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The (m, n)-fuzzy set and its application in BCK-algebras

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Abstract. The concept of the (m, n)-fuzzy set is introduced and compared with other types of fuzzy sets. Some operations for the (m, n)-fuzzy set are introduced, and their properties are investigated. We define (m, n)-fuzzy subalgebras in BCK-algebras and BCI-algebras and study their properties. A given (m, n)-fuzzy subalgebra is used to create a new (m, n)-fuzzy subalgebra. The intersection of two (m, n)-fuzzy subalgebras to be a (m, n)-fuzzy subalgebra is proved, and an example is given to show that the union of two (m, n)-fuzzy subalgebras may not be a (m, n)-fuzzy subalgebra. The (m, n)-cutty set is used to obtain the characterization of (m, n)-fuzzy subalgebra. The homomorphic image and preimage of (m, n)-fuzzy subalgebra is discussed. It turns out that intuitionistic fuzzy subalgebra is a subclass of (m, n)-fuzzy subalgebra.

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1. Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (See [1]) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$. As a generalization of fuzzy set, Atanassov [2] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all fields (See [3, 4, 5, 6] for applications in algebraic structures). In 2013, Yager [7, 8, 9] introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [10]), UP-algebras (See [11]) and topological spaces (See [12]). Senapati et al. [13] introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [14]). Ibrahim et al. [15] introduced $(3, 2)$-fuzzy sets and applied it to topological spaces.

In this paper, we introduce the concept of the (m, n)-fuzzy set which is the supclass of intuitionistic fuzzy set, Pythagorean fuzzy set, $(3, 2)$-fuzzy set, Fermatean fuzzy set and n-Pythagorean fuzzy set, and compared with them. We introduce some operations for the (m, n)-fuzzy set, and investigate their properties. The (m, n)-fuzzy set is applied to BCK-algebras and BCI-algebras. We introduce the (m, n)-fuzzy subalgebra in BCK-algebras and BCI-algebras and investigate their properties. Using the given (m, n)-fuzzy subalgebra, we make a new (m, n)-fuzzy subalgebra. We prove that the intersection of two (m, n)-fuzzy subalgebras is also a (m, n)-fuzzy subalgebra and provide an example to show that the union of two (m, n)-fuzzy subalgebras may not be a (m, n)-fuzzy subalgebra. The (m, n)-cutty set is used to obtain the characterization of (m, n)-fuzzy subalgebra. We show that intuitionistic fuzzy subalgebra is a subclass of (m, n)-fuzzy subalgebra. We consider the homomorphic image and preimage of (m, n)-fuzzy subalgebra.

2. Preliminaries

If a set X has a special element "0" and a binary operation " $*$ " satisfying the conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$,

then we say that X is a *BCI-algebra* (See [16, 17]). If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra* (See [16, 17]). The order relation " $\leq$ " in a BCK/BCI-algebra X is defined as follows:

$$(\forall x, y \in X)(x \leq y\ \Leftrightarrow\ x * y = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions (See [16, 17]):

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,. \tag{2.4}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X (See [16, 17]), if $x * y \in S$ for all $x, y \in S$.

Every ideal A of a BCK/BCI-algebra X satisfies the next assertion (See [16, 17]).

$$(\forall x, y \in X)\,(x \leq y,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.5}$$

Let X and Y be BCK/BCI-algebras. A mapping $\Psi : X \to Y$ is call a *homomorphism* (See [16, 17]), if it satisfies:

$$(\forall x, y \in X)(\Psi(x * y) = \Psi(x) * \Psi(y)). \tag{2.6}$$

Let $\mu_\Omega : X \to [0, 1]$ and $\gamma_\Omega : X \to [0, 1]$ be fuzzy sets in a set X. The structure $\Omega := \{\langle x, \mu_\Omega(x), \gamma_\Omega(x)\rangle \mid x \in X\}$ is called

- an *intuitionistic fuzzy set* in X (See [2]), if it satisfies:

$$(\forall x \in X)(0 \leq \mu_\Omega(x) + \gamma_\Omega(x) \leq 1), \tag{2.7}$$

- a *Pythgorean fuzzy set* in X (See [7]), if it satisfies:

$$(\forall x \in X)(0 \leq (\mu_\Omega(x))^2 + (\gamma_\Omega(x))^2 \leq 1), \tag{2.8}$$

- a $(3, 2)$-*fuzzy set* in X (See [15]), if it satisfies:

$$(\forall x \in X)(0 \leq (\mu_\Omega(x))^3 + (\gamma_\Omega(x))^2 \leq 1), \tag{2.9}$$

- a *Fermatean fuzzy set* in X (See [13]), if it satisfies:

$$(\forall x \in X)(0 \leq (\mu_\Omega(x))^3 + (\gamma_\Omega(x))^3 \leq 1), \tag{2.10}$$

- an n-*Pythagorean fuzzy set*, where $n \in \mathbb{N}$, in X (See [18]), if it satisfies:

$$(\forall x \in X)(0 \leq (\mu_\Omega(x))^n + (\gamma_\Omega(x))^n \leq 1). \tag{2.11}$$

3. (m, n)-fuzzy sets

Definition 3.1. Let $f : X \to [0, 1]$ and $g : X \to [0, 1]$ be fuzzy sets in a set X. If there exists $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that

$$(\forall x \in X)((f(x))^m + (g(x))^n \leq 1), \tag{3.1}$$

then the structure

$$\mathcal{C} := \{\langle x, f(x), g(x)\rangle \mid x \in X\} \tag{3.2}$$

is called the (m, n)-*fuzzy set* on X.

In what follows, we use the notations $f^m(x)$ and $g^n(x)$ instead of $(f(x))^m$ and $(g(x))^n$, respectively, and the (m, n)-fuzzy set on X in (3.2) is simply denoted by $\mathcal{C} := (X, f, g)$. The collection of (m, n)-fuzzy sets on X is denoted by $\mathcal{F}^n_m(X)$.

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a set and define fuzzy sets $f : X \to [0, 1]$ and $g : X \to [0, 1]$ as follows:

X	0	a	b	c	d
$f(x)$	0.93	0.74	0.92	0.55	0.67
$g(x)$	0.87	0.43	0.79	0.66	0.58

Then $\mathcal{C} := (X, f, g)$ is a $(5, n)$-fuzzy set on X for $n \geq 9$. But it is not a $(5, n)$-fuzzy set on X for $n \leq 8$ because of $0.93^5 + 0.87^8 = 1.02390004084 > 1$.

The (m, n)-fuzzy set varies according to (m, n) as shown in the table below.

(m, n)	(m, n)-fuzzy set	References
$(1, 1)$	Intuitionistic fuzzy set	[2]
$(2, 2)$	Pythagorean fuzzy set	[7]
$(3, 2)$	$(3, 2)$-fuzzy set	[15]
$(3, 3)$	Fermatean fuzzy set	[13]
(n, n)	n-Pythagorean fuzzy set	[18]

Remark 3.3. The (m, n)-fuzzy set is not any of intuitionistic fuzzy set, Pythagorean fuzzy set, $(3, 2)$-fuzzy set, Fermatean fuzzy set, and n-Pythagorean fuzzy set.

Example 3.4. Consider the $(5, n)$-fuzzy set $\mathcal{C} := (X, f, g)$ on X for $n \geq 9$ in Example 3.2. It is not an intuitionistic fuzzy set because of $f(0) + g(0) = 0.93 + 0.87 = 1.8 > 1$. Since $f^2(b) + g^2(b) = 0.92^2 + 0.79^2 = 1.4705 > 1$, we know that $\mathcal{C} := (X, f, g)$ is not a Pythagorean fuzzy set on X. Since $f^3(b) + g^2(b) = 0.92^3 + 0.79^2 = 1.402788 > 1$, we know that $\mathcal{C} := (X, f, g)$ is not a $(3, 2)$-fuzzy set on X. Because of $f^3(0) + g^3(0) = 0.93^3 + 0.87^3 = 1.46286 > 1$, we know that $\mathcal{C} := (X, f, g)$ is not a Fermatean fuzzy set on X. Finally, $\mathcal{C} := (X, f, g)$ is not a 5-Pythagorean fuzzy set on X since $f^5(0) + g^5(0) = 0.93^5 + 0.87^5 = 1.19410929 > 1$.

Proposition 3.5. *Let $\mathcal{C} := (X, f, g)$ be an (m, n)-fuzzy set on a set X. For every $(m_1, n_1) \in \mathbb{N} \times \mathbb{N}$, if $m \leq m_1$ and $n \leq n_1$, then $\mathcal{C} := (X, f, g)$ is an (m_1, n_1)-fuzzy set on X.*

Proof. It is strightforward because if $m \leq m_1$ and $n \leq n_1$, then

$$f^{m_1}(x) + g^{n_1}(x) \leq f^m(x) + g^n(x) \leq 1$$

for all $x \in X$. □

For every $(m_1, n_1) \in \mathbb{N} \times \mathbb{N}$ with $m \leq m_1$ and $n \leq n_1$, the (m_1, n_1)-fuzzy set may not be an (m, n)-fuzzy set as seen in the following example.

Example 3.6. In Example 3.2, $\mathcal{C} := (X, f, g)$ is a $(5, 9)$-fuzzy set on X. But it is not a $(2, 5)$-fuzzy set on X since $f^2(b) + g^5(b) = 0.92^2 + 0.79^5 = 1.1541056399 > 1$.

Definition 3.7. We define a binary relation " $\precapprox$ " and the equality " $=$ " in $\mathcal{F}_m^n(X)$ as follows:

$$\mathcal{C}_1 \precapprox \mathcal{C}_2 \;\Leftrightarrow\; f_1 \leq f_2,\, g_1 \geq g_2, \tag{3.3}$$

$$\mathcal{C}_1 = \mathcal{C}_2 \;\Leftrightarrow\; f_1 = f_2,\, g_1 = g_2 \tag{3.4}$$

for all $\mathcal{C}_1 := (X, f_1, g_1)$, $\mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}_m^n(X)$.

The notation $\mathcal{C}_1 \precnapprox \mathcal{C}_2$ means $\mathcal{C}_1 \precapprox \mathcal{C}_2$ and $\mathcal{C}_1 \neq \mathcal{C}_2$. It is clear that $(\mathcal{F}_m^n(X), \precapprox)$ is a poset.

Definition 3.8. For all $\mathcal{C}_1 := (X, f_1, g_1)$, $\mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}_m^n(X)$, we define the union ($\Cup$) and the intersection ($\Cap$) as follows:

$$\mathcal{C}_1 \Cup \mathcal{C}_2 = (X, f_1 \cup f_2, g_1 \cap g_2), \tag{3.5}$$

$$\mathcal{C}_1 \Cap \mathcal{C}_2 = (X, f_1 \cap f_2, g_1 \cup g_2), \tag{3.6}$$

where

$$f_1 \cup f_2 : X \to [0,1],\ x \mapsto \max\{f_1(x), f_2(x)\},$$
$$f_1 \cap f_2 : X \to [0,1],\ x \mapsto \min\{f_1(x), f_2(x)\},$$
$$g_1 \cup g_2 : X \to [0,1],\ x \mapsto \max\{g_1(x), g_2(x)\},$$
$$g_1 \cap g_2 : X \to [0,1],\ x \mapsto \min\{g_1(x), g_2(x)\}.$$

It is clear that the union ($\Cup$) and the intersection ($\Cap$) are associative binary operations in $\mathcal{F}^n_m(X)$.

Example 3.9. Let $X = \{0, a, b, c\}$ be a set and define $(3,2)$-fuzzy sets $\mathcal{C}_1 := (X, f_1, g_1)$ and $\mathcal{C}_2 := (X, f_2, g_2)$ on X by the tables below:

X	0	a	b	c
$f_1(x)$	0.93	0.74	0.82	0.55
$g_1(x)$	0.17	0.43	0.19	0.66

and

X	0	a	b	c
$f_2(x)$	0.85	0.84	0.69	0.75
$g_2(x)$	0.37	0.25	0.48	0.36

respectively. Then the union $\mathcal{C}_1 \Cup \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by the table below.

X	0	a	b	c
$(f_1 \cup f_2)(x)$	0.93	0.84	0.82	0.75
$(g_1 \cap g_2)(x)$	0.17	0.25	0.19	0.36

Also, the intersection $\mathcal{C}_1 \Cap \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by the table below.

X	0	a	b	c
$(f_1 \cap f_2)(x)$	0.85	0.74	0.69	0.55
$(g_1 \cup g_2)(x)$	0.37	0.43	0.48	0.66

Proposition 3.10. *Let* $\mathcal{C}_1 := (X, f_1, g_1)$, $\mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}^n_m(X)$. *Then*

$$\mathcal{C}_1 \Cap \mathcal{C}_2 = \mathcal{C}_2 \Cap \mathcal{C}_1, \tag{3.7}$$
$$\mathcal{C}_1 \Cup \mathcal{C}_2 = \mathcal{C}_2 \Cup \mathcal{C}_1, \tag{3.8}$$
$$(\mathcal{C}_1 \Cap \mathcal{C}_2) \Cup \mathcal{C}_2 = \mathcal{C}_2, \tag{3.9}$$
$$(\mathcal{C}_1 \Cup \mathcal{C}_2) \Cap \mathcal{C}_2 = \mathcal{C}_2. \tag{3.10}$$

Proof. Staightforward. □

Proposition 3.11. *Every element of* $\mathcal{F}^n_m(X)$ *is idempotent under the binary operators "*$\Cap$*" and "*$\Cup$*".*

Proof. Staightforward. □

Theorem 3.12. *$(\mathcal{F}^n_m(X), \Cup, \mathcal{C}_{01})$ and $(\mathcal{F}^n_m(X), \Cap, \mathcal{C}_{10})$ are commutative monoids where $\mathcal{C}_{01} := (X, \tilde{0}, \tilde{1})$ and $\mathcal{C}_{10} := (X, \tilde{1}, \tilde{0})$ with $\tilde{0} : X \to [0, 1],\ x \mapsto 0$ and $\tilde{1} : X \to [0, 1],\ x \mapsto 1$.*

Proof. Staightforward. □

Definition 3.13. The *complement* of $\mathcal{C} := (X, f, g) \in \mathcal{F}^n_m(X)$ is denoted by $\mathcal{C}^c := (X, f^c, g^c)$ and is defined to be an (n, m)-fuzzy set $\mathcal{C}^c = (X, g, f)$.

Example 3.14. Consider a $(3, 2)$-fuzzy set $\mathcal{C} := (X, f, g)$ on $X = \{0, a, b, c\}$ which is defined by the table below.

X	0	a	b	c
$f(x)$	0.76	0.34	0.85	0.47
$g(x)$	0.57	0.53	0.39	0.67

Then its complement $\mathcal{C}^c := (X, f^c, g^c)$ is given as follows:

X	0	a	b	c
$f^c(x)$	0.57	0.53	0.39	0.67
$g^c(x)$	0.76	0.34	0.85	0.47

Proposition 3.15. *If $\mathcal{C}_1 := (X, f_1, g_1), \mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}^n_m(X)$, then $(\mathcal{C}_1 \Cap \mathcal{C}_2)^c = \mathcal{C}_1^c \Cup \mathcal{C}_2^c$ and $(\mathcal{C}_1 \Cup \mathcal{C}_2)^c = \mathcal{C}_1^c \Cap \mathcal{C}_2^c$.*

Proof. For a given $\mathcal{C}_1 := (X, f_1, g_1), \mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}^n_m(X)$, we have

$$\begin{aligned}(\mathcal{C}_1 \Cap \mathcal{C}_2)^c &= (X, (f_1 \cap f_2)^c, (g_1 \cup g_2)^c)\\ &= (X, g_1 \cup g_2, f_1 \cap f_2)\\ &= (X, g_1, f_1) \Cup (X, g_2, f_2)\\ &= (X, f_1^c, g_1^c) \Cup (X, f_2^c, g_2^c)\\ &= \mathcal{C}_1^c \Cup \mathcal{C}_2^c.\end{aligned}$$

The same way induces $(\mathcal{C}_1 \Cup \mathcal{C}_2)^c = \mathcal{C}_1^c \Cap \mathcal{C}_2^c$. □

4. (m, n)-fuzzy subalgebras of BCK/BCI-algebras

In what follows, let X represent the BCK-algebra or BCI-algebra, and m and n are natural numbers unless otherwise specified.

Definition 4.1. An (m, n)-fuzzy set $\mathcal{C} := (X, f, g)$ is called an *(m, n)-fuzzy subalgebra* of X, if it satisfies:

$$(\forall x, y \in X)\left(\begin{array}{l} f^m(x * y) \geq \min\{f^m(x), f^m(y)\} \\ g^n(x * y) \leq \max\{g^n(x), g^n(y)\}\end{array}\right). \tag{4.1}$$

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with a binary operation "$*$" in the table below.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then X is a BCK-algebra (See [17]). Define an (m,n)-fuzzy set $\mathcal{C} := (X, f, g)$ by the tables below:

X	0	a	b	c
$f(x)$	0.87	0.65	0.65	0.76
$g(x)$	0.17	0.43	0.43	0.56

It is routine to verify that $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X for all $(m,n) \in \mathbb{N} \times \mathbb{N}$ with $(m,n) \notin \{(1,1),(1,2),(2,1)\}$.

Lemma 4.3. *Every (m,n)-fuzzy subalgebra $\mathcal{C} := (X, f, g)$ satisfies:*

$$(\forall x \in X)(f^m(0) \geq f^m(x),\ g^n(0) \leq g^n(x)). \tag{4.2}$$

Proof. The combination of (III) and (4.1) results in the following.

$$f^m(0) = f^m(x * x) \geq \min\{f^m(x), f^m(x)\} = f^m(x),$$
$$g^n(0) = g^n(x * x) \leq \max\{g^n(x), g^n(x)\} = g^n(x)$$

for all $x \in X$. □

Theorem 4.4. *If $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X, then the set*

$$X_{\mathcal{C}} := \{x \in X \mid f^m(x) = f^m(0),\ g^n(x) = g^n(0)\}$$

is a subalgebra of X.

Proof. If $x, y \in X_{\mathcal{C}}$, then $f^m(x) = f^m(0)$, $f^m(y) = f^m(0)$, $g^n(x) = g^n(0)$ and $g^n(y) = g^n(0)$. It follows from (4.1) that

$$f^m(x * y) \geq \min\{f^m(x), f^m(y)\} = f^m(0)$$

and

$$g^n(x * y) \leq \max\{g^n(x), g^n(y)\} = g^n(0).$$

By combining this and Lemma 4.3, we derive $f^m(x*y) = f^m(0)$ and $g^n(x*y) = g^n(0)$, and so $x * y \in X_{\mathcal{C}}$. Hence $X_{\mathcal{C}}$ is a subalgebra of X. □

Given an (m,n)-fuzzy set $\mathcal{C} := (X, f, g)$, we define a new (m,n)-fuzzy set $\mathcal{C}^* := (X, f^*, g^*)$ on X as follows:

$$f^* : X \to [0,1],\ x \mapsto \tfrac{f(x)}{\sup\{f(x) \mid x \in X\}},$$
$$g^* : X \to [0,1],\ x \mapsto \tfrac{g(x)}{\inf\{g(x) \mid x \in X\}},$$

where $\inf\{g(x) \mid x \in X\} \neq 0$.

Theorem 4.5. *If $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X with $g(0) \neq 0$, then $\mathcal{C}^* := (X, f^*, g^*)$ is an (m,n)-fuzzy subalgebra of X.*

Proof. If $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X, then $\sup\{f(x) \mid x \in X\} = f(0)$ and $\inf\{g(x) \mid x \in X\} = g(0) \neq 0$. Thus we have

$$\begin{aligned} f^{*m}(x*y) &= \left(\tfrac{f(x*y)}{\sup\{f(x*y) \mid x*y \in X\}}\right)^m = \left(\tfrac{f(x*y)}{f(0)}\right)^m = \tfrac{f^m(x*y)}{f^m(0)} \\ &\geq \tfrac{1}{f^m(0)} \min\{f^m(x), f^m(y)\} \\ &= \min\left\{\tfrac{f^m(x)}{f^m(0)}, \tfrac{f^m(y)}{f^m(0)}\right\} = \min\left\{\left(\tfrac{f(x)}{f(0)}\right)^m, \left(\tfrac{f(y)}{f(0)}\right)^m\right\} \\ &= \min\{f^{*m}(x), f^{*m}(y)\} \end{aligned}$$

and

$$\begin{aligned} g^{*n}(x*y) &= \left(\tfrac{g(x*y)}{\inf\{g(x*y) \mid x*y \in X\}}\right)^n = \left(\tfrac{g(x*y)}{g(0)}\right)^n = \tfrac{g^n(x*y)}{g^n(0)} \\ &\leq \tfrac{1}{g^n(0)} \max\{g^n(x), g^n(y)\} \\ &= \max\left\{\tfrac{g^n(x)}{g^n(0)}, \tfrac{g^n(y)}{g^n(0)}\right\} = \max\left\{\left(\tfrac{g(x)}{g(0)}\right)^n, \left(\tfrac{g(y)}{g(0)}\right)^n\right\} \\ &= \max\{g^{*n}(x), g^{*n}(y)\} \end{aligned}$$

for all $x, y \in X$. So $\mathcal{C}^* := (X, f^*, g^*)$ is an (m,n)-fuzzy subalgebra of X. □

Theorem 4.6. *If $\mathcal{C}_1 := (X, f_1, g_1)$ and $\mathcal{C}_2 := (X, f_2, g_2)$ are (m,n)-fuzzy subalgebras of X, then their intersection $\mathcal{C}_1 \Cap \mathcal{C}_2 = (X, f_1 \cap f_2, g_1 \cup g_2)$ is also an (m,n)-fuzzy subalgebra of X.*

Proof. For every $x, y \in X$, we have

$$\begin{aligned} (f_1 \cap f_2)^m(x*y) &= \min\{f_1^m(x*y), f_2^m(x*y)\} \\ &\geq \min\{\min\{f_1^m(x), f_1^m(y)\}, \min\{f_2^m(x), f_2^m(y)\}\} \\ &= \min\{\min\{f_1^m(x), f_2^m(x)\}, \min\{f_1^m(y), f_2^m(y)\}\} \\ &= \min\{(f_1 \cap f_2)^m(x), (f_1 \cap f_2)^m(y)\} \end{aligned}$$

and

$$\begin{aligned} (g_1 \cup g_2)^n(x*y) &= \max\{g_1^n(x*y), g_2^n(x*y)\} \\ &\leq \max\{\max\{g_1^n(x), g_1^n(y)\}, \max\{g_2^n(x), g_2^n(y)\}\} \\ &= \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(y), g_2^n(y)\}\} \\ &= \max\{(g_1 \cup g_2)^n(x), (g_1 \cup g_2)^n(y)\}. \end{aligned}$$

Then $\mathcal{C}_1 \Cap \mathcal{C}_2 = (X, f_1 \cap f_2, g_1 \cup g_2)$ is an (m,n)-fuzzy subalgebra of X. □

The following example shows that the union of two (m,n)-fuzzy subalgebras may not be an (m,n)-fuzzy subalgebra.

Example 4.7. Let $X = \{0, a, b, c\}$ be a set with a binary operation "$*$" in the table below.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then X is a BCI-algebra (See [17]). Let's define $\mathcal{C}_1 := (X, f_1, g_1), \mathcal{C}_2 := (X, f_2, g_2) \in \mathcal{F}_3^2(X)$ in the table below, respectively.

X	0	a	b	c
$f_1(x)$	0.78	0.63	0.47	0.47
$g_1(x)$	0.24	0.58	0.33	0.58

and

X	0	a	b	c
$f_2(x)$	0.78	0.53	0.53	0.69
$g_2(x)$	0.24	0.56	0.63	0.63

Then $\mathcal{C}_1 := (X, f_1, g_1)$ and $\mathcal{C}_2 := (X, f_2, g_2)$ are $(3, 2)$-fuzzy subalgebras of X. The union $\mathcal{C}_1 \Cup \mathcal{C}_2 = (X, f_1 \cup f_2, g_1 \cap g_2)$ of $\mathcal{C}_1 := (X, f_1, g_1)$ and $\mathcal{C}_2 := (X, f_2, g_2)$ is calculated as follows:

X	0	a	b	c
$(f_1 \cup f_2)(x)$	0.78	0.63	0.53	0.69
$(g_1 \cap g_2)(x)$	0.24	0.56	0.33	0.58

and it is not a $(3, 2)$-fuzzy subalgebra of X because of

$$\begin{aligned}(f_1 \cup f_2)^3(c * a) &= (f_1 \cup f_2)^3(b) = \max\{f_1^3(b), f_2^3(b)\} = \max\{0.47^3, 0.53^3\} \\ &= 0.53^3 \ngeq 0.63^3 = \min\{(f_1 \cup f_2)^3(c), (f_1 \cup f_2)^3(a)\}\end{aligned}$$

and/or

$$\begin{aligned}(g_1 \cap g_2)^2(b * a) &= (g_1 \cap g_2)^2(c) = \min\{g_1^2(c), g_2^2(c)\} = \min\{0.58^2, 0.63^2\} \\ &= 0.58^2 \nleq 0.33^2 = \max\{(g_1 \cap g_2)^2(b), (g_1 \cap g_2)^2(a)\}.\end{aligned}$$

Let $\mathcal{C} := (X, f, g) \in \mathcal{F}_m^n(X)$. For every $(s, t) \in [0, 1] \times [0, 1]$ with $0 \leq s^m + t^n \leq 1$, we consider the set

$$\mathcal{C}_{(s,t)} := \mathcal{C}_s \cap \mathcal{C}_t \tag{4.3}$$

which is called the (m, n)-*cutty set* of $\mathcal{C} := (X, f, g)$ where

$$\mathcal{C}_s := \{x \in X \mid f^m(x) \geq s\} \text{ and } \mathcal{C}_t := \{x \in X \mid g^n(x) \leq t\}.$$

Proposition 4.8. *Let* $\mathcal{C} := (X, f, g)$, $\mathcal{D} := (X, \xi, \eta) \in \mathcal{F}_m^n(X)$. *Then*

$$\mathcal{C} \precapprox \mathcal{D} \;\Rightarrow\; \mathcal{C}_{(s,t)} \subseteq \mathcal{D}_{(s,t)}, \tag{4.4}$$

$$(\forall(\alpha, \beta) \in [0, 1] \times [0, 1])(\alpha \leq s,\ \beta \geq t \;\Rightarrow\; \mathcal{C}_{(s,t)} \subseteq \mathcal{C}_{(\alpha,\beta)}). \tag{4.5}$$

Proof. Assume that $\mathcal{C} \precapprox \mathcal{D}$ and let $x \in \mathcal{C}_{(s,t)}$. Then $f \leq \xi$ and $g \geq \eta$, that is, $f(x) \leq \xi(x)$ and $g(x) \geq \eta(x)$ for all $x \in X$. It follows that $s \leq f^m(x) \leq \xi^m(x)$ and $t \geq g^n(x) \geq \eta^n(x)$. Thus $x \in \mathcal{D}_{(s,t)}$. This proves (4.4).

Now let $(\alpha, \beta) \in [0,1] \times [0,1]$ be such that $\alpha \leq s$ and $\beta \geq t$. If $x \in \mathcal{C}_{(s,t)}$, then $f^m(x) \geq s \geq \alpha$ and $g^n(x) \leq t \leq \beta$. Thus $x \in \mathcal{C}_{(\alpha,\beta)}$. So $\mathcal{C}_{(s,t)} \subseteq \mathcal{C}_{(\alpha,\beta)}$. □

Theorem 4.9. *If $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X, then its (m,n)-cutty set $\mathcal{C}_{(s,t)}$ is a subalgebra of X.*

Proof. Let $x, y \in \mathcal{C}_{(s,t)}$. Then $f^m(x) \geq s$, $f^m(y) \geq s$, $g^n(x) \leq t$ and $g^n(y) \leq t$. It follows from (4.1) that

$$f^m(x * y) \geq \min\{f^m(x), f^m(y)\} \geq s$$

and

$$g^n(x * y) \leq \max\{g^n(x), g^n(y)\} \leq t.$$

Thus $x * y \in \mathcal{C}_{(s,t)}$. So $\mathcal{C}_{(s,t)}$ is a subalgebra of X. □

Theorem 4.10. *For a given $\mathcal{C} := (X, f, g) \in \mathcal{F}_m^n(X)$, if its (m,n)-cutty set $\mathcal{C}_{(s,t)}$ is a subalgebra of X for every $(s,t) \in [0,1] \times [0,1]$ with $0 \leq s^m + t^n \leq 1$, then $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X.*

Proof. Assume that $\mathcal{C}_{(s,t)}$ is a subalgebra of X for every $(s,t) \in [0,1] \times [0,1]$ with $0 \leq s^m + t^n \leq 1$. For every $x, y \in X$, we put $s_x := f^m(x)$, $s_y := f^m(y)$, $t_x := g^n(x)$ and $t_y := g^n(y)$. Then $x, y \in \mathcal{C}_{(s,t)}$ for $s := \min\{s_x, s_y\}$ and $t := \max\{t_x, t_y\}$. Thus $x * y \in \mathcal{C}_{(s,t)}$. It follows that

$$f^m(x * y) \geq s = \min\{s_x, s_y\} = \min\{f^m(x), f^m(y)\}$$

and

$$g^n(x * y) \leq t = \max\{t_x, t_y\} = \max\{g^n(x), g^n(y)\}.$$

So $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X. □

Definition 4.11. Let $\mathcal{C} := (X, f, g)$ and $\mathcal{D} := (Y, \xi, \eta)$ be (m,n)-fuzzy sets on X and Y respectively. Let $\Psi : X \to Y$ be a mapping from a set X to a set Y.

(i) The *preimage* of $\mathcal{D} := (Y, \xi, \eta)$ under Ψ is defined to be the (m,n)-fuzzy set $\Psi^{-1}(\mathcal{D}) := \big(X, \Psi^{-1}(\xi), \Psi^{-1}(\eta)\big)$ on X where

$$\Psi^{-1}(\xi) : X \to [0,1],\ x \mapsto \xi(\Psi(x))$$

and

$$\Psi^{-1}(\eta) : X \to [0,1],\ x \mapsto \eta(\Psi(x)).$$

(ii) The *image* of $\mathcal{C} := (X, f, g)$ under Ψ is defined to be the (m,n)-fuzzy set $\Psi(\mathcal{C}) = (Y, \Psi(f), \Psi(g))$ on Y where

$$\Psi(f) : Y \to [0,1],\ y \mapsto \begin{cases} \sup\limits_{x \in \Psi^{-1}(y)} f(x) & \text{if } \Psi^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Psi(g) : Y \to [0,1],\ y \mapsto \begin{cases} \inf\limits_{x \in \Psi^{-1}(y)} g(x) & \text{if } \Psi^{-1}(y) \neq \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 4.12. *Let $\Psi : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{D} := (Y, \xi, \eta)$ is an (m,n)-fuzzy subalgebra of Y, then its preimage $\Psi^{-1}(\mathcal{D}) := \big(X, \Psi^{-1}(\xi), \Psi^{-1}(\eta)\big)$ under Ψ is an (m,n)-fuzzy subalgebra of X.*

Proof. For every $x, y \in X$, we have

$$\begin{aligned}
\Psi^{-1}(\xi)^m(x*y) &= (\Psi^{-1}(\xi)(x*y))^m \\
&= (\xi(\Psi(x*y)))^m \\
&= (\xi(\Psi(x)*\Psi(y)))^m = \xi^m(\Psi(x)*\Psi(y)) \\
&\geq \min\{\xi^m(\Psi(x)), \xi^m(\Psi(y))\} \\
&= \min\{(\xi(\Psi(x)))^m, (\xi(\Psi(y)))^m\} \\
&= \min\{(\Psi^{-1}(\xi)(x))^m, (\Psi^{-1}(\xi)(y))^m\} \\
&= \min\{\Psi^{-1}(\xi)^m(x), \Psi^{-1}(\xi)^m(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\Psi^{-1}(\eta)^n(x*y) &= (\Psi^{-1}(\eta)(x*y))^n \\
&= (\eta(\Psi(x*y)))^n \\
&= (\eta(\Psi(x)*\Psi(y)))^n = \eta^n(\Psi(x)*\Psi(y)) \\
&\leq \max\{\eta^n(\Psi(x)), \eta^n(\Psi(y))\} \\
&= \max\{(\eta(\Psi(x)))^n, (\eta(\Psi(y)))^n\} \\
&= \max\{(\Psi^{-1}(\eta)(x))^n, (\Psi^{-1}(\eta)(y))^n\} \\
&= \max\{\Psi^{-1}(\eta)^n(x), \Psi^{-1}(\eta)^n(y)\}.
\end{aligned}$$

Then $\Psi^{-1}(\mathcal{D}) := \big(X, \Psi^{-1}(\xi), \Psi^{-1}(\eta)\big)$ is an (m,n)-fuzzy subalgebra of X. □

Theorem 4.13. *Let $\Psi : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If $\mathcal{C} := (X, f, g)$ is an (m,n)-fuzzy subalgebra of X, then its image $\Psi(\mathcal{C}) = (Y, \Psi(f), \Psi(g))$ under Ψ is an (m,n)-fuzzy subalgebra of Y.*

Proof. For every $y_1, y_2 \in Y$, we have

$$\{x_1 * x_2 \in X \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\} \subseteq \{x \in X \mid x \in \Psi^{-1}(y_1 * y_2)\}.$$

Then we get

$$\begin{aligned}
\Psi(f)^m(y_1*y_2) &= (\Psi(f)(y_1*y_2))^m \\
&= \big(\sup\{f(x) \mid x \in \Psi^{-1}(y_1*y_2)\}\big)^m \\
&\geq \big(\sup\{f(x_1*x_2) \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\}\big)^m \\
&= \sup\{f^m(x_1*x_2) \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\} \\
&\geq \sup\{\min\{f^m(x_1)*f^m(x_2)\} \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\} \\
&= \min\{\sup\{f^m(x_1) \mid x_1 \in \Psi^{-1}(y_1)\}, \sup\{f^m(x_2) \mid x_2 \in \Psi^{-1}(y_2)\}\} \\
&= \min\{\Psi(f)^m(y_1), \Psi(f)^m(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\Psi(g)^n(y_1*y_2) &= (\Psi(g)(y_1*y_2))^n \\
&= \big(\inf\{g(x) \mid x \in \Psi^{-1}(y_1*y_2)\}\big)^n \\
&\leq \big(\inf\{g(x_1*x_2) \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\}\big)^n \\
&= \inf\{g^n(x_1*x_2) \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\} \\
&\leq \inf\{\max\{g^n(x_1)*g^n(x_2)\} \mid x_1 \in \Psi^{-1}(y_1), x_2 \in \Psi^{-1}(y_2)\} \\
&= \max\{\inf\{g^n(x_1) \mid x_1 \in \Psi^{-1}(y_1)\}, \inf\{g^n(x_2) \mid x_2 \in \Psi^{-1}(y_2)\}\} \\
&= \max\{\Psi(g)^n(y_1), \Psi(g)^n(y_2)\}.
\end{aligned}$$

Thus $\Psi(\mathcal{C}) = (Y, \Psi(f), \Psi(g))$ is an (m,n)-fuzzy subalgebra of Y. □

Finally, we discuss the relationship between an intuitionistic fuzzy subalgebra and an (m, n)-fuzzy subalgebra.

Theorem 4.14. *Every intuitionistic fuzzy subalgebra is an* (m, n)*-fuzzy subalgebra.*

Proof. Let $\mathcal{C} := (X, f, g)$ be an intuitionistic fuzzy subalgebra of X. Then $f(x*y) \geq \min\{f(x), f(y)\}$ and $g(x*y) \leq \max\{g(x), g(y)\}$ for all $x, y \in X$. We consider the following cases:

(1) $f(x) \geq f(y)$ and $g(x) \geq g(y)$,
(2) $f(x) \geq f(y)$ and $g(x) < g(y)$,
(3) $f(x) < f(y)$ and $g(x) \geq g(y)$,
(4) $f(x) < f(y)$ and $g(x) < g(y)$.

The first case implies $f^m(x) \geq f^m(y)$ and $g^n(x) \geq g^n(y)$, and then

$$\begin{aligned} f^m(x*y) &= (f(x*y))^m \geq (\min\{f(x), f(y)\})^m \\ &= (f(y))^m = f^m(y) = \min\{f^m(x), f^m(y)\} \end{aligned}$$

and

$$\begin{aligned} g^n(x*y) &= (g(x*y))^n \leq (\max\{g(x), g(y)\})^n \\ &= (g(x))^n = g^n(x) = \max\{g^n(x), g^n(y)\} \end{aligned}$$

for all $x, y \in X$. In the rest of the cases, the condition (4.1) can be derived in the same way. Thus $\mathcal{C} := (X, f, g)$ is an (m, n)-fuzzy subalgebra of X. □

The converse of Theorem 4.14 may not be tue. In fact, the (m, n)-fuzzy subalgebra $\mathcal{C} := (X, f, g)$ of X for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ with $(m, n) \notin \{(1, 1), (1, 2), (2, 1)\}$ in Example 4.2 is not an intuitionistic fuzzy subalgebra of X because of $f(c) + g(c) = 1.32 > 1$.

5. Conclusion

As the the supclass of intuitionistic fuzzy set, Pythagorean fuzzy set, $(3, 2)$-fuzzy set, Fermatean fuzzy set and n-Pythagorean fuzzy set, we introduced the notion of the (m, n)-fuzzy set and applied it to BCK/BCI-algebras. We gave some operations for the (m, n)-fuzzy set, and investigated their properties. We introduced the (m, n)-fuzzy subalgebra in BCK/BCI-algebras and investigated several properties. Using the given (m, n)-fuzzy subalgebra, we established a new (m, n)-fuzzy subalgebra. We proved that the intersection of two (m, n)-fuzzy subalgebras is also a (m, n)-fuzzy subalgebra and provided an example to show that the union of two (m, n)-fuzzy subalgebras may not be a (m, n)-fuzzy subalgebra. We used the (m, n)-cutty set to obtain the characterization of (m, n)-fuzzy subalgebra. We shown that intuitionistic fuzzy subalgebra is a subclass of (m, n)-fuzzy subalgebra, and considered the homomorphic image and preimage of (m, n)-fuzzy subalgebra.

The idea of this paper and the results obtained will be used for the study of various types of logical algebra in the future. And considering research on soft set theory and rough set theory etc. based on $(3, 2)$-fuzzy set is also a subject of future research. It also attempts to explore the role of source in solving problems that include uncertainty.

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