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On fuzzy strongly semiregular spaces

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Abstract. In this paper, the notion of fuzzy strongly semiregular space is introduced and studied. It is established that fuzzy strongly semiregular spaces are fuzzy quasi- regular spaces and fuzzy almost P-spaces. The conditions under which fuzzy strongly semiregular spaces become fuzzy Baire spaces, are also obtained. It is observed that fuzzy strongly semiregular and fuzzy hyperconnected spaces are fuzzy semi-$P-$spaces and fuzzy ∂-spaces. It is found that fuzzy globally disconnected and fuzzy quasi-regular spaces are fuzzy strongly semiregular spaces. A condition under which fuzzy G_δ-sets become fuzzy Baire dense sets in fuzzy strongly semiregular spaces is also obtained.

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1. Introduction

The potential of fuzzy notion, introduced by Zadeh [1] in 1965 to describe vagueness mathematically in its very abstractness , was realized by many researchers. In 1968, Chang [2] introduced the notion of fuzzy topological spaces. Subsequently, many researchers have been worked in this area and other related areas which have applications in different fields based on this concept. In recent years, there has been a growing trend to introduce and study different kinds of fuzzy topological spaces.

In 1937, Stone [3] had introduced the idea of semiregular spaces in classical topology and obtained their characterizations. In 1969, Singal and Arya [4] introduced a new class of separation axiom (named almost-regular space) in topological spaces which is weaker than regularity but it is equivalent to semiregular spaces due to Stone. Nayar [5] has carried out a study on semi regular spaces in 2008. In 1981, Azad [6] defined the notions of fuzzy semi-open set, fuzzy semi-closed set, fuzzy regular-open set, fuzzy regular-closed set, fuzzy regular spaces and fuzzy semi regular spaces. The notion of fuzzy semiregular spaces in the sense of Sostak was introduced and studied by Kim and Ko [7]. Fuzzy semi regularity in fuzzy topological spaces was also studied by Ajmal and Tyagi [8]. In 1998, Noiri and Jafari [9] introduced and studied the notion of strong semi-regularity in classical topology.

The aim of this paper is to introduce and study the notion of fuzzy strongly semi regular spaces. In Section 3, several characterizations of fuzzy strongly semi regular spaces are established. In Section 4, it is established that fuzzy strongly semiregular spaces are fuzzy quasi- regular spaces and fuzzy almost P-spaces. The conditions under which fuzzy strongly semiregular spaces become fuzzy Baire spaces, are also obtained. It is observed that fuzzy strongly semiregular and fuzzy hyperconnected spaces are fuzzy semi-P-spaces and fuzzy ∂-spaces. It is found that fuzzy globally disconnected and fuzzy quasi-regular spaces are fuzzy strongly semiregular spaces and fuzzy G_δ-sets in fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular spaces are fuzzy second category sets.

In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications[10, 11]. Many authors [12, 13] redefined the classical topological concepts via soft topological structure. Recently, Şenel et al. [14] applied the concept of octahedron sets to multi-criteria group decision making problems. Lee et al. [15] formed the concrete category of cubic relations and morphisms between them and studied some of its categorical structures in the sense of the topological universe. On these lines, there is a need and scope of investigation considering different types of fuzzy spaces such as fuzzy semi regular spaces and fuzzy strongly semi regular spaces for applying some fuzzy topological concepts to information science and decision-making problems. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self-contained. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang(1968). Let X be a non-empty set and I the unit interval $[0, 1]$. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$ for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$ for all $x \in X$. The *fuzzy interior*, the *fuzzy closure* and the *complement* of a fuzzy set λ in a fuzzy topological space (X, T) are defined respectively as follows:

(i) $int(\lambda) = \bigvee\{\mu : \mu \leq \lambda,\ \mu \in T\}$,

(ii) $cl(\lambda) = \bigwedge\{\mu : \lambda \leq \mu, 1 - \mu \in T\}$,

(iii) $\lambda'(x) = 1 - \lambda(x)$ for all $x \in X$.

For a family $(\lambda_j)_{j\in J}$ of fuzzy sets in (X, T), where J denotes an index set, the *union* $\psi = \bigvee_{j\in J} \lambda_j$ and the *intersection* $\delta = \bigwedge_{j\in J} \lambda_j$, are defined respectively as follows: for each $x \in X$,

(iv) $\psi(x) = sup_{j\in J}\{\lambda_j(x) : x \in X\}$,

(v) $\delta(x) = inf_{j\in J}\{\lambda_j(x)/x \in X\}$.

Lemma 2.1 (Lemma 3.2, [6])**.** *For a fuzzy set λ of a fuzzy topological space X,*

(1) $1 - int(\lambda) = cl(1 - \lambda)$,

(2) $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called a:

(i) *fuzzy regular-open set* in X, if $\lambda = intcl(\lambda)$ and *fuzzy regular-closed set* in X, if $\lambda = clint(\lambda)$ [6],

(ii) *fuzzy semi-open set* in X, if $\lambda \leq clint(\lambda)$ and *fuzzy semi-closed set* in (X, T), if $intcl(\lambda) \leq \lambda$ [6],

(iii) *fuzzy G_δ-set* in X, if $\lambda = \bigwedge_{j=1}^{\infty} \lambda_j$, where $\lambda_j \in T$ for $j \in J$ and *fuzzy F_σ-set* in X, if $\lambda = \bigvee_{j=1}^{\infty} \mu_j$, where $1 - \mu_j \in T$ for $j \in J$ [16],

(iv) *fuzzy semi-G_δ-set* in X if $\lambda = \bigwedge_{j=1}^{\infty} \lambda_j$, where each $\lambda_j^{'}$ is a fuzzy semi-open set in X and *fuzzy semi-F_σ-set* in X, if $\lambda = \bigvee_{j=1}^{\infty} \mu_j$, where each $\mu_j^{'}$ is a fuzzy semi-closed set in X [17],

(v) *fuzzy dense set* in X, if there exists no fuzzy closed set μ in X such that $\lambda < \mu < 1$, i.e., $cl(\lambda) = 1$ [18],

(vi) *fuzzy nowhere dense set* in X, if there exists no non-zero fuzzy open set μ in X such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) = 0$ [18],

(vii) *fuzzy first category set* in X, if $\lambda = \bigvee_{j=1}^{\infty} \lambda_j$, where each λ_j is a fuzzy nowhere dense set in X and any other fuzzy set in X is said to *be of fuzzy second category* [18],

(viii) *fuzzy residual set* in X, if $1 - \lambda$ is a fuzzy first category set in X [19],

(ix) *fuzzy σ-nowhere dense set* in X, if λ is a fuzzy F_σ-set with $int(\lambda) = 0$ [20],

(x) *fuzzy somewhere dense set* in X, if there exists a non-zero fuzzy open set μ in X such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) \neq 0$ [21] and $1 - \lambda$ is called a *fuzzy complement of fuzzy somewhere dense set* (briefly, fuzzy cs dense set) in X [22],

(xi) *fuzzy δ-open set* in X, if $\lambda = \bigvee_{j \in J} \lambda_j$, where each λ_j is a fuzzy regular open set in X and *fuzzy δ-closed set* in X, if $1 - \lambda$ is a fuzzy δ-open set in X [23],

(xii) *fuzzy locally closed set* (resp. *fuzzy A-set*) in X, if $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy closed set(resp. fuzzy regular closed) in X [24],

(xiii) *fuzzy Baire set* in X, if $\lambda = \mu \wedge \delta$, where μ is a non-zero fuzzy open set and δ is a fuzzy residual set in X [25],

(xiv) *fuzzy pseudo-open set* in X, if $\lambda = \mu \vee \delta$, where μ is a non-zero fuzzy open set and δ is a fuzzy first category set in X [26],

(xv) *fuzzy simply open set* in X, if $Bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$ is a fuzzy nowhere dense set in X [27],

(xvi) *fuzzy Baire dense set* in X, if for a non-zero fuzzy open set μ in X, $\lambda \wedge \mu$ is a fuzzy second category set in X [28].

Definition 2.3. A fuzzy topological space (X, T) is called a:

(i) *fuzzy regular space*, if for each $\lambda \in T$, there is $(\lambda_j)_{j \in J} \subset T$ such that $cl(\lambda_j) \leq \lambda$ for each $j \in J$ and $\lambda = \bigvee_{j \in J} \lambda_j$ [6],

(ii) *fuzzy semi regular* , if the collection of all fuzzy regular open sets in X forms a base for T [6],

(iii) *fuzzy Baire space*, if $int(\bigvee_{j=1}^{\infty} \lambda_j) = 0$, where each λ_j is a fuzzy nowhere dense set in X [19],

(iv) *fuzzy quasi-regular space*, if for each $\lambda \in T$, there exists a fuzzy regular closed set μ in (X, T) such that $\mu \leq \lambda$ [29],

(v) *fuzzy P-space*, if for each fuzzy G_δ-set in X, $G_\delta \in T$ [30],

(vi) *fuzzy almost P-space*, if for each non-zero fuzzy G_δ-set λ in X, $int(\lambda) \neq 0$ [31],

(vii) *fuzzy semi-P-space*, if each fuzzy G_δ-set in X is a fuzzy semi-open set in X [32],

(viii) *fuzzy globally disconnected space*, if each fuzzy semi-open set in X is a fuzzy open set in X [33],

(ix) *fuzzy hyperconnected space* if every non-null fuzzy open set in X is fuzzy dense in X [34],

(x) *fuzzy ∂-space*, if each fuzzy G_δ-set in X is a fuzzy simply open set in X [35],

(xi) *fuzzy strongly regular space*, if for each fuzzy semi-open set λ in X, there exist $(\lambda_j)_{j\in J} \subset T$ such that $\lambda = \bigvee_{j\in J} \lambda_j$ and $cl(\lambda_j) \leq \lambda$ [36].

(xii) it fuzzy first category space, if $1_X = \bigvee_{n=1}^{\infty} \lambda_n$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a family of fuzzy nowhere dense set in X. A fuzzy topological space which is not of fuzzy first category is said to *be of fuzzy second category* [18].

Theorem 2.4 (Theorem 5.6, [6]). *In a fuzzy topological space,*

(1) *the closure of a fuzzy open set is a fuzzy regular closed set,*

(2) *the interior of a fuzzy closed set is a fuzzy regular open set.*

Theorem 2.5 (Proposition 3.5, [17]). *If λ is a fuzzy residual set in a fuzzy topological space (X, T), then λ is a fuzzy semi-G_δ-set in (X, T).*

Theorem 2.6 (Proposition 4.3, [19]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *X is a fuzzy Baire space,*

(2) *$int(\lambda) = 0$ for every fuzzy first category set λ in X,*

(3) *$cl(\mu) = 1$ for every fuzzy residual set μ in X.*

Theorem 2.7 (Proposition 3.1, [20]). *If λ is a fuzzy σ-nowhere dense set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy residual set in X.*

Theorem 2.8 (Proposition 3.5, [28]). *If λ is a fuzzy somewhere dense set in a fuzzy topological space (X, T), then there exists a fuzzy regular closed set η in X such that $\eta \leq cl(\lambda)$.*

Theorem 2.9 (Proposition 3.12, [25]). *If λ is a fuzzy Baire set in a fuzzy topological space (X, T), then $\lambda = \mu \wedge \delta$, where $\mu \in T$ and δ is a fuzzy semi-G_δ-set in X.*

Theorem 2.10 (Proposition 3.16, [27]). *If each fuzzy simply open set λ is a fuzzy open set in a fuzzy topological space (X, T), then λ is a fuzzy semi-open set in X.*

Theorem 2.11 (Proposition 4.1, [31]). *If a fuzzy topological space (X, T) is a fuzzy almost P-space, then X is a fuzzy second category space.*

Theorem 2.12 (Proposition 3.6, [33]). *If λ is a fuzzy residual set in a fuzzy globally disconnnected space (X, T), then λ is a fuzzy G_δ-set in X).*

Theorem 2.13 (Proposition 3.5, [37]). *If λ is a fuzzy residual set in a fuzzy topological space (X, T), then there exists a fuzzy G_δ-set μ in X such that $\mu \leq \lambda$.*

Theorem 2.14 (Proposition 3.1, [31]). *A fuzzy topological space (X, T) is a fuzzy almost P-space if and only if the only fuzzy F_σ-set λ in X such that $cl(\lambda) = 1$ is 1_X.*

Theorem 2.15 (Proposition 3.11, [38]). *If $clint(\mu) = 1$ for each fuzzy G_δ-set μ in a fuzzy topological space (X, T), then X is a fuzzy ∂-space.*

Theorem 2.16 (Proposition 5.1, [39]). *If a fuzzy topological space (X, T) is a fuzzy hyperconnected and fuzzy P-space, then X is a fuzzy Baire space.*

Theorem 2.17 (Proposition 5.6, [39]). *If λ is a fuzzy G_δ-set in a fuzzy Baire and fuzzy P-space (X, T), then λ is a fuzzy second category set in X.*

Theorem 2.18 (Proposition 4.6, [40]). *If λ is a fuzzy Baire set in a fuzzy globally disconnected, fuzzy Baire and fuzzy P-space (X, T), then λ is a fuzzy Baire dense set in X.*

3. Fuzzy strongly semiregular spaces

Motivated by the works of Jafari and Noiri [9] on strongly semiregular spaces in classical topology, the notion of fuzzy strongly semiregular space is introduced as follows.

Definition 3.1. A fuzzy topological space (X, T) is called a *fuzzy strongly semiregular space*, if for each fuzzy semi-open set λ in X, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$.

Example 3.2. Let $X = \{a, b, c\}$ and let $\alpha,\ \beta,\ \gamma,\ \delta$ and θ be the fuzzy sets in X defined as follows:

$$\alpha(a) = 0.5,\ \alpha(b) = 0.6,\ \alpha(c) = 0.4;\ \beta(a) = 0.6,\ \beta(b) = 0.4,\ \beta(c) = 0.5;$$

$$\gamma(a) = 0.5,\ \gamma(b) = 0.4,\ \gamma(c) = 0.6;\ \delta(a) = 0.7,\ \delta(b) = 0.6,\ \delta(c) = 0.8;$$

$$\theta(a) = 0.6,\ \theta(b) = 0.5,\ \theta(c) = 0.6.$$

Then $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \beta \wedge \gamma, \alpha \vee (\beta \wedge \gamma), \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that:

$cl(\alpha) = 1 - \gamma = \alpha,\ cl(\beta) = 1,\ cl(\gamma) = 1 - \alpha = \gamma,\ cl(\alpha \vee \beta) = 1,$
$cl(\alpha \vee \gamma) = 1 - (\alpha \wedge \beta) = \alpha \vee \gamma,\ cl(\beta \vee \gamma) = 1,\ cl(\alpha \wedge \beta) = 1 - (\alpha \vee \gamma) = \alpha \wedge \beta,$
$cl(\beta \wedge \gamma) = 1 - [\alpha \vee (\beta \wedge \gamma)] = \beta \wedge \gamma,\ cl(\alpha \vee [\beta \wedge \gamma]) = 1 - (\beta \wedge \gamma) = \alpha \vee (\beta \wedge \gamma),$
$cl(\alpha \vee \beta \vee \gamma) = 1;$
$int(1 - \alpha) = \gamma,\ int(1 - \beta) = 0,\ int(1 - \gamma) = \alpha; int[1 - (\alpha \vee \beta)] = 0,$
$int(1 - [\alpha \vee \gamma]) = \alpha \wedge \beta,\ int(1 - [\beta \vee \gamma]) = 0,\ int(1 - [\alpha \wedge \beta]) = \alpha \vee \gamma,$
$int(1 - [\beta \wedge \gamma]) = \alpha \vee (\beta \wedge \gamma),\ int[1 - (\alpha \vee [\beta \wedge \gamma])] = \beta \wedge \gamma,\ int(1 - [\alpha \vee \beta \vee \gamma]) = 0.$

Also by computation, one can find that $clint(\theta) = 1$ and $clint(\delta) = 1$ and thus δ and θ are the fuzzy semi-open sets in X. By computation, one can find that the fuzzy regular closed sets in X are $1 - \alpha, 1 - \gamma, 1 - (\alpha \vee \gamma), 1 - (\alpha \wedge \beta), 1 - (\beta \wedge \gamma)$ and $1 - [\alpha \vee \beta \vee \gamma]$ and the fuzzy semi-open sets are all the fuzzy open sets and δ and θ and for each fuzzy semi-open set λ in X, there exists a fuzzy regular closed set $\eta[= 1 - \alpha, 1 - \gamma, 1 - (\alpha \vee \gamma), 1 - (\alpha \wedge \beta), 1 - (\beta \wedge \gamma)$ and $1 - (\alpha \vee \beta \vee \gamma)]$ in X such that $\eta \leq \lambda$. So X is a fuzzy strongly semiregular space.

Proposition 3.3. *If a fuzzy topological space (X, T) is a fuzzy strongly semi-regular space, then for each fuzzy semi-closed set δ in X, there exists a fuzzy regular open set η in X such that $\delta \leq \eta$.*

Proof. Suppose δ is a fuzzy semi-closed set in X. Then $1-\delta$ is a fuzzy semi-open set in X. As X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq 1-\delta$. Thus $\delta \leq 1-\mu$. Let $\eta = 1-\mu$. Then η is a fuzzy regular open set in X and $\delta \leq \eta$. □

Proposition 3.4. *If δ is a fuzzy closed set in a fuzzy strongly semiregular space (X,T), then there exists a fuzzy regular open set η in X such that $\delta \leq \eta$.*

Proof. Suppose δ is a fuzzy closed set in X. Then clearly, δ is a fuzzy semi closed set in X. Since X is a fuzzy strongly semi regular space, by Proposition 3.3, there exists a fuzzy regular open set η in X such that $\delta \leq \eta$. □

Proposition 3.5. *If λ is a fuzzy open set in a fuzzy strongly semiregular space (X,T), then there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$.*

Proof. Suppose λ is a fuzzy open set in X. Then clearly, λ is a fuzzy semi-open set in X. Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$. □

Remark 3.6. In view of Propositions 3.4 and 3.5, one will have the following result.

If λ is a fuzzy open set in a fuzzy strongly semiregular space (X,T), then there exists a fuzzy regular closed set μ and a fuzzy regular open set η in X such that $\mu \leq \lambda \leq \eta$ [for, $\mu \leq \lambda \leq cl(\lambda) \leq \eta$].

Proposition 3.7. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X,T), there exist a family $(\delta_n)_{n\in\mathbb{N}}$ of fuzzy regular closed sets in X such that $\bigwedge_{n=1}^{\infty} \delta_n \leq \lambda$.*

Proof. Suppose λ is a fuzzy residual set in X. Then, by Theorem 2.5, λ is a fuzzy semi-G_δ- set in X and thus $\lambda = \bigwedge_{n=1}^{\infty} \mu_n$, where $(\mu_n)_{n\in\mathbb{N}}$ is a family of fuzzy semi-open sets in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.5, there exists a fuzzy regular closed set δ_n in X such that $\delta_n \leq \mu_n$. So $\bigwedge_{n=1}^{\infty} \delta_n \leq \bigwedge_{n=1}^{\infty} \mu_n$ and $\bigwedge_{n=1}^{\infty} \delta_n \leq \lambda$. □

Corollary 3.8. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X,T), then there exist a fuzzy closed set μ in X such that $\mu \leq \lambda$.*

Proof. Suppose λ is a fuzzy residual set in X. Then by Proposition 3.7, there exist a family $(\delta_n)_{n\in\mathbb{N}}$ of fuzzy regular closed sets in X such that $\bigwedge_{n=1}^{\infty} \delta_n \leq \lambda$. Since fuzzy regular closed sets are fuzzy closed sets in a fuzzy topological space, $(\delta_n)_{n\in\mathbb{N}}$ is a family of fuzzy closed sets in X. Let $\mu = \bigwedge_{n=1}^{\infty} \delta_n$. Then μ is a fuzzy closed set in X and $\mu \leq \lambda$. □

Proposition 3.9. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X,T), then $int(\lambda) \neq 0$.*

Proof. Suppose λ is a fuzzy residual set in X. Then by Corollary 3.8, there exist a fuzzy closed set μ in X such that $\mu \leq \lambda$. This implies that $int(\mu) \leq int(\lambda)$. Thus by Theorem 2.4, $int(\mu)$ is a fuzzy regular open set in X.So $int(\mu)$ is a fuzzy open set in X. Hence $int(\mu) \leq int(\lambda)$ implies that $int(\lambda) \neq 0$. □

Corollary 3.10. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X,T), then λ is a fuzzy somewhere dense set in X.*

Proof. Suppose λ is a fuzzy residual set in a fuzzy strongly semiregular space X. Then by Proposition 3.9, $int(\lambda) \neq 0$. Since $int(\lambda) \leq intcl(\lambda)$, $int(\lambda) \neq 0$. Thus $intcl(\lambda) \neq 0$. So λ is a fuzzy somewhere dense set in X. □

Proposition 3.11. *If μ is a fuzzy first category set in a fuzzy strongly semiregular space (X, T), there exist a family $(\gamma_n)_{n\in\mathbb{N}}$ of fuzzy regular open sets in X such that $\mu \leq \bigvee_{n=1}^{\infty} \gamma_n$.*

Proof. Suppose μ is a fuzzy first category set in X. Then $1-\mu$ is a fuzzy residual set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.7, there exist a family $(\delta_n)_{n\in\mathbb{N}}$ of fuzzy regular closed sets in X such that $\bigwedge_{n=1}^{\infty} \delta_n \leq 1-\mu$. Thus $\mu \leq 1 - \bigwedge_{n=1}^{\infty} \delta_n = \bigvee_{n=1}^{\infty}(1-\delta_n)$. Let $\gamma_n = 1-\delta_n$. Then $(\gamma_n)_{n\in\mathbb{N}}$ is a family of fuzzy regular open sets in X and $\mu \leq \bigvee_{n=1}^{\infty} \gamma_n$. □

Corollary 3.12. *If μ is a fuzzy first category set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy open set γ in X such that $\mu \leq \gamma$.*

Proof. Suppose μ is a fuzzy first category set in X. Then by Proposition 3.11, there exist a family $(\gamma_n)_{n\in\mathbb{N}}$ of fuzzy regular open sets in X such that $\mu \leq \bigvee_{n=1}^{\infty} \gamma_n$. Since fuzzy regular open sets are fuzzy open sets, $(\gamma_n)_{n\in\mathbb{N}}$ is a family of fuzzy open sets in X and $\mu \leq \gamma$. □

The following propositions show that fuzzy first category sets, fuzzy σ-nowhere dense sets are not fuzzy dense sets in fuzzy strongly semiregular spaces.

Proposition 3.13. *If μ is a fuzzy first category set in a fuzzy strongly semiregular space (X, T), then μ is not a fuzzy dense set in X.*

Proof. Suppose μ is a fuzzy first category set in X. Then by Corollary 3.12, there exist a fuzzy open set γ in X such that $\mu \leq \gamma$. Thus $cl(\mu) \leq cl(\gamma)$. By Theorem 2.4, $cl(\gamma)$ is a fuzzy regular closed set in X. So $cl(\gamma)$ is a fuzzy closed set in X. Let $\eta = cl(\gamma)$. Then there exist a fuzzy closed set η in X such that $\mu \leq cl(\mu) \leq \eta$. Thus μ is not a fuzzy dense set in X. □

Proposition 3.14. *If λ is a fuzzy σ-nowhere dense set in a fuzzy strongly semiregular space (X, T), then λ is not a fuzzy dense set in X.*

Proof. Suppose λ is a fuzzy σ-nowhere dense set in X. Then by Theorem 2.7, $1-\lambda$ is a fuzzy residual set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.9, $int(1-\lambda) \neq 0$. Thus by Lemma 2.1, $1 - cl(\lambda) \neq 0$. So $cl(\lambda) \neq 1$. Hence it follows that the fuzzy σ-nowhere dense set λ is not a fuzzy dense set in X. □

Remark 3.15. In view of the Proposition 3.14, one will have the following result.

Fuzzy F_σ-sets with zero interior in fuzzy strongly semiregular spaces are not fuzzy dense sets.

Proposition 3.16. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy regular closed set η in X such that $\eta \leq cl(\lambda)$.*

Proof. Suppose λ is a fuzzy residual set in X. Since X is a fuzzy strongly semiregular space, by Corollary 3.10, λ is a fuzzy somewhere dense set in X. Then by Theorem 2.8, there exists a fuzzy regular closed set η in X such that $\eta \leq cl(\lambda)$. □

In classical topology, each semi-open set is the union of regular closed sets in a strongly semi-regular space whereas each fuzzy semi-open set lies between fuzzy regular closed sets in fuzzy strongly semiregular spaces. For, consider the following proposition.

Proposition 3.17. *If λ is a fuzzy semi-open set in a fuzzy strongly semiregular space (X, T), then there exist fuzzy regular closed sets η_1 and η_2 in X such that $\eta_1 \leq \lambda \leq \eta_2$.*

Proof. Suppose λ is a fuzzy semi-open set in X. Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set η_1 in X such that $\eta_1 \leq \lambda$. Then, $\eta_1 \leq \lambda \leq cl(\lambda)$. Since the closure of a fuzzy semi-open set is a fuzzy regular closed set, $cl(\lambda)$ is a fuzzy regular closed set in X. Let $\eta_2 = cl(\lambda)$. Then $\eta_1 \leq \lambda \leq \eta_2$. Thus fuzzy semi-open sets lies between fuzzy regular closed sets in fuzzy strongly semiregular spaces. □

Proposition 3.18. *If η is a fuzzy regular closed set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy regular closed set γ in X such that $\gamma \leq \eta$.*

Proof. Suppose η is a fuzzy regular closed set in X. Then $clint(\eta) = \eta$. Since $int[int(\eta)] = int(\eta), int(\eta) \leq \eta = clint(\eta) = cl(int[int(\eta)])$. Thus $int(\eta) \leq clint[int(\eta)]$. So $int(\eta)$ is a fuzzy semi-open set in X. Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set γ in X such that $\gamma \leq int(\eta)$. Hence $\gamma \leq int(\eta) \leq \eta$. □

Corollary 3.19. *If η is a fuzzy regular closed set in a fuzzy strongly semiregular space (X, T), then $\eta \geq \bigvee_{n=1}^{\infty} \gamma_n$, where $(\gamma_n)_{n \in \mathbb{N}}$ is a decreasing sequence of fuzzy regular closed sets in X.*

Proof. Suppose η is a fuzzy regular closed set in X. Then by Proposition 3.18, there exists a family $(\gamma_n)_{n \in \mathbb{N}}$ of fuzzy regular closed sets in X such that $\cdots \leq \gamma_3 \leq \gamma_2 \leq \gamma_1 \leq \eta$. Thus $\eta \geq \bigvee_{n=1}^{\infty} \gamma_n$, where $(\gamma_n)_{n \in \mathbb{N}}$ is a decreasing sequence of fuzzy regular closed sets in X. □

Proposition 3.20. *If θ is a fuzzy regular open set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy regular open set ρ in X such that $\theta \leq \rho$.*

Proof. Suppose θ is a fuzzy regular open set in X. Then $1 - \theta$ is a fuzzy regular closed set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.18, there exists a fuzzy regular closed set γ in X such that $\gamma \leq 1 - \theta$. Thus $\theta \leq 1 - \gamma$. Let $\rho = 1 - \gamma$. Then ρ is a fuzzy regular open set in X and $\theta \leq \rho$. □

Corollary 3.21. *If θ is a fuzzy regular open set in a fuzzy strongly semiregular space (X, T), then $\theta \leq \bigwedge_{n=1}^{\infty} \delta_n)$, where $(\delta_n)_{n \in \mathbb{N}}$ is an increasing sequence of fuzzy regular open sets in X.*

Proof. Suppose θ is a fuzzy regular open set in X. Then by Proposition 3.20, there exist a family $(\delta_n)_{n \in \mathbb{N}}$ of fuzzy regular open sets in X such that $\theta \leq \delta_1 \leq \delta_2 \leq \delta_3 \cdots$. Thus $\theta \leq \bigwedge_{n=1}^{\infty} \delta_n$, where $(\delta_n)_{n \in \mathbb{N}}$ is an increasing sequence of fuzzy regular open sets in X. □

Proposition 3.22. *If λ is a fuzzy G_δ-set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy closed set μ in X such that $\mu \leq \lambda$.*

Proof. Suppose λ is a fuzzy G_δ-set in X. Then $\lambda = \bigwedge_{n=1}^{\infty} \lambda_n$, where $(\lambda_n)_{n\in\mathbb{N}} \subset T$. Since X is a fuzzy strongly semiregular space, by Proposition 3.5, for each $n \in \mathbb{N}$, there exist fuzzy regular closed sets μ_n in X such that $\mu_n \leq \lambda_n$. Thus $\bigwedge_{n=1}^{\infty} \mu_n \leq \bigwedge_{n=1}^{\infty} \lambda_n$. Since each fuzzy regular closed set is a fuzzy closed set, $(\mu_n)_{n\in\mathbb{N}}$ is a family of fuzzy closed sets in X. So $\bigwedge_{n=1}^{\infty} \mu_n$ is a fuzzy closed set in X. Let $\mu = \bigwedge_{n=1}^{\infty} \mu_n$. Then for the fuzzy $G_\delta-$set λ, there exists a fuzzy closed set μ in X such that $\mu \leq \lambda$. □

Proposition 3.23. *If δ is a fuzzy F_σ-set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy open set γ in X such that $\delta \leq \gamma$.*

Proof. Suppose δ is a fuzzy F_σ-set in X. Then $1 - \delta$ is a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.22, there exists a fuzzy closed set μ in X such that $\mu \leq 1 - \delta$. Thus $\delta \leq 1 - \mu$. Let $\gamma \leq 1 - \mu$. Then clearly, $\gamma \in T$. Thus there exists a fuzzy open set γ in X such that $\delta \leq \gamma$. □

Proposition 3.24. *If a fuzzy simply open set λ is a fuzzy open set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$.*

Proof. Suppose λ is a fuzzy simply open set in X. Then by the hypothesis, $\lambda \in T$. Thus by Theorem 2.10, λ is a fuzzy semi-open set in X). Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$. □

Proposition 3.25. *If λ is a fuzzy Baire set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy open set μ and a fuzzy closed set γ in X such that $\mu \wedge \gamma \leq \lambda$.*

Proof. Suppose λ is a fuzzy Baire set in X. Then by Theorem 2.9, $\lambda = \mu \wedge \delta$, where $\mu \in T$ and δ is a fuzzy semi-G_δ-set in X. Since δ is a fuzzy semi-G_δ-set, $\delta = \bigwedge_{n=1}^{\infty} \delta_n$, where $(\delta_i)_{n\in\mathbb{N}}$ is a family of fuzzy semi-open sets in X. Thus $\lambda = \mu \wedge (\bigwedge_{n=1}^{\infty} \delta_n)$. Since X is a fuzzy strongly semiregular space, for each $n \in \mathbb{N}$, there exist a fuzzy regular closed set γ_n in X such that $\gamma_n \leq \delta_n$. So $\bigwedge_{n=1}^{\infty} \gamma_n \leq \bigwedge_{n=1}^{\infty} \delta_n$. Hence $\bigwedge_{n=1}^{\infty} \gamma_n \leq \delta$. Let $\gamma = \bigwedge_{n=1}^{\infty} \gamma_n$. Then $\gamma \leq \delta$ and $\mu \wedge \gamma \leq \mu \wedge \delta = \lambda$. Since each fuzzy regular closed set is a fuzzy closed set, $(\gamma_n)_{n\in\mathbb{N}}$ is a family of fuzzy closed sets in X. Thus $\bigwedge_{n=1}^{\infty} \gamma_n$ is a fuzzy closed set in X. So γ is a fuzzy closed set in X. Hence by the hypothesis, there exist a fuzzy open set μ and a fuzzy closed set γ in X such that $\mu \wedge \gamma \leq \lambda$. □

Remark 3.26. It should be noted that if λ is a fuzzy Baire set in a fuzzy strongly semiregular space (X, T), then λ need not be a fuzzy semi-open set in X. For, in the above proof, $\gamma = \bigwedge_{n=1}^{\infty} \delta_n$, where $(\delta_n)_{n\in\mathbb{N}} \subset T$ and then $\lambda = \mu \wedge \delta$, where $\mu \in T$, need not be a fuzzy semi-open set in fuzzy topological spaces, whereas in ordinary topological setting the intersection of a semi-open set with an open set is a semi-open set (See Lemma 1, [41]).

Corollary 3.27. *If λ is a fuzzy Baire set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy locally closed set η in X such that $\eta \leq \lambda$.*

Proof. The proof follows from the definition of fuzzy locally closed set and Proposition 3.25. □

Proposition 3.28. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy G_δ-set μ and a fuzzy closed set δ in X such that $\delta \leq \mu \leq \lambda$.*

Proof. Suppose λ is a fuzzy residual set in X. Then by Theorem 2.13, there exists a fuzzy G_δ-set μ in X such that $\mu \leq \lambda$. Thus by Proposition 3.22, there exists a fuzzy closed set δ in X such that $\delta \leq \mu$. So $\delta \leq \mu \leq \lambda$. □

Proposition 3.29. *If γ is a fuzzy first category set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy F_σ-set η and a fuzzy open set λ in X such that $\gamma \leq \eta \leq \lambda$.*

Proof. Suppose γ is a fuzzy first category set in a fuzzy strongly semiregular space X. Then clearly, $1 - \gamma$ is a fuzzy residual set in X. Thus by Proposition 3.28, there exist a fuzzy G_δ-set μ and a fuzzy closed set δ in X such that $\delta \leq \mu \leq 1 - \gamma$. So $1 - \delta \geq 1 - \mu \geq \gamma$. Let $\eta = 1 - \mu$ and $\lambda = 1 - \delta$. Then η is a fuzzy F_σ-set and λ is a fuzzy open set in X. Thus by the hypothesis, there exist a fuzzy F_σ-set η and a fuzzy open set λ in X such that $\gamma \leq \eta \leq \lambda$. □

Proposition 3.30. *If μ is a fuzzy first category set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy δ-open γ in X such that $\mu \leq \gamma$.*

Proof. Suppose μ is a fuzzy first category set in X). Since (X, T) is a fuzzy strongly semiregular space, by Proposition 3.11, there exist a family $(\gamma_n)_{n \in \mathbb{N}}$ of fuzzy regular open sets in X such that $\mu \leq \bigvee_{n=1}^{\infty} \gamma_n$. Let $\gamma = \bigvee_{n=1}^{\infty} \gamma_n$. Then γ is a fuzzy δ-open in X. Thus $\mu \leq \gamma$. □

Proposition 3.31. *If λ is a fuzzy residual set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy δ-closed set η in X such that $\eta \leq \lambda$.*

Proof. Suppose λ is a fuzzy residual set in X. Then $1 - \lambda$ is a fuzzy first category set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.30, there exists a fuzzy δ -open γ in X such that $1 - \lambda \leq \gamma$. Thus $1 - \gamma \leq \lambda$. Let $\eta = 1 - \gamma$. Then η is a fuzzy δ-closed set in X. Thus $\eta \leq \lambda$. □

Remark 3.32. It is to be mentioned that any fuzzy regular open set is fuzzy δ-open and any fuzzy δ-open set is a fuzzy open set in a fuzzy topological space. Furthermore, if a fuzzy set λ is fuzzy semi-open in a fuzzy topological space X, then $cl(\lambda) = cl\delta(\lambda)$ (See [38]).

Proposition 3.33. *If a fuzzy topological space (X, T) is a fuzzy strongly semi-regular space, then for each fuzzy semi-closed set δ in X, there exists a fuzzy δ-open set η in X such that $\delta \leq \eta$.*

Proof. Let δ be a fuzzy semi-closed set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.3, there exists a fuzzy regular open set η in X such that $\delta \leq \eta$. Since each fuzzy regular open set is a fuzzy δ-open set, η is a fuzzy δ-open in X and $\delta \leq \eta$. □

Proposition 3.34. *If α is a fuzzy pseudo open-set in a fuzzy strongly semiregular space (X, T), then there exist a fuzzy open set β in X such that $\alpha \leq \beta$.*

Proof. Suppose α is a fuzzy pseudo open-set in X. Then $\alpha = \mu \vee \gamma$, where μ is a non-zero fuzzy open set and γ is a fuzzy first category set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.29, there exist a fuzzy F_σ-set η and a fuzzy open set λ in X such that $\gamma \leq \eta \leq \lambda$. Thus $\mu \vee \gamma \leq \mu \vee \eta \leq \mu \vee \lambda$ and $\alpha \leq \mu \vee \lambda$. Let $\beta = \mu \vee \lambda$. Then clearly, $\alpha \leq \beta$. Furthermore, $\beta \in T$.. □

Proposition 3.35. *If λ is a fuzzy semi-open set in a fuzzy strongly semi-regular space (X, T), then there exists a fuzzy regular open set η in X such that $\eta \leq \lambda$.*

Proof. Suppose λ is a fuzzy semi-open set in X. Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$. Then $int(\mu) \leq \mu \leq \lambda$. Since μ is a fuzzy regular closed set in X, $\mu = clint(\mu)$. On the other hand, $intcl(int(\mu)) = int(clint(\mu)) = int(\mu)$. Thus $int(\mu)$ is a fuzzy regular open set in X. Let $\eta = int(\mu)$. Then clearly, $\mu \leq \lambda$. Moreover, η is a fuzzy regular open set in X. □

Corollary 3.36. *If λ is a fuzzy open set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy regular open set η in X such that $\eta \leq \lambda$.*

Proof. Since each fuzzy open set is a fuzzy semi-open set in a fuzzy topological space, the proof follows from Proposition 3.35. □

Proposition 3.37. *If λ is a fuzzy set in a fuzzy strongly semiregular space (X, T), then there exist fuzzy regular open sets η_1 and η_2 in X such that $\eta_1 \leq \lambda \leq \eta_2$.*

Proof. Suppose λ is a fuzzy set in a fuzzy strongly semiregular space (X, T). Then clearly, int $(\lambda) \in T$. Thus by Corollary 3.36, there exists a fuzzy regular open set η_1 in X such that $\eta_1 \leq int(\lambda)$. Since cl(λ) is a fuzzy closed set in X, by Proposition 3.4, there exists a fuzzy regular open set η_2 in X such that $cl(\lambda) \leq \eta_2$. Thus $\eta_1 \leq int(\lambda) \leq \lambda \leq cl(\lambda) \leq \eta_2$. So $\eta_1 \leq \lambda \leq \eta_2$. □

Proposition 3.38. *If λ is a fuzzy σ-nowhere dense set in a fuzzy strongly semiregular space (X, T), then there exists a fuzzy open γ in X such that $\lambda \leq \gamma$.*

Proof. Suppose λ is a fuzzy σ-nowhere dense set in X. Then by Theorem 2.7, 1-λ is a fuzzy residual set in X. Since X is a fuzzy strongly semiregular space, by corollary 3.8, there exist a fuzzy closed set μ in X such that $\mu \leq 1 - \lambda$. Thus $\lambda \leq 1 - \mu$. Let $\gamma = 1 - \mu$. Then clearly, $\lambda \leq \gamma$ and $\sigma \in T$. □

4. Fuzzy strongly semiregular spaces and other fuzzy topological spaces

The inter-relations between fuzzy strongly regular spaces, fuzzy strongly semiregular spaces, fuzzy quasi-regular spaces and fuzzy almost P-spaces are established in this section.
The conditions under which fuzzy Baire sets in fuzzy strongly semi regular spaces become fuzzy Baire dense sets are obtained.

Proposition 4.1. *If a fuzzy topological space (X, T) is a fuzzy strongly semiregular space, then (X, T) is a fuzzy quasi-regular space.*

Proof. Let $\lambda \in T$. Since each fuzzy open set is a fuzzy semi-open set, λ is a fuzzy semi-open set in X. Since X is a fuzzy strongly semiregular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$. Thus X is a fuzzy quasi-regular space. □

Proposition 4.2. *If a fuzzy topological space (X, T) is a fuzzy globally disconnected and fuzzy quasi regular space, then X is a fuzzy strongly semi regular space.*

Proof. Suppose X is a fuzzy globally disconnected and fuzzy quasi regular space, and let λ be a fuzzy semi-open set in X). Since X is a fuzzy globally disconnected space, $\lambda \in T$. Since X is a fuzzy quasi regular space, there exists a fuzzy regular closed set μ in X such that $\mu \leq \lambda$. Then X is a fuzzy strongly semiregular space. □

Proposition 4.3. *If a fuzzy first category space (X, T) is a fuzzy strongly semiregular space, then $\bigvee_{n=1}^{\infty} \eta_n = 1$, where $(\eta_n)_{n\in\mathbb{N}}$ is a family of fuzzy regular open sets in X.*

Proof. Suppose a fuzzy first category space X is a fuzzy strongly semiregular space. It is clear that 1 is a fuzzy first category st in X. Then $\bigvee_{n=1}^{\infty}(\lambda_n) = 1$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a family of fuzzy nowhere dense sets in X. Thus $intcl(\lambda_n) = 0$, i.e., $intcl(\lambda_n) \leq \lambda_n$ for each $n \in \mathbb{N}$. So $(\lambda_n)_{n\in\mathbb{N}}$ is a family of fuzzy semi-closed sets in X. By the hypothesis and Proposition 3.3, for each $n \in \mathbb{N}$, there exist fuzzy regular open set η_n in X such that $\lambda_n \leq \eta_n$. This implies that $\bigvee_{n=1}^{\infty} \lambda_n \leq \bigvee_{n=1}^{\infty} \eta_n$ and $1 \leq \bigvee_{n=1}^{\infty} \eta_n$. Hence $\bigvee_{n=1}^{\infty} \eta_n) = 1$, where $(\eta_n)_{n\in\mathbb{N}}$ is a family of fuzzy regular open sets in X. □

The following proposition gives a condition for fuzzy strongly semiregular spaces to become fuzzy Baire spaces by means of fuzzy G_δ-sets.

Proposition 4.4. *If each fuzzy G_δ-set is a fuzzy dense set in a fuzzy strongly semiregular space (X, T), then X is a fuzzy Baire space.*

Proof. Let λ be a fuzzy residual set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.28, there exist a fuzzy G_δ-set μ and a fuzzy closed set δ in X such that $\delta \leq \mu \leq \lambda$. Then $cl(\delta) \leq cl(\mu) \leq cl(\lambda)$. By the hypothesis, μ is a fuzzy dense set in X and thus $cl(\mu) = 1$. So $1 \leq cl(\lambda)$, i.e., $cl(\lambda) = 1$. Hence by Theorem 2.6, X is a fuzzy Baire space. □

The following proposition gives a condition for fuzzy strongly semiregular spaces to become fuzzy Baire spaces by means of fuzzy F_σ-sets.

Proposition 4.5. *If $int(\mu) = 0$ for each fuzzy F_σ-set μ in a fuzzy strongly semiregular space (X, T), then X is a fuzzy Baire space.*

Proof. Let γ be a fuzzy first category set in X. Since (X, T) is a fuzzy strongly semiregular space, by Proposition 3.29, there exist a fuzzy $F_\sigma-$set η and a fuzzy open set λ in X such that $\gamma \leq \eta \leq \lambda$. Then $int(\gamma) \leq int(\eta) \leq int(\lambda) = \lambda$. Thus By the hypothesis, $int(\eta) = 0$. So $int(\gamma) = 0$. Hence by Theorem 2.6, (X, T) is a fuzzy Baire space. □

Proposition 4.6. *If (X, T) is a fuzzy strongly semiregular space, then X is a fuzzy almost P-space*

Proof. Let λ be a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular space, by Proposition 3.22, there exists a fuzzy closed set μ in X such that $\mu \leq \lambda$. Then $int(\mu) \leq int(\lambda)$. By Lemma 2.1, $int(\mu)$ is a fuzzy regular open set and thus a fuzzy open set in X. So $int(\lambda) \neq 0$. Hence X is a fuzzy almost P-space. □

Proposition 4.7. *If (X, T) is a fuzzy strongly semiregular space , then the only fuzzy F_σ-set λ such that $cl(\lambda) = 1$ is 1_X.*

Proof. Suppose X is a fuzzy strongly semiregular space. Then by Proposition 4.6, X is a fuzzy almost P-space. Thus by Theorem 2.14, the only fuzzy F_σ-set λ such that $cl(\lambda) = 1$ is 1_X. □

Proposition 4.8. *If λ is a fuzzy G_δ-set is a fuzzy strongly semiregular and fuzzy hyperconnected space (X, T), then λ is a fuzzy semi-open set in X.*

Proof. Suppose λ is a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular space, by Proposition 4.6, X is a fuzzy almost P-space. Then $int(\lambda) \neq 0$. Since X is a fuzzy hyperconnected space and $int(\lambda) \in T$, $int(\lambda)$ is a fuzzy dense set in X. Thus $clint(\lambda) = 1$. So $\lambda \leq clint(\lambda)$. Hence λ is a fuzzy semi-open set in X. □

Proposition 4.9. *If a fuzzy topological space (X, T) is a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular space, then X is a fuzzy P-space.*

Proof. Let λ be a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular and fuzzy hyperconnected space, by Proposition 4.8, λ is a fuzzy semi-open set in X. Since X is a fuzzy globally disconnected space, $\lambda \in T$. Then X is a fuzzy P-space. □

Proposition 4.10. *If a fuzzy topological space (X, T) is a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular space, then X is a fuzzy Baire space.*

Proof. The proof follows from Proposition 4.8 and Theorem 2.16. □

Proposition 4.11. *If a fuzzy topological space (X, T) is a fuzzy hyperconnected fuzzy strongly semi regular and fuzzy P-space, then X is a fuzzy Baire space.*

Proof. Let λ be a fuzzy G_δ-set in X. Since X is a fuzzy P-space, $\lambda \in T$. Since X is a fuzzy hyperconnected space, λ is a fuzzy dense set in X. Then λ is a fuzzy dense set in X. Thus, by Proposition 4.4, X is a fuzzy Baire space. □

The following propositions show that fuzzy strongly semiregular and fuzzy hyperconnected spaces are fuzzy semi-P-spaces and fuzzy ∂-spaces.

Proposition 4.12. *If a fuzzy topological space (X, T) is a fuzzy strongly semiregular and fuzzy hyperconnected space, then X is a fuzzy semi-P-space .*

Proof. Let λ be a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular and fuzzy hyperconnected space, by Proposition 4.8, λ is a fuzzy semi-open set in X. Then (X, T) is a fuzzy semi-P-space. □

Proposition 4.13. *If a fuzzy topological space (X, T) is a fuzzy strongly semiregular and fuzzy hyperconnected space, then X is a fuzzy ∂-space.*

Proof. Let λ be a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular space, by Proposition 4.6, X is a fuzzy almost P-space. Then $int(\lambda) \neq 0$. Since X is a fuzzy hyperconnected space and $int(\lambda) \in T$, $int(\lambda)$ is a fuzzy dense set in X. Thus $clint(\lambda) = 1$. So by Theorem 2.15, X is a fuzzy ∂-space. □

The following proposition shows that fuzzy G_δ-sets are having non-zero interiors in fuzzy strongly semiregular spaces.

Proposition 4.14. *If λ is a non-zero fuzzy G_δ-set in a fuzzy strongly semiregular space (X, T), then $int(\lambda) \neq 0$.*

Proof. Assume that λ is a non-zero fuzzy G_δ-set with $int(\lambda) = 0$. Then $1 - \lambda$ is a fuzzy F_σ-set in X such that $cl(1 - \lambda) = 1$. Since X is a fuzzy strongly semi regular space, by Proposition 4.6, $1 - \lambda = 1$. Thus $\lambda = 0$. This is a contradiction. So $int(\lambda) \neq 0$. □

Corollary 4.15. *If λ is a fuzzy G_δ-set in a fuzzy strongly semiregular space (X, T), then λ is a fuzzy somewhere dense set in X.*

Corollary 4.16. *If μ is a fuzzy $F_\sigma-$set in a fuzzy strongly semiregular space (X, T), then μ is a fuzzy cs dense set in X.*

Remark 4.17. In view of corollary 4.16, one will have the following result.
Fuzzy G_δ-sets in fuzzy strongly semiregular spaces are fuzzy somewhere dense sets.

Proposition 4.18. *If a fuzzy topological space (X, T) is a fuzzy strongly regular space, then X is a fuzzy strongly semiregular space.*

Proof. Let λ be a fuzzy semi-open set in X. Since X is a fuzzy strongly regular space, there exist a family $(\delta_j)_{j\in J}$ of fuzzy open sets in X such that $\lambda = \bigvee_{j\in J} \delta_j$ and $cl(\delta_j) \leq \lambda$ for each $j \in J$. Then by Lemma 2.1, $(cl(\delta_j))_{j\in J}$ is a family of fuzzy regular closed sets in X. Thus X is a fuzzy strongly semiregular space. □

The following proposition shows that fuzzy G_δ-sets in fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular spaces are not fuzzy first category sets.

Proposition 4.19. *If λ is a fuzzy G_δ-set in a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semiregular space (X, T), then λ is a fuzzy second category set in X.*

Proof. Suppose λ is a fuzzy G_δ-set in X. Since X is a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semiregular space, by Propositions 4.9 and 4.10, X is a fuzzy Baire and fuzzy P-space. Then by Theorem 2.17, λ is a fuzzy second category set in X. □

The following proposition gives a condition for fuzzy G_δ-sets in fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular spaces to become fuzzy Baire dense sets.

Proposition 4.20. *Let (X, T) be a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semiregular space. If λ is a fuzzy G_δ-set in X such that $\lambda \leq \mu$ for each $\mu \in T$, then λ is a fuzzy Baire dense set in X.*

Proof. Suppose λ is a fuzzy G_δ-set in X such that $\lambda \leq \mu$ for each $\mu \in T$. Since X is a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semiregular space, by Propositions 4.9 and 4.10, X is a fuzzy Baire and fuzzy P-space. Then Proposition 4.19, λ is a fuzzy second category set in X. Then by the hypothesis, $\lambda \wedge \mu = \lambda$. Note that λ is a non-zero fuzzy open set in X. Thus $\lambda \wedge \mu$ is a fuzzy second category set in X. So λ is a fuzzy Baire dense set in X. □

The following proposition give conditions under which fuzzy Baire sets become fuzzy Baire dense sets in fuzzy strongly semi regular spaces.

Proposition 4.21. *If λ is a fuzzy Baire set in a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular space (X, T), then λ is a fuzzy Baire dense set in X.*

Proof. Suppose λ is a fuzzy Baire set in X. Since X is a fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular space, by Propositions 4.9 and 4.10, X is a fuzzy Baire and fuzzy P-space. Then X is a fuzzy globally disconnected, fuzzy Baire and fuzzy P-space. Thus by Theorem 2.18, λ is a fuzzy Baire dense set in X. □

Proposition 4.22. *If (X, T) is a fuzzy strongly semiregular space, then X is a fuzzy second category space.*

Proof. The proof follows from Proposition 4.6 and Theorem 2.11. □

Proposition 4.23. *If μ is a fuzzy F_σ-set in a fuzzy strongly semiregular and fuzzy hyperconnected space (X, T), then λ is a fuzzy nowhere dense set in X.*

Proof. Suppose μ is a fuzzy F_σ-set in X. Then $1 - \mu$ is a fuzzy G_δ-set in X. Since X is a fuzzy strongly semiregular space, by Proposition 4.6, X is a fuzzy almost P-space. Thus $int(1 - \mu) \neq 0$. Since X is a fuzzy hyperconnected space and $int(1 - \mu) \in T$, $int(1 - \mu)$ is a fuzzy dense set in X. So $clint(1 - \mu) = 1$. This implies that $1 - intcl(\mu) = 1$, i.e., $intcl(\mu) = 0$. Hence λ is a fuzzy nowhere dense set in X. □

Remark 4.24. The inter-relations between fuzzy strongly regular space, fuzzy strongly semi regular spaces, fuzzy quasi-regular spaces and fuzzy almost P-spaces can be stated as follows:

Fuzzy Strongly Regular Spaces
↓
Fuzzy Strongly Semiregular Spaces
↙ ↘
Fuzzy Quasi-Regular Spaces   Fuzzy Almost P-Spaces

5. Conclusion

In this paper, the concept of fuzzy strongly semiregular space is introduced by means of fuzzy semi-open sets and fuzzy regular closed sets. Several characterizations of fuzzy strongly semi regular spaces are established. It is obtained that fuzzy residual sets, fuzzy G_δ-sets are fuzzy somewhere dense sets and fuzzy first category sets are not fuzzy dense sets and fuzzy F_σ-sets with zero interior are not fuzzy dense sets in fuzzy strongly semiregular spaces. Further it is established that fuzzy locally closed sets are subsets of fuzzy Baire sets in fuzzy strongly semiregular spaces. It is obtained that fuzzy G_δ-sets are fuzzy semi-open sets in fuzzy strongly semiregular and fuzzy hyperconnected spaces. It is shown that fuzzy globally disconnected and fuzzy quasi-regular spaces are fuzzy strongly semiregular spaces. It is obtained that fuzzy G_δ-sets in fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular spaces are fuzzy second category sets. A condition under which fuzzy G_δ-sets become fuzzy Baire dense sets in fuzzy strongly semiregular spaces is also obtained. It is found that fuzzy Baire sets in fuzzy globally disconnected, fuzzy hyperconnected and fuzzy strongly semi regular spaces are fuzzy Baire dense sets. The inter-relations between fuzzy strongly semiregular spaces, fuzzy quasi-regular spaces and fuzzy almost P-spaces are also established in this paper.

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