Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (x), pp. 1–x
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

New aggregation operators of intuitionistic trapezoidal fuzzy multi-numbers based on Archimedean norms

İrfan Deli, Hümeyra Karadöl

Received 29 September 2023; Revised 10 December 2023; Accepted 4 January 2024

Abstract. In trapezoidal intuitionistic fuzzy multi numbers, an aggregation operator performs the task of aggregate the described information to generate a ranking of alternatives which are two critical tasks: a flexible and superior tool for the first task and an effective tool for the second task is aggregation operator. Therefore, in the study, we give new aggregation operators, based on Archimedean T-norm and T-conorm operations which is called Archimedean norms operator for aggregating trapezoidal intuitionistic fuzzy multi-information. Then, we develop a multi criteria decision making method based on the given operators. Finally, the feasibility and effectiveness of the method is demonstrated via a numerical example.

2010 AMS Classification: 03E72, 08A72

Keywords: Fuzzy sets, Fuzzy number, Intuitionistic trapezoidal fuzzy multi numbers, Archimedean T-norm and T-conorm, Multiple criteria decision making.

Corresponding Author: İrfan Deli (irfandeli20@gmail.com)

1. Introduction

Throughout our daily life including development of modern science and artificial intelligence technology, more uncertain data processing problems urgently need to be solved and uncertain values need to be modelled. Therefore, we always encounter the uncertain values such as; good, very good, very very good, bad, very bad, hot, very hot, very very hot, cold, very cold, very very cold, and so on. These linguistic terms differ from individual to individual, from time to time and from environment to environment, and it is very difficult to model and solve an event or problem under uncertainty. For this, interval mathematics, probability theory, fuzzy set theory [1], intuitionistic fuzzy set theory [2] with strong applications, different theories are presented. The most comprehensive of set theory is fuzzy set theory proposed by

Zadeh [1] in 1965. Fuzzy sets and especially fuzzy numbers which are a fuzzy set on R real numbers have been studied increasingly by many authors to express more abundant and flexible information than classical sets and fuzzy sets. For example: Ban et al. [3] found nearest trapezoidal approximation and the nearest symmetric trapezoidal approximation to a given fuzzy number, with respect to the average Euclidean distance, preserving the value and ambiguity. Cheng [4] developed a centroid-based distance method was suggested for ranking fuzzy numbers and Wang et al. [5] showed incorrect and have led to some misapplications of the method and updated the concept of centroid for fuzzy numbers. Wei and Chen [6] proposed a new similarity measure between generalized fuzzy numbers by combining the concepts of geometric distance, the perimeter and the height of generalized fuzzy numbers for calculating the degree of similarity between generalized fuzzy numbers. The proposed theory which contain uncertain information have gained much attention from past and latter researchers for applications in various fields in [7, 8].

Since intuitionistic fuzzy sets proposed by Atanassov [2] are successful to handle the uncertain situations of data in the decision making problems they have great practical potential in a variety of areas. For example; Wang and Xin [9] gave the axiom definition of distance measure between intuitionistic fuzzy sets. Luo and Zhao [10] defined a new distance measure between intuitionistic fuzzy sets, which is based on a matrix norm and a strictly increasing (or decreasing) binary function and applied to pattern recognitions. Szmidt and Kacprzyk [11] showed how to calculate distances for intuitionistic fuzzy sets not only from a mathematical point of view but also of an intuitive appeal making use of all the relevant information. Park et al. [12] proposed some measures applied to pattern recognitions. Liang and Shi [13] proposed new similarity measures and proved the relationships between some similarity measures. Garg [14] developed improved cosine similarity measure for an intu- itionistic fuzzy sets by considering the inter- action between the pairs of the membership degrees. based on the Hausdor metric, are suggested. Recently, Lei et al. [15] defined two subtraction and division operations of intuitionistic fuzzy numbers (IFNs), and develop a sequence of general integrals dealing with continuous intuitionistic fuzzy data based on Archimedean t-conorm and t-norm. Xia et al. [16] proposed some operations on intuitionistic fuzzy sets under Archimedean t-conorm and t-norms, Also, intuitionistic fuzzy sets, including Archimedean t-conorm and t-norm, gained attention of many researchers in [17, 18, 19, 10, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Recently, Uluçay et al. [31] defined trapezoidal fuzzy multi-number which are more than one with the possibility of the same or the different membership functions allowing the repeated occurrences of any element. Deli and Keleş [32] and Deli and Keleş [33] developed new methods to solve multi-criteria decision-making problems using trapezoidal fuzzy multi-numbers under distance measures and similarity based on the concept of value and ambiguity of trapezoidal fuzzy multi-numbers. Then, Sahin et al. [34] developed a novel approach based on multi-criteria decision making(MCDM) trapezoidal fuzzy multi-number by defining Dice vector similarity and weighted Dice vector similarity measure. Also, Sahin et al. [35] presented some similarity measures for trapezoidal fuzzy multi numbers such as; Jaccard similarity measure, weighted Jacard similarity measure, Cosine similarity measure, weighted cosine similarity measure, Hybrid vector similarity measure and weighted Hybrid vector similarity measure. Also, Kesen and Deli [36, 37] initiated Archimedean T-norm and T-conorm operations of trapezoidal fuzzy multi-numbers to aggregating trapezoidal fuzzy multi-numbers. Moreover, Uluçay et al. [38] definedintuitionistic trapezoidal fuzzy multi numbers which are more than one with the possibility of the same or the different membership functions allowing the repeated occurrences of any element under intuitionistic fuzzy numbers.

An aggregation operator performs the task of generate a ranking of alternatives in uncertainly information and considering the increasing complexity of decision-making situations, it is imperative to extend aggregation operators for fusing uncertain information with the different forms of attribute values. As we know, no studies about archimedean norms on intuitionistic trapezoidal fuzzy multi numbers have been conducted until now. This study focuses on the development of intuitionistic fuzzy multi numbers and aims to design a managerial decision-making solving method by generalized Kesen and Deli [36, 37]. Some operational principles of intuitionistic fuzzy multi numbers on account of the Archimedean t-norm and t-conorm are initiated, on which two intuitionistic fuzzy multi numbers operators are established by taking various weight forms. Moreover, we explore the aggregation operators' idempotency, boundedness, and monotonicity, as well as analyze some particular forms of these operators. Furthermore, these aggregation operators are employed to design a method of deriving an overall performance from evaluation of experts with intuitionistic fuzzy multi numbers. This paper is derived from the second author's master's thesis [39]supervised by the first author.

2. Preliminaries

Definition 2.1 ([1]). Let X be a non-empty set. A *fuzzy set* F on X is defined as:

$$F = \{\langle x, \mu_F(x)\rangle : x \in X\},$$

where $\mu_F : X \to [0,1]$ is a mapping.

Definition 2.2 ([40]). Let X be a non-empty set. A *multi-fuzzy set* G on X is defined as:

$$G = \{\langle x, \mu_G^1(x), \mu_G^2(x), ..., \mu_G^i(x), ...\rangle : x \in X\},$$

where $\mu_G^i : X \to [0,1]$ is a mapping for all $i \in \{1,2,...,p\}$.

Definition 2.3 ([31]). Let $\eta_A^i \in [0,1]$ $i \in \{1,2,...,p\}$ and $a,b,c,d \in R$ such that $a \le b \le c \le d$. Then a *trapezoidal fuzzy multi-number* (TFM-number) $A = \langle (a,b,c,d); \eta_A^1, \eta_A^2, ..., \eta_A^P\rangle$ is a special fuzzy multi-set on the real number set R, whose membership functions are defined as:

$$\mu_A^i(x) = \begin{cases} (x-a)\eta_A^i/(b-a) & a \le x < b \\ \eta_A^i & b \le x < c \\ (d-x)\eta_A^i/(d-c) & c \le x < d \\ 0 & otherwise \end{cases}$$

Note that the set of all TFM-number on $\mathbb{R}^+$ will be denoted by $\mho(\mathbb{R}^+)$.

Definition 2.4 ([41]). A function $\tau : [0,1]\times[0,1]\to[0,1]$ is called a *t-norm*, if it satisfies the following four conditions:
(i) $\tau(1,x)=x$ for all $x\in[0,1]$,
(ii) $\tau(x,y)=\tau(y,x)$ for all $x,\ y\in[0,1]$,
(iii) $\tau(x,\tau(y,z))=\tau(\tau(x,y),z)$ for all $x,\ y,\ z\in[0,1]$,
(iv) if $x\le x'$ and $y\le y'$, then $\tau(x,y)\le\tau(x',y')$, where $x,\ y,\ x',\ y'\in[0,1]$.

Definition 2.5 ([41]). A function $s : [0,1]\times[0,1]\to[0,1]$ is called a *t-conorm*, if it satisfies the following four conditions:
(i) $s(0,x)=x$ for all $x\in[0,1]$,
(ii) $s(x,y)=s(y,x)$ for all $x,\ y\in[0,1]$,
(iii) $s(x,s(y,z))=s(s(x,y),z)$ for all $x,\ y,\ z\in[0,1]$,
(iv) if $x\le x'$ and $y\le y'$ then $s(x,y)\le s(x',y')$, where $x,\ y,\ x',\ y'\in[0,1]$.

Definition 2.6 ([41]). A t-norm function $\tau(x,y)$ is called an *Archimedean t-norm*, if it is continuous and $\tau(x,x)<x$ for all $x\in(0,1)$. An Archimedean t-norm is called a *strict Archimedean t-norm*, if it is strictly increasing in each variable for $x,y\in(0,1)$. A t-conorm function $s(x,y)$ is called Archimedean t-conorm if it is continuous and $s(x,x)>x$ for all $x\in(0,1)$. An Archimedean t-conorm is called a *strict Archimedean t-conorm*, if it is strictly increasing in each variable for $x,y\in(0,1)$.

In [42], a strict Archimedean t-norm s is expressed and Archimedean t-conorm T is expressed via function h and g, respectively as: for all $x,\ Y,\ T\in[0,1]$,

$$s(x,y)=h^{-1}(h(x)+h(y)),$$

and

$$\tau(x,y)=g^{-1}(g(x)+g(y))\quad with\quad h(t)=g(1-t),$$

where g is continuous Archimedean t-norm and strictly decreasing function such that $g:[0,1]\to[0,1]$ and $g(1)=0$.

In [8, 43], according to specific forms of function g, some well-known t-conorms and t-norms are given as follows.

(1) Let $g_1(t)=-Int$. Then $h_1(t)=-ln(1-t)$, $g_1^{-1}(t)=e^{-t}, h_1^{-1}(t)=1-e^{-t}$ and Algebraic t-conorm and t-norm are obtained as follows:

$$s_A=(x,y)=x+y-xy,\tau(x,y)=xy.$$

(2) Let $g_2(t)=ln(\frac{2-t}{t})$. Then $h_2(t)=ln(\frac{2-(1-t)}{1-t})$, $g_2^{-1}(t)=\frac{2}{e^t+1}$, $h_2^{-1}(t)=1-\frac{2}{e^t+1}$ and we can get Einstein t-conorm and t-norm:

$$s_E(x,y)=\frac{x+y}{1+xy},\ \tau_E(x,y)=\frac{xy}{1+(1-x)(1-y)}.$$

(3) Let $g_3(t)=ln(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma\in(0,+\infty)$. Then we have $h_3(t)=ln(\frac{\gamma+(1-\gamma)(1-t)}{1-t})$, $g_3^{-1}(t)=\frac{\gamma}{e^t+\gamma-1}$, $h_3^{-1}(t)=1-\frac{\gamma}{e^t+\gamma-1}$ and Hamacher t-conorm and t-norm are obtained as follows:

$$s_H(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy},\quad \gamma\in(0,+\infty)$$
$$\tau_H(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)},\quad \gamma\in(0,+\infty).$$

(4) Let $g_4(t) = ln(\frac{\gamma-1}{\gamma^t-1})$, $\gamma \in (0,+\infty)$. Then $h_4(t) = ln(\frac{\gamma}{(1-\gamma)^{(t)}-1})$, $g_4^{-1}(t) = \log_\gamma^{\frac{\gamma-1+e^t}{e^{(t)}}}$, $h_4^{-1}(t) = \log_\gamma^{\frac{\gamma+e^t}{e^t(1-\gamma)}}$ and get Frank t-conorm and t-norm:

$$\begin{aligned} s_F(x,y) = & \quad 1 - ln_\gamma(1 + \tfrac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1}), \quad \gamma \in (1,+\infty) \\ \tau_F(x,y) = & \quad ln_\gamma(1 + \tfrac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1}), \quad \gamma \in (1,+\infty). \end{aligned}$$

Definition 2.7 ([16, 41, 43]). Let $\grave{A} = (\theta_{\grave{A}}, \sigma_{\grave{A}})$, $\grave{A} = (\rho_{\grave{A}_1}, \sigma_{\grave{A}_1})$ and $\grave{A}_2 = (\rho_{\grave{A}_2}, \sigma_{\grave{A}_2})$ be three intuitionistic fuzzy elements. Then the operations of intuitionistic fuzzy elements based on Archimedean t-norm and Archimedean t-conorm is defined as:

(i) $\grave{A}_1 \oplus \grave{A}_2 = (s(\rho_{\grave{A}_1}, \rho_{\grave{A}_1}), \tau(\delta_{\grave{A}_1}, \delta_{\grave{A}_2}))$
$= \Big(h^{-1}(h(\rho_{\grave{A}_1}) + h(\rho_{\grave{A}_2})), g^{-1}(g(\delta_{\grave{A}_1}) + g(\delta_{\grave{A}_2}))\Big)$,

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \Big(\tau(\rho_{\grave{A}_1}, \rho_{\grave{A}_1}), s(\delta_{\grave{A}_1}, \delta_{\grave{A}_2})\Big)$
$= \Big(g^{-1}(g(\rho_{\grave{A}_1}) + g(\rho_{\grave{A}_2})), h^{-1}(h(\delta_{\grave{A}_1}) + h(\delta_{\grave{A}_2}))\Big)$,

(iii) $\lambda\grave{A} = (h^{-1}(\lambda h(\rho_{\grave{A}}), g^{-1}(\lambda g(\delta_{\grave{A}})), \lambda > 0$,

(iv) $\grave{A}^\lambda = (g^{-1}(\lambda g(\rho_{\grave{A}}), h^{-1}(\lambda h(\delta_{\grave{A}})), \lambda > 0$.

Especially, in [16, 41, 43, 44] the operational laws based on the function g are given as:

(1) for $g_1(t) = -log(t)$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \Big(\rho_{\grave{A}_1} + \rho_{\grave{A}_2} - \rho_{\grave{A}_1}\rho_{\grave{A}_2}, \delta_{\grave{A}_1}\delta_{\grave{A}_2}\Big)$,

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \Big(\rho_{\grave{A}_1}\rho_{\grave{A}_2}, \delta_{\grave{A}_1} + \delta_{\grave{A}_2} - \delta_{\grave{A}_1}\delta_{\grave{A}_2}\Big)$,

(iii) $\lambda\grave{A} = \Big(1 - (1-\rho_{\grave{A}})^\lambda, \delta_{\grave{A}}^\lambda, \Big), \lambda > 0$,

(iv) $\grave{A}^\lambda = \Big(\rho_{\grave{A}}^\lambda, 1 - (1-\delta_{\grave{A}})^\lambda\Big), \lambda > 0$,

(2) for $g_2(t) = -log(\frac{2-t}{t})$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \Big(\frac{\rho_{\grave{A}_1}+\rho_{\grave{A}_2}}{1+\rho_{\grave{A}_1}\rho_{\grave{A}_2}}, \frac{\delta_{\grave{A}_1}\delta_{\grave{A}_2}}{1+(1-\delta_{\grave{A}_1})(1-\delta_{\grave{A}_2})}\Big)$,

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \Big(\frac{\rho_{\grave{A}_1}\rho_{\grave{A}_2}}{1+(1-\rho_{\grave{A}_1})(1-\rho_{\grave{A}_2})}, \frac{\delta_{\grave{A}_1}+\delta_{\grave{A}_2}}{1+\delta_{\grave{A}_1}\delta_{\grave{A}_2}}\Big)$,

(iii) $\lambda\grave{A} = \Big(\frac{(1+\rho_{\grave{A}}^\lambda-(1-\rho_{\grave{A}})^\lambda}{(1+\rho_{\grave{A}})^\lambda+(1-\rho_{\grave{A}})^\lambda}, \frac{2\delta_{\grave{A}}^\lambda}{(2-\delta_{\grave{A}})^\lambda+\delta_{\grave{A}}^\lambda}\Big), \lambda > 0$,

(iv) $\grave{A}^\lambda = \Big(\frac{2\rho_{\grave{A}}^\lambda}{(2-\rho_{\grave{A}})^\lambda+\rho_{\grave{A}}^\lambda}, \frac{(1+\delta_{\grave{A}})^\lambda-(1-\delta_{\grave{A}})^\lambda}{(1+\delta_{\grave{A}})^\lambda+(1-\delta_{\grave{A}})^\lambda}\Big), \lambda > 0$,

(3) for $g_3(t) = log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma \in (0,+\infty)$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \Big(\frac{\rho_{\grave{A}_1}+\rho_{\grave{A}_2}-\rho_{\grave{A}_1}\rho_{\grave{A}_2}-(1-\gamma)\rho_{\grave{A}_1}\rho_{\grave{A}_2}}{1-(1-\gamma)\rho_{\grave{A}_1}\rho_{\grave{A}_2}}, \frac{\delta_{\grave{A}_1}\delta_{\grave{A}_2}}{\gamma+(1-\gamma)(\delta_{\grave{A}_1}+\delta_{\grave{A}_2}-\delta_{\grave{A}_1}\delta_{\grave{A}_2}})\Big)$,

(ii) $\grave{A}_1 \otimes \grave{A}_2 = (\frac{\rho_{\grave{A}_1}\rho_{\grave{A}_2}}{\gamma+(1-\gamma)(\rho_{\grave{A}_1}+\rho_{\grave{A}_2}-\rho_{\grave{A}_1}\rho_{\grave{A}_2}}), \frac{\delta_{\grave{A}_1}+\delta_{\grave{A}_2}-\delta_{\grave{A}_1}\delta_{\grave{A}_2}-(1-\gamma)\delta_{\grave{A}_1}\delta_{\grave{A}_2}}{1-(1-\gamma)\delta_{\grave{A}_1}\delta_{\grave{A}_2}}\Big)$,

(iii) $\lambda\grave{A} = \left(\frac{(1+(\gamma-1)\rho_{\grave{A}})^{\lambda}-(1-\rho_{\grave{A}})^{\lambda}}{(1+(\gamma-1)\rho_{\grave{A}})^{\lambda}+(\gamma-1)(1-\rho_{\grave{A}})^{\lambda}}, \frac{\gamma\delta_{\grave{A}\lambda}}{(1+(\gamma-1)(1-\delta_{\grave{A}}))^{\lambda}+(\gamma-1)\delta_{\grave{A}\lambda}} \right), \lambda > 0,$

(iv) $\grave{A}^{\lambda} = \left(\frac{\gamma\rho_{\grave{A}\lambda}}{(1+(\gamma-1)(1-\rho_{\grave{A}}))^{\lambda}+(\gamma-1)\rho_{\grave{A}\lambda}}, \frac{(1+(\gamma-1)\delta_{\grave{A}})^{\lambda}-(1-\delta_{\grave{A}})^{\lambda}}{(1+(\gamma-1)\delta_{\grave{A}})^{\lambda}+(\gamma-1)(1-\delta_{\grave{A}})^{\lambda}} \right), \lambda > 0,$

(4) for $g_4(t) = ln(\frac{\gamma-1}{\gamma^t-1}), \gamma \in (1, \infty)$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \left(1-\ln_{\gamma}(1+\frac{(\gamma^{1-\rho_{\grave{A}_1}}-1)(\gamma^{1-\rho_{\grave{A}_2}}-1)}{\gamma-1}), ln_{\gamma}(1+\frac{(\gamma^{\delta_{\grave{A}_1}}-1)(\gamma^{\delta_{\grave{A}_2}}-1)}{\gamma-1} \right), \lambda > 1,$

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \left(ln_{\gamma}(1+\frac{(\gamma^{\rho_{\grave{A}_1}}-1)(\gamma^{\rho_{\grave{A}_2}}-1)}{\gamma-1}, 1-\ln_{\gamma}(1+\frac{(\gamma^{1-\delta_{\grave{A}_1}}-1)(\gamma^{1-\delta_{\grave{A}_2}}-1)}{\gamma-1}) \right), \lambda > 1,$

(iii) $\lambda\grave{A} = \left(1 - ln_{\gamma}(1 + \frac{(\gamma^{1-\rho_{\grave{A}}}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}), ln_{\gamma}(1 + \frac{(\gamma^{\delta_{\grave{A}}}-1)^{\lambda}}{(y-1)^{\lambda-1}}) \right), \lambda > 0, \lambda > 1,$

(iv) $\grave{A}^{\lambda} = \left(ln_{\gamma}(1 + \frac{(\gamma^{\rho_{\grave{A}}}-1)^{\lambda}}{(y-1)^{\lambda-1}}), 1 - ln_{\gamma}(1 + \frac{(\gamma^{1-\delta_{\grave{A}}}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}), \lambda > 0, \lambda > 1. \right.$

Definition 2.8 ([2])**.** Let X be a non-empty set. An *intuitionistic fuzzy set* $\mathbb{IF}$ on X is defined as:

$$\mathbb{IF} = \{\langle x, \mu_{\mathbb{IF}}(x), \nu_{\mathbb{IF}}(x)\rangle : x \in X\},$$

where $\mu_{\mathbb{IF}}(x), \nu_{\mathbb{IF}}(x) : X \to [0, 1]$ such that $0 \leq \mu_{\mathbb{IF}}(x) + \mu_{\mathbb{IF}}(x) \leq 1$ for $x \in X$.

Definition 2.9 ([38])**.** Let $\rho^i_{\grave{A}}, \delta^i_{\grave{A}} \in [0, 1]$ $(i \in \{1, 2, \cdots, P\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then a *trapezoidal intuitionistic fuzzy multi-numbers* (TIFM-numbers) $\grave{A} = \langle[a, b, c, d]; (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}})\rangle$ is a special intuitionistic fuzzy multi-set on the real number set $\mathbb{R}$, whose membership functions and non-membership functions are defined, respectively as:

$$\mu^i_{\grave{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\rho^i_{\grave{A}}, & a \leq x < b \\ \rho^i_{\grave{A}}, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)}\rho^i_{\grave{A}}, & c < x \leq d \\ 0, & otherwise, \end{cases} \quad and \quad \nu^i_{\grave{A}}(x) = \begin{cases} \frac{(b-x)+\delta^i_{\grave{A}}(x-a)}{(b-a)}, & a \leq x < b \\ \delta^i_{\grave{A}}, & b \leq x \leq c \\ \frac{(x-c)+\delta^i_{\grave{A}}(d-x)}{(d-c)}, & c < x \leq d \\ 1, & otherwise, \end{cases}$$

Note that the set of all TIFM-numbers on $[0, 1]$ will be denoted by Λ.

Let $\grave{A}$, $\grave{B} \in \Lambda$ and $\gamma \neq 0$, where $\grave{A} = \langle[a_1, b_1, c_1, d_1]; (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}})\rangle$, $\grave{B} = \langle[a_2, b_2, c_2, d_2]; (\rho^1_{\grave{B}}, \rho^2_{\grave{B}}, \cdots, \rho^P_{\grave{B}}), (\delta^1_{\grave{B}}, \delta^2_{\grave{B}}, \cdots, \delta^P_{\grave{B}})\rangle$. Then $\grave{A} + \grave{B}$, $\grave{A} \cdot \grave{B}$, $\gamma\grave{A}$ and $\grave{A}^{\gamma}$ are respectively defined as follows:

(i) $\grave{A}+\grave{B} = \langle[a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2]; (s(\rho^1_{\grave{A}}, \rho^1_{\grave{B}}), s(\rho^2_{\grave{A}}, \rho^2_{\grave{B}}), \cdots, s(\rho^P_{\grave{A}}, \rho^P_{\grave{B}})),$
$(t(\delta^1_{\grave{A}}, \delta^1_{\grave{B}}), t(\delta^2_{\grave{A}}, \delta^2_{\grave{B}}), \cdots, t(\delta^P_{\grave{A}}, \delta^P_{\grave{B}}))\rangle,$

(ii) $\grave{A} \cdot \grave{B} = \langle[a_1a_2, b_1b_2, c_1c_2, d_1d_2]; (t(\rho^1_{\grave{A}}, \rho^1_{\grave{B}}), \cdots, t(\rho^P_{\grave{A}}, \rho^P_{\grave{B}})),$
$(s(\delta^1_{\grave{A}}, \delta^1_{\grave{B}}), \cdots, s(\delta^P_{\grave{A}}, \delta^P_{\grave{B}}))\rangle,$

(iii) $\gamma\grave{A} = \langle[\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; (1 - (1 - \rho^1_{\grave{A}})^{\gamma}, 1 - (1 - \rho^2_{\grave{A}})^{\gamma}), 1 - (1 - \rho^P_{\grave{A}})^{\gamma}),$
$((\delta^1_{\grave{A}})^{\gamma}, (\delta^2_{\grave{A}})^{\gamma}, \cdots, (\delta^P_{\grave{A}})^{\gamma})\rangle$ $(\gamma \geq 0)$,

(iv) $\grave{A}^{\gamma} = \langle[a_1^{\gamma}, b_1^{\gamma}, c_1^{\gamma}, d_1^{\gamma}]; ((\rho^1_{\grave{A}})^{\gamma}, (\rho^2_{\grave{A}})^{\gamma}, \cdots, (\rho^P_{\grave{A}})^{\gamma}),$
$(1 - (1 - \delta^1_{\grave{A}})^{\gamma}, 1 - (1 - \delta^2_{\grave{A}})^{\gamma}, \cdots, 1 - (1 - \delta^P_{\grave{A}})^{\gamma})\rangle$ $(\gamma \geq 0)$,

where s is s-norm and t is a t-norm.

Definition 2.10 ([38])**.** Let $\grave{A}_j = \langle [a_j, b_j, c_j, d_j]; (\rho^1_{\grave{A}_j}, \rho^2_{\grave{A}_j}, \cdots, \rho^P_{\grave{A}_j}), (\delta^1_{\grave{A}_j}, \delta^2_{\grave{A}_j}, \cdots, \delta^P_{\grave{A}_j})\rangle j \in I_n$ be a collection of TIFM-numbers and let $S\grave{A}_w : \Lambda^n \to \Lambda$, if

$$SA_w(\grave{A}_1, \grave{A}_2, \grave{A}_3, \cdots, \grave{A}_n) = \sum_{j=1}^{n} w_j \grave{A}_j$$

then $S\grave{A}_w$ is called a *TIFM-number weighted arithmetic operator of dimension n*, where $w = (w_1, w_2, w_3, \cdots, w_n)^T$ is the weight vector of $\grave{A}_j, j \in I_n$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 2.11. Let $\grave{A} = \langle (a, b, c, d); (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}})\rangle$ be a TIFM-numbers. Then

(i) *1 score value* of TIFM-number $\grave{A}$ based on [45, 46], denoted by $\bar{S}_1(\grave{A})$, is defined as:

$$\bar{S}_1(\grave{A}) = (\frac{d-a+c-b}{2}) \times \sum_{i=1}^{P} \frac{1-(\delta^i_{\grave{A}})^2}{2-(\rho_{\grave{A}} i)^2-(\delta^i_{\grave{A}})^2}, \tag{2.1}$$

(ii) *2 score value* of TIFM-number $\grave{A}$ based on [45, 47], denoted by $\bar{S}_2(\grave{A})$, is defined as:

$$\bar{S}_2(\grave{A}) = (\frac{d-a+c-b}{2}) \times \sum_{i=1}^{P} \frac{e^{(\rho^i_{\grave{A}})^2-(\delta^i_{\grave{A}})^2}}{2-(\rho^i_{\grave{A}})^2-(\delta^i_{\grave{A}})^2}, \tag{2.2}$$

(iii) *3 score value* of TIFM-number $\grave{A}$ based on Zhang [45, 48, 49, 50], denoted by $\bar{S}_3(\grave{A})$, is defined as:

$$\bar{S}_3(\grave{A}) = (\frac{d-a+c-b}{2}) \times \sum_{i=1}^{P} \left(\frac{1}{2} + \sqrt{(\rho^i_{\grave{A}})^2+(\delta^i_{\grave{A}})^2} (\frac{1}{2} - \arcsin(\frac{\delta^i_{\grave{A}}}{\sqrt{(\rho^i_{\grave{A}})^2+(\delta^i_{\grave{A}})^2}})\right), \tag{2.3}$$

(iv) *4 score value* of TIFM-number $\grave{A}$ based on [45, 51], denoted by $\bar{S}_4(\grave{A})$, is defined as:

$$\bar{S}_4(\grave{A}) = (\frac{d-a+c-b}{2}) \times \sum_{i=1}^{P} \left((\rho^i_{\grave{A}})^2-(\delta^i_{\grave{A}})^2\right) + \left(\frac{e^{(\rho^i_{\grave{A}})^2-(\delta^i_{\grave{A}})^2}}{e^{(\rho^i_{\grave{A}})^2-(\delta^i_{\grave{A}})^2}+1} - \frac{1}{2}\right). \tag{2.4}$$

Let $\grave{A} = \langle (a, b, c, d); (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}})\rangle$ and $\grave{B} = \langle (a, b, c, d); (\rho^1_{\grave{B}}, \rho^2_{\grave{B}}, \cdots, \rho^P_{\grave{B}}), (\delta^1_{\grave{B}}, \delta^2_{\grave{B}}, \cdots, \delta^P_{\grave{B}})\rangle$ be two TIFM-numbers. Then comparison of $\grave{A}$ and $\grave{B}$ is given as:

(1) $\bar{S}_j(\grave{A}) < \bar{S}_j(\grave{B})$, then $\grave{A} < \grave{B}$ (j=1,2,3,4),
(2) $\bar{S}_j(\grave{A}) = \bar{S}_j(\grave{B})$, then $\grave{A} = \grave{B}$ (j=1,2,3,4).

Definition 2.12. Let $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ for i=1,2 be two TIFM-numbers. Then the operations of TIFM-numbers under Archimedean t-norm and Archimedean t-conorm are defined as:

(i) the *sum* of $\grave{A}_1$ and $\grave{A}_2$, denoted by $\grave{A}_1 \oplus \grave{A}_2$, is defined as:

$\grave{A}_1 \oplus \grave{A}_2$

$= \langle (s(a_1,a_2), s(b_1,b_2), s(c_1,c_2), s(d_1,d_2)); s(\rho^1_{\grave{A}_1},\rho^1_{\grave{A}_2}), s(\rho^2_{\grave{A}_1},\rho^2_{\grave{A}_2}),$
$\cdots, s(\rho^P_{\grave{A}_1},\rho^P_{\grave{A}_2}), (t(\delta^1_{\grave{A}_1},\delta^1_{\grave{A}_2}), t(\delta^2_{\grave{A}_1},\delta^2_{\grave{A}_2}), \cdots, t(\delta^P_{\grave{A}_1},\delta^P_{\grave{A}_2}))\rangle$
$= \langle (h_1^{-1}(h_1(a_1)+h_1(a_2)), h_1^{-1}(h_1(b_1)+h_1(b_2)),$
$h_1^{-1}(h_1(c_1)+h_1(c_2)), h_1^{-1}(h_1(d_1)+h_1(d_2)));$
$h_1^{-1}(h_1(\rho^1_{\grave{A}_1})+h_1(\rho^1_{\grave{A}_2})), h_1^{-1}(h_1(\rho^2_{\grave{A}_1})+h_1(\rho^2_{\grave{A}_2})), \cdots, h_1^{-1}(h_1(\rho^P_{\grave{A}_1})+h_1(\rho^P_{\grave{A}_2})),$
$g_1^{-1}(g_1(\delta^1_{\grave{A}_1})+g_1(\delta^1_{\grave{A}_2})), g_1^{-1}(g_1(\delta^2_{\grave{A}_1})+g_1(\delta^2_{\grave{A}_2})), \cdots, g_1^{-1}(g_1(\delta^P_{\grave{A}_1})+g_1(\delta^P_{\grave{A}_2}))\rangle,$

(ii) the *product* of $\grave{A}_1$ and $\grave{A}_2$, denoted by $\grave{A}_1 \otimes \grave{A}_2$, is defined as:
$\grave{A}_1 \otimes \grave{A}_2$
$= \langle (\tau(a_1,a_2), \tau(b_1,b_2), \tau(c_1,c_2), \tau(d_1,d_2))); (\tau(\rho^1_{\grave{A}_1},\rho^1_{\grave{A}_2}), \tau(\rho^2_{\grave{A}_1},\rho^2_{\grave{A}_2}),$
$\cdots, \tau(\rho^P_{\grave{A}_1},\rho^P_{\grave{A}_2})), s(\delta^1_{\grave{A}_1},\delta^1_{\grave{A}_2}), s(\delta^2_{\grave{A}_1},\delta^2_{\grave{A}_2}), \cdots, s(\delta^P_{\grave{A}_1},\delta^P_{\grave{A}_2}))\rangle$
$= \langle (g_1^{-1}(g_1(\rho^1_{\grave{A}_1})+g_1(\rho^1_{\grave{A}_2})), g_1^{-1}(g_1(\rho^2_{\grave{A}_1})+g_1(\rho^2_{\grave{A}_2})), \cdots, g_1^{-1}(g_1(\rho^P_{\grave{A}_1})+g_1(\rho^P_{\grave{A}_2}),$
$(h_1^{-1}(h_1(\delta^1_{\grave{A}_1})+h_1(\delta^1_{\grave{A}_2})), h_1^{-1}(h_1(\delta^2_{\grave{A}_1})+h_1(\delta^2_{\grave{A}_2})), \cdots, h_1^{-1}(h_1(\delta^P_{\grave{A}_1})+h_1(\delta^P_{\grave{A}_2}))\rangle,$

(iii) the *scalar multiple* of $\grave{A}_1$, denoted by $\lambda\grave{A}_i$, is defined as:
$\lambda\grave{A}_i$
$= \langle (h_1^{-1}(\lambda h_1(a_i)), h_1^{-1}(\lambda h_1(b_i)), h_1^{-1}(\lambda h_1(c_i)), h_1^{-1}(\lambda h_1(d_i))),$
$(g_1^{-1}(\lambda g_1(a_i)), g_1^{-1}(\lambda g_1(b_i)), g_1^{-1}(\lambda g_1(c_i)), g_1^{-1}(\lambda g_1(d_i)));$
$h_1^{-1}(\lambda h_1(\rho^1_{\grave{A}_i})), h_1^{-1}(\lambda h_1(\rho^2_{\grave{A}_i})), \cdots, h_1^{-1}(\lambda h_1(\rho^P_{\grave{A}_i})),$
$(g_1^{-1}(\lambda g_1(\delta^1_{\grave{A}_i})), g_1^{-1}(\lambda g_1(\delta^2_{\grave{A}_i})), \cdots, g_1^{-1}(\lambda g_1(\delta^P_{\grave{A}_i}))\rangle,$

(iv) the *scalar power* of $\grave{A}_1$, denoted by $\grave{A}_i^\lambda$, is defined as:
$\grave{A}_i^\lambda$
$= \langle (g_1^{-1}(\lambda g_1(a_i)), g_1^{-1}(\lambda g_1(b_i)), g_1^{-1}(\lambda g_1(c_i)), g_1^{-1}(\lambda g_1(d_i))),$
$((h_1^{-1}(\lambda h_1(a_i)), h_1^{-1}(\lambda h_1(b_i)), h_1^{-1}(\lambda h_1(c_i)), h_1^{-1}(\lambda h_1(d_i))));$
$g_1^{-1}(\lambda g_1(\rho^1_{\grave{A}_i})), g_1^{-1}(\lambda g_1(\rho^2_{\grave{A}_i})), \cdots, g_1^{-1}(\lambda g_1(\rho^P_{\grave{A}_i})),$
$(h_1^{-1}(\lambda h_1(\delta^1_{\grave{A}_i})), h_1^{-1}(\lambda h_1(\delta^2_{\grave{A}_i})), \cdots, h_1^{-1}(\lambda h_1(\delta^P_{\grave{A}_i})))\rangle.$

Especially, under Archimedean t-norm and Archimedean t-conorm, the operational laws based on the function g are given as:

(1) for $g_1(t) = -log(t)$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \langle (a_1+a_2-a_1a_2, b_1+b_2-b_1b_2, c_1+c_2-c_1c_2, d_1+d_2-d_1d_2);$
$(\rho^1_{\grave{A}_1}+\rho^1_{\grave{A}_2}-\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}, \rho^2_{\grave{A}_1}+\rho^2_{\grave{A}_2}-\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}, \cdots,$
$\rho^P_{\grave{A}_1}+\rho^P_{\grave{A}_2}-\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}), (\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}, \delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}, \cdots, \delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2})\rangle,$

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \langle (a_1a_2, b_1b_2, c_1c_2, d_1d_2); (\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}, \rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}, \cdots, \rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}),$
$(\delta^1_{\grave{A}_1}+\delta^1_{\grave{A}_2}-\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}, \delta^2_{\grave{A}_1}+\delta^2_{\grave{A}_2}-\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}, \cdots,$
$\delta^P_{\grave{A}_1}+\delta^P_{\grave{A}_2}-\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}))\rangle,$

(iii) $\lambda\grave{A}_i = \langle (1-(1-a_i)^\lambda, 1-(1-b_i)^\lambda, 1-(1-c_i)^\lambda, 1-(1-d_i)^\lambda);$
$1-(1-\rho^1_{\grave{A}_i})^\lambda, 1-(1-\rho^2_{\grave{A}_i})^\lambda, \cdots,$
$1-(1-\rho^P_{\grave{A}_i})^\lambda), ((\delta^1_{\grave{A}_i})^\lambda, (\delta^2_{\grave{A}_i})^\lambda, \cdots, (\delta^P_{\grave{A}_i})^\lambda)\rangle, \quad \lambda > 0,$

(iv) $\grave{A}_i^\lambda = \langle ((a_i)^\lambda, (b_i)^\lambda, (c_i)^\lambda, (d_i)^\lambda); (\rho^1_{\grave{A}_i})^\lambda, (\rho^2_{\grave{A}_i})^\lambda, \cdots,$
$(\rho^P_{\grave{A}_i})^\lambda, (1-(1-\delta^1_{\grave{A}_i})^\lambda, 1-(1-\delta^2_{\grave{A}_i})^\lambda, \cdots, 1-(1-\delta^P_{\grave{A}_i})^\lambda))\rangle, \quad \lambda > 0,$

(2) for $g_2(t) = -log(\frac{2-t}{t})$,

(i) $\grave{A}_1 \oplus \grave{A}_2 = \langle (\frac{a_1+a_2}{1+a_1a_2}, \frac{b_1+b_2}{1+b_1b_2}, \frac{c_1+c_2}{1+c_1c_2}, \frac{d_1+d_2}{1+d_1d_2}); \frac{\rho^1_{\grave{A}_1}+\rho^1_{\grave{A}_2}}{1+\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}}, \frac{\rho^2_{\grave{A}_1}+\rho^2_{\grave{A}_2}}{1+\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}}, \cdots, \frac{\rho^P_{\grave{A}_1}+\rho^P_{\grave{A}_2}}{1+\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}},$
$(\frac{\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}}{1+(1-\delta^1_{\grave{A}_1})(1-\delta^1_{\grave{A}_2})}, \frac{\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}}{1+(1-\delta^2_{\grave{A}_1})(1-\delta^2_{\grave{A}_2})}, \cdots, \frac{\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}}{1+(1-\delta^P_{\grave{A}_1})(1-\delta^P_{\grave{A}_2})})\rangle,$

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \langle (\frac{a_1a_2}{1+(1-a_1)(1-a_2)}, \frac{b_1b_2}{1+(1-b_1)(1-b_2)}, \frac{c_1c_2}{1+(1-c_1)(1-c_2)}, \frac{d_1d_2}{1+(1-d_1)(1-d_2)});$
$\frac{\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}}{1+(1-\rho^1_{\grave{A}_1})(1-\rho^1_{\grave{A}_2})}, \frac{\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}}{1+(1-\rho^2_{\grave{A}_1})(1-\rho^2_{\grave{A}_2})}, \cdots, \frac{\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}}{1+(1-\rho^P_{\grave{A}_1})(1-\rho^P_{\grave{A}_2})},$
$(\frac{\delta^1_{\grave{A}_1}+\delta^1_{\grave{A}_2}}{1+\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}}, \frac{\delta^2_{\grave{A}_1}+\delta^2_{\grave{A}_2}}{1+\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}}, \cdots, \frac{\delta^P_{\grave{A}_1}+\delta^P_{\grave{A}_2}}{1+\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}})\rangle,$

(iii) $\lambda\grave{A}_i = \langle (\frac{(1+a_i)^\lambda-(1-a_i)^\lambda}{(1+a_i)^\lambda+(1-a_i)^\lambda}, \frac{(1+b_i)^\lambda-(1-b_i)^\lambda}{(1+b_i)^\lambda+(1-b_i)^\lambda}, \frac{(1+c_i)^\lambda-(1-c_i)^\lambda}{(1+c_i)^\lambda+(1-c_i)^\lambda}, \frac{(1+d_i)^\lambda-(1-d_i)^\lambda}{(1+d_i)^\lambda+(1-d_i)^\lambda});$
$\frac{(1+\rho^1_{\grave{A}_i})^\lambda-(1-\rho^1_{\grave{A}_i})^\lambda}{(1+\rho^1_{\grave{A}_i})^\lambda+(1-\rho^1_{\grave{A}_i})^\lambda}, \frac{(1+\rho^2_{\grave{A}_i})^\lambda-(1-\rho^2_{\grave{A}_i})^\lambda}{(1+\rho^2_{\grave{A}_i})^\lambda+(1-\rho^2_{\grave{A}_i})^\lambda}, \cdots, \frac{(1+\rho^P_{\grave{A}_i})^\lambda-(1-\rho^P_{\grave{A}_i})^\lambda}{(1+\rho^P_{\grave{A}_i})^\lambda+(1-\rho^P_{\grave{A}_i})^\lambda},$
$(\frac{2(\delta^1_{\grave{A}_i})^\lambda}{(2-\delta^1_{\grave{A}_i})^\lambda+(\delta^1_{\grave{A}_i})^\lambda}, \frac{2(\delta^2_{\grave{A}_i})^\lambda}{(2-\delta^2_{\grave{A}_i})^\lambda+(\delta^2_{\grave{A}_i})^\lambda}, \cdots, \frac{2(\delta^P_{\grave{A}_i})^\lambda}{(2-\delta^P_{\grave{A}_i})^\lambda+(\rho^P_{\grave{A}_i})^\lambda})\rangle,$

(iv) $\grave{A}_i^\lambda = \langle (\frac{2(a_i)^\lambda}{(2-a_i)^\lambda+(a_i)^\lambda}, \frac{2(b_i)^\lambda}{(2-b_i)^\lambda+(b_i)^\lambda}, \frac{2(c_i)^\lambda}{(2-c_i)^\lambda+(c_i)^\lambda}, \frac{2(d_i)^\lambda}{(2-d_i)^\lambda+(d_i)^\lambda});$
$\frac{2(\rho^1_{\grave{A}_i})^\lambda}{(2-\rho^1_{\grave{A}_i})^\lambda+(\rho^1_{\grave{A}_i})^\lambda}, \frac{2(\rho^2_{\grave{A}_i})^\lambda}{(2-\rho^2_{\grave{A}_i})^\lambda+(\rho^2_{\grave{A}_i})^\lambda}, \cdots, \frac{2(\rho^P_{\grave{A}_i})^\lambda}{(2-\rho^P_{\grave{A}_i})^\lambda+(\rho^P_{\grave{A}_i})^\lambda},$
$(\frac{(1+\delta^1_{\grave{A}_i})^\lambda-(1-\delta^1_{\grave{A}_i})^\lambda}{(1+\delta^1_{\grave{A}_i})^\lambda+(1-\delta^1_{\grave{A}_i})^\lambda}, \frac{(1+\delta^2_{\grave{A}_i})^\lambda-(1-\delta^2_{\grave{A}_i})^\lambda}{(1+\delta^2_{\grave{A}_i})^\lambda+(1-\delta^2_{\grave{A}_i})^\lambda}, \cdots, \frac{(1+\delta^P_{\grave{A}_i})^\lambda-(1-\delta^P_{\grave{A}_i})^\lambda}{(1+\delta^P_{\grave{A}_i})^\lambda+(1-\delta^P_{\grave{A}_i})^\lambda})\rangle,$

(3) for $g_3(t) = log(\frac{\gamma+(1-\gamma)t}{t}), \quad \gamma \in (0, +\infty),$

(i) $\grave{A}_1 \oplus \grave{A}_2 = \langle (\frac{a_1+a_2-a_1a_2-(1-\gamma)a_1a_2}{1-(1-\gamma)a_1a_2}, \frac{b_1+b_2-b_1b_2-(1-\gamma)b_1b_2}{1-(1-\gamma)b_1b_2},$
$\frac{c_1+c_2-c_1c_2-(1-\gamma)c_1c_2}{1-(1-\gamma)c_1c_2}, \frac{d_1+d_2-d_1d_2-(1-\gamma)d_1d_2}{1-(1-\gamma)d_1d_2});$
$= \frac{\rho^1_{\grave{A}_1}+\rho^1_{\grave{A}_2}-\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}-(1-\gamma)\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}}{1-(1-\gamma)\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}}, \frac{\rho^2_{\grave{A}_1}+\rho^2_{\grave{A}_2}-\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}-(1-\gamma)\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}}{1-(1-\gamma)\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}}, \cdots,$
$\frac{\rho^P_{\grave{A}_1}+\rho^P_{\grave{A}_2}-\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}-(1-\gamma)\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}}{1-(1-\gamma)\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}}, (\frac{\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}}{\gamma+(1-\gamma)(\delta^1_{\grave{A}_1}+\delta^1_{\grave{A}_2}-\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2})},$
$\frac{\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}}{\gamma+(1-\gamma)(\delta^2_{\grave{A}_1}+\delta^2_{\grave{A}_2}-\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2})}, \cdots, \frac{\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}}{\gamma+(1-\gamma)(\delta^P_{\grave{A}_1}+\delta^P_{\grave{A}_2}-\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2})})\rangle,$

(ii) $\grave{A}_1 \otimes \grave{A}_2 = \langle (\frac{a_1a_2}{\gamma+(1-\gamma)(a_1+a_2-a_1a_2)}, \frac{b_1b_2}{\gamma+(1-\gamma)(b_1+b_2-b_1b_2)},$
$\frac{c_1c_2}{\gamma+(1-\gamma)(c_1+c_2-c_1c_2)}, \frac{d_1d_2}{\gamma+(1-\gamma)(d_1+d_2-d_1d_2)});$
$\frac{\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2}}{\gamma+(1-\gamma)(\rho^1_{\grave{A}_1}+\rho^1_{\grave{A}_2}-\rho^1_{\grave{A}_1}\rho^1_{\grave{A}_2})}, \frac{\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2}}{\gamma+(1-\gamma)(\rho^2_{\grave{A}_1}+\rho^2_{\grave{A}_2}-\rho^2_{\grave{A}_1}\rho^2_{\grave{A}_2})}, \cdots,$
$\frac{\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2}}{\gamma+(1-\gamma)(\rho^P_{\grave{A}_1}+\rho^P_{\grave{A}_2}-\rho^P_{\grave{A}_1}\rho^P_{\grave{A}_2})}, (\frac{\delta^1_{\grave{A}_1}+\delta^1_{\grave{A}_2}-\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}-(1-\gamma)\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}}{1-(1-\gamma)\delta^1_{\grave{A}_1}\delta^1_{\grave{A}_2}},$
$\frac{\delta^2_{\grave{A}_1}+\delta^2_{\grave{A}_2}-\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}-(1-\gamma)\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}}{1-(1-\gamma)\delta^2_{\grave{A}_1}\delta^2_{\grave{A}_2}}, \cdots, \frac{\delta^P_{\grave{A}_1}+\delta^P_{\grave{A}_2}-\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}-(1-\gamma)\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}}{1-(1-\gamma)\delta^P_{\grave{A}_1}\delta^P_{\grave{A}_2}})\rangle,$

(iii) $\lambda\grave{A}_i = \langle (\frac{(1+(\gamma-1)a_i)^\lambda-(1-a_i)^\lambda}{(1+(\gamma-1)a_i)^\lambda+(\gamma-1)(1-a_i)^\lambda}, \frac{(1+(\gamma-1)b_i)^\lambda-(1-b_i)^\lambda}{(1+(\gamma-1)b_i)^\lambda+(\gamma-1)(1-b_i)^\lambda},$
$\frac{(1+(\gamma-1)c_i)^\lambda-(1-c_i)^\lambda}{(1+(\gamma-1)c_i)^\lambda+(\gamma-1)(1-c_i)^\lambda}, \frac{(1+(\gamma-1)d_i)^\lambda-(1-d_i)^\lambda}{(1+(\gamma-1)d_i)^\lambda+(\gamma-1)(1-d_i)^\lambda});$
$\frac{(1+(\gamma-1)\rho^1_{\grave{A}_i})^\lambda-(1-\rho^1_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\rho^1_{\grave{A}_i})^\lambda+(\gamma-1)(1-\rho^1_{\grave{A}_i})^\lambda}, \frac{(1+(\gamma-1)\rho^2_{\grave{A}_i})^\lambda-(1-\rho^2_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\rho^2_{\grave{A}_i})^\lambda+(\gamma-1)(1-\rho^2_{\grave{A}_i})^\lambda}, \cdots,$
$\frac{(1+(\gamma-1)\rho^P_{\grave{A}_i})^\lambda-(1-\rho^P_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\rho^P_{\grave{A}_i})^\lambda+(\gamma-1)(1-\rho^P_{\grave{A}_i})^\lambda},$

$$(\frac{\gamma(\delta^1_{\grave{A}_i})^\lambda}{(1+(\gamma-1)(1-\delta^1_{\grave{A}_i}))^\lambda+(\gamma-1)(\delta^1_{\grave{A}_i})^\lambda}, \frac{\gamma(\delta^2_{\grave{A}_i})^\lambda}{(1+(\gamma-1)(1-\delta^2_{\grave{A}_i}))^\lambda+(\gamma-1)(\delta^2_{\grave{A}_i})^\lambda}, \cdots, \frac{\gamma(\delta^P_{\grave{A}_i})^\lambda}{(1+(\gamma-1)(1-\delta^P_{\grave{A}_i}))^\lambda+(\gamma-1)(\delta^P_{\grave{A}_i})^\lambda})\rangle,\ \lambda>0,$$

(iv) $$\grave{A}_i^\lambda = \langle(\frac{\gamma(a_i)^\lambda}{(1+(\gamma-1)(1-a_i))^\lambda+(\gamma-1)(a_i)^\lambda}, \frac{\gamma(b_i)^\lambda}{(1+(\gamma-1)(1-b_i))^\lambda+(\gamma-1)(b_i)^\lambda}, \frac{\gamma(c_i)^\lambda}{(1+(\gamma-1)(1-c_i))^\lambda+(\gamma-1)(c_i)^\lambda}, \frac{\gamma(d_i)^\lambda}{(1+(\gamma-1)(1-d_i))^\lambda+(\gamma-1)(d_i)^\lambda}); \frac{\gamma(\rho^1_{\grave{A}_i})^\lambda}{(1+(\gamma-1)(1-\rho^1_{\grave{A}_i}))^\lambda+(\gamma-1)(\rho^1_{\grave{A}_i})^\lambda}, \frac{\gamma(\rho^2_{A_i})^\lambda}{(1+(\gamma-1)(1-\rho^2_{A_i}))^\lambda+(\gamma-1)(\rho^2_{A_i})^\lambda}, \cdots, \frac{\gamma(\rho^P_{\grave{A}_i})^\lambda}{(1+(\gamma-1)(1-\rho^P_{\grave{A}_i}))^\lambda+(\gamma-1)(\rho^P_{\grave{A}_i})^\lambda}, (\frac{(1+(\gamma-1)\rho^1_{\grave{A}_i})^\lambda-(1-\delta^1_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\delta^1_{\grave{A}_i})^\lambda+(\gamma-1)(1-\delta^1_{\grave{A}_i})^\lambda}, \frac{(1+(\gamma-1)\delta^2_{\grave{A}_i})^\lambda-(1-\delta^2_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\delta^2_{\grave{A}_i})^\lambda+(\gamma-1)(1-\delta^2_{\grave{A}_i})^\lambda}, \cdots, \frac{(1+(\gamma-1)\delta^P_{\grave{A}_i})^\lambda-(1-\delta^P_{\grave{A}_i})^\lambda}{(1+(\gamma-1)\delta^P_{\grave{A}_i})^\lambda+(\gamma-1)(1-\delta^P_{\grave{A}_i})^\lambda})\rangle,\ \lambda>0,$$

(4) for $g_4(t) = ln(\frac{\gamma-1}{\gamma^t-1}),\ \gamma\in(1,\infty)$,

(i) $$\grave{A}_1\oplus\grave{A}_2 = \langle(1-log_\gamma(1+\frac{(\gamma^{1-a_1}-1)(\gamma^{1-a_2}-1)}{\gamma-1}, 1-log_\gamma(1+\frac{(\gamma^{1-b_1}-1)(\gamma^{1-b_2}-1)}{\gamma-1}, 1-log_\gamma(1+\frac{(\gamma^{1-c_1}-1)(\gamma^{1-c_2}-1)}{\gamma-1}, 1-log_\gamma(1+\frac{(\gamma^{1-d_1}-1)(\gamma^{1-d_2}-1)}{\gamma-1}); 1-log_\gamma(1+\frac{(\gamma^{1-\rho^1_{\grave{A}_1}}-1)(\gamma^{1-\rho^1_{\grave{A}_2}}-1)}{\gamma-1}), 1-log_\gamma(1+\frac{(\gamma^{1-\rho^2_{\grave{A}_1}}-1)(\gamma^{1-\rho^2_{\grave{A}_2}}-1)}{\gamma-1}), \cdots, 1-log_\gamma(1+\frac{(\gamma^{1-\rho^P_{\grave{A}_1}}-1)(\gamma^{1-\rho^P_{\grave{A}_2}}-1)}{\gamma-1}), log_\gamma(1+\frac{(\gamma^{\delta^2_{\grave{A}_1}}-1)(\gamma^{\delta^2_{\grave{A}_2}}-1)}{\gamma-1}), \cdots, log_\gamma(1+\frac{(\gamma^{\delta^P_{\grave{A}_1}}-1)(\gamma^{\delta^P_{\grave{A}_2}}-1)}{\gamma-1}))\rangle,\ \gamma>1,$$

(ii) $$\grave{A}_1\otimes\grave{A}_2 = \langle(log_\gamma(1+\frac{(\gamma^{a_1}-1)(\gamma^{a_2}-1)}{\gamma-1}, log_\gamma(1+\frac{(\gamma^{b_1}-1)(\gamma^{b_2}-1)}{\gamma-1}, log_\gamma(1+\frac{(\gamma^{c_1}-1)(\gamma^{c_2}-1)}{\gamma-1}, log_\gamma(1+\frac{(\gamma^{d_1}-1)(\gamma^{d_2}-1)}{\gamma-1}); log_\gamma(1+\frac{(\gamma^{\rho^1_{\grave{A}_1}}-1)(\gamma^{\rho^1_{\grave{A}_2}}-1)}{\gamma-1}), log_\gamma(1+\frac{(\gamma^{\rho^2_{\grave{A}_1}}-1)(\gamma^{\rho^2_{\grave{A}_2}}-1)}{\gamma-1}), \cdots, log_\gamma(1+\frac{(\gamma^{\rho^P_{\grave{A}_1}}-1)(\gamma^{\rho^P_{\grave{A}_2}}-1)}{\gamma-1}), (1-log_\gamma(1+\frac{(\gamma^{1-\delta^1_{\grave{A}_1}}-1)(\gamma^{1-\delta^1_{\grave{A}_2}}-1)}{\gamma-1}), 1-log_\gamma(1+\frac{(\gamma^{1-\delta^2_{\grave{A}_1}}-1)(\gamma^{1-\delta^2_{\grave{A}_2}}-1)}{\gamma-1}), \cdots, 1-log_\gamma(1+\frac{(\gamma^{1-\delta^P_{\grave{A}_1}}-1)(\gamma^{1-\delta^P_{\grave{A}_2}}-1)}{\gamma-1}))\rangle,\ \gamma>1,$$

(iii) $$\lambda\grave{A}_i = \langle(1-log_\gamma(1+\frac{(\gamma^{1-a_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-log_\gamma(1+\frac{(\gamma^{1-b_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-log_\gamma(1+\frac{(\gamma^{1-c_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-log_\gamma(1+\frac{(\gamma^{1-d_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}})); 1-log_\gamma(1+\frac{(\gamma^{1-\rho^1_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-log_\gamma(1+\frac{(\gamma^{1-\rho^2_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \cdots, 1-log_\gamma(1+\frac{(\gamma^{1-\rho^P_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), (log_\gamma(1+\frac{(\gamma^{\delta^1_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), log_\gamma(1+\frac{(\gamma^{\delta^2_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \cdots, log_\gamma(1+\frac{(\gamma^{\delta^P_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}))\rangle, \lambda>0,\ \gamma>1,$$

(iv) $$\grave{A}_i^\lambda = \langle(log_\gamma(1+\frac{(\gamma^{a_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), log_\gamma(1+\frac{(\gamma^{b_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), log_\gamma(1+\frac{(\gamma^{c_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), log_\gamma(1+\frac{(\gamma^{d_i}-1)^\lambda}{(\gamma-1)^{\lambda-1}})); log_\gamma(1+\frac{(\gamma^{\rho^1_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), log_\gamma(1+\frac{(\gamma^{\rho^2_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),$$

$$\cdots, log_\gamma(1+\frac{(\gamma^{\rho^P_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), (1-log_\gamma(1+\frac{(\gamma^{1-\delta^1_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}),$$
$$1-log_\gamma(1+\frac{(\gamma^{1-\delta^2_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \cdots, 1-log_\gamma(1+\frac{(\gamma^{1-\delta^P_{\grave{A}_i}}-1)^\lambda}{(\gamma-1)^{\lambda-1}}))\rangle, \lambda>0,\ \gamma>1.$$

3. TIFM aggregation operators based on Archimedean t-conorm and t-norm

Definition 3.1. Let $\grave{A}_i = \langle(a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of TIFM-numbers and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of the TIFM-numbers $\grave{A}_i (i = 1, 2, \cdots, n)$. Then *intuitionistic trapezoidal fuzzy multi weighted averaging operator* of $\grave{A}_i$ $(i = 1, 2, \cdots, n)$ based on Archimedean t-norm and Archimedeant-conorm, denoted by $\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n)$, is defined as:

$$\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) = \bigoplus_{i=1}^{n}(w_i\grave{A}_i),$$

where w_i indicates the importance degree of $\grave{A}_i$ $(i = 1, 2, \cdots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Theorem 3.2. *Let* $\grave{A}_i = \langle(a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of TIFM-numbers,* $\grave{A} = \langle(a, b, c, d); (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}})\rangle$ *be a TIFM-number and* λ, λ_1, $\lambda_2 > 0$. *The aggregated value by using the ATS-IFWA operator is also an IFV, and*

(3.1)

$$\begin{aligned}\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) =&\langle(h^{-1}(\sum_{i=1}^{n} w_i h(a_i)), h^{-1}(\sum_{i=1}^{n} w_i h(b_i)), h^{-1}(\sum_{i=1}^{n} w_i h(c_i)), h^{-1}(\sum_{i=1}^{n} w_i h(d_i)));\\ &h^{-1}(\sum_{i=1}^{n} w_i h(\rho^1_{\grave{A}_i})), h^{-1}(\sum_{i=1}^{n} w_i h(\rho^2_{\grave{A}_i})), \cdots, h^{-1}(\sum_{i=1}^{n} w_i h(\rho^P_{\grave{A}_i})),\\ &(g^{-1}(\sum_{i=1}^{n} w_i g(\delta^1_{\grave{A}_i})), g^{-1}(\sum_{i=1}^{n} w_i g(\delta^2_{\grave{A}_i})), \cdots, g^{-1}(\sum_{i=1}^{n} w_i g(\delta^P_{\grave{A}_i}))\rangle\end{aligned}$$

which has been investigated by Beliakov et al. [52], *Xu and Yager* [53], *Wu and Cai* [54] *and next we give a further study.*

Definition 3.3. Let $\grave{A}_i = \langle(a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of TIFM-numbers and $w = (w_1, w_2, ..., w_n)^T$ weight vector of $\grave{A}_i$ $(i = 1, 2, \cdots, n)$ where $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) = \bigoplus_{i=1}^{n}(w_i\grave{A}_i)$$

then $\mathbb{TIFMA}$ is called an *Archimedean t-conorm* and *t-norm* based intuitionistic trapezoidal fuzzy multi weighted averaging ($\mathbb{TIFMA}$) operator.

Theorem 3.4 ([55]). *Let* $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, ..., \delta^P_{\grave{A}_i})\rangle$,

$$\grave{B}_i = \langle (\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i); (\rho^1_{\grave{B}_i}, \rho^2_{\grave{B}_i}, \cdots, \rho^P_{\grave{B}_i}), (\delta^1_{\grave{B}_i}, \delta^2_{\grave{B}_i}, ..., \delta^P_{\grave{B}_i})\rangle \text{ (i=1,2,...,n)}$$

be two collection of TIFM-numbers. If

$$a_i \le \dot{a}_i, b_i \le \dot{b}_i, c_i \le \dot{c}_i, d_i \le \dot{d}_i,$$

$$\rho^1_{\grave{A}_i} \le \rho^1_{\grave{B}_i}, \rho^2_{\grave{A}_i} \le \rho^2_{\grave{\bar{B}}_i}, ..., \rho^P_{\grave{A}_i} \le \rho^P_{\grave{B}_i}$$

and

$$\delta^1_{\grave{B}_i} \le \delta^1_{\grave{A}_i}, \delta^2_{\grave{B}_i} \le \delta^2_{\grave{A}_i}, ..., \delta^P_{\grave{B}_i} \le \delta^P_{\grave{A}_i},$$

then

$$\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, ..., \grave{A}_n) \le \mathbb{TIFMA}(\grave{B}_1, \grave{B}_2, ..., \grave{B}_n)$$

For any $a, b \in \mathbb{R}$, we will denote $max\{a, b\}$ and $min\{a, b\}$ by $a \vee b$ and $a \wedge b$ respectively.

Theorem 3.5. *Let* $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of TFM-numbers. Let* $\grave{A}^-$ *and* $\grave{A}^+$ *be given by* $(i = 1, 2, \cdots, n)$

$$\grave{A}^- \le \mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) \le \grave{A}^+,$$

where

$$\begin{aligned}&\langle (\textstyle\bigvee_{i=1}^n a_i, \bigvee_{i=1}^n b_i, \bigvee_{i=1}^n c_i, \bigvee_{i=1}^n d_i);\\ &\quad (s(\rho^1_{\grave{A}_i}), s(\rho^2_{\grave{A}_i}), \cdots, s(\rho^P_{\grave{A}_i}), (\tau(\delta^1_{\grave{B}_i}), \tau(\delta^2_{\grave{B}_i}), \cdots, \tau(\delta^P_{\grave{B}_i})\rangle,\\ &\langle (\textstyle\bigvee_{i=1}^n a_i, \bigvee_{i=1}^n b_i, \bigvee_{i=1}^n c_i, \bigvee_{i=1}^n d_i); (\bigvee_{i=1}^n \rho^1_{\grave{A}_i}, \bigvee_{i=1}^n \rho^2_{\grave{A}_i}, \cdots, \bigvee_{i=1}^n \rho^P_{\grave{A}_i}),\\ &\quad (\textstyle\bigwedge_{i=1}^n \delta^1_{\grave{B}_i}, \bigwedge_{i=1}^n \delta^2_{\grave{B}_i}, \cdots, \bigwedge_{i=1}^n \delta^P_{\grave{B}_i})\rangle\end{aligned}$$

and

$$\begin{aligned}\grave{A}^- &= \langle (\textstyle\bigwedge_{i=1}^n a_i, \bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i, \bigwedge_{i=1}^n d_i;\\ &\qquad \tau(\rho^1_{\grave{A}_i}), \tau(\rho^2_{\grave{A}_i}), \cdots, \tau(\rho^P_{\grave{A}_i}), (s(\delta^1_{\grave{B}_i}), s(\delta^2_{\grave{B}_i}), \cdots, s(\delta^P_{\grave{B}_i})\rangle,\\ \grave{A}^- &= \langle (\textstyle\bigwedge_{i=1}^n a_i, \bigwedge_{i=1}^n b_i, \bigwedge_{i=1}^n c_i, \bigwedge_{i=1}^n d_i; (\bigwedge_{i=1}^n \rho^1_{\grave{A}_i}, \bigwedge_{i=1}^n \rho^2_{\grave{A}_i}, \cdots, \bigwedge_{i=1}^n \rho^P_{\grave{A}_i}),\\ &\qquad (\textstyle\bigvee_{i=1}^n \delta^1_{\grave{B}_i}, \bigvee_{i=1}^n \delta^2_{\grave{B}_i}, \cdots, \bigvee_{i=1}^n \delta^P_{\grave{B}_i})\rangle.\end{aligned}$$

If we choose the additive generator g in different forms, then some specific intuitionistic trapezoidal fuzzy aggregation operators can be obtained (See [55]).

Theorem 3.6. *Let* $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i})\rangle$ $(i = 1, 2, \cdots, n)$ *be a collection of TIFM-numbers and* $w = (w_1, w_2, ..., w_n)^T$ *weight vector of* $\grave{A}_i$ $(i = 1, 2, \cdots, n)$ *where* $w_i \ge 0$ $(i = 1, 2, \cdots, n)$ *and* $\sum_{i=1}^n w_i = 1$. *If* $r > 0$, *then*

$$\mathbb{TIFMA}(r\grave{A}_1, r\grave{A}_2, \cdots, r\grave{A}_n) = r\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n).$$

If the additive generator g is assigned different forms, then some specific intuitionistic fuzzy aggregation operators can be obtained (See [55]).

Result 1 If $g(t) = -logt$, then the $\mathbb{TIFMA}$ operator reduces to the following:

$$\begin{aligned}\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) =& \langle(\bigoplus_{i=1}^{n}(w_ia_i), \bigoplus_{i=1}^{n}(w_ib_i), \bigoplus_{i=1}^{n}(w_ic_i), \bigoplus_{i=1}^{n}(w_id_i));\\ & 1-\prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}, 1-\prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}, \cdots, 1-\prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i},\\ & 1-\prod_{i=1}^{n}(1-\delta^1_{\grave{A}_i})^{w_i}, 1-\prod_{i=1}^{n}(1-\delta^2_{\grave{A}_i})^{w_i}, \cdots, 1-\prod_{i=1}^{n}(1-\delta^P_{\grave{A}_i})^{w_i}\rangle\\ =& \langle(1-\prod_{i=1}^{n}(1-a_i)^{w_i}, 1-\prod_{i=1}^{n}(1-b_i)^{w_i}, 1-\prod_{i=1}^{n}(1-c_i)^{w_i}, 1-\prod_{i=1}^{n}(1-d_i)^{w_i});\\ & 1-\prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}, 1-\prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}, \cdots, 1-\prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i},\\ & \prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}, \prod_{i=1}^{n}(\delta^2_{\grave{A}_i})^{w_i}, \cdots, \prod_{i=1}^{n}(\delta^P_{\grave{A}_i})^{w_i}\rangle\end{aligned}$$

Result 2 If $g(t) = log(\frac{2-t}{t})$, then the $\mathbb{TIFMA}$ operator reduces to the following:

$$\begin{aligned}\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) =& \langle(\frac{\prod_{i=1}^{n}(1+a_i)^{w_i}-\prod_{i=1}^{n}(1-a_i)^{w_i}}{\prod_{i=1}^{n}(1+a_i)^{w_i}+\prod_{i=1}^{n}(1-a_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+b_i)^{w_i}-\prod_{i=1}^{n}(1-b_i)^{w_i}}{\prod_{i=1}^{n}(1+b_i)^{w_i}+\prod_{i=1}^{n}(1-b_i)^{w_i}},\\ & \frac{\prod_{i=1}^{n}(1+c_i)^{w_i}-\prod_{i=1}^{n}(1-c_i)^{w_i}}{\prod_{i=1}^{n}(1+c_i)^{w_i}+\prod_{i=1}^{n}(1-c_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+d_i)^{w_i}-\prod_{i=1}^{n}(1-d_i)^{w_i}}{\prod_{i=1}^{n}(1+d_i)^{w_i}+\prod_{i=1}^{n}(1-d_i)^{w_i}});\\ & \frac{\prod_{i=1}^{n}(1+\rho^1_{\grave{A}_i})^{w_i}-\prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+\rho^1_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}}, \frac{\prod_{i=1}^{n}(1+\rho^2_{\grave{A}_i})^{w_i}-\prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+\rho^2_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}}, \cdots,\\ & \frac{\prod_{i=1}^{n}(1+\rho^P_{\grave{A}_i})^{w_i}-\prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+\rho^P_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i}}, (\frac{2\prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(2-\delta^1_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}},\\ & \frac{2\prod_{i=1}^{n}(\delta^2_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(2-\delta^2_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}}, \cdots, \frac{2\prod_{i=1}^{n}(\delta^P_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(2-\delta^P_{\grave{A}_i})^{w_i}+\prod_{i=1}^{n}(\delta^P_{\grave{A}_i})^{w_i}})\rangle\end{aligned}$$

which is called an *Einstein* $\mathbb{TIFMA}$*operator*.

Result 3 If $g(t) = log(\frac{\gamma+(1-\gamma)t}{t}), \gamma \in (0, +\infty)$, then the $\mathbb{TIFMA}$ operator reduces to the following:

$$
\begin{aligned}
&\mathbb{TIFMA}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n)\\
=&\langle(\frac{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{w_i} - \prod_{i=1}^{n}(1-a_i)^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)a_i)^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-a_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{w_i} - \prod_{i=1}^{n}(1-b_i)^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)b_i)^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-b_i)^{w_i}},\\
&\frac{\prod_{i=1}^{n}(1+(\gamma-1)c_i)^{w_i} - \prod_{i=1}^{n}(1-c_i)^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)c_i)^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-c_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+(\gamma-1)d_i)^{w_i} - \prod_{i=1}^{n}(1-d_i)^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)d_i)^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-d_i)^{w_i}});\\
&\frac{\prod_{i=1}^{n}(1+(\gamma-1)\rho^1_{\grave{A}_i})^{w_i} - \prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\rho^1_{\grave{A}_i})^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-\rho^1_{\grave{A}_i})^{w_i}}, \frac{\prod_{i=1}^{n}(1+(\gamma-1)\rho^2_{\grave{A}_i})^{w_i} - \prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\rho^2_{A_i})^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-\rho^2_{\grave{A}_i})^{w_i}}, \cdots,\\
&\frac{\prod_{i=1}^{n}(1+(\gamma-1)\rho^P_{\grave{A}_i})^{w_i} - \prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\rho^P_{\grave{A}_i})^{w_i} + (\gamma-1)\prod_{i=1}^{n}(1-\rho^P_{\grave{A}_i})^{w_i}}, \frac{\gamma\prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\delta^1_{\grave{A}_i}))^{w_i} + (\gamma-1)\prod_{i=1}^{n}(\delta^1_{\grave{A}_i})^{w_i}},\\
&\frac{\gamma\prod_{i=1}^{n}(\delta^2_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\delta^2_{\grave{A}_i}))^{w_i} + (\gamma-1)\prod_{i=1}^{n}(\delta^2_{\grave{A}_i})^{w_i}}, \cdots, \frac{\gamma\prod_{i=1}^{n}(\delta^P_{\grave{A}_i})^{w_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\delta^P_{\grave{A}_i}))^{w_i} + (\gamma-1)\prod_{i=1}^{n}(\delta^P_{\grave{A}_i})^{w_i}}\rangle
\end{aligned}
$$

which is called a *Hammer* $\mathbb{TIFMA}$ *operator*.

Result 4 If $g(t) = log(\frac{\gamma-1}{\gamma^t-1}), \gamma \in (1, +\infty)$, then the $\mathbb{TIFMA}$ operator reduces to the following:

$$
\begin{aligned}
\mathbb{ITMA}(A_1, A_2, \cdots, A_n) =&\langle(1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-a_i}-1)^{w_i}}{\gamma-1}), 1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-b_i}-1)^{w_i}}{\gamma-1}),\\
&1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-c_i}-1)^{w_i}}{\gamma-1}), 1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-d_i}-1)^{w_i}}{\gamma-1}));\\
&1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-\rho^1_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}), 1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-\rho^2_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}), \cdots,\\
&1 - log_\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{1-\rho^P_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}), \ln\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{\delta^1_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}),\\
&\ln\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{\delta^2_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}), \cdots, \ln\gamma(1 + \frac{\prod_{i=1}^{n}(\gamma^{\delta^P_{\grave{A}_i}}-1)^{w_i}}{\gamma-1}))\rangle
\end{aligned}
$$

which is called a *Frank* $\mathbb{TIFMA}$ *operator*.

We give following definition inspired by geometric mean.

Definition 3.7. Let $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i}) \rangle$ $(i = 1, 2, \cdots, n)$ be a collection of TIFM-numbers and $w = (w_1, w_2, ..., w_n)^T$ weight vector of $\grave{A}_i$ $(i = 1, 2, \cdots, n)$ such that $\sum_{i=1}^{n} w_i = 1$. If

$$\mathbb{TIFMG}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) = \bigotimes_{i=1}^{n} (\grave{A}_i)^{w_i},$$

then $\mathbb{TIFMG}$ is called an *Archimedean t-cornorm* and *t-norm* based intuitionistic trapezoidal fuzzy multi geometric ($\mathbb{TIFMG}$) operator.

Theorem 3.8. *Let* $\grave{A}_i = \langle (a_i, b_i, c_i, d_i); (\rho^1_{\grave{A}_i}, \rho^2_{\grave{A}_i}, \cdots, \rho^P_{\grave{A}_i}), (\delta^1_{\grave{A}_i}, \delta^2_{\grave{A}_i}, \cdots, \delta^P_{\grave{A}_i}) \rangle$ $(i = 1, 2)$ *be a collection of TIFM-numbers,* $\grave{A} = \langle (a, b, c, d); (\rho^1_{\grave{A}}, \rho^2_{\grave{A}}, \cdots, \rho^P_{\grave{A}}), (\delta^1_{\grave{A}}, \delta^2_{\grave{A}}, \cdots, \delta^P_{\grave{A}}) \rangle$ *be a TIFM-number and* λ, λ_1, $\lambda_2 > 0$ *.The aggregated value by using the* $\mathbb{TIFMA}$ *operator is also an IFV, and The aggregated value by using the* $\mathbb{TIFMG}$ *operator is also a TIFM-number and*

$$\begin{aligned}\mathbb{TIFMG}(\grave{A}_1, \grave{A}_2, \cdots, \grave{A}_n) = &\bigotimes_{i=1}^{n} (\grave{A}_i)^{w_i} \\ =&\langle (g^{-1}(\sum_{i=1}^{n} w_i g(a_i)), g^{-1}(\sum_{i=1}^{n} w_i g(b_i)), g^{-1}(\sum_{i=1}^{n} w_i g(c_i)), g^{-1}(\sum_{i=1}^{n} w_i g(d_i))); \\ &g^{-1}(\sum_{i=1}^{n} w_i g(\rho^1_{\grave{A}_i})), g^{-1}(\sum_{i=1}^{n} w_i g(\rho^2_{\grave{A}_i})), \cdots, g^{-1}(\sum_{i=1}^{n} w_i g(\rho^P_{\grave{A}_i}), \\ &h^{-1}(\sum_{i=1}^{n} w_i h(\delta^1_{\grave{A}_i})), h^{-1}(\sum_{i=1}^{n} w_i h(\delta^2_{\grave{A}_i})), \cdots, h^{-1}(\sum_{i=1}^{n} w_i h(\delta^P_{\grave{A}_i})))\rangle.\end{aligned}$$

Proof. The operator can be easily proven similar to the $\mathbb{TIFMA}$ operator. This is why no need to prove $\mathbb{TIFMG}$ operator here again. □

If the additive generator g is assigned different forms then following intuitionistic trapezoidal fuzzy aggregation operators can be obtained (See [55]).

Result 1 If $g(t) = -logt$, then the $\mathbb{TIFMG}$ operator reduces to following:

$$\begin{aligned}\mathbb{TIFMG}(\grave{A}_1, \grave{A}_2, ..., \grave{A}_n) =&\langle (\prod_{i=1}^{n} (a_i)^{w_i}, \prod_{i=1}^{n} (b_i)^{w_i}, \prod_{i=1}^{n} (c_i)^{w_i}, \prod_{i=1}^{n} (d_i)^{w_i}); \\ &\prod_{i=1}^{n} (\rho^1_{\grave{A}_i})^{w_i}, \prod_{i=1}^{n} (\rho^2_{\grave{A}_i})^{w_i}, ..., \prod_{i=1}^{n} (\rho^P_{\grave{A}_i})^{w_i}, \\ &1 - \prod_{i=1}^{n} (1 - \delta^1_{\grave{A}_i})^{w_i}, 1 - \prod_{i=1}^{n} (1 - \delta^2_{\grave{A}_i})^{w_i}, ..., 1 - \prod_{i=1}^{n} (1 - \delta^P_{\grave{A}_i})^{w_i} \rangle.\end{aligned}$$

Similarly, some new operators based on $log(\frac{2-t}{t})$, $g(t) = log(\frac{\gamma+(1-\gamma)t}{t}), \gamma \in (0, +\infty)$, $g(t) = log(\frac{\gamma-1}{\gamma^t-1}), \gamma \in (1, +\infty)$ can be given.

4. An approach to trapezoidal fuzzy multi-attribute Decision Making

For a decision making problem under intuitionistic trapezoidal fuzzy multi environment, let $\dot{X}$ be set of alternatives and U be set of attributes. In order to evaluate the performance of the alternative $\dot{X}_i$ with regards to u_j ,an expert is required to provide information that the alternative $\dot{X}_i$ satisfies the attribute u_j. This information can be expressed by $\alpha_{ij} = \langle(a_{ij}, b_{ij}, c_{ij}, d_{ij}); (\rho^1_{\alpha_{ij}}, \rho^2_{\alpha_{ij}}, \cdots, \rho^P_{\alpha_{ij}}), (\delta^1_{\alpha_{ij}}, \delta^2_{\alpha_{ij}}, \cdots, \delta^P_{\alpha_{ij}})\rangle$ which denote the degrees that the alternative $\dot{X}_i$ satisfies the attribute u_j with the condition that $0 < \rho^1_{\alpha_{ij}}, \rho^2_{\alpha_{ij}}, ..., \rho^P_{\alpha_{ij}} < 1$ and $0 < \delta^1_{\alpha_{ij}}, \delta^2_{\alpha_{ij}}, ..., \delta^P_{\alpha_{ij}} < 1$. When all the performances of the alternatives are provided, the intuitionistic trapezoidal fuzzy multi decision matrix $C = (\alpha_{ij})_{mxn}$ can be constructed. To obtain the ranking of the alternatives, the steps can be given as:

Step 1 Construct intuitionistic trapezoidal fuzzy multi decision matrix $C = (\alpha_{ij})_{mxn}$ based on Table 1:

Table 1. TIFM-numbers for linguistic terms [36]

Linguistic terms	Linguistic values of TIFM-numbers
Absolutely low(AL)	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
Very Very Low(VVL)	$\langle(0.05, 0.10, 0.15, 0.20); (0.1, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.1)\rangle$
Very Low(VL)	$\langle(0.15, 0.20, 0.25, 0.40); (0.2, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
Low(L)	$\langle(0.10, 0.20, 0.25, 0.30); (0.1, 0.3, 0.3, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
Fairly low(FL)	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
Medium(M)	$\langle(0.25, 0.30, 0.35, 0.40); (0.1, 0.3, 0.7, 0.4), (0.6, 0.3, 0.4, 0.7)\rangle$
Fairly high(FH)	$\langle(0.25, 0.35, 0.40, 0.45); (0.1, 0.3, 0.2, 0.5), (0.2, 0.5, 0.4, 0.1)\rangle$
High(H)	$\langle(0.40, 0.45, 0.50, 0.55); (0.1, 0.3, 0.2, 0.9), (0.2, 0.5, 0.3, 0.1)\rangle$
Very High(VH)	$\langle(0.45, 0.55, 0.65, 0.75); (0.1, 0.6, 0.2, 0.4), (0.6, 0.3, 0.4, 0.1)\rangle$
Very Very High(VVH)	$\langle(0.50, 0.65, 0.70, 0.80); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
Absolutely high(AH)	$\langle(0.70, 0.85, 0.90, 1.00; (0.1, 0.3, 0.3, 0.4), (0.1, 0.3, 0.4, 0.5)\rangle$

Step 2 Transform the intuitionistic trapezoidal fuzzy multi decision matrix $C = (c_{ij})_{mxn}$ into the normalized intuitionistic trapezoidal fuzzy multi decision matrix $K = (k_{ij})_{mxn}$, where

$$k_{ij} = \begin{cases} \langle(a_{ij}, b_{ij}, c_{ij}, d_{ij}); (\rho^1_{\alpha_{ij}}, \rho^2_{\alpha_{ij}}, \cdots, \rho^P_{\alpha_{ij}}), (\delta^1_{\alpha_{ij}}, \delta^2_{\alpha_{ij}}, \cdots, \delta^P_{\alpha_{ij}})\rangle, & \text{for benefit attribute } u_j; \\ \langle(a_{ij}, b_{ij}, c_{ij}, d_{ij}); (\delta^1_{\alpha_{ij}}, \delta^2_{\alpha_{ij}}, \cdots, \delta^P_{\alpha_{ij}}), (\rho^1_{\alpha_{ij}}, \rho^2_{\alpha_{ij}}, \cdots, \rho^P_{\alpha_{ij}})\rangle, & \text{for cost attribute } u_j. \end{cases}$$

Step 3 Insert the weigted vector $w = (w_1, w_2, ..., w_n)$ such that $w_j \geq 0$ $j = (1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$. In here, w_j is weigted of u_j $j = (1, 2, \cdots, n)$;

Step 4 Obtain values b_i $(i = 1, 2, \cdots, m)$ for $\dot{X}_i$ by the operators as:
$b_i = \mathbb{TIFMA}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in})$ or $b_i = \mathbb{TIFMG}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}) i = 1, 2, \cdots, m$

Step 5 Calculate the score values $\bar{S}_p(b_i)$(p=1,2,3,4) $i = (1, 2, \cdots, m)$ and rank the alternatives. The alternative with the highest score value is the best alternative.

Illustrative Example

Example 4.1. Assume that $\dot{X} = \{\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5\}$ be production of five different solar power plants depends on seasonal conditions. Then, we examine their performance according to the criterias set $U = \{u_1 = \text{wind speed}, u_2 = \text{cloudiness}, u_3 = \text{amount of solar radiation on the earth}, u_4 = \text{surface angle of the panels}\}$. In here, 4 different membership function was used for different season (spring, summer, autumn, winter). The solutions were given as follows:

Step 1 We constructed intuitionistic trapezoidal fuzzy multi decision matrix $C = (\alpha_{ij})_{mxn}$ based on Table 1 as:

Table 2. The intuitionistic trapezoidal fuzzy multi decision matrix $C = (\alpha_{ij})_{mxn}$

	u_1
$\dot{X}_1$	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_2$	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
$\dot{X}_3$	$\langle(0.25, 0.30, 0.35, 0.40); (0.1, 0.3, 0.7, 0.4), (0.6, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_4$	$\langle(0.45, 0.55, 0.65, 0.75); (0.1, 0.6, 0.2, 0.4), (0.6, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_5$	$\langle(0.10, 0.20, 0.25, 0.30); (0.1, 0.3, 0.3, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
	u_2
$\dot{X}_1$	$\langle(0.15, 0.20, 0.25, 0.40); (0.2, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
$\dot{X}_2$	$\langle(0.05, 0.10, 0.15, 0.20); (0.1, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_3$	$\langle(0.05, 0.10, 0.15, 0.20); (0.1, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_4$	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
$\dot{X}_5$	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
	u_3
$\dot{X}_1$	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
$\dot{X}_2$	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_3$	$\langle(0.25, 0.35, 0.40, 0.45); (0.1, 0.3, 0.2, 0.5), (0.2, 0.5, 0.4, 0.1)\rangle$
$\dot{X}_4$	$\langle(0.70, 0.85, 0.90, 1.00); (0.1, 0.3, 0.3, 0.4), (0.1, 0.3, 0.4, 0.5)\rangle$
$\dot{X}_5$	$\langle(0.45, 0.55, 0.65, 0.75); (0.1, 0.6, 0.2, 0.4), (0.6, 0.3, 0.4, 0.1)\rangle$
	u_4
$\dot{X}_1$	$\langle(0.10, 0.20, 0.25, 0.30); (0.1, 0.3, 0.3, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_2$	$\langle(0.15, 0.20, 0.25, 0.40); (0.2, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
$\dot{X}_3$	$\langle(0.40, 0.45, 0.50, 0.55); (0.1, 0.3, 0.2, 0.9), (0.2, 0.5, 0.3, 0.1)\rangle$
$\dot{X}_4$	$\langle(0.50, 0.65, 0.70, 0.80); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_5$	$\langle(0.05, 0.10, 0.15, 0.20); (0.1, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.1)\rangle$

Step 2 We transformed the intuitionistic trapezoidal fuzzy multi decision matrix $C = (c_{ij})_{mxn}$ into the normalized intuitionistic trapezoidal fuzzy multi decision matrix $K = (k_{ij})_{mxn}$ as:

TABLE 3. The normalized intuitionistic trapezoidal fuzzy multi decision matrix $K = (k_{ij})_{mxn}$

	u_1
$\dot{X}_1$	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_2$	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
$\dot{X}_3$	$\langle(0.25, 0.30, 0.35, 0.40); (0.1, 0.3, 0.7, 0.4), (0.6, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_4$	$\langle(0.45, 0.55, 0.65, 0.75); (0.1, 0.6, 0.2, 0.4), (0.6, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_5$	$\langle(0.10, 0.20, 0.25, 0.30); (0.1, 0.3, 0.3, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
	u_2
$\dot{X}_1$	$\langle(0.15, 0.20, 0.25, 0.40); (0.2, 0.3, 0.4, 0.5), (0.2, 0.3, 0.5, 0.4)\rangle$
$\dot{X}_2$	$\langle(0.05, 0.10, 0.15, 0.20); (0.2, 0.3, 0.4, 0.1), (0.1, 0.3, 0.5, 0.4)\rangle$
$\dot{X}_3$	$\langle(0.05, 0.10, 0.15, 0.20); (0.2, 0.3, 0.4, 0.1), (0.1, 0.3, 0.5, 0.4)\rangle$
$\dot{X}_4$	$\langle(0.15, 0.20, 0.25, 0.30); (0.2, 0.7, 0.4, 0.3), (0.1, 0.5, 0.2, 0.4)\rangle$
$\dot{X}_5$	$\langle(0.01, 0.05, 0.10, 0.15); (0.5, 0.3, 0.4, 0.7), (0.1, 0.3, 0.2, 0.4)\rangle$
	u_3
$\dot{X}_1$	$\langle(0.15, 0.20, 0.25, 0.30); (0.1, 0.5, 0.2, 0.4), (0.2, 0.7, 0.4, 0.3\rangle$
$\dot{X}_2$	$\langle(0.01, 0.05, 0.10, 0.15); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.7)\rangle$
$\dot{X}_3$	$\langle(0.25, 0.35, 0.40, 0.45); (0.1, 0.3, 0.2, 0.5), (0.2, 0.5, 0.4, 0.1)\rangle$
$\dot{X}_4$	$\langle(0.70, 0.85, 0.90, 1.00; (0.1, 0.3, 0.3, 0.4), (0.1, 0.3, 0.4, 0.5)\rangle$
$\dot{X}_5$	$\langle(0.45, 0.55, 0.65, 0.75); (0.1, 0.6, 0.2, 0.4), (0.6, 0.3, 0.4, 0.1)\rangle$
	u_4
$\dot{X}_1$	$\langle(0.10, 0.20, 0.25, 0.30); (0.1, 0.3, 0.3, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_2$	$\langle(0.15, 0.20, 0.25, 0.40); (0.2, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle$
$\dot{X}_3$	$\langle(0.40, 0.45, 0.50, 0.55); (0.1, 0.3, 0.2, 0.9), (0.2, 0.5, 0.3, 0.1)\rangle$
$\dot{X}_4$	$\langle(0.50, 0.65, 0.70, 0.80); (0.1, 0.3, 0.2, 0.4), (0.5, 0.3, 0.4, 0.1)\rangle$
$\dot{X}_5$	$\langle(0.05, 0.10, 0.15, 0.20); (0.1, 0.3, 0.5, 0.4), (0.2, 0.3, 0.4, 0.1)\rangle$

Step 3 We inserted the weigted vector $w = (w_1, w_2, \cdots, w_n)$ such that $w_j \geq 0$ $j = (1, 2, \cdots, n)$ and $\sum\limits_{j=1}^{n} w_j = 1$ as: $w = (0.4, 0.3, 0.2, 0.1)$.

Step 4 We obtained values b_i $(i = 1, 2, \cdots, 5)$ for $\dot{X}_i$ by the $\mathbb{TIFMA}$ operators as:

$$b_i = \mathbb{TIFMA}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}) = (\bigoplus_{j=1}^{n})(w_i \odot \alpha_{ij}) \quad i = 1, 2, \cdots, 5 \tag{4.1}$$

$$b_i = \mathbb{TIFMG}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}) = (\bigotimes_{i=1}^{n})(\alpha_{ij})^{w_i} \quad i = 1, 2, \cdots, 5. \tag{4.2}$$

or

Table 4. The values based on $\mathbb{TIFMA}$

b_1	$\langle(0.0914, 0.1431, 0.1933, 0.2777); (0.1312, 0.3456, 0.3144, 0.4000), (0.3162, 0.3554, 0.4000, 0.4397)\rangle$
b_2	$\langle(0.0939, 0.1422, 0.1924, 0.2541); (0.1105, 0.3881, 0.3371, 0.4000), (0.2402, 0.4210, 0.4000, 0.2690)\rangle$
b_3	$\langle(0.2126, 0.2740, 0.3247, 0.3754); (0.1000, 0.3000, 0.5307, 0.5164), (0.3104, 0.3497, 0.3887, 0.2178)\rangle$
b_4	$\langle(0.4501, 0.5814, 0.6628, 0.7941); (0.1000, 0.4941, 0.2211, 0.4000), (0.3757, 0.3484, 0.2155, 0.3492)\rangle$
b_5	$\langle(0.1562, 0.2403, 0.3113, 0.3880); (0.1000, 0.3741, 0.2764, 0.4000), (0.4732, 0.3000, 0.4000, 0.1793)\rangle$

Table 5. The values based on $\mathbb{TIFMG}$

b_1	$\langle(0.0488, 0.1149, 0.1733, 0.2479); (0.1231, 0.3323, 0.2564, 0.4277), (0.2367, 0.4091, 0.4319, 0.5116)\rangle$
b_2	$\langle(0.0628, 0.1231, 0.1786, 0.2380); (0.1320, 0.3680, 0.2699, 0.2639), (0.2456, 0.5012, 0.4000, 0.4835)\rangle$
b_3	$\langle(0.1617, 0.2317, 0.2889, 0.3434); (0.1231, 0.3000, 0.4064, 0.2993), (0.3719, 0.3672, 0.4231, 0.4865)\rangle$
b_4	$\langle(0.3573, 0.4504, 0.5247, 0.6073); (0.1231, 0.5104, 0.2670, 0.3669), (0.3865, 0.3672, 0.3459, 0.2915)\rangle$
b_5	$\langle(0.0632, 0.1507, 0.2185, 0.2811); (0.1621, 0.3446, 0.3174, 0.4731), (0.4022, 0.3000, 0.3459, 0.2031)\rangle$

Step 5 We calculate the score values based on Table 4, $\bar{S}_1(b_i)$ $i = (1, 2, \cdots, 5)$ as:

$\bar{S}_1(b_1) = 0.2304$

$\bar{S}_1(b_2) = 0.2094$

$\bar{S}_1(b_3) = 0.2214$

$\bar{S}_1(b_4) = 0.2238$

$\bar{S}_1(b_5) = 0.2974$

and all the alternatives ranked as; $b_5 > b_1 > b_4 > b_3 > b_2$.

Similarity, based on Table 5, we have

$\bar{S}^1_1(b_1) = 0.2463$

$\bar{S}^1_1(b_2) = 0.2168$

$\bar{S}^1_1(b_3) = 0.2277$

$\bar{S}^1_1(b_4) = 0.1947$

$\bar{S}^1_1(b_5) = 0.2882$

and all the alternatives ranked as; $b_5 > b_1 > b_3 > b_2 > b_4$

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İRFAN DELI (irfandeli@kilis.edu.tr)
KILISLI MUALLIM RIFAT FACULTY OF EDUCATION, 7 ARALIK UNIVERSITY, 79000 KILIS, TÜRKIYE, irfandeli@kilis.edu.tr

HÜMEYRA KARADÖL (kara46israfil@gmail.com)
Kilisli Muallim Rıfat Faculty of Education, 7 Aralık University, 79000 Kilis, Türkiye, kara46israfil@gmail.com