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Regular nearness semigroups

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Abstract. This paper concerned with basic concepts and some results on regular semigroup on weak nearness approximation spaces. Also, it is given an example related to the subject.

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1. Introduction

Semigroups, as the basic algebraic structure were used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In the abstract theory of semigroups, the notion of regular element was first investigated by Thierrin [1] as a generalization in the semigroup theory of the concept of inverse element in the group theory. This subject of regular element has been effectively used in the ideal theory of semigroups in Miller and Clifford [2]. Semigroups in which all the elements are regular, is known to be regular semigroups. The regular semigroups that the definition is copied from Von Neumann's definition of a regular ring in 1936 [3] are particulary possible to study in many areas. Ideal theory has an important role in semigroup theory. Researchers have studied different kinds of ideals in semigroups such as quasi-ideals, bi-ideals, interior ideals and so on. The concept of interior ideals in semigroup is studied by Szasz [4].

Rough set theory studied by Pawlak can be seen as a new mathematical approach to uncertainty [5]. The rough set idea is about assumption that every object of the universe of discourse we deal with have some information. Afterwards, Peters defined an indiscernibility relation by using the features of the objects to determine the nearness of the objects [6] in 2002 as a generalization of rought set theory. Moreover, he studied approach theory of the nearness of non-empty sets resembling each other [7, 8]. In 2012, İnan and Öztürk investigated the concept of nearness semigroups [9] and other algebraic approaches of near sets. Tekin defined bi nearness ideals and nearness quasi ideal in semirings [10, 11].

Reader can be found other nearness algebraic structures [12, 13, 14]. In this paper, we introduced the concept of nearness regular semigroups, nearness interior ideals and also studied some properties. Furthermore, we investigated some features of bi nearness ideals and quasi nearness ideals in regular semigroups.

2. Preliminaries

A regular semigroup is a semigroup S in which every element is regular that is for each element $a \in S$ there exist an element $x \in S$ such that $a = axa$ that is $a \in aSa$ [15].

Definition 2.1 ([4]). Let S be a semigroup. A subsemigroup I of S is called an *interior ideal* of S, if $SIS \subseteq I$.

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where the nearness approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r}.

Definition 2.2 ([9]). Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space, let S be a near semigroup and let I a nonempty subset of S. If $N_r(B)^*(I)$ is a left (right,two sided) ideal of S, then I is called a *near left* (*right*, *two sided*) *ideal* of S.

Definition 2.3 ([13]). Let $\mathcal{O}$ be a set of sample objects, let $\mathcal{F}$ be a set of the probe functions, let $\sim_{B_r}$ be an indiscernibility relation and let N_r be a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a *weak nearness approximation space.*

Theorem 2.4 ([13]). *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and let $X,\ Y \subset \mathcal{O}$. Then the following statements hold:*

(1) $X \subseteq N_r\,(B)^*\,X$,
(2) $N_r\,(B)^*\,(X \cup Y) = N_r\,(B)^*\,X \cup N_r\,(B)^*\,Y$,
(3) $X \subseteq Y$ *implies* $N_r\,(B)^*\,X \subseteq N_r\,(B)^*\,Y$,
(4) $N_r\,(B)^*\,(X \cap Y) \subseteq N_r\,(B)^*\,X \cap N_r\,(B)^*\,Y$.

Lemma 2.5 ([12]). *Let S be a nearness semiring. The following properties hold:*

(1) *if* $X, Y \subseteq S$, *then* $(N_r\,(B)^*\,X) + (N_r\,(B)^*\,Y) \subseteq N_r\,(B)^*\,(X + Y)$,
(2) *if* $X, Y \subseteq S$, *then* $(N_r\,(B)^*\,X)(N_r\,(B)^*\,Y) \subseteq N_r\,(B)^*\,(XY)$.

Theorem 2.6 ([12]). *Let S be a nearness semiring, let $\sim_{B_r}$ be a complete congruence indiscernibility relation on S and let $X,\ Y$ be two non-empty subsets of S. Then the following properties hold:*

(1) $(N_r\,(B)^*\,X) + (N_r\,(B)^*\,Y) = N_r\,(B)^*\,(X + Y)$,
(2) $(N_r\,(B)^*\,X)(N_r\,(B)^*\,Y) = N_r\,(B)^*\,(XY)$.

Definition 2.7 ([16]). Let $S \subseteq \mathcal{O}$, where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is weak nearness approximation spaces. Then S is called a *semigroup on $\mathcal{O}$* (in short, *nearness semigroup*), if it satisfies the following conditions: for all $x,\ y \in S$,

(i) $xy \in N_r(B)^* S$,
(ii) $(xy)z = x(yz)$ property holds in $N_r(B)^* S$.

Definition 2.8 ([10]). Let S be a nearness semiring and let Q be a nearness subsemigroup of S, where $Q \neq S$.
(i) Q is said to be *quasi-nearness ideal*, if $QS \cap SQ \subseteq N_r(B)^* Q$.
(ii) Q is called a *quasi upper-near ideal* of S, if $(N_r(B)^* Q)S \cap S(N_r(B)^* Q) \subseteq N_r(B)^* Q$.

Definition 2.9 ([11]). Let S be a nearness semiring and let A be a nearness subsemigroup of S.
(i) A is called a *bi-nearness ideal*, if $ASA \subseteq N_r(B)^* A$.
(ii) A is called a *bi-upper-near ideal* of S, if $(N_r(B)^* A)S(N_r(B)^* A) \subseteq N_r(B)^* A$.

Definition 2.10 ([17]). Let S be a nearness semigroup and let Q be a nearness subsemigroup of S.
(i) Q is called a *quasi-nearness ideal* of S, if $QS \cap SQ \subseteq N_r(B)^* Q$.
(ii) Q is called a *quasi upper-near ideal* of S, if $(N_r(B)^* Q)S \cap S(N_r(B)^* Q) \subseteq N_r(B)^* Q$.

Lemma 2.11 ([17]). *Let S be a nearness semigroup. If S is commutative, then each quasi-nearness ideal of S is two-sided nearness ideal of S.*

3. Regular nearness semigroups

Definition 2.2 can be restated as follow for weak nearness approximation space by considering Definition 2.3.

Definition 3.1. Let S be a semigroup on weak nearness approximation space $\mathcal{O}$ and let I be a non-empty subset of S.
(i) I is called a *right* (*left*) *nearness ideal* of S, if $IS \subseteq N_r(B)^* I$ ($SI \subseteq N_r(B)^* I$).
(ii) I is called a *right* (*left*) *upper-near nearness ideal* of S, if $(N_r(B)^* I)S \subseteq N_r(B)^* I(S(N_r(B)^* I) \subseteq N_r(B)^* I)$.

A nearness ideal of S is both left as well as right nearness ideal.

Definition 3.2. Let S be a nearness semigroup and let I be a nearness subsemigroup of S.
(i) I is called a *bi-nearness ideal* of S, if $ISI \subseteq N_r(B)^* I$.
(ii) I is called a *bi-upper-near ideal* of S, if $(N_r(B)^* I)S(N_r(B)^* I) \subseteq N_r(B)^* I$.

Definition 3.3. Let S be a nearness semigroup and let I be a nearness subsemigroup of S.
(i) I is called an *interior nearness ideal* of S, if $SIS \subseteq N_r(B)^* I$.
(ii) I is called an *interior upper-near ideal* of S, if $S(N_r(B)^* I)S \subseteq N_r(B)^* I$.

Example 3.4. Let $\mathcal{O} = \{o, p, q, s, t, u, v, w, x, y, z\}$ be a set of perceptual objects where $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ is a set of probe functions and $S = \{t, u, v, x\} \subset \mathcal{O}$. For $r = 1$, values of the probe functions

$$\chi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$
$$\chi_2 : \mathcal{O} \to V_2 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$
$$\chi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_3, \beta_4, \beta_5\}$$

are given in the following table.

	o	p	q	s	t	u	v	w	x	y	z
χ_1	β_1	β_2	β_3	β_4	β_2	β_4	β_3	β_4	β_4	β_1	β_1
χ_2	β_1	β_1	β_2	β_3	β_4	β_5	β_2	β_1	β_2	β_4	β_3
χ_3	β_1	β_3	β_1	β_1	β_4	β_5	β_4	β_5	β_4	β_1	β_3

Now, we determine the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ for $\mathcal{O}$.

$$\begin{aligned}
[o]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(o) = \beta_1\} = \{o, y, z\} \\
&= [y]_{\chi_1} = [z]_{\chi_1}, \\
[p]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(p) = \beta_2\} = \{p, t\} \\
&= [t]_{\chi_1}, \\
[q]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(q) = \beta_3\} = \{q, v\} \\
&= [v]_{\chi_1}, \\
[s]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(s) = \beta_4\} = \{s, u, w, x\} \\
&= [u]_{\chi_1} = [w]_{\chi_1} = [x]_{\chi_1}.
\end{aligned}$$

Then we get $\xi_{\chi_1} = \left\{[o]_{\chi_1}, [p]_{\chi_1}, [q]_{\chi_1}, [s]_{\chi_1}\right\}$.

$$\begin{aligned}
[o]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(o) = \beta_1\} = \{o, p, w\} \\
&= [p]_{\chi_2} = [w]_{\chi_2}, \\
[q]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(q) = \beta_2\} = \{q, v, x\} \\
&= [v]_{\chi_2} = [x]_{\chi_2}, \\
[s]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(s) = \beta_3\} = \{s, z\} \\
&= [z]_{\chi_2}, \\
[t]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(t) = \beta_4\} = \{t, y\} \\
&= [w]_{\chi_2}, \\
[u]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(u) = \beta_5\} = \{u\}.
\end{aligned}$$

Thus we have $\xi_{\chi_2} = \left\{[o]_{\chi_2}, [q]_{\chi_2}, [s]_{\chi_2}, [t]_{\chi_2}, [u]_{\chi_2}\right\}$.

$$\begin{aligned}
[o]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(o) = \beta_1\} = \{o, q, s, y\} \\
&= [q]_{\chi_3} = [s]_{\chi_3} = [y]_{\chi_3}, \\
[p]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(p) = \beta_3\} = \{p, z\} \\
&= [z]_{\chi_3}, \\
[t]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(t) = \beta_4\} = \{t, v, x\} \\
&= [v]_{\chi_3} = [x]_{\chi_3}, \\
[u]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(u) = \beta_5\} = \{u, w\} \\
&= [w]_{\chi_3}.
\end{aligned}$$

So we obtain $\xi_{\chi_3} = \left\{[o]_{\chi_3}, [p]_{\chi_3}, [t]_{\chi_3}, [u]_{\chi_3}\right\}$, and for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Hence we can write

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\chi_i} \cap S \neq \varnothing} [x]_{\chi_i} \\
&= [p]_{\chi_1} \cup [q]_{\chi_1} \cup [s]_{\chi_1} \cup [q]_{\chi_2} \cup [t]_{\chi_2} \cup [u]_{\chi_2} \cup [t]_{\chi_3} \cup [u]_{\chi_3} \\
&= \{p, q, s, t, u, v, w, x, y\}.
\end{aligned}$$

Considering the following table of operation:

$\cdot$	t	u	v	x
t	s	v	u	w
u	x	w	t	v
v	w	x	s	u
x	u	t	w	s

Therefore $(S, \cdot)$ is a semigroup on $\mathcal{O}$, i.e., $(S, \cdot)$ is a nearness semigroup. Afterwards, it is taken $I = \{u, v, x\} \subseteq S$.

$$\begin{aligned}
N_1(B)^* I &= \bigcup_{[x]_{\chi_i} \cap I \neq \varnothing} [x]_{\chi_i} \\
&= [q]_{\chi_1} \cup [s]_{\chi_1} \cup [q]_{\chi_2} \cup [u]_{\chi_2} \cup [t]_{\chi_3} \cup [u]_{\chi_3} \\
&= \{q, s, t, u, v, w, x\}.
\end{aligned}$$

In this case, I is a nearness subsemigroup of S, and satisfies the condition $SIS \subseteq N_r(B)^* I$. Then I is an interior nearness ideal of S.

Lemma 3.5. *Let S be a nearness semigroup. If $N_r(B)^* (N_r(B)^* I) = N_r(B)^* I$, then every nearness ideal I of S is an interior nearness ideal I of S.*

Proof. Let S be a nearness semigroup and let I be a nearness ideal of S. Then we have

$$SI \subseteq N_r(B)^* I \Rightarrow SIS \subseteq (N_r(B)^* I)S.$$

By using Theorem 2.4 (2) and Lemma 2.5 (2), we get

$$\begin{aligned} SIS &\subseteq (N_r(B)^* I)S \\ &\subseteq (N_r(B)^* I)(N_r(B)^* S) \\ &\subseteq N_r(B)^* (IS) \\ &\subseteq N_r(B)^* (N_r(B)^* I) \\ &= N_r(B)^* I. \end{aligned}$$

Thus $SIS \subseteq N_r(B)^* I$ and I is an interior nearness ideal of S. □

Lemma 3.6. *Let S be a commutative nearness semigroup. If $N_r(B)^* \left(N_r(B)^* I\right) = N_r(B)^* I$, then each quasi-nearness ideal I of S is a bi-nearness ideal I of S*

Proof. Let S be a commutative nearness semigroup and let I be a quasi-nearness ideal of S. We show that I is bi-nearness ideal of S. It is obvious that

$$ISI = (ISI) \cap (ISI) = I(SI) \cap (IS)I \subseteq S(SI) \cap (IS)S \subseteq (SS)I \cap I(SS).$$

Then by Theorem 2.4 (1) and Lemma 2.5 (2), we have

$$\begin{aligned} (SS)I \cap I(SS) &\subseteq (N_r(B)^* S)(N_r(B)^* I) \cap (N_r(B)^* I)(N_r(B)^* S) \\ &\subseteq (N_r(B)^* (SI)) \cap (N_r(B)^* (IS)). \end{aligned}$$

Thus from Lemma 2.11, we get

$$\begin{aligned} (N_r(B)^* (SI)) \cap (N_r(B)^* (IS)) &\subseteq N_r(B)^* \left(N_r(B)^* I\right) \cap N_r(B)^* \left(N_r(B)^* I\right) \\ &= N_r(B)^* \left(N_r(B)^* I\right) \\ &= N_r(B)^* I. \end{aligned}$$

So, $ISI \subseteq N_r(B)^* I$ and I is a bi-nearness ideal of S. □

Definition 3.7. Let S be a nearness semigroup. The element $a \in S$ is called a *regular element*, if there exists $x \in S$ so that the property $axa = a$ holds in $N_r(B)^* S$. S is called a *nearness regular semigroup*, if all its elements of S are regular.

Example 3.8. Let $\mathcal{O} = \{o, p, q, s, t, u, v, w, x, y, z\}$ be a set of perceptual objects where $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ is a set of probe functions and $S = \{s, t, w, x\} \subset \mathcal{O}$. For $r = 1$, values of the probe functions

$$\begin{aligned} \chi_1 &: \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}, \\ \chi_2 &: \mathcal{O} \to V_2 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}, \\ \chi_3 &: \mathcal{O} \to V_3 = \{\beta_2, \beta_3, \beta_4, \beta_5\} \end{aligned}$$

are given in the following table.

	o	p	q	s	t	u	v	w	x	y	z
χ_1	β_1	β_1	β_2	β_3	β_2	β_3	β_2	β_4	β_3	β_4	β_5
χ_2	β_1	β_2	β_2	β_3	β_3	β_1	β_4	β_4	β_5	β_1	β_2
χ_3	β_2	β_3	β_4	β_5	β_4	β_3	β_3	β_5	β_4	β_6	β_2

Next, it is found the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ for $\mathcal{O}$.

$$\begin{aligned}
[o]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(o) = \beta_1\} = \{o, p\} \\
&= [p]_{\chi_1}\,, \\
[q]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(q) = \beta_2\} = \{q, t, v\} \\
&= [t]_{\chi_1} = [v]_{\chi_1}\,, \\
[s]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(s) = \beta_3\} = \{s, u, x\} \\
&= [u]_{\chi_1} = [x]_{\chi_1}\,, \\
[w]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(w) = \beta_4\} = \{w, y\} \\
&= [y]_{\chi_1}\,, \\
[z]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(z) = \beta_5\} = \{z\}.
\end{aligned}$$

Then we get $\xi_{\chi_1} = \left\{[o]_{\chi_1}, [q]_{\chi_1}, [s]_{\chi_1}, [w]_{\chi_1}, [z]_{\chi_1}\right\}$.

$$\begin{aligned}
[o]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(o) = \beta_1\} = \{o, u, y\} \\
&= [u]_{\chi_2} = [y]_{\chi_2}\,, \\
[p]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(p) = \beta_2\} = \{p, q, z\} \\
&= [q]_{\chi_2} = [z]_{\chi_2}\,, \\
[s]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(s) = \beta_3\} = \{s, t\} \\
&= [t]_{\chi_2}\,, \\
[v]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(v) = \beta_4\} = \{v, w\} \\
&= [w]_{\chi_2}\,, \\
[x]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(x) = \beta_5\} = \{x\}.
\end{aligned}$$

Thus we have $\xi_{\chi_2} = \left\{[o]_{\chi_2}, [p]_{\chi_2}, [s]_{\chi_2}, [v]_{\chi_2}, [x]_{\chi_2}\right\}$.

$$\begin{aligned}
[o]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(o) = \beta_2\} = \{o, z\} \\
&= [z]_{\chi_3}\,, \\
[p]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(p) = \beta_3\} = \{p, u, v\} \\
&= [u]_{\chi_3} = [v]_{\chi_3}\,, \\
[q]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(q) = \beta_4\} = \{q, t, x\} \\
&= [t]_{\chi_3} = [x]_{\chi_3}\,, \\
[s]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(s) = \beta_5\} = \{s, w\} \\
&= [w]_{\chi_3}\,, \\
[y]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(y) = \beta_6\} = \{y\}.
\end{aligned}$$

So we obtain $\xi_{\chi_3} = \left\{ [o]_{\chi_3}, [p]_{\chi_3}, [q]_{\chi_3}, [s]_{\chi_3}, [y]_{\chi_3} \right\}$, and for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Hence it can be written

$$\begin{aligned} N_1(B)^* S &= \bigcup_{[x]_{\chi_i} \cap S \neq \varnothing} [x]_{\chi_i} \\ &= [q]_{\chi_1} \cup [s]_{\chi_1} \cup [w]_{\chi_1} \cup [s]_{\chi_2} \cup [v]_{\chi_2} \cup [x]_{\chi_2} \cup [q]_{\chi_3} \cup [s]_{\chi_3} \\ &= \{q, s, t, u, v, w, x, y\}. \end{aligned}$$

Let "$\cdot$" be a binary operation on $N_1(B)^* S$ as given in the following table

$\cdot$	q	s	t	u	v	w	x	y
q	w	u	v	y	x	t	s	q
s	u	t	v	x	s	u	w	y
t	v	v	s	w	t	x	u	q
u	y	x	w	v	u	t	s	q
v	x	s	t	u	v	w	x	s
w	t	u	x	t	w	s	v	w
x	s	w	u	s	x	v	t	y
y	q	y	q	q	s	w	y	x

Then "$\cdot$" be an operation of perceptual objects on $S \subseteq \mathcal{O}$.

$\cdot$	s	t	w	x
s	t	v	u	w
t	v	s	x	u
w	u	x	s	v
x	w	u	v	t

In that case, $(S, \cdot)$ is a semigroup on $\mathcal{O}$. Next, for $s \in S$ there exists $t \in S$ so that the property $sts = s$ holds in $N_r(B)^* S$. Thus S is a regular nearness semigroup.

Theorem 3.9. *Let S be a regular nearness semigroup, let $\sim_{B_r}$ be a complete congruence indiscernibility relation on S and let I be an interior nearness ideal of S. If $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then $N_r(B)^* I = N_r(B)^*(SIS)$.*

Proof. Let S be a regular nearness semigroup and let I be an interior nearness ideal of S. Since I is an interior nearness ideal of S, $SIS \subseteq N_r(B)^* I$. Then by Theorem 2.6 (2), we have

$$N_r(B)^*(SIS) \subseteq N_r(B)^*(N_r(B)^* I).$$

Thus by the hypothesis, $N_r(B)^*(SIS) \subseteq N_r(B)^* I$.

On the other hand, let $a \in N_r(B)^* I$. In this case, $[a]_{B_r} \cap I \neq \varnothing$. Then there exists an element $x \in [a]_{B_r}$ and $x \in I$. Since S is regular, for $x \in S$, there exists $y \in S$ so that the property $xyx = x$ holds. From here, $x \in xSxSx \subseteq S(SIS)S \subseteq S(N_r(B)^* I)S$, for I is an interior nearness ideal of S. Thus $x \in S(N_r(B)^* I)S$. In this case, we get that $x \in [a]_{B_r}$ and $x \in S(N_r(B)^* I)S$. So $x \in [a]_{B_r} \cap S(N_r(B)^* I)S$. Hence we have $[a]_{B_r} \cap S(N_r(B)^* I)S \neq \varnothing$ and $a \in N_r(B)^*(S(N_r(B)^* I)S)$. Since

$\sim_{B_r}$ is a complete congruence indiscernibility relation and from Theorem 2.6 (2),

$$\begin{aligned} N_r(B)^*(S(N_r(B)^* I)S) &= (N_r(B)^* S)(N_r(B)^*(N_r(B)^* I))(N_r(B)^* S) \\ &= (N_r(B)^* S)(N_r(B)^* I)(N_r(B)^* S) \\ &= N_r(B)^*(SIS). \end{aligned}$$

Thereby, $a \in N_r(B)^*(SIS)$ and we get $N_r(B)^* I \subseteq N_r(B)^*(SIS)$. Consequently, $N_r(B)^* I = N_r(B)^*(SIS)$. □

Theorem 3.10. *Let S be a nearness semigroup and let $\{I_i | i \in \Delta\}$ set of interior nearness ideals of S such that $N_r(B)^*(N_r(B)^* I_i) = N_r(B)^* I_i$ for all $i \in \Delta$ with index set Δ. If $N_r(B)^*(\bigcap_{i\in\Delta} I_i) = \bigcap_{i\in\Delta} N_r(B)^* I_i$, then $\bigcap_{i\in\Delta} I_i = \varnothing$ or $\bigcap_{i\in\Delta} I_i$ is an interior nearness ideal of S.*

Proof. Let $\bigcap_{i\in\Delta} I_i = I$. Now, we show that I is either empty or an interior nearness ideal of S. Assume that I is non-empty. Since I_i is an interior nearness ideals of S for all $i \in \Delta$, I is nearness subsemigroup of S and $SI_iS \subseteq N_r(B)^* I_i$ for all $i \in \Delta$. Then

$$SIS = S(\bigcap_{i\in\Delta} I_i)S \subseteq \bigcap_{i\in\Delta} \left(N_r(B)^* I_i\right)$$

for all $i \in \Delta$. Since $\bigcap_{i\in\Delta} N_r(B)^* I_i = N_r(B)^*(\bigcap_{i\in\Delta} I_i)$, we obtain $SIS \subseteq N_r(B)^* I$. Thus I is an interior nearness ideal of S. □

Theorem 3.11. *Let S be a regular nearness semigroup, let $\sim_{B_r}$ be a complete congruence indiscernibility relation on S and let I be a bi-nearness ideal of S. If $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then $N_r(B)^* I = N_r(B)^*(ISI)$.*

Proof. Let S be a regular nearness semigroup and I be a bi-nearness ideal of S. Then by Theorem 2.6 (2), we have

$$ISI \subseteq N_r(B)^* I \Rightarrow N_r(B)^*(ISI) \subseteq N_r(B)^*(N_r(B)^* I) = N_r(B)^* I.$$

Thus we get $N_r(B)^*(ISI) \subseteq N_r(B)^* I$.

Otherwise, let $x \in N_r(B)^* I$. From here, $[x]_{B_r} \cap A \neq \varnothing$ and there exists an element $a \in [x]_{B_r}$ and $a \in I$. Since S is regular, for $a \in S$, there exists $b \in S$ so that the property $aba = a$ satisfies on S. In this case, $a \in aSaSa \subseteq IS(ISI) \subseteq IS(N_r(B)^* I)$. Then, $a \in IS(N_r(B)^* I)$. From here, $a \in [x]_{B_r}$ and $a \in ISN_r(B)^* I$. Thus $a \in [x]_{B_r} \cap IS(N_r(B)^* I)$ and we have $[x]_{B_r} \cap IS(N_r(B)^* I) \neq \varnothing$. In this way, $x \in N_r(B)^*(IS(N_r(B)^* I))$. Since $\sim_{B_r}$ is a complete congruence indiscernibility relation, from Theorem 2.6 (2), we have

$$\begin{aligned} N_r(B)^*(IS(N_r(B)^* I)) &= (N_r(B)^* I)(N_r(B)^* S)(N_r(B)^*(N_r(B)^* I)) \\ &= (N_r(B)^* I)(N_r(B)^* S)(N_r(B)^* I) \\ &= N_r(B)^*(ISI). \end{aligned}$$

In this case, $x \in N_r(B)^*(ISI)$. So $N_r(B)^* I \subseteq N_r(B)^*(ISI)$. Finally, $N_r(B)^* I = N_r(B)^*(ISI)$. □

Theorem 3.12. *Let S be a commutative regular nearness semigroup, let $\sim_{B_r}$ be a complete congruence indiscernibility relation on S and let I be a quasi-nearness ideal of S. If $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then $N_r(B)^* I = N_r(B)^*(ISI)$.*

Proof. By Lemma 3.6, quasi-nearness ideal I is a bi-nearness ideal of S. Then from Theorem 3.11, $N_r(B)^* I = N_r(B)^*(ISI)$. □

4. Conclusion

As a recent study of nearness ideals, this paper introduces regular nearness semigroup and gives an example about subject. Also, it is given that some properties about ideals of a regular nearness semigroup. Afterward, we study relations among them. We believe that these properties will be more useful theoretical developments for nearness semigroup theory.

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