm \widetilde{G})^{\ddagger}(x) = \widetilde{F}^{\ddagger}(x) \pm \widetilde{G}^{\ddagger}(x) \tag{3.5}$$

for $x \in I$, where we assume that H-differences $\widetilde{F} - \widetilde{G}$ in the equations on "$-$" of (3.4) and (3.5).

Proof. It is obvious for sums from Definition 2.20. We prove only

$$(\widetilde{F} - \widetilde{G})^{\dagger}(x) = \widetilde{F}^{\dagger}(x) - \widetilde{G}^{\dagger}(x), \quad x \in I. \tag{3.6}$$

The other case can be shown in the same way. Since $\widetilde{F} - \widetilde{G}$ exists, there is some $\widetilde{W} : I \to E^2(\mathbb{R})$ such that

$$\widetilde{F}(x) = \widetilde{G}(x) + \widetilde{W}(x),$$
$$\widetilde{F}(x \pm h) = \widetilde{G}(x \pm h) + \widetilde{W}(x \pm h)$$

where $h > 0$ is a crisp number. Then, we have

$$\begin{aligned}\widetilde{F}(x+h) - \widetilde{F}(x) &= (\widetilde{G}(x+h) + \widetilde{W}(x+h)) - (\widetilde{G}(x) + \widetilde{W}(x)) \\ &= (\widetilde{G}(x+h) - \widetilde{G}(x)) + (\widetilde{W}(x+h) - \widetilde{W}(x))\end{aligned} \tag{3.7}$$

by virtue of Lemma 3.5 mentioned later. Similarly, we also have

$$\widetilde{F}(x) - \widetilde{F}(x-h) = (\widetilde{G}(x) - \widetilde{G}(x-h)) + (\widetilde{W}(x) - \widetilde{W}(x-h)). \tag{3.8}$$

Since $\widetilde{F}$ and $\widetilde{G}$ are T2-differentiable on I from the assumption, $\widetilde{F}(x+h) - \widetilde{F}(x)$, $\widetilde{F}(x) - \widetilde{F}(x-h)$, $\widetilde{G}(x+h) - \widetilde{G}(x)$ and $\widetilde{G}(x) - \widetilde{G}(x-h)$ exist for any $x \in I$. Thus the following derivative in d_{HY} exists:

$$(\widetilde{F} - \widetilde{G})^{\dagger}(x) = \lim_{h \downarrow 0} \frac{\widetilde{W}(x+h) - \widetilde{W}(x)}{h} = \lim_{h \downarrow 0} \frac{\widetilde{W}(x) - \widetilde{W}(x-h)}{h}, \quad x \in I.$$

This limit is $\widetilde{F}^{\dagger}(x) - \widetilde{G}^{\dagger}(x)$, $x \in I$, from (3.7) and (3.8). So we obtain (3.6). This completes the proof. □

Lemma 3.5. *Let $u_j, v_j \in E^1(\mathbb{R})$ $(j = 1, 2)$. Suppose that $(u_1 + v_1) - (u_2 + v_2)$, $u_1 - u_2$ and $v_1 - v_2$ exist. Then, the distributive law for T1-H-differences in the following sense hold:*

$$(u_1 + v_1) - (u_2 + v_2) = (u_1 - u_2) + (v_1 - v_2).$$

Moreover, the same result holds for type-2 fuzzy numbers.

Proof. We prove only for type-1 fuzzy numbers, since the definition of T2-H-differences for type-2 fuzzy numbers is essentially equal to that for type-1 fuzzy numbers. Then, by noting that $-$ is the H-difference, the α-cut discussion implies that

$$\begin{aligned}[(u_1 + v_1) - (u_2 + v_2)]_{\alpha} &= [u_1 + v_1]_{\alpha} - [u_2 + v_2]_{\alpha} \\ &= ([u_1]_{\alpha} + [v_1]_{\alpha}) - ([u_2]_{\alpha} + [v_2]_{\alpha}) \\ &= ([u_1]_{\alpha} - [u_2]_{\alpha}) + ([v_1]_{\alpha} - [v_2]_{\alpha}) \\ &= [u_1 - u_2]_{\alpha} + [v_1 - v_2]_{\alpha} \\ &= [(u_1 - u_2) + (v_1 - v_2)]_{\alpha}\end{aligned}$$

for any $\alpha \in [0, 1]$, and finish the proof. □

Remark 3.6. We can generally have the similar results to Theorem 3.4 for any order $N \in \mathbb{N} \cup \{0\}$.

Theorem 3.7. *Let $f : I \to \mathbb{R}$ be differentiable, let $\widetilde{G} : I \to E^2(\mathbb{R})$ be T2-differentiable on I and let $x \in I$.*

(1) *If $f(x)f'(x) > 0$ and $\widetilde{G}$ is (1)-T2-differentiable, then $f\widetilde{G}$ is (1)-T2-differentiable and*

$$(f\widetilde{G})^{\dagger}(x) = f'(x)\widetilde{G}(x) + f(x)\widetilde{G}^{\dagger}(x). \tag{3.9}$$

(2) *if $f(x)f'(x) < 0$ and $\widetilde{G}$ is (2)-T2-differentiable, then $f\widetilde{G}$ is (2)-T2-differentiable and*

$$(f\widetilde{G})^{\ddagger}(x) = f'(x)\widetilde{G}(x) + f(x)\widetilde{G}^{\ddagger}(x). \tag{3.10}$$

3.3. **Differentiation of composite T2FNV functions.** We finally consider the composition of crisp and type-1 / type-2 fuzzy functions and its derivative.

Theorem 3.8. *Let $f : I \to \mathbb{R}$ be differentiable and $\widetilde{g} : \mathbb{R} \to E^1(\mathbb{R})$ T1-differentiable. We consider the type-1 fuzzy composite function $\widetilde{g} \circ f : I \to E^1(\mathbb{R})$. Suppose that T1-H-differences*

$$(\widetilde{g} \circ f)(x+h) - (\widetilde{g} \circ f)(x) \quad \text{and} \quad (\widetilde{g} \circ f)(x) - (\widetilde{g} \circ f)(x-h)$$

exist for any $x \in I$ and $h > 0$ sufficiently small. Then $\widetilde{g} \circ f$ is T1-differentiable on I and

$$(\widetilde{g} \circ f)^{\dagger} = \widetilde{g}^{\dagger}(f(x))f'(x), \tag{3.11}$$

$$(\widetilde{g} \circ f)^{\ddagger} = \widetilde{g}^{\ddagger}(f(x))f'(x). \tag{3.12}$$

Moreover, the same result holds for $\widetilde{G} : \mathbb{R} \to E^2(\mathbb{R})$.

Proof. We prove only for type-1 $\widetilde{g}$ because we can prove for type-2 $\widetilde{G}$ in the same way. Let $h > 0$ be a crisp number. The α-cut set of the right difference quotient of $(\widetilde{g} \circ f)$ is

$$\begin{aligned}
&\left[\frac{(\widetilde{g} \circ f)(x+h) - (\widetilde{g} \circ f)(x)}{h}\right]_{\alpha} \\
&= \left[\frac{(\widetilde{g} \circ f)_{-,\alpha}(x+h) - (\widetilde{g} \circ f)_{-,\alpha}(x)}{h}, \ \frac{(\widetilde{g} \circ f)_{+,\alpha}(x+h) - (\widetilde{g} \circ f)_{+,\alpha}(x)}{h}\right] \\
&= \left[\frac{\widetilde{g}_{-,\alpha}(f(x+h)) - \widetilde{g}_{-,\alpha}(f(x))}{h}, \ \frac{\widetilde{g}_{+,\alpha}(f(x+h)) - \widetilde{g}_{+,\alpha}(f(x))}{h}\right] \\
&= \left[\frac{\widetilde{g}_{-,\alpha}(f(x+h)) - \widetilde{g}_{-,\alpha}(f(x))}{f(x+h) - f(x)} \frac{f(x+h) - f(x)}{h}, \right. \\
&\qquad\qquad \left. \frac{\widetilde{g}_{+,\alpha}(f(x+h)) - \widetilde{g}_{+,\alpha}(f(x))}{f(x+h) - f(x)} \frac{f(x+h) - f(x)}{h}\right].
\end{aligned}$$

Similarly, the α-cut set of the left difference quotient of $(\widetilde{g} \circ f)$ is

$$\begin{aligned}
&\left[\frac{(\widetilde{g} \circ f)(x) - (\widetilde{g} \circ f)(x-h)}{h}\right]_{\alpha} \\
&= \left[\frac{\widetilde{g}_{-,\alpha}(f(x)) - \widetilde{g}_{-,\alpha}(f(x-h))}{f(x) - f(x-h)} \frac{f(x) - f(x-h)}{h}, \right. \\
&\qquad\qquad \left. \frac{\widetilde{g}_{+,\alpha}(f(x)) - \widetilde{g}_{+,\alpha}(f(x-h))}{f(x) - f(x-h)} \frac{f(x) - f(x-h)}{h}\right].
\end{aligned}$$

Then the rightmost-hand sides of the above two equations converge as $h \downarrow 0$. In fact, $f(x \pm h) \to f(x)$ as $h \downarrow 0$, since f is continuous on I by virtue of the differentiability of f on I. Hence (3.11) can be obtained. (3.12) can be similarly obtained. This completes the proof. □

4. Application to T2FIVPs

We consider T2FDEs and actually solve a concrete T2FIVP in this section. Both "†" and "‡" are collectively denoted as "$\mathscr{D}$".

We consider T2FIVPs on $I = [0, r]$ for some $r > 0$ or $I = [0, +\infty)$:

$$\begin{cases} \mathscr{D}\widetilde{Y}(x) + a\widetilde{Y}(x) = 0, \\ \widetilde{Y}(0) = \widetilde{U} \in T^2(\mathbb{R}), \end{cases} \tag{4.1}$$

where a is a crisp constant. Let us define the meaning of a T2FIVP having the solution. $T^1(\mathbb{R})$ stands for the space of triangular type-1 fuzzy numbers on $\mathbb{R}$ (see Appendix A.1).

Definition 4.1. Let $\widetilde{Y} : I \to T^2(\mathbb{R})$ be T2-differentiable. We denote the β-plane of $\widetilde{Y} = \widetilde{Y}(x)$ by

$$[\widetilde{Y}(x)]_\beta = \left\langle \underline{Y_\beta}(x),\ \overline{Y_\beta}(x) \right\rangle.$$

Then $\widetilde{Y}$ is the (i, j)*-type-2 fuzzy solution* of (4.1), $(i, j) \in \{1, 2\}^2$, if and only if, for each $\beta \in [0, 1]$,

(i) $\mathscr{D}_i\underline{Y_\beta}, \mathscr{D}_i\overline{Y}_\beta$ exist on I, and

(ii) $\underline{Y_\beta}$ and $\overline{Y_\beta} : I \to T^1(\mathbb{R})$ satisfy

$$\begin{cases} [\mathscr{D}_i\underline{Y_\beta}(x)]_\alpha + [a\underline{Y_\beta}(x)]_\alpha = 0, \quad x \in I, \\ \underline{Y_\beta}(0) = \underline{U_\beta} \in T^1(\mathbb{R}) \end{cases}$$

and

$$\begin{cases} [\mathscr{D}_i\overline{Y_\beta}(x)]_\alpha + [a\overline{Y_\beta}(x)]_\alpha = 0, \quad x \in I, \\ \overline{Y_\beta}(0) = \overline{U_\beta} \in T^1(\mathbb{R}) \end{cases}$$

for any $\alpha \in [0, 1]$ respectively.

Remark 4.2. (1) The type-2 fuzzy solution generally becomes the type-1 fuzzy solution if $\beta = 1$; it becomes the crisp solution if $\alpha = \beta = 1$.

(2) By virtue of Proposition 2.8, the fuzzy solution $\widetilde{Y}$ can be represented by using its (α, β)-cut sets.

We treat T2FDEs with positive crisp coefficients.

Problem 4.3. Let $\widetilde{Y} : [0, 1] \to T^2(\mathbb{R})$. Then, solve T2FIVP:

$$\begin{cases} \mathscr{D}\widetilde{Y}(x) + 3\widetilde{Y}(x) = 0, & (4.2) \\ \widetilde{Y}(0) = \widetilde{1} \in T^2(\mathbb{R}). & (4.3) \end{cases}$$

Solution. *The* (α, β)*-cut-set equation of (4.2) is*

$$\left\langle [\mathscr{D}\underline{Y_\beta}(x)]_\alpha + 3[\underline{Y_\beta}(x)]_\alpha,\ [\mathscr{D}\overline{Y_\beta}(x)]_\alpha + 3[\overline{Y_\beta}(x)]_\alpha \right\rangle = 0, \quad \alpha \in [0, 1].$$

This can be reduced to the following two T1FIVPs:

$$\begin{cases} [\underline{Y_\beta}^\dagger(x)]_\alpha + 3[\underline{Y_\beta}(x)]_\alpha \\ \qquad = [\underline{Y}'_{\alpha,\beta,-}(x) + 3\underline{Y}_{\alpha,\beta,-}(x),\ \underline{Y}'_{\alpha,\beta,+}(x) + 3\underline{Y}_{\alpha,\beta,+}(x)] = 0, & (4.4) \\ \underline{Y_\beta}(0) = \underline{1_\beta} \in E^1(\mathbb{R}), & (4.5) \end{cases}$$

and

$$\begin{cases} [\overline{Y_\beta}^\dagger(x)]_\alpha + 3[\overline{Y_\beta}(x)]_\alpha \\ \qquad = [\overline{Y}'_{\alpha,\beta,-}(x) + 3\overline{Y}_{\alpha,\beta,-}(x),\ \overline{Y}'_{\alpha,\beta,+}(x) + 3\overline{Y}_{\alpha,\beta,+}(x)] = 0, & (4.6) \\ \overline{Y_\beta}(0) = \overline{1_\beta} \in E^1(\mathbb{R}). & (4.7) \end{cases}$$

(4.4) can be solved in the form of

$$\underline{Y}_{\alpha,\beta,\pm}(x) = \underline{C}_\pm e^{-3x}, \tag{4.8}$$

and (4.6) can also be solved in the form of

$$\overline{Y}_{\alpha,\beta,\pm}(x) = \overline{C}_\pm e^{-3x}, \tag{4.9}$$

where these four equations represent two equations, one for the upper sign and the other for the lower sign.

From (2.9)-(2.11), the (α,β)-cut set of $\widetilde{1} \in T^2(\mathbb{R})$ is calculated as

(i) *if* $0 < \alpha \le \beta/2$*:*

$$\widetilde{1}^\alpha_\beta = \left\langle \left[\frac{\alpha}{2-\beta},\ \frac{\alpha}{\beta}\right],\ \left[2-\frac{\alpha}{\beta},\ 2-\frac{\alpha}{2-\beta}\right] \right\rangle,$$

(ii) *if* $\beta/2 < \alpha \le 1-\beta/2$*:*

$$\widetilde{1}^\alpha_\beta = \left\langle \left[\frac{\alpha}{2-\beta},\ 1-\frac{1-\alpha}{2-\beta}\right],\ \left[1+\frac{1-\alpha}{2-\beta},\ 2-\frac{\alpha}{2-\beta}\right] \right\rangle,$$

(iii) *if* $1-\beta/2 < \alpha \le 1$*:*

$$\widetilde{1}^\alpha_\beta = \left\langle \left[1-\frac{1-\alpha}{\beta},\ 1-\frac{1-\alpha}{2-\beta}\right],\ \left[1+\frac{1-\alpha}{2-\beta},\ 1+\frac{1-\alpha}{\beta}\right] \right\rangle.$$

Then we have

(i) *if* $0 < \alpha \le \beta/2$*:*

$$[\widetilde{Y}(x)]^\alpha_\beta = \left\langle \left[\frac{\alpha}{2-\beta}e^{-3x},\ \frac{\alpha}{\beta}e^{-3x}\right],\ \left[\left(2-\frac{\alpha}{\beta}\right)e^{-3x},\ \left(2-\frac{\alpha}{2-\beta}\right)e^{-3x}\right] \right\rangle,$$

(ii) *if* $\beta/2 < \alpha \le 1-\beta/2$*:*

$$\begin{aligned} &[\widetilde{Y}(x)]^\alpha_\beta \\ &= \left\langle \left[\frac{\alpha}{2-\beta}e^{-3x},\ \left(1-\frac{1-\alpha}{2-\beta}\right)e^{-3x}\right],\right. \\ &\qquad\qquad \left.\left[\left(1+\frac{1-\alpha}{2-\beta}\right)e^{-3x},\ \left(2-\frac{\alpha}{2-\beta}\right)e^{-3x}\right] \right\rangle, \end{aligned}$$

(iii) *if* $1-\beta/2 < \alpha \le 1$*:*

$$[\widetilde{Y}(x)]_\beta^\alpha = \left\langle \left[\left(1 - \frac{1-\alpha}{\beta}\right) e^{-3x},\ \left(1 - \frac{1-\alpha}{2-\beta}\right) e^{-3x} \right], \left[\left(1 + \frac{1-\alpha}{2-\beta}\right) e^{-3x},\ \left(1 + \frac{1-\alpha}{\beta}\right) e^{-3x} \right] \right\rangle.$$

Thus the fuzzy solution of (4.2) with (4.3) is given as

$$\widetilde{Y}(x) = \bigcup_{\beta\in[0,1]} \beta \bigcup_{\alpha\in[0,1]} \alpha[\widetilde{Y}(x)]_\beta^\alpha.$$

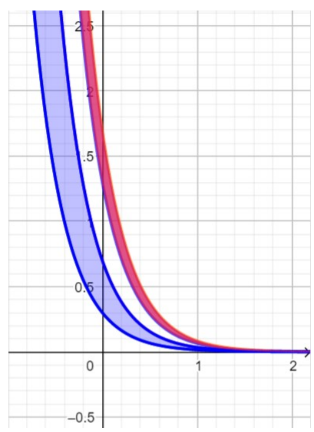

FIGURE 10. (1/2, 1/3)-cut set of fuzzy solution:

$$[\widetilde{Y}(x)]_{\beta=1/3}^{\alpha=1/2} = \left\langle \left[\frac{3}{10}e^{-3x},\ \frac{7}{10}e^{-3x}\right],\quad \left[\frac{13}{10}e^{-3x},\ \frac{17}{10}e^{-3x}\right] \right\rangle$$

5. Conclusion

We have introduced a new type-2 fuzzy numbers, i.e., triangular shaped type-2 fuzzy numbers (Definition 2.17). The feature of this is that the footprint of type-2 fuzzy number $\widetilde{A}$ consists of two isosceles parallelograms. Moreover, let $\widetilde{A}$ pass through the origin for the sake of clarity. If the $(1,1)$-cut set of $\widetilde{A}$ is $a \in \mathbb{R}$, we should note that 0-cut sets $S_{\widetilde{A}}(a/2|0)$ and $S_{\widetilde{A}}(3a/2|0)$ of the vertical slice of $\widetilde{A}$ is $[0,1]$. We can consider and set a variety of type-2 fuzzy numbers, but this paper believes that triangular shaped type-2 fuzzy numbers are natural forms.

We have found (α,β)-cuts of a triangular shaped type-2 fuzzy number ((2.9)-(2.11)), and so they can be used as formulas in applied problems. They will be useful for applications of type-2 fuzzy theory.

Furthermore, we have given an application to type-2 fuzzy differential equation theory (Section 4). Generally, the coefficients of differential equations and initial values are calculated based on given data, but we have solved the model in case that method is based on human subjective judgement. This "human subjective judgement" leads to ambiguity, so it is inevitably necessary to consider fuzzy differential equation theory. We should then treat appropriate fuzzy numbers, but we proposed to use triangular shaped type-2 fuzzy numbers for them in this paper. For example, in the case of differential equations where coefficients and initial values are determined empirically, there may be a difference in measurement accuracy between expert and new observers. In this case, it is convenient that the membership functions of both can be obtained in a simultaneous comparative manner. In such cases, type-2 fuzzy numbers can be applied.

Type-2 fuzzy theory can be applied to various fields. The exploration of other applications of triangular shaped type-2 fuzzy numbers is our future task.

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Appendix A. Necessary items on type-1 fuzzy theory

In this appendix, the notations, definitions and theorems on type-1 fuzzy numbers are written.

A.1. **Type-1 fuzzy numbers.**

Definition A.1 (See [15, 17, 18, 23])**.** A fuzzy set $u : \mathbb{R} \to [0, 1]$ is the *type-1 fuzzy number* if and only if

- (i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
- (ii) u is fuzzy convex, i.e., $u(tx + (1 - t)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$ and $t \in [0, 1]$;
- (iii) u is upper semi-continuous;
- (iv) the support $\operatorname{supp}(u) := \operatorname{cl}(\{x \in \mathbb{R} : u(x) > 0\})$ of u is compact, where $\operatorname{cl}(S)$ stands for the closure of the crisp set S.

It is well known that the argument of fuzzy sets can be reduced to that of the level cut sets, in the study of fuzzy theory so far. The α-cut set $[A]_\alpha$ of a type-1 fuzzy set A on X is defined by

$$\begin{aligned} [A]_\alpha &:= \{x \in X : A(x) \geq \alpha\}, \quad && \alpha \in (0, 1]; \\ [A]_0 &:= \operatorname{supp}(u), && \alpha = 0, \end{aligned}$$

and these level cut sets make up the original type-1 fuzzy set A:

$$A = \bigcup_{\alpha \in [0,1]} \alpha[A]_\alpha$$

where $\alpha[A]_\alpha : X \to \{0, \alpha\}$ is a type-1 fuzzy set. Thus, it is sufficient to consider and argue α-cut sets for most problems. In particular, the type-1 fuzzy number is the fuzzy set whose arbitrary α-cut set is a bounded and closed interval, so we put the following notation.

We denote the α-cut set of a type-1 fuzzy number $u : \mathbb{R} \to [0, 1]$ by

$$[u]_\alpha = [u_{-,\alpha}, u_{+,\alpha}] \tag{A.1}$$

for any $\alpha \in [0, 1]$. (A.1) is called the parametric form of u.

Remark A.2. We write 0 for crisp zero in this paper and its α-cut set is represented as the closed interval $[0, 0]$. In the same way, we regard a crisp real number r as the closed interval $[r, r]$ and bring crisp numbers into the group of fuzzy numbers.

Remark A.3 (See [22])**.** Each end of $[u]_\alpha = [u_{-,\alpha}, u_{+,\alpha}]$ should satisfy the followings:

(1) $u_{-,\alpha}$ is bounded, monotone increasing, left-continuous with respect to $\alpha \in (0, 1]$ and right-continuous on $\alpha = 0$,

(2) $u_{+,\alpha}$ is bounded, monotone decreasing, left-continuous with respect to $\alpha \in (0, 1]$ and right-continuous on $\alpha = 0$,

(3) $u_{-,\alpha} \leq u_{+,\alpha}$ for any $\alpha \in [0, 1]$.

In short, $[u]_\alpha$ that does not satisfy any of the above property cannot be called the fuzzy number.

We hereafter omit the description "$\alpha \in [0, 1]$" when we argue α-cut sets or parametric forms.

Definition A.4 (See [15, 23, 24])**.** Let u, v be type-1 fuzzy numbers on $\mathbb{R}$. We denote the *α-cut sets* of u, v by

$$[u]_\alpha = [u_{-,\alpha}, u_{+,\alpha}], \quad [v]_\alpha = [v_{-,\alpha}, v_{+,\alpha}] \tag{A.2}$$

respectively. Then, $u = v$ if and only if $[u]_\alpha = [v]_\alpha$, i.e.,

$$u_{-,\alpha} = v_{-,\alpha} \quad \text{and} \quad u_{+,\alpha} = v_{+,\alpha}$$

for any $\alpha \in [0, 1]$.

Definition A.5 (See [15, 17, 18, 24])**.** Let u, v be type-1 fuzzy numbers on $\mathbb{R}$ and $k \in \mathbb{R}$. We denote the α-cut sets of them by (A.2). The*sum* $u + v$ of u and v is defined by

$$[u + v]_\alpha := [u]_\alpha + [v]_\alpha,$$

i.e.,

$$[u_{-,\alpha}, u_{+,\alpha}] + [v_{-,\alpha}, v_{+,\alpha}] = [u_{-,\alpha} + v_{-,\alpha},\ u_{+,\alpha} + v_{+,\alpha}].$$

Moreover, the *scalar multiple* ku of u is defined by

$$[ku]_\alpha := k[u]_\alpha = \begin{cases} [ku_{-,\alpha}, ku_{+,\alpha}], & k \geq 0; \\ [ku_{+,\alpha}, ku_{-,\alpha}], & k < 0. \end{cases}$$

Definition A.6 (See [15, 17, 18, 24])**.** Let u, v be type-1 fuzzy numbers on $\mathbb{R}$. We denote the α-cut sets of them by (A.2). The *product* uv of u and v is defined by

$$[uv]_\alpha = [u]_\alpha [v]_\alpha = \left[\min_{i,j \in \{-,+\}} u_{i,\alpha} v_{j,\alpha},\ \max_{i,j \in \{-,+\}} u_{i,\alpha} v_{j,\alpha} \right]$$

for any $\alpha \in [0, 1]$.

Definition A.7 (See [15, 17, 18])**.** Let $u,\ v$ be type-1 fuzzy numbers on $\mathbb{R}$. We denote the α-cut sets of them by (A.2). The *fuzzy Hausdorff distance* d_{H} of u and v is defined by

$$d_{\mathrm{H}}(u,v) := \sup_{\alpha\in[0,1]} \max\{|u_{-,\alpha} - v_{-,\alpha}|, |u_{+,\alpha} - v_{+,\alpha}|\}.$$

We write $E^1(\mathbb{R})$ for the type-1 fuzzy number space equipped with the d_{H}-topology. In particular, we denote by $T^1(\mathbb{R})$ the space of triangular type-1 fuzzy numbers.

A.2. **Type-1 fuzzy-number-valued functions.** Let I be an interval which is a proper subset of $\mathbb{R}$ or let $I = \mathbb{R}$. We can consider the type-1 fuzzy function $F : E^1(I) \to E^1(\mathbb{R})$ by using Zadeh's extension principle for a crisp function $f : I \to \mathbb{R}$. We set $E^1(I) = I$ in this paper, i.e., we consider crisp-variable type-1 fuzzy-number-valued functions exclusively. The α-cut set of $F : I \to E^1(\mathbb{R})$ is represented by

$$[F(x)]_\alpha := [F_{-,\alpha}(x),\ F_{+,\alpha}(x)]$$

for all $x \in I$ and any $\alpha \in [0,1]$.

We define the difference of type-1 fuzzy numbers as the following sense, so as to consider the difference quotient of F.

Definition A.8 (See [15, 17, 18, 24])**.** Let $u, v \in E^1(\mathbb{R})$. If there exists some $w \in E^1(\mathbb{R})$ such that $u = v + w$, we write $w = u - v$ and call it the *T1-H-difference* of u and v.

Remark A.9. For any $u \in E^1(\mathbb{R})$, $-u$ stands for $0 - u$, i.e.,

$$[-u]_\alpha = [-u_{-,\alpha}, -u_{+,\alpha}],$$

whereas

$$[(-1)u]_\alpha = [-u_{+,\alpha}, -u_{-,\alpha}].$$

We should remark that, in general,

$$u + (-1)v \neq u - v$$

for any $u, v \in E^1(\mathbb{R})$.

We adopt, in this paper, dagger † and double dagger ‡ to denote fuzzy derivatives in the sense of Hukuhara. We use prime $'$ as the crisp derivative notation.

Definition A.10 ([10, 15, 17])**.** Let $F : I \to E^1(\mathbb{R})$ and $h > 0$ be a crisp number. F is T1-differentiable in the first form at some $x_0 \in I$ if and only if there exist $F(x_0 + h) - F(x_0)$ and $F(x_0) - F(x_0 - h)$ satisfying that the fuzzy limit

$$F^\dagger(x_0) := \lim_{h\downarrow 0} \frac{F(x_0 + h) - F(x_0)}{h} = \lim_{h\downarrow 0} \frac{F(x_0) - F(x_0 - h)}{h}$$

exists. Moreover, F is T1-differentiable in the second form at some $x_0 \in I$ if and only if there exist $F(x_0) - F(x_0 + h)$ and $F(x_0 - h) - F(x_0)$ satisfying that the fuzzy limit

$$F^\ddagger(x_0) := \lim_{h\downarrow 0} \frac{F(x_0) - F(x_0 + h)}{-h} = \lim_{h\downarrow 0} \frac{F(x_0 - h) - F(x_0)}{-h}$$

exists. Here the above differences (resp. limits) are due to the meaning of T1-Hukuhara (resp. d_{H}). If F is T1-differentiable in both senses at any $x \in I$, $F^{\dagger}$ and $F^{\ddagger}$ is called the (1)-T1-derivative and (2)-T1-derivative of F, respectively.

Remark A.11. In addition to the above two forms, it is possible to consider the following two forms, i.e.,

(1) the third form:

$$\lim_{h\downarrow 0}\frac{F(x_0+h)-F(x_0)}{h}=\lim_{h\downarrow 0}\frac{F(x_0-h)-F(x_0)}{-h}, \tag{A.3}$$

(2) the fourth form:

$$\lim_{h\downarrow 0}\frac{F(x_0)-F(x_0+h)}{-h}=\lim_{h\downarrow 0}\frac{F(x_0)-F(x_0-h)}{h}. \tag{A.4}$$

However, it is known ([12], Theorem 7) that (A.3) becomes the crisp number if there exist $F(x_0+h)-F(x_0)$ and $F(x_0-h)-F(x_0)$. (A.4) is also so if $F(x_0)-F(x_0+h)$ and $F(x_0)-F(x_0-h)$. We shall thus ignore the third and fourth forms.

Remark A.12. We mention the validity of our dagger symbols $\dagger, \ddagger$ meaning as fuzzy derivatives. First of all, we can avoid confusion between the crisp derivative and the fuzzy derivative by using $\dagger$ and $\ddagger$. Secondly, if we want to the number of the differential order to 2 or less, we can use $\dagger, \ddagger$ in the same sense as prime $'$ to clarify what kind of fuzzy derivative it is. Moreover, the acronym for a letter is often used to represent a mathematical concept or quantity, such as Δ (**D**elta) for differences or D for derivatives. From this point of view, "dagger" has an appropriate acronym **D** to represent derivatives.

By using this definition repeatedly, we find that the nth-order T1-derivative of F has 2^n forms. For example, when applied to fuzzy differential equations, we need to choose the most appropriate solution from these 2^n solutions.

Theorem A.13 ([14]). *Let $F : I \to E^1(\mathbb{R})$ be T1-differentiable on I. Then the parametric forms of its T1-derivatives are given by*

(1) *the first parametric form:*

$$[F^{\dagger}(x)]_{\alpha}=[F'_{-,\alpha}(x), F'_{+,\alpha}(x)],$$

(2) *the second parametric form:*

$$[F^{\ddagger}(x)]_{\alpha}=[F'_{+,\alpha}(x), F'_{-,\alpha}(x)].$$

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