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Bi-interior ideals and Fuzzy bi-interior ideals of Γ-semigroups

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Abstract. In this paper, we introduce the notion of a bi-interior ideal, fuzzy bi-interior ideal of Γ-semigroup and study the properties of bi-interior ideals, minimal bi-interior ideal and fuzzy bi-interior ideal. We characterize the (fuzzy)bi-interior ideal of Γ-semigroup and regular Γ-semigroup in terms of (fuzzy) bi-interior ideal of Γ-semigroup.

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1. Introduction

Rao [1] introduced the concept of a bi-interior ideal for a semigroup as a generalization of bi-ideals, interior ideals, quasi ideals and a further generalization of ideals in semigroups. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of one sided ideal of any algebraic structure is a generalization of notion of an ideal. In 1952, the concept of bi-ideals was introduced by Good and Hughes [2] for semigroups. The notion of bi-ideals in rings and semgroups were introduced by Lajos and Szasz [3, 4]. Quasi ideals are generalization of right ideals and left ideals whereas bi-ideals are generalization of quasi ideals.

Steinfeld [5] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [6, 7] introduced the concept of quasi ideal for a semigroup. Rao introduced the concept of bi-interior ideal for a semigroup.

Henriksen [8], and Shabir and Batod [9] studied ideals in semgroups. Rao [10] introduced the notion of left (right) bi-quasi ideal of semigroup, Γ-semigroup and studied the properties of left bi-quasi ideals. We characterize the left bi-quasi simple Γ-semigroup and regular Γ-semigroup using left bi-quasi ideals of Γ-semigroup.

The fuzzy set theory was developed by Zadeh in 1965 [11]. The fuzzification of algebraic structure was introduced by Rosenfeld [12] and he introduced the notion of fuzzy subgroups in 1971. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. Rao [15] studied fuzzy soft Γ-semiring, fuzzy soft k-ideal over Γ-semiring, T-fuzzy ideals of ordered Γ-semirings. Kuroki [13] studied fuzzy interior ideals in semigroups.

In this paper, we introduce the notion of bi-interior ideal, fuzzy bi-interior ideal of Γ-semigroup and we characterize the regular Γ-semigroup in terms of fuzzy bi-interior ideals of Γ-semigroup.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 ([1])**.** A *semigroup* is an algebraic system $(S, .)$ consisting of a non-empty set S together with an associative binary operation "·".

Definition 2.2 ([1])**.** A *subsemigroup* T of S is a non-empty subset T of S such that $TT \subseteq T$.

Definition 2.3 ([1])**.** A non-empty subset T of S is called a *left* (*right*) *ideal* of S, if $ST \subseteq T(TS \subseteq T)$.

Definition 2.4 ([1])**.** A non-empty subset T of a semigroup S is called an *ideal* of S, if it is both a left ideal and a right ideal of$S$.

Definition 2.5 ([1])**.** A non-empty Q of a semigroup S is called a *quasi ideal* of S, if $QS \cap SQ \subseteq Q$.

Definition 2.6 ([1])**.** A sub semigroup T of a semigroup S is called a *bi-ideal* of S, if $TST \subseteq T$.

Definition 2.7 ([1])**.** A sub semigroup T of a semigroup S is called an *interior ideal* of S, if $STS \subseteq T$.

Definition 2.8 ([1])**.** An element a of a semigroup S is called a *regular element*, if there exists an element b of S such that $a = aba$.

Definition 2.9 ([1])**.** A semigroup S is called a *regular semigroup*, if every element of S is a regular element.

Definition 2.10 ([14])**.** Let M and Γ be non-empty sets. Then we call M a Γ-semigroup, if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (the image of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x,\ y,\ z \in M$ and $\alpha,\ \beta \in \Gamma$.

Definition 2.11 ([14])**.** A non-empty subset A of $\Gamma-$semigroup M is called

(i) a Γ-subsemigroup of M if $A\Gamma A \subseteq A$,
(ii) a *quasi ideal* of M, if $A\Gamma M \cap M\Gamma A \subseteq A$,
(iii) a *bi-ideal* of M, if $A\Gamma M\Gamma A \subseteq A$,
(iv) an *interior ideal* of M, if $M\Gamma A\Gamma M \subseteq A$,
(v) a *left* (*right*) *ideal* of M, if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an *ideal*, if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition 2.12 ([14])**.** AΓ-semigroup M is said to be *left* (*right*) *singular*, if for each $a \in M$, there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b)$ for all $b \in M$.

Definition 2.13 ([14])**.** A Γ-semigroup M is said to be *commutative*, if $a\alpha b = b\alpha a$, for all $a,\ b \in M$ for all $\alpha \in \Gamma$.

Definition 2.14 ([14])**.** Let M be a Γ-semigroup. An element $a \in M$ is said to be an *idempotent* of M, if there exist $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α *-idempotent*.

Definition 2.15 ([14])**.** Let M be a Γ-semigroup. If every element of M is an idempotent of M, then M is said to be *band*.

Definition 2.16 ([14])**.** Let M be a Γ-semigroup. An element $a \in M$ is said to be a *regular element* of M, if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.17 ([14])**.** Let M be a Γ-semigroup. If every element of M, is a regular element of M, then M is said to be a *regular* Γ*-semigroup*.

Definition 2.18 ([15])**.** Let M be a non-empty set. A mapping $\mu : M \to [0,1]$ is called a *fuzzy subset* of M.

Definition 2.19 ([15])**.** If μ is a fuzzy subset of M for $t \in [0,1]$, then the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a *level subset* of M with respect to a fuzzy subset μ.

Definition 2.20 ([15])**.** A fuzzy subset $\mu : M \to [0,1]$ is a non-empty fuzzy subset if μ is not a constant function.

Definition 2.21 ([15])**.** For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition 2.22 ([15])**.** Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$

Definition 2.23 ([15])**.** Let f and g be fuzzy subsets of S. Then $f \cup g, f \cap g$ are fuzzy subsets of S defined by $f \cup g(x) = \max\{f(x), g(x)\},\ f \cap g(x) = \min\{f(x), g(x)\}$ for all $x \in S$. And $f \circ g$ is defined by: for all $z \in S$,

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy,\ x,y\in S} & \{\min\{f(x), g(y)\}\} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.24 ([10]). A fuzzy subset μ of a Γ-semigroup M is called

(i) a *fuzzy* Γ-*subsemigroup* of M, if $\mu(x\alpha y) \geq min \ \{\mu(x), \mu(y)\}$ for all $x, \ y \in M, \alpha \in \Gamma$,
(ii) a *fuzzy left* (*right*) *ideal* of M, if $\mu(x\alpha y) \geq \mu(y) \ (\mu(x))$ for all $x, \ y \in M, \ \alpha \in \Gamma$,
(iii) a *fuzzy ideal* of M, if $\mu(x\alpha y) \geq max \ \{\mu(x), \mu(y)\}$ for all $x, \ y \in M, \ \alpha \in \Gamma$,
(iv) a *fuzzy bi-ideal* of M, if $\mu \circ \chi_M \circ \mu \subseteq \mu$ for all $x, \ y \in M$,
(v) a *fuzzy quasi-ideal*, if $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$ for all $x, \ y \in M$.

3. Bi-interior ideals of Γ-semgroups

In this section, we introduce the notion of a bi-interior ideal as a generalization of bi-ideal and interior ideal of Γ-semigroup and study the properties of bi-interior ideals of a Γ-semigroup.

Definition 3.1. A non-empty subset B of a Γ-semigroup M is said to be *bi-interior ideal* of M, if B is a Γ-subsemigroup of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.

Definition 3.2. A Γ-semigroup M is said to be *bi-interior simple* Γ-*semgroup*, if M has no bi-interior ideals other than M itself.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.3. *Let M be a $\Gamma-$semigroup. Then the following hold.*

(1) *Every left ideal is a bi-interior ideal of M.*
(2) *Every right ideal is a bi-interior ideal of M.*
(3) *Every quasi ideal is a bi-interior ideal of M.*
(4) *If A and B are bi-interior ideals of M then $A\Gamma B$ and $B\Gamma A$ are bi-interior ideals of M.*
(5) *Every ideal is a bi-interior ideal of M.*
(6) *If B is a bi-interior ideal of M then $B\Gamma M$ and $M\Gamma B$ are bi-interior ideals of M.*

Theorem 3.4. *Every bi-ideal of a Γ-semigroup M is a bi-interior ideal of M.*

Proof. Let B be a bi-ideal of M. Then $B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma M\Gamma B \cap M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B \subseteq B$. So every bi-ideal of M is a bi-interior ideal of M. □

Theorem 3.5. *Every interior ideal of a Γ-semigroup M is a bi-interior ideal of M.*

Proof. Let I be an interior ideal of the M. Then $I\Gamma M\Gamma I \cap M\Gamma I\Gamma M \subseteq M\Gamma I\Gamma M \subseteq I$. Thus I is a bi-interior ideal of M. □

Theorem 3.6. *Let M be a simple Γ-semigroup. Every bi-interior ideal is a bi-ideal of M.*

Proof. Let M be a simple Γ-semigroup and B be a bi-interior ideal of M. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of M. Since M is a simple Γ-semigroup, we have $M\Gamma B\Gamma M = M$. Thus $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. So $M \cap B\Gamma M\Gamma B \subseteq B$. Hence $B\Gamma M\Gamma B \subseteq B$. □

Theorem 3.7. *Let M be a Γ-semigroup. Then M is a bi-interior simple Γ-semigroup if and only if $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$ for all $a \in M$.*

Proof. Suppose M is a bi-interior simple Γ-semigroup and let $a \in M$. We know that $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a$ is a bi-interior ideal of M. Then $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$ for all $a \in M$.

Conversely, suppose that $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$ for all $a \in M$. Let B be a bi-interior ideal of Γ-semigroup M and let $a \in B$. we have

$$M = M\Gamma a\Gamma M \cap a\Gamma M\Gamma a \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B.$$

Thus $M = B$. So M is a bi-interior simple Γ-semigroup. □

Theorem 3.8. *If L is a minimal left ideal and R is a minimal right ideal of a Γ-semigroup M, then $B = R\Gamma L$ is a minimal bi-interior ideal of M.*

Proof. Obviously, $B = R\Gamma L$ is a bi-interior ideal of M. Suppose A is a bi-interior ideal of M such that $A \subseteq B$. Since L is a left ideal of M, we get

$$M\Gamma A \subseteq M\Gamma B = M\Gamma R\Gamma L \subseteq L.$$

Similarly, we can prove $A\Gamma M \subseteq R$. Then $M\Gamma A = L,\ A\Gamma M = R$. Thus we have

$$B = A\Gamma M\Gamma M\Gamma A \subseteq A\Gamma M\Gamma A,\ B = R\Gamma L = R\Gamma M\Gamma A \subseteq M\Gamma A \subseteq M\Gamma A\Gamma M.$$

So $B \subseteq A\Gamma M\Gamma A \cap M\Gamma A\Gamma M \subseteq A$. Hence $A = B$. Therefore B is a minimal bi-interior ideal of M. □

Theorem 3.9. *The intersection of a bi-interior ideal B of a Γ-semigroup M and a Γ-subsemigroup A of M is a bi-interior ideal of M.*

Proof. Let B be a bi-interior ideal B of M and A be a Γ-subsemigroup of M. Suppose $C = B \cap A$. Since A is a Γ-subsemigroup of M and $C \subseteq A$, we get

$$C\Gamma A\Gamma C \subseteq A\Gamma A\Gamma A \subseteq A. \quad (3.1)$$

It is clear that $C\Gamma A\Gamma C \subseteq B\Gamma A\Gamma B \subseteq B\Gamma M\Gamma B$. Then $M\Gamma C\Gamma M \subseteq M\Gamma B\Gamma M$. Thus

$$C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B.$$

So by (3.1), $C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq C\Gamma A\Gamma C \subseteq A$. Hence we have

$$C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq B \cap A = C.$$

Therefore the result holds. □

Theorem 3.10. *Let A and C be Γ-subsemigroups of M and $B = A\Gamma C$. If A is the left ideal, then B is a bi-interior ideal of M.*

Proof. Let A and C be $\Gamma-$subsemigroups of M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then $B\Gamma M\Gamma B = A\Gamma C\Gamma M\Gamma A\Gamma C = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$. Thus we have

$$B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B\Gamma M\Gamma B \subseteq B.$$

So B is a bi-interior ideal of M. □

Corollary 3.11. *Let A and C be Γ-subsemigroups of Γ-semigroup M and $B = A\Gamma C$. If C is a right ideal, then B is a bi-interior ideal of M.*

Theorem 3.12. *Let M be a $\Gamma-$semigroup and T be a non-empty subset of M. Then every Γ-semigroup of T containing $T\Gamma M\Gamma T \cup M\Gamma T\Gamma M$ is a bi-interior ideal of M.*

Proof. Let B be a Γ-semigroup of T containing $T\Gamma M\Gamma T \cup M\Gamma T\Gamma M$. Then clearly,

$$B\Gamma M\Gamma B \subseteq T\Gamma M\Gamma T \subseteq T\Gamma M\Gamma T \cup M\Gamma T\Gamma M \subseteq B$$

. Thus $B\Gamma M\Gamma B \cap M\Gamma T\Gamma M \subseteq B$. So B is a bi-interior ideal of M. □

Theorem 3.13. *Let M be a regular Γ-semigroup. Then every bi-interior ideal of M is an ideal of M.*

Proof. Let B be a bi-interior ideal of M. Then clearly,

$$B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B.$$

Since M is regular, $B\Gamma M \subseteq B\Gamma M\Gamma B$. Thus $B\Gamma M \subseteq M\Gamma B\Gamma M$. So we have

$$B\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B.$$

Similarly, we can show that $M\Gamma B \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B$. □

Theorem 3.14. *Let M be a Γ-semigroup. Then the following statements are equivalent:*

(1) *M is a bi-interior simple $\Gamma-$semgroup,*
(2) *$M\Gamma a = M$ for all $a \in M$,*
(3) *$< a >= M$ for all $a \in M$, where $< a >$ is the smallest bi-interior ideal generated by a.*

Proof. Let M be a $\Gamma-$semigroup.

(1) ⇒ (2): Suppose M is a bi-interior simple Γ-semgroup, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Thus by Theorem 3.3 , B is a bi-interior ideal of M. So $B = M$. Hence $M\Gamma a = M$ for all $a \in M$.

(2) ⇒ (3): Suppose $M\Gamma a = M$ for all $a \in M$. Then $M\Gamma a \subseteq< a >\subseteq M$. Thus $M \subseteq< a >\subseteq M$. So $M =< a >$.

(3) ⇒ (1): Suppose $< a >$ is the smallest bi-interior ideal of aΓ-M generated by a, $< a >= M$, A is the bi-interior ideal and $a \in A$. Then $< a >\subseteq A \subseteq M$. Thus $M \subseteq A \subseteq M$. So $A = M$. Hence M is a bi-interior simple Γ-semgroup. □

Theorem 3.15. *If B is a bi-interior ideal of a Γ-semigroup M, T is a Γ-subsemigroup of M and $T \subseteq B$, then $B\Gamma T$ is a bi-interior ideal of M.*

Proof. Obviously, $B\Gamma T$ is a subsemigroup of the semigroup $(M, +)$. Then we have

$$(B\Gamma T)\Gamma B \subseteq B\Gamma M\Gamma B,\ (B\Gamma T)\Gamma B \subseteq M\Gamma B\Gamma M.$$

Thus $B\Gamma T\Gamma B \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma B \subseteq B$. So $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a Γ-semigroup of M.

On the other hand, we have

$$M\Gamma B\Gamma T\Gamma M \subseteq M\Gamma B\Gamma M \text{ and } B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B.$$

Then $M\Gamma B\Gamma T\Gamma M \cap B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma T$ is a bi-interior ideal of M. □

Theorem 3.16. *Let B be a bi-ideal of a Γ-semigroup M and I be an interior ideal of M. Then $B \cap I$ is a bi-interior ideal of M.*

Proof. Suppose B is a bi-ideal of M and I is an interior ideal of M. Obviously, $B \cap I$ is a $\Gamma-$subsemigroup of M. Then we have

$$(B \cap I)\Gamma M\Gamma(B \cap I) \subseteq B\Gamma M\Gamma B \subseteq B \text{ and } M\Gamma(B \cap I)\Gamma M \subseteq M\Gamma I\Gamma M \subseteq I.$$

Thus $(B \cap I)\Gamma M\Gamma(B \cap I) \cap M\Gamma(B \cap I)\Gamma M \subseteq B \cap I$. So $B \cap I$ is a bi-interior ideal of M. □

Theorem 3.17. *If B is a minimal bi-interior ideal of a Γ-semigroup M, then any two non-zero elements of B generate the same right ideal of M.*

Proof. Let B be a minimal bi-interior ideal of M and $x \in B$. Then $(x)_R \cap B$ is a bi-interior ideal of M. Then $(x)_R \cap B \subseteq B$. Since B is a minimal bi-interior ideal of M, $(x)_R \cap B = B$. Thus $B \subseteq (x)_R$. Suppose $y \in B$. Then $(y)_R \subseteq (x)_R$. Similarly, we can prove $(x)_R \subseteq (y)_R$. Thus $(x)_R = (y)_R$. □

Corollary 3.18. *If B is a minimal bi-interior ideal of a Γ-semigroup M, then any two non-zero elements of B generate the same left ideal of M.*

Theorem 3.19. *Let M be a Γ-semigroup and T be a subsemigroup of M. Then every subsemigroup of T containing $M\Gamma T\Gamma M \cap T\Gamma M\Gamma T$ is a bi-interior ideal of M.*

Proof. Let C be a subsemigroup of T containing $M\Gamma T\Gamma M \cap T\Gamma M\Gamma T$. Then we get

$$M\Gamma C\Gamma M \cap C\Gamma M\Gamma C \subseteq M\Gamma T\Gamma M \cap T\Gamma M\Gamma T \subseteq C.$$

Thus C is a bi-interior ideal of Γ-semigroup. □

Theorem 3.20. *Let M be a Γ-semigroup. If $M = M\Gamma a$ for all $a \in M$. Then every bi-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a bi-interior ideal of M and let $a \in B$. Then

$$\begin{aligned} & M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B \\ \Rightarrow & M\Gamma a \subseteq M\Gamma B, \\ \Rightarrow & M \subseteq M\Gamma B \subseteq M \\ \Rightarrow & M\Gamma B = M \\ \Rightarrow & B\Gamma M\Gamma B = B\Gamma M \\ \Rightarrow & M\Gamma B \cap B\Gamma M \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Thus B is a quasi ideal of M. □

Theorem 3.21. *Let M be a Γ-semigroup. Then M is a regular Γ-semigroup if and only if $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$ for all bi-interior ideals B of M.*

Proof. Let B be a bi-interior ideal of M and let $x \in M$. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$ and there exists $y \in M$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x \in B\Gamma M\Gamma B$. Thus $x \in M\Gamma B\Gamma M \cap B\Gamma M\Gamma B$. So $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$.

Conversely, suppose that $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$ for all bi-interior ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a bi-interior ideal of M. Thus $M\Gamma(R\cap L)\Gamma M\cap R\cap L\Gamma M\Gamma R\cap L = R\cap L$. On the other hand, we get

$$\begin{aligned} R\cap L &\subseteq R\cap L\Gamma M\Gamma R\cap L \\ &\subseteq R\Gamma M\Gamma L \\ &\subseteq R\Gamma L \\ &\subseteq R\cap L \text{ (Since } R\Gamma L\subseteq L \text{ and } R\Gamma L\subseteq R). \end{aligned}$$

So $R\cap L = R\Gamma L$. Hence M is a regular Γ-semigroup. □

Theorem 3.22. *Let M be a Γ-semigroup. B is a bi-interior simple Γ-subsemigroup of M, then B is minimal bi-interior ideal.*

Proof. Suppose that B is a bi-interior simple $\Gamma-$ subsemigroup of M. Let C be a bi-interior ideal of M and $C\subseteq B$. Then we have

$$\begin{aligned} &C\Gamma B\Gamma C\cap B\Gamma C\Gamma B\subseteq C\Gamma M\Gamma C\cap M\Gamma C\Gamma M\subseteq C \\ \Rightarrow\ &C\Gamma M\Gamma C\cap M\Gamma C\Gamma M\subseteq B\Gamma M\Gamma B\cap M\Gamma B\Gamma M\subseteq B \\ \Rightarrow\ &C\Gamma B\Gamma C\cap B\Gamma C\Gamma B\subseteq C\cap B\subseteq C \\ \Rightarrow\ &B = C. \end{aligned}$$

Thus B is a minimal bi-interior ideal of M. □

Theorem 3.23. *Let $\{B_\lambda \mid \lambda\in A\}$ be a family of bi-interior ideals of a Γ-semigroup M is a bi-interior ideal of M. Then $\bigcap_{\lambda\in A} B_\lambda$ is a bi-interior ideal of M.*

Proof. Let $B = \bigcap\limits_{\lambda\in A} B_\lambda$. Then B is a $\Gamma-$ subsemigroup M. Since B_λ is a bi-interior ideal of M for all $\lambda\in A$, we have

$$B_\lambda\Gamma M\Gamma B_\lambda\cap M\Gamma B_\lambda\cap M\subseteq B_\lambda,\quad \text{for all } \lambda\in A.$$

Thus we get

$$(\bigcap_{\lambda\in A} B_\lambda)\Gamma M\Gamma(\bigcap_{\lambda\in A} B_\lambda)\cap M\Gamma(\bigcap_{\lambda\in A} B_\lambda)\cap M\subseteq(\bigcap_{\lambda\in A} B_\lambda).$$

So $B\Gamma M\Gamma B\cap M\Gamma B\Gamma M\subseteq B$. Hence B is a bi-interior ideal of M. □

Theorem 3.24. *Let B be a bi-interior ideal of a Γ-semigroup M, let e be β-idempotent and let $e\Gamma B\subseteq B$. Then $e\Gamma B$ is a bi-interior ideal of M.*

Proof. Let B be a bi-interior ideal of M. and suppose $x\in B\cap e\Gamma M$. Then $x\in B$ and $x = e\alpha y, \alpha\in\Gamma, y\in M$. On the other hand, we have

$$x = e\alpha y = e\beta e\alpha y = e\beta(e\alpha y) = e\beta x\in e\Gamma B.$$

Thus $B\cap e\Gamma M\subseteq e\Gamma B$, $e\Gamma B\subseteq B$ and $e\Gamma B\subseteq e\Gamma M$. So $e\Gamma B\subseteq B\cap e\Gamma M$. Hence $e\Gamma B = B\cap e\Gamma M$. Therefore $e\Gamma B$ is a bi-interior ideal of M. □

Corollary 3.25. *Let M be a Γ-semigroup and e be α-idempotent. Then $e\Gamma M$ and $M\Gamma e$ are bi-interior ideals of M.*

Theorem 3.26. *Let e and f be α-idempotent and β-idempotent of a Γ-semigroup M respectively. Then $e\Gamma M\Gamma f$ is a bi-interior ideal of M.*

Proof. Suppose e and f be α-idempotent and β-idempotent of M respectively. Then $e\Gamma M\Gamma f \subseteq e\Gamma M$ and $e\Gamma M\Gamma f \subseteq M\Gamma f$. Thus $e\Gamma M\Gamma f \subseteq e\Gamma M \cap M\Gamma f$. Let $a \in e\Gamma M \cap M\Gamma f$. Then $a = e\alpha c = d\beta f,\ c, d \in M$. On the other hand, we get

$$a = e\alpha c = e\alpha e\alpha c = e\alpha a = e\alpha d\beta f \in e\Gamma M\Gamma f.$$

Thus $e\Gamma M \cap M\Gamma f \subseteq e\Gamma M\Gamma f$. So $e\Gamma M \cap M\Gamma f = e\Gamma M\Gamma f$. Hence $e\Gamma M\Gamma f$ is a bi-interior ideal of M. □

Theorem 3.27. *Let B be Γ-subsemigroup of a regular Γ-semigroup M. Then B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of M if and only if B is a bi-interior ideal of M.*

Proof. Suppose $B = R\Gamma L$, where R is right ideal of M and L is a left ideal of M. Then we have

$$B\Gamma M\Gamma B = R\Gamma L\Gamma M\Gamma R\Gamma L \subseteq R\Gamma L.$$

Thus we get

$$M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma L = B.$$

So B is a bi-interior ideal of M.

Conversely, suppose that B is a bi-interior ideal of M. By Theorem 3.21, $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then $R = B\Gamma M$ is a right ideal of M and $L = M\Gamma B$ is a left ideal of M. On the other hand, we have

$$B\Gamma M \cap M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B.$$

Thus $B\Gamma M \cap M\Gamma B \subseteq B$. So $R \cap L \subseteq B$. Furthermore, we get

$$B \subseteq B\Gamma M = R \text{ and } B \subseteq M\Gamma B = L.$$

Hence $B \subseteq R \cap L$. Since M is a regular Γ-semigroup, $B = R \cap L = R\Gamma L$. Therefore the sufficient condition holds. □

The following theorem is a necessary and sufficient condition for Γ-semigroup M to be regular using bi-interior ideal.

Theorem 3.28. *M is a regular $\Gamma-$semigroup if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$ for any bi-interior ideal B, ideal I and left ideal L of M.*

Proof. Suppose M is a regular $\Gamma-$semigroup, B, I and L are bi-interior ideal, ideal and left ideal of M respectively. Let $a \in B \cap I \cap L$. Then by the hypothesis, $a \in a\Gamma M\Gamma a$. Thus we get

$$a \in a\Gamma M\Gamma a \subseteq a\Gamma M\Gamma a\Gamma M\Gamma a\Gamma M\Gamma a \subseteq B\Gamma I\Gamma B \subseteq B\Gamma M\Gamma B,$$

$$a \in a\Gamma M\Gamma a \subseteq a\Gamma M\Gamma a\Gamma M\Gamma a\Gamma M\Gamma a \subseteq M\Gamma B\Gamma M.$$

So $a \in B\Gamma M\Gamma B \cap M\Gamma B\Gamma M = B$. Hence $B \cap I \cap L \subseteq B$.

Conversely, suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$ for any bi-interior ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be a left ideal of M. Then by the assumption, $R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L$. Thus $R\Gamma L \subseteq R,\ R\Gamma L \subseteq L$. So $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$. Therefore M is a regular Γ-semigroup. □

Theorem 3.29. *If Γ-semigroup M is a left (right) simple Γ-semigroup, then every bi-interior ideal of M is a right (left) ideal of M.*

Proof. Let M be a left simple Γ-semigroup and let B be a bi-interior of M. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Thus $M\Gamma B = M$. So we get

$$\begin{aligned}
&M\Gamma B\Gamma M = M\Gamma M \subseteq M\\
\Rightarrow &B\Gamma M\Gamma B = B\Gamma M\\
\Rightarrow &M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = M \cap B\Gamma M = B\Gamma M\\
\Rightarrow &B\Gamma M \subseteq M\Gamma B\Gamma M\\
\Rightarrow &B\Gamma M \subseteq B\Gamma M\Gamma B\\
\Rightarrow &B\Gamma M \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B\\
\Rightarrow &M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B\Gamma M \subseteq B.
\end{aligned}$$

Hence every bi-interior ideal is a right ideal of M. Similarly, we can prove for right simple Γ-semigroup M. □

Theorem 3.30. *Let M be a Γ-semigroup. If B is a bi-interior ideal of M and T is a non-empty subset of B such that $B\Gamma T$ is an additive subsemigroup of M, then $B\Gamma T$ is a bi-interior ideal of M.*

Proof. Suppose B is a bi-interior ideal of M and T is a non-empty subset of B such that $B\Gamma T$ is an additive subsemigroup of M. The clearly, $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Thus $(M\Gamma B\Gamma M \cap B\Gamma M\Gamma B)\Gamma T \subseteq B\Gamma T$. So $M\Gamma B\Gamma M\Gamma T \cap B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $M\Gamma B\Gamma T\Gamma M \cap B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma T$. Therefore $B\Gamma T$ is a bi-interior ideal of M. □

Theorem 3.31. *Let B be a Γ-subsemigroup of a Γ-semigroup M. If B is a bi-interior ideal of M, then B is a bi-quasi ideal of M.*

Proof. Suppose B is a bi-interior ideal of M. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Thus we have

$$M\Gamma B \cap B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$$

and

$$B\Gamma M \cap B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B.$$

So B is a bi-quasi ideal of M. □

Theorem 3.32. *Let M be a Γ-semigroup. If B is a bi-interior ideal of M and T is a non-empty subset of B such that $B\Gamma T$ is a subsemigroup of M, then $B\Gamma T$ is a bi-interior ideal of M.*

Proof. Suppose B is a bi-interior ideal of M and T is a non-empty subset of B such that $B\Gamma T$ is a subsemigroup of M. Then clearly, $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Thus $(M\Gamma B\Gamma M \cap B\Gamma M\Gamma B)\Gamma T \subseteq B\Gamma T$. So $M\Gamma B\Gamma M\Gamma T \cap B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T$. Hence we have

$$M\Gamma B\Gamma T\Gamma M \cap B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M\Gamma T \cap B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T.$$

Therefore $B\Gamma T$ is a bi-interior ideal of M. □

Theorem 3.33. *Let M be a regular Γ-semigroup. If B is a bi-interior ideal of M, then B is a bi-ideal of M.*

Proof. Suppose B is a bi-interior ideal of M. Then $B\Gamma M\Gamma B \cap M\Gamma B\Gamma M = B$. Thus we get

$$B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B\Gamma M \subseteq M\Gamma B\Gamma M.$$

Thus $B\Gamma M\Gamma B \cap M\Gamma B\Gamma B = B$. So $B\Gamma M\Gamma B = B$. Hence B is a bi-ideal of M. □

Theorem 3.34. *Let M be a regular Γ-semigroup. If B is a bi-interior ideal of M and B is itself a regular Γ-subsemigroup of M, then any bi-interior ideal of B is a bi-interior ideal of M.*

Proof. Let B be a bi-interior ideal of M and A be any bi-interior ideal of B. Then $B\Gamma M\Gamma B \cap M\Gamma B\Gamma M = B,\ A\Gamma B\Gamma A \cap B\Gamma A\Gamma B = A$. Thus we have

$$A\Gamma M\Gamma A \cap M\Gamma A\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M = B.$$

Now let $a \in A$. Then there exist $b \in M, \alpha, \beta \in \Gamma$ such that

$$a = a\alpha b\beta a = a\alpha(b\beta a) \in A\Gamma B.$$

Thus $a \in B\Gamma A$. Since B is a bi-interior ideal of M, we get

$$A\Gamma M\Gamma A \subseteq A\Gamma B\Gamma M\Gamma B\Gamma A \subseteq A\Gamma B\Gamma A.$$

So we have

$$\begin{aligned} A\Gamma M\Gamma A \cap M\Gamma A\Gamma M &\subseteq A\Gamma B\Gamma A \cap M\Gamma B\Gamma M \\ &\subseteq A\Gamma B\Gamma A \cap B \\ &\subseteq A \cap B \subseteq A. \end{aligned}$$

Hence $A = A\Gamma M\Gamma A \cap M\Gamma A\Gamma M$. Therefore A is a bi-interior ideal of M. □

4. Fuzzy bi-interior ideals of $\Gamma-$semigroups

In this section, we introduce the notion of a fuzzy bi-interior ideal as a generalization of a fuzzy bi-ideal and a fuzzy interior ideal of Γ-semigroup M. We study the properties of fuzzy bi-interior ideal.

Definition 4.1. A fuzzy subset μ of a Γ-semigroup M is called a *fuzzy bi-interior ideal* of M, if μ is a fuzzy Γ-subsemigroup and $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$

Example 4.2. Let $\mathbb{Q}$ be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Q} \right\}$ and $M = \Gamma$. Ternary operation is defined as the usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in \mathbb{Q} \right\}$. Then M is a Γ-semigroup and A is a bi-interior ideal but not a bi-ideal of semigroup M.

Define $\mu : M \to [0,1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

Then μ is a fuzzy bi-interior ideal of M.

Theorem 4.3. *Every fuzzy left ideal of a Γ-semigroup M is a fuzzy bi-interior ideal of M.*

Proof. Let μ be a fuzzy left ideal of and let $x \in M, \alpha, \beta \in \Gamma$. Then we get

$$\begin{aligned}\chi_M \circ \mu(x) &= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu(b)\}\}\\ &= \sup_{x=a\alpha b}\{\min\{1, \mu(b)\}\}\\ &= \sup_{x=a\alpha b}\{\mu(b)\}\\ &\leq \sup_{x=a\alpha b}\{\mu(ab)\}\\ &= \sup_{x=a\alpha b}\{\mu(x)\}\\ &= \mu(x).\end{aligned}$$

Thus we have

$$\begin{aligned}\mu \circ \chi_M \circ \mu(x) &= \sup_{x=u\alpha v\beta s}\{\min\{\mu(u), \chi_M \circ \mu(v\beta s)\}\}\\ &\leq \sup_{x=u\alpha v\beta s}\{\min\{\mu(u), \mu(v\beta s)\}\}\\ &= \mu(x).\end{aligned}$$

So we get

$$\begin{aligned}\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) &= \min\{\chi_M \circ \mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\}\\ &\leq \min\{\chi_M \circ \mu \circ \chi_M(x), \mu(x)\}\\ &\leq \mu(x).\end{aligned}$$

Hence $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) \subseteq \mu$. Therefore μ is a fuzzy bi-interior ideal of M. □

Theorem 4.4. *Every fuzzy right ideal of a* Γ*-semigroup* M *is a fuzzy bi-interior ideal of* M*.*

Proof. Let μ be a fuzzy right ideal of the Γ-semigroup M and let $x \in M, \alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned}\mu \circ \chi_M(x) &= \sup_{x=a\alpha b}\{\min\{\mu(a), \chi_M(b)\}\}\\ &= \sup_{x=a\alpha b}\{\mu(a)\}\\ &\leq \sup_{x=a\alpha b}\{\mu(ab)\}\\ &= \mu(x).\end{aligned}$$

Thus we get

$$\begin{aligned}\mu \circ \chi_M \circ \mu(x) &= \sup_{x=u\alpha v\beta s}\{\min\{\mu \circ \chi_M(uv), \mu(s)\}\}\\ &\leq \sup_{x=u\alpha v\beta s}\{\min\{\mu(u\alpha v), \mu(s)\}\}\\ &= \mu(x).\end{aligned}$$

So we have

$$\begin{aligned}\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) &= \min\{\chi_M \circ \mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\}\\ &\leq \min\{\chi_M \circ \mu \circ \chi_M(x), \mu(x)\}\\ &\leq \mu(x).\end{aligned}$$

Hence μ is a fuzzy bi-interior ideal of $\Gamma-$ semigroup M. □

Corollary 4.5. *Every fuzzy ideal of* Γ*-semigroup* M *is a fuzzy bi-interior ideal of* M.

Theorem 4.6. *Let* M *be a* Γ*-semigroup and* μ *be a non-empty fuzzy subset of* M. *A fuzzy subset* μ *is a fuzzy bi-interior ideal of* M *if and only if the level subset* μ_t *of* μ *is a bi-interior ideal of* M *for every* $t \in [0, 1]$, *where* $\mu_t \neq \phi$.

Proof. Suppose μ is a fuzzy bi-interior ideal of M and let $\mu_t \neq \phi$ for $t \in [0, 1]$. Let $x \in M\mu_t M \cap \mu_t M\mu_t$. Then $x = bau = cde$, where $b, u, d \in M, a, c, e \in \mu_t$. Then $\chi_M \circ \mu \circ \chi_M(x) \geq t$ and $\mu \circ \chi_M \circ \mu(x) \geq t$. Thus $\mu(x) \geq t$. So $x \in \mu_t$. Hence μ_t is a bi-interior ideal of M.

Conversely, suppose that μ_t is a bi-interior ideal of M for all $t \in Im(\mu)$. Let $x,\ y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$. Thus we have

$$M\mu_l M \cap \mu_l M\mu_l \subseteq \mu_t \text{ for all } l \in Im(\mu).$$

Now let $t = \min\{Im(\mu)\}$. Then $M\mu_t M \cap \mu_t M\mu_t \subseteq \mu_t$. Thus we get

$$\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu.$$

So μ is a fuzzy bi-interior ideal of the $\Gamma-$semigroup M. □

Theorem 4.7. *Let* I *be a non-empty subset of a* Γ*-semigroup* M *and* χ_I *be the characteristic function of* I. *Then* I *is a bi-interior ideal of* M *if and only if* χ_I *is a fuzzy bi-interior ideal of* Γ*-semigroup* M.

Proof. Suppose I is a bi-interior ideal of M. Obviously, χ_I is a fuzzy Γ-subsemigroup of M. Then $M\Gamma I\Gamma M \cap I\Gamma M\Gamma I \subseteq I$. Thus we have

$$\begin{aligned}\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I &= \chi_{M\Gamma I\Gamma M} \cap \chi_{I\Gamma M\Gamma I}\\ &= \chi_{M\Gamma I\Gamma M\cap I\Gamma M\Gamma I}\\ &\subseteq \chi_I.\end{aligned}$$

So χ_I is a fuzzy bi-interior ideal of M.

Conversely, suppose that χ_I is a fuzzy bi-interior ideal of M. Then I is a Γ-subsemigroup of M. Thus we have

$$\begin{aligned}&\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I\\ \Rightarrow&\chi_{M\Gamma I\Gamma M} \cap \chi_{I\Gamma M\Gamma I} \subseteq \chi_I\\ \Rightarrow&\chi_{M\Gamma I\Gamma M\cap I\Gamma M\Gamma I} \subseteq \chi_I.\end{aligned}$$

So $M\Gamma I\Gamma M \cap I\Gamma M\Gamma I \subseteq I$. Hence I is a bi-interior ideal of M. □

Theorem 4.8. *If* μ *and* λ *are fuzzy bi-interior ideals of* Γ*-semigroup* M, *then* $\mu \cap \lambda$ *is a fuzzy bi-interior ideal of* M.

Proof. Suppose μ and λ are fuzzy bi-interior ideals of M and let $x \in M, \alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned}
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu \cap \lambda(b)\}\} \\
&= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b}\{\min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b}\{\min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\chi_M \circ \mu(x). \chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x),
\end{aligned}$$

$$\begin{aligned}
\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \lambda(a)\}, \min\{\chi_M \circ \mu(b\beta c), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \chi_M \circ \mu(b\beta c)\}, \min\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= min\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

Thus $\chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda$ and $\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda$. Similarly, $\chi_M \circ \mu \cap \lambda \circ \chi_M = \chi_M \circ \mu \circ \chi_M \cap \chi_M \circ \lambda \circ \chi_M$. So we get

$$\begin{aligned}
&\chi_M \circ \mu \cap \lambda \circ \chi_M \cap \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda \\
&= (\chi_M \circ \mu \circ \chi_M) \cap (\mu \circ \chi_M \circ \mu) \cap (\chi_M \circ \lambda \circ \chi_M) \cap (\lambda \circ \chi_M \circ \lambda) \\
&\subseteq \mu \cap \lambda.
\end{aligned}$$

Hence $\mu \cap \lambda$ is a fuzzy bi-interior ideal of M. □

Theorem 4.9. *Let M be a Γ-semigroup. Then M is regular if and only if $\mu = \chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu$ for any fuzzy bi-interior ideal μ of M.*

Proof. Suppose M is regular and let μ be a fuzzy left bi-interior ideal of M and let $x,\ y \in M, \alpha, \beta \in \Gamma$. Then $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Thus

$$\chi_M \circ \mu \circ \chi_M(x) = \sup_{x=x\alpha y\beta x}\{\min\{\chi_M(x\alpha y), \mu(x)\}\} = \mu(x),$$

$$\begin{aligned}
\mu \circ \chi_M \circ \mu(x) &= \sup_{x=x\alpha y\beta x}\{\min\{\mu(x), \chi_M \circ \mu(y\beta x)\}\} \\
&= \sup_{x=x\alpha y\beta x}\{\min\{\mu(x), \sup_{y\beta x=r\delta s}\{\min\{\chi_M(r), \mu(s)\}\}\}\} \\
&= \sup_{x=x\alpha y\beta x}\{\min\{\mu(x), \sup_{y\beta x=r\delta s}\{\min\{1, \mu(s)\}\}\} \\
&\geq \sup_{x=x\alpha y\beta x}\{\min\{\mu(x), \mu(x)\}\} \\
&= \mu(x).
\end{aligned}$$

Similarly, $\chi_M \circ \mu \circ \chi_M \supseteq \mu$. So $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu = \mu$.

Conversely, suppose that $\mu = \chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu$ for any fuzzy bi-interior ideal μ of M. Let B be a bi-interior ideal of M. Then by Theorem 4.7, χ_B is a fuzzy bi-interior ideal of M. Thus we have

$$\chi_B = \chi_M \circ \chi_B \circ \chi_M \cap \chi_B \circ \chi_M \circ \chi_B = \chi_{M\Gamma B\Gamma M} \cap \chi_{B\Gamma M\Gamma B}.$$

So $B = M\Gamma B\Gamma M \cap B\Gamma M\Gamma B$. Hence by Theorem 3.20, M is a regular Γ-semigroup. □

Theorem 4.10. *Let M be a regular Γ-semigroup. Then μ is a fuzzy bi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Proof. Suppose μ is a fuzzy bi-interior ideal of M and let $x \in M$. Then we get

$$\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu.$$

Assume that $\chi_M \circ \mu(x) > \mu(x)$. Since M is regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then we have

$$\begin{aligned}
\mu \circ \chi_M \circ \mu(x) &= \sup_{x=x\alpha y\beta x} \{\min\{\mu(x), \chi_M \circ \mu(y\beta x)\}\ ) \\
&> \sup_{x=x\alpha y\beta x} \{\min\{\mu(x), \mu(y\beta x)\}\} \\
&= \mu(x).
\end{aligned}$$

Which is a contradiction. Thus $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$. So μ is a fuzzy quasi ideal of M.

The proof of the converse is obvious. □

Theorem 4.11. *Let M be a semigroup. Then M is regular if and only if $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$ for every fuzzy bi-interior ideal μ and every fuzzy ideal of M.*

Proof. Suppose M is regular and let $x \in M$. Then there exists $y \in M$ such that $x = xyx$. Thus we have

$$\begin{aligned}
\mu \circ \gamma \circ \mu(x) &= \sup_{x=xyx} \{\min\{\mu \circ \gamma(xy), \mu(x)\}\} \\
&= \min\{\sup_{xy=xyxy} \{\min\{\mu(x), \gamma(yxy)\}, \mu(x)\}\} \\
&\geq \min\{\min\{\mu(x), \gamma(x)\}, \mu(x\} \\
&= \min\{\mu(x), \gamma(x)\} = \mu \cap \gamma(x), \\
\gamma \circ \mu(x) &= \sup_{x=xyx} \{\min\{\gamma(xy), \mu(x)\}\} \\
&\geq \min\{\gamma(x), \mu(x)\} = \mu \cap \gamma(x).
\end{aligned}$$

So $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu$ $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma$. Hence $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \gamma \cap \mu \circ \gamma \circ \mu$.

Conversely, suppose that the necessary condition holds. Let μ be a fuzzy bi-interior ideal of semigroup M. Then we have

$$\mu \cap \chi_M \subseteq \chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu,\ \mu \subseteq \chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu.$$

Thus M is a regular semigroup. □

5. Conclusion

As a further generalization of ideals, we introduced the notion of a bi-interior ideal of $\Gamma-$semigroup as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal, interior ideal, bi-interior ideal of $\Gamma-$semigroup and studied some of their properties. We introduced the notion of a bi-interior simple $\Gamma-$semigroup and characterized the bi-interior simple $\Gamma-$semigroup, regular $\Gamma-$semigroup using bi-interior ideals of $\Gamma-$semigroup. In continuiation of this paper, we study prime bi-interior ideals, maximal and minimal bi-interior ideals of $\Gamma-$semigroup. In this paper, we introduced the notion of a fuzzy bi-interior ideal in a $\Gamma-$semigroup and characterized the regular $\Gamma-$semigroup in terms of fuzzy bi-interior ideal. In continuation of this paper, we propose to study the fuzzy tri ideal and fuzzy soft tri ideal of semirings.

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