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Some remarks on fuzzy open hereditarily irresolvable spaces

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Abstract. In this paper, the notion of fuzzy open hereditarily irresolvability of fuzzy topological spaces is characterized by means of fuzzy simply*-open sets, fuzzy B*-sets and fuzzy pre-open sets possessing fuzzy Baire property. A condition for fuzzy open hereditarily irresolvable spaces to become fuzzy quasi-submaximal spaces is obtained. Also conditions under which fuzzy D-Baire spaces and fuzzy semi-P-spaces become fuzzy open hereditarily irresolvable spaces are obtained. A condition under which fuzzy open hereditarily irresolvability coincides with fuzzy strong irresolvability of fuzzy topological spaces is obtained.

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Keywords: Fuzzy somewhere dense set, Fuzzy simply*-open set, Fuzzy B*-set, Fuzzy quasi-submaximal space, Weak fuzzy O_z-space, Fuzzy nodec space, Fuzzy weakly Baire space, Fuzzy GID space, Fuzzy semi-P-space.

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1. Introduction

The concept of fuzzy sets as a new approach for modelling uncertainties was introduced by Zadeh [1] in the year 1965. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. In 1968, Chang [2]introduced the concept of fuzzy topological space. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

At the beginning of twentieth century, the problem of resolvability of a topological space became a matter of intense research. Research in this area stems from the papers of Hewitt[3] and Katetov[4]. In 1969, El'Kin [5] introduced open hereditarily irresolvable spaces in classical topology. Motivated by the above works on resolvability, the concepts of resolvability, irresolvability and open hereditarily irresolvability of fuzzy topological spaces were introduced and studied by Thangaraj and Balasubramanian[6] in 2002. The notion of fuzzy simply* open sets by means of fuzzy open sets and fuzzy nowhere dense sets in fuzzy topological spaces was introduced and studied by Thangaraj and Dinakaran [7]. The notion of fuzzy B*-sets in fuzzy topological spaces was introduced ands tudied by Thangaraj and Dharmasaraswathi [8].

The purpose of this paper is to study more deeply the notion of fuzzy open hereditarily irresolvable spaces. In this paper, the notion of fuzzy open hereditarily irresolvability of fuzzy topological spaces is characterized by means of fuzzy simply*-open sets, fuzzy B*-sets and fuzzy pre-open sets possessing fuzzy Baire property. It is found that fuzzy dense sets are fuzzy α-open sets and fuzzy nowhere dense sets are fuzzy α-closed sets in fuzzy open hereditarily irresolvable spaces. It is established that fuzzy open hereditarily irresolvable and fuzzy nodec space are fuzzy quasi-submaximal spaces and fuzzy open hereditarily irresolvable and fuzzy nodef spaces, are fuzzy DG_δ-spaces. The conditions under which fuzzy D-Baire spaces and fuzzy semi-P-spaces become fuzzy open hereditarily irresolvable spaces are also obtained. It is established that fuzzy weakly Baire and weak fuzzy Oz-spaces are not fuzzy open hereditarily irresolvable spaces. It is also obtained that fuzzy open hereditarily irresolvable spaces are fuzzy GID spaces. A condition under which fuzzy open hereditarily irresolvability coincides with fuzzy strong irresolvability of fuzzy topological spaces is obtained in this work.

In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications [9, 10, 11, 12]. Many authors redefined the classical topological concepts via soft topological structure. Recently, Şenel et al. [13] applied the concept of octahedron sets proposed by Lee et al. [14] to multi-criteria group decision making problems. Octahedron sets are a very useful generalization of fuzzy sets where one is allowed to extend the output through a subinterval of [0,1] and a number from [0,1]. Lee et al. [15] studied neighborhood structures, closures and interiors, and continuities based on cubic sets. On these lines, there is a need and scope of investigation considering different types of fuzzy sets such as fuzzy Baire sets, fuzzy simply* open sets for applying some fuzzy topological concepts to information science and decision-making problems.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self - contained. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0,1]. A *fuzzy set* λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$. For any fuzzy set λ and a family $(\lambda_j)_{j\in J}$ of fuzzy sets in X, where J denotes an index set, the *complement* $\lambda^{'}$ of λ, the *intersection* $\bigwedge_{j\in J}\lambda_j$ and the *union* $\bigvee_{j\in J}\lambda_j$ of $(\lambda_j)_{j\in J}$ (See [1]) respectively defined as follows: for each $x \in X$,

(i) $\lambda^{'}(x) = 1 - \lambda(x)$,
(ii) $\bigwedge_{j\in J}\lambda_j = inf_{j\in J}\lambda_j(x)$,
(iii) $\bigvee_{j\in J}\lambda_j = sup_{j\in J}\lambda_j(x)$.

Definition 2.1 ([2])**.** A *fuzzy topology* is a family T of fuzzy sets in X which satisfies the following conditions:

(i) $0_X \in T$ and $1_X \in T$,
(ii) if $\lambda,\ \mu \in T$, then $\lambda \wedge \mu \in T$,
(iii) if $\lambda_j \in T$ for each $j \in J$, then $\bigvee_j \lambda_i \in T$, where J is an index set.

The pair (X, T) is called a *fuzzy topological space* (briefly, fts). Members of T are called *fuzzy open sets* of X and their complements are *fuzzy closed sets*.

Definition 2.2 ([2])**.** Let (X, T) be an fts and λ be any fuzzy set in X. Then the *interior* and the *closure* of λ are defined respectively as follows:

(i) $int(\lambda) = \bigvee\{\mu : \mu \leq \lambda, \mu \in T\}$,
(ii) $cl(\lambda) = \bigwedge\{\mu : \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([16])**.** *For a fuzzy set λ of an fts X,*

(1) $1 - int(\lambda) = cl(1 - \lambda)$,
(2) $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.4. A fuzzy set λ in an fts (X, T) is called a:

(i) *fuzzy regular-open*, if $\lambda = intcl(\lambda)$ and *fuzzy regular-closed* if $\lambda = clint(\lambda)$ [16],
(ii) *fuzzy semi-open*, if $\lambda \leq clint(\lambda)$ and *fuzzy semi-closed*, if $intcl(\lambda) \leq \lambda$ [16],
(iii) *fuzzy pre-open*, if $\lambda \leq intcl(\lambda)$ and *fuzzy pre-closed*, if $clint(\lambda) \leq \lambda$ [17],
(iv) *fuzzy α-open*, if $\lambda \leq intclint(\lambda)$ and *fuzzy α-closed*, if $clintcl(\lambda) \leq \lambda$ [17],
(v) *fuzzy β-open*, if $\lambda \leq clintcl(\lambda)$ and *fuzzy β-closed*, if $intclint(\lambda) \leq \lambda$ [18],
(vi) *fuzzy G_δ-set*, in X if $\lambda = \bigwedge_{j=1}^{\infty} \lambda_j$, where $\lambda_j \in T$ for $j \in J$ and *fuzzy F_σ-set* in X, if $\lambda = \bigvee_{i=1}^{\infty} \lambda_j$, where $1 - \lambda_j \in T$ for $j \in J$ [19],
(vii) *fuzzy dense set* in X, if there exists no fuzzy closed set μ in X such that $\lambda < \mu < 1$, i.e.,, $cl(\lambda) = 1$ [20],
(viii) *fuzzy nowhere dense set* in X, if there exists no non-zero fuzzy open set μ in X such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) = 0$ [20],
(ix) *fuzzy first category set* in X, if $\lambda = \bigvee_{j=1}^{\infty} \lambda_j$, where each λ_j is a fuzzy nowhere dense set in X and any other fuzzy set in X is said to be of *fuzzy second category* [20],
(x) *fuzzy residual set* in X, if $1 - \lambda$ is a fuzzy first category set in X [21],
(xi) *fuzzy simply open set* in X, if $Bd(\lambda)$ is a fuzzy nowhere dense set in X, i.e., $cl(\lambda) \wedge cl(1 - \lambda)$ is a fuzzy nowhere dense set in X [22],
(xii) *fuzzy somewhere dense set* in X, if there exists a non-zero fuzzy open set μ in X such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) \neq 0$ and $1 - \lambda$ is called a *fuzzy cs dense set* in X [23],
(xiii) *fuzzy simply* open set* in X, if $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X and $1 - \lambda$ is called a *fuzzy simply* closed set* in X [7],
(xiv) *fuzzy pseudo-open set* in X, if $\lambda = \mu \vee \delta$, where μ is a non-zero fuzzy open set and δ is a fuzzy first category set in X [24],
(xv) *fuzzy σ-boundary set* in X, if $\lambda = \bigvee_{j=1}^{\infty} \mu_j$, where $\mu_j = cl(\lambda_j) \wedge (1 - \lambda_j)$ andeach λ_j is a fuzzy regular open set in X [25].

Definition 2.5 ([8]). A fuzzy set λ in an fts (X, T) is called a *fuzzy B^* set*, if λ is a fuzzy set with fuzzy Baire property in X such that $intcl(\lambda) \neq 0$.

Definition 2.6 ([26]). Let (X, T) be an fts. Then a fuzzy set λ in X is said to *have the property of fuzzy Baire*, if $\lambda = (\mu \wedge \delta) \vee \eta$, where μ is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in X.

Definition 2.7. An fts (X, T) is called a:

(i) *fuzzy submaximal space*, if for each fuzzy set λ in X) such that $cl(\lambda) = 1$, $\lambda \in T$ [19],

(ii) *fuzzy strongly irresolvable space*, if for each fuzzy set λ in X, $cl[int(\lambda) \vee int(1 - \lambda)] = 1$ [27],

(iii) *fuzzy globally disconnected space*, if each fuzzy semi-open set in X is fuzzy open in X [28],

(iv) *fuzzy nodec space*, if each fuzzy nowhere dense set is a fuzzy closed set in X [29],

(v) *fuzzy nodef space*, if each fuzzy nowhere dense set is a fuzzy F_σ-set in X [30],

(vi) *fuzzy quasi-submaximal space*, if for each fuzzy dense set λ in X, the fuzzy boundary of λ is a fuzzy nowhere dense set in X [31],

(vii) *fuzzy DG_δ-space*, if each fuzzy dense (but not fuzzy open) set in X is a fuzzy G_δ-set in X [30],

(viii) *fuzzy D-Baire space*, if every fuzzy first category set in X is a fuzzy nowhere dense set in X [32],

(ix) *fuzzy weakly Baire space*, if $int(\bigvee_{j=1}^{\infty} \mu_j)) = 0$, where $\mu_j = cl(\lambda_j) \wedge (1 - \lambda_j)$ and each λ_j is a fuzzy regular open sets in X [25],

(x) *weak fuzzy O_z-space*, if for each fuzzy F_σ-set δ in X, $cl(\delta)$ is a fuzzy G_δ-set in X [33],

(xi) *fuzzy almost P-space*, if for each non-zero fuzzy G_δ-set λ in X, $int(\lambda) \neq 0$ [34],

(xii) *fuzzy semi P-space*, if each fuzzy G_δ-set in X is a fuzzy semi-open set in X [35],

(xiii) *fuzzy GID space*, if for each fuzzy dense and fuzzy G_δ-set λ in X, $clint(\lambda) = 1$ [36],

(xiv) *fuzzy ultraconnected space*, if whenever λ and μ are two non-zero fuzzy closed sets in X, $\lambda \nleq 1 - \mu$ [37],

(xv) *fuzzy hyperconnected space*, if every non null fuzzy open set in X is fuzzy dense in X [38],

(xvi) *fuzzy perfectly disconnected space*, if for any two non-zero fuzzy sets λ and μ in X with $\lambda \leq 1 - \mu$, $cl(\lambda) \leq 1 - cl(\mu)$ [39].

Theorem 2.8 ([16]). *In an fts X,*

(1) *The closure of a fuzzy open set in $X$is a fuzzy regular closed set in X,*

(2). *The interior of a fuzzy closed set in X is a fuzzy regular open set in X.*

Theorem 2.9 ([7]). *If λ is a fuzzy simply* open set in an fts (X, T), then $int(\lambda) \neq 0$.*

Theorem 2.10 ([8]). *If λ is a B^*-set in an fts (X, T), then $int(\lambda) \neq 0$.*

Theorem 2.11 ([40]). *If λ is a non-zero fuzzy pre-open set with fuzzy Baire property in an fts (X, T), then λ is a fuzzy B^*-set in X.*

Theorem 2.12 ([6]). *If an fts (X, T) is a fuzzy open hereditarily irresolvable space, then $cl(\lambda) = 1$ implies that $clint(\lambda) = 1$, where λ is a non-zero fuzzy set in X.*

Theorem 2.13 ([41]). *If λ is a fuzzy dense and fuzzy G_δ-set in an fts (X, T), then λ is a fuzzy residual set in X.*

Theorem 2.14 ([7]). *If λ is a fuzzy simply* open set in an fts (X, T), then λ is not a fuzzy simply open set in X.*

Theorem 2.15 ([31]). *For an fts (X, T), the following are equivalent:*
1. *X is a fuzzy submaximal space,*
2. *each fuzzy pre-open set is an fuzzy open set in X.*

Theorem 2.16 ([7]). *If λ is a fuzzy simply* open set in an fts (X, T), then λ is not a fuzzy nowhere dense set in X.*

Theorem 2.17 ([39]). *In a fuzzy perfectly disconnected space (X, T), 0_X and 1_X are the only two fuzzy simply open sets in X.*

Theorem 2.18 ([25]). *Let (X, T) be an fts. Then the following are equivalent:*
1. *X is a fuzzy weakly Baire space,*
2. *$int(\lambda) = 0$ for every fuzzy σ-boundary set λ in X.*
3. *$cl(\mu) = 1$ for every fuzzy co-σ-boundary set μ in X.*

Theorem 2.19 ([33]). *If λ is a fuzzy σ-boundary set in a weak fuzzy O_z-space (X, T), then λ is a fuzzy somewhere dense set in X.*

Theorem 2.20 ([36]). *Let (X, T) be an fts. Then the following are equivalent:*
1. *(X, T) is a fuzzy GID space,*
2. *each fuzzy dense and fuzzy G_δ-set in X is fuzzy semi-open in X.*

Theorem 2.21 ([27]). *Let (X, T) be an fts. Then X is a fuzzy strongly irresolvable space if and only if each fuzzy set λ in X is a fuzzy simply open set in X.*

Theorem 2.22 ([7]). *If λ is a fuzzy simply* open set in an fts (X, T) such that $clint(\lambda) = 1$, then λ is a fuzzy simply open set in X.*

3. Fuzzy open hereditarily irresolvable spaces

In [6], the notion of fuzzy open hereditarily irresolvable spacesis introduced by means of fuzzy somewhere dense sets. In this section, several characterizations of fuzzy open hereditarily irresolvable spaces are established.

Definition 3.1 ([6]). An fts (X, T) is called a *fuzzy open hereditarily irresolvable space*, if $intcl(\lambda) \neq 0$ implies $int(\lambda) \neq 0$, where λ is a non-zero fuzzy set in X.

Proposition 3.2. *If each fuzzy set in an fts X is a fuzzy simply*-open set in X, then (X, T) is a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a fuzzy set in X and suppose λ is a fuzzy simply*-open set in X. Then $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X. On the other hand, we have

$$intcl(\mu \vee \delta) = int[cl(\mu) \vee cl(\delta)]$$

$$\geq intcl(\mu) \vee intcl(\delta)$$
$$= intcl(\mu) \vee 0$$
$$= intcl(\mu) \geq int(\mu)$$
$$= \mu.$$

Thus $intcl(\lambda) = intcl(\mu \vee \delta) \neq 0$. So λ is a fuzzy somewhere dense set in X. By Theorem 2.9, $int(\lambda) \neq 0$. Hence X is a fuzzy open hereditarily irresolvable space. □

Remark 3.3. It should be noted in view of Theorem 2.16, that a fuzzy set in X should not be a fuzzy nowhere dense set since fuzzy simply* open sets are not fuzzy nowhere dense sets in an fts.

Example 3.4. Let $X = \{a, b, c\}$ and $I = [0, 1]$. Consider the fuzzy sets $\alpha, \beta, \gamma, \delta, \theta, \eta, \omega$ and ρ in X defined as follows:

$$\alpha(a) = 0.2,\ \alpha(b) = 0.6,\ \alpha(c) = 1,\ \beta(a) = 1,\ \beta(b) = 0.8,\ \beta(c) = 0.4,$$

$$\gamma(a) = 1, \gamma(b) = 1,\ \gamma(c) = 0.6,\ \delta(a) = 0.8,\ \delta(b) = 0.6,\ \delta(c) = 1,$$

$$\theta(a) = 1,\ \theta(b) = 0.8,\ \theta(c) = 0.6,\ \eta(a) = 0.8,\ \eta(b) = 0.6,\ \eta(c) = 0.4,$$

$$\omega(a) = 0.2,\ \omega(b) = 0.6,\ \omega(c) = 0.6,\ \rho(a) = 0.8,\ \rho(b) = 0.6,\ \rho(c) = 0.6.$$

Then $T = \{0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1\}$ is a fuzzy topology on X. By computation, one can find that

$$int(1 - \alpha) = 0,\ int(1 - \beta) = 0,\ int(1 - [\alpha \vee \beta]) = 0,\ int(1 - [\alpha \wedge \beta]) = 0,$$

$$cl(\alpha) = 1,\ cl(\beta) = 1,\ cl(\alpha \vee \beta) = 1,\ cl(\alpha \wedge \beta) = 1,\ cl(\delta) = 1,\ cl(1 - \delta) = 1 - \beta.$$

Also, $intcl(1 - \delta) = int(1 - \beta) = 1 - cl(\beta) = 1 - 1 = 0$. Thus

$$1 - \alpha,\ 1 - \beta,\ 1 - [\alpha \vee \beta],\ 1 - [\alpha \wedge \beta],\ 1 - \delta$$

are fuzzy nowhere dense sets in X. On the other hand, we have

$$\alpha = \alpha \vee (1 - \beta),\ \beta = \beta \vee (1 - \alpha),$$
$$\alpha \vee \beta = (\alpha \vee \beta) \vee (1 - [\alpha \wedge \beta]),$$
$$\alpha \wedge \beta = (\alpha \wedge \beta) \vee (1 - [\alpha \vee \beta]),$$
$$\delta = \alpha \vee (1 - \alpha),\ \theta = \beta \vee (1 - \beta),$$
$$\eta = (\alpha \wedge \beta) \vee (1 - \alpha),$$
$$\omega = (\alpha \wedge \beta) \vee (1 - \beta),$$
$$\rho = (\alpha \wedge \beta) \vee (1 - [\alpha \wedge \beta]).$$

So each fuzzy set in X is a fuzzy simply*-open set in X. Hence (X, T) is a fuzzy open hereditarily irresolvable space. [For, $intcl(\lambda) \neq 0$ for any non-zero fuzzy set λ $(= \alpha, \beta, [\alpha \vee \beta], [\alpha \wedge \beta], \delta, \theta, \eta, \omega$ and $\rho)$ in X, $int(\lambda) \neq 0$ implies that (X, T) is a fuzzy open hereditarily irresolvable space.]

Proposition 3.5. *If each fuzzy set defined on X is a fuzzy B^*-set in an fts (X, T), then X is a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a fuzzy set in X and suppose λ is a fuzzy B^*-set in X. Then, λ is a fuzzy set with fuzzy Baire property in (X, T) such that $intcl(\lambda) \neq 0$. Now, $intcl(\lambda) \neq 0$ implies that λ is a fuzzy somewhere dense set in X. By Theorem 2.10, $int(\lambda) \neq 0$ for the fuzzy B^*-set λ in X. Thus for the fuzzy somewhere dense set λ, $int(\lambda) \neq 0$. So (X, T) is a fuzzy open hereditarily irresolvable space. □

Proposition 3.6. *If each non-zero fuzzy set in a fts X is a fuzzy pre-open set with fuzzy Baire property in X, then X is a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a non-zero fuzzy in X and suppose λ is a pre-open set in X with with fuzzy Baire property. Then $\lambda \leq intcl(\lambda)$. Thus $intcl(\lambda) \neq 0$. This implies that λ is a fuzzy somewhere dense set in X.Soand then λ is a fuzzy set in X with fuzzy Baire propert in (X, T) such that $intcl(\lambda) \neq 0$. By Theorem 2.11, λ is a fuzzy B^*-set in X. Hence λ is a fuzzy B^*-set in X. Therefore by Proposition 3.5, X is a fuzzy open hereditarily irresolvable space. □

Proposition 3.7. *If λ is a fuzzy dense set in a fuzzy open hereditarily irresolvable space (X, T), then λ is a fuzzy α-open set in X.*

Proof. Suppose λ is a fuzzy dense set in X. Then $cl(\lambda) = 1$. Since X is a fuzzy open hereditarily irresolvable space, by Theorem 2.12, $clint(\lambda) = 1$. Now $int[clint(\lambda)] = int[1] = 1$. Thus it follows clearly that $\lambda \leq intclint(\lambda)$. So λ is a fuzzy α-open set in X. □

Proposition 3.8. *If λ is a fuzzy nowhere dense set in a fuzzy open hereditarily irresolvable space (X, T), then λ is a fuzzy α-closed set in X.*

Proof. Suppose λ is a fuzzy nowhere dense set in X. Then $intcl(\lambda) = 0$. Since $int(\lambda) \leq intcl(\lambda), int(\lambda) = 0$, by Lemma 2.3, $cl(1 - \lambda) = 1 - int(\lambda) = 1$. Since X is a fuzzy open hereditarily irresolvable space, by Proposition 3.7, $1 - \lambda$ is a fuzzy α-open set in X. Thus λ is a fuzzy α-closed set in X. □

Remark 3.9. In view of Propositions 3.7 and 3.8, we have the following results.

(1) If λ is a fuzzy dense set in a fuzzy open hereditarily irresolvable space (X, T), then λ is a fuzzy semiopen set and a fuzzy pre-open set in X and then λ is a fuzzy β-open set in X.
(2) If λ is a fuzzy nowhere dense set in a fuzzy open hereditarily irresolvable space (X, T), then λ is a fuzzy semi-closed set and a fuzzy pre-closed set in X and thenλ is a fuzzy β-closed set in X.

Proposition 3.10. *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy open hereditarily irresolvable space (X, T), then λ is a fuzzy residual and fuzzy α-open set in X.*

Proof. Suppose λ is a fuzzy dense and fuzzy G_δ-set in X. Then by Theorem 2.13, λ is a fuzzy residual set in X. Since (X, T) is a fuzzy open hereditarily irresolvable space, by Proposition 3.7, λ is a fuzzy α-open set in X. Thus λ is a fuzzy residual and fuzzy α-open set in X. □

Proposition 3.11. *If a fuzzy set λ is a fuzzy pre-open set in a fuzzy open hereditarily irresolvable space (X, T), then $int(\lambda) \neq 0$.*

Proof. Suppose λ is a fuzzy pre-open set in X. Then $\lambda \leq intcl(\lambda)$ and $intcl(\lambda) \neq 0$ [For, $intcl(\lambda) = 0$ will imply that $\lambda = 0$, a contradiction]. Since X is a fuzzy open hereditarily irresolvable space, $intcl(\lambda) \neq 0$. Thus $int(\lambda) \neq 0$. □

Proposition 3.12. *If a fuzzy set λ is a fuzzy regular open set in a fuzzy open hereditarily irresolvable space (X, T), then $int(\lambda) \neq 0$.*

Proof. Suppose λ is a fuzzy regular open set in X. Then $\lambda = intcl(\lambda)$ and $intcl(\lambda) \neq 0$ [For, $intcl(\lambda) = 0$ will imply that $\lambda = 0$, a contradiction]. Since X is a fuzzy open hereditarily irresolvable space, $intcl(\lambda) \neq 0$. Thus $int(\lambda) \neq 0$. □

Proposition 3.13. *If a fuzzy set λ is a fuzzy regular closed set in a fuzzy open hereditarily irresolvable space (X, T), then $cl(\lambda) \neq 1$.*

Proof. Suppose λ is a fuzzy regular closed set in X. Then $1 - \lambda$ is a fuzzy regular open set in X. Since X is a fuzzy open hereditarily irresolvable space, by Proposition 3.12, $int(1 - \lambda) \neq 0$. By Lemma 2.3, $1 - cl(\lambda) \neq 0$. Thus $cl(\lambda) \neq 1$. □

4. Fuzzy open hereditarily irresolvable spaces and other fuzzy topological spaces

The following proposition gives a condition under which fuzzy dense sets in fuzzy open hereditarily irresolvable spaces are fuzzy simply open sets.

Proposition 4.1. *If λ is a fuzzy dense set in a fuzzy open hereditarily irresolvable and fuzzy nodec space (X, T), then λ is a fuzzy simply open set in X.*

Proof. Suppose λ is a fuzzy dense set in X. Then $cl(\lambda) = 1$. Since (X, T) is a fuzzy open hereditarily irresolvable space, by Theorem 2.12, $clint(\lambda) = 1$, i.e., $1 - clint(\lambda) = 0$. By Lemma 2.3, $intcl(1 - \lambda) = 0$. Thus $1 - \lambda$ is a fuzzy nowhere dense set in X. Since X) is a fuzzy nodec space, $1 - \lambda$ is a fuzzy closed set in X. So λ is a fuzzy open set in X. Hence λ is a fuzzy open and fuzzy dense set in X. On the other hand, we get

$$\begin{aligned} intcl[Bd(\lambda)] &= intcl[cl(\lambda) \wedge cl(1 - \lambda)] \\ &= intcl[1 \wedge cl(1 - \lambda)] \\ &= intcl[cl(1 - \lambda)] \\ &= intcl[1 - \lambda] \\ &= 1 - clint(\lambda) \\ &= 1 - cl(\lambda) \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Therefore λ is a fuzzy simply open set in X. □

Remark 4.2. In view of Proposition 4.1 and Proposition 3.7, one will have the following result: "Fuzzy dense sets in fuzzy open hereditarily irresolvable and fuzzy nodec spaces are fuzzy simply open and fuzzy α-open sets".

Proposition 4.3. *If (X, T) is a fuzzy open hereditarily irresolvable and fuzzy nodec space, then X is not a fuzzy perfectly disconnected space.*

Proof. Let $\lambda(\neq 1_X)$ be a fuzzy dense set in X. Since X is a fuzzy open hereditarily irresolvable and nodec space, by Proposition 4.1, λ is a fuzzy simply open set in X. Then, by Theorem 2.17, X is not a fuzzy perfectly disconnected space. □

Proposition 4.4. *If (X, T) is a fuzzy open hereditarily irresolvable and fuzzy nodec space, then X is a fuzzy quasi-submaximal space.*

Proof. Let λ be a fuzzy dense set in X. Since (X, T) is a fuzzy open hereditarily irresolvable and fuzzy nodec space, by Proposition 4.1, λ is a fuzzy simply open set in X. Then $intcl[Bd(\lambda)] = 0$. Thus the fuzzy boundary of λ is a fuzzy nowhere dense set in X. So X is a fuzzy quasi-submaximal space. □

It is ascertained that open hereditarily irresolvable spaces need not be fuzzy submaximal spaces. For, in Example 3.4, (X, T) is a fuzzy open hereditarily irresolvable space but not a fuzzy submaximal space, since $cl(\delta) = 1$ and δ is not a fuzzy open set in X.

The following proposition gives a condition under which open hereditarily irresolvable spaces become fuzzy submaximal spaces.

Proposition 4.5. *If (X, T) is a fuzzy open hereditarily irresolvable and fuzzy globally disconnected space, then (X, T) is a fuzzy submaximal space.*

Proof. Let λ be a fuzzy dense set in X. Then $cl(\lambda) = 1$. Since X is a fuzzy open hereditarily irresolvable space, by Theorem 2.12, $clint(\lambda) = 1$. Then it follows clearly that $\lambda \leq clint(\lambda)$. Thus λ is a fuzzy semi-open set in X. Since X is a fuzzy globally disconnected space, the fuzzy semi-open set λ is a fuzzy open set in X. Thus, λ is a fuzzy open set in X. So X is a fuzzy submaximal space. □

Remark 4.6. It should be noted that fuzzy submaximal spaces are fuzzy quasi-submaximal spaces but fuzzy quasi-submaximal spaces need not be fuzzy submaximal spaces [31] and fuzzy nodec spaces need not be globally disconnected spaces [28].

Proposition 4.7. *If λ is a fuzzy pre-open set in a fuzzy open hereditarily irresolvable and fuzzy globally disconnected space (X, T), then λ is a fuzzy open set in X.*

Proof. Suppose λ is a fuzzy pre-open set in X. Since X is a fuzzy open hereditarily irresolvable and fuzzy globally disconnected space, by Proposition 4.5, X is a fuzzy submaximal space. Then by Theorem 2.10, λ is a fuzzy open set in X. □

Proposition 4.8. *If (X, T) is a fuzzy open hereditarily irresolvable and fuzzy nodef space, then (X, T) is a fuzzy DG_δ-space.*

Proof. Let λ be a fuzzy dense (but not fuzzy open) set in X. Then $cl(\lambda) = 1$. Since X is a fuzzy open hereditarily irresolvable space, by Theorem 2.12, $clint(\lambda) = 1$, i.e., $1 - clint(\lambda) = 0$. By Lemma 2.3, $intcl(1 - \lambda) = 0$. Thus $1 - \lambda$ is a fuzzy nowhere dense set in X. Since X is a fuzzy nodef space, $1 - \lambda$ is a fuzzy F_σ-set in X. So λ is a fuzzy G_δ-set in X. Hence λ is a fuzzy G_δ-set in X. Therefore X is a fuzzy DG_δ-space. □

Proposition 4.9. *If λ is a fuzzy pseudo-open set in a fuzzy D-Baire space (X, T), then λ is a fuzzy simply*-open set in (X, T).*

Proof. Suppose λ is a fuzzy pseudo-open set in X. Then $\lambda = \mu \vee \delta$, where $\mu \in T$ and δ is a fuzzy first category set in X. Since (X, T) is a fuzzy D-Baire space, δ is a fuzzy nowhere dense set in X. Thus λ is a fuzzy simply* open set in X. □

It should be noted that fuzzy Baire and fuzzy open hereditarily irresolvable spaces are fuzzy D-Baire spaces and fuzzy D-Baire spaces are fuzzy Baire spaces [32]. The following proposition gives a condition for fuzzy D-Baire spaces to become fuzzy open hereditarily irresolvable spaces.

Proposition 4.10. *If each fuzzy set in a fuzzy D-Baire space (X, T) is a fuzzy pseudo-open set in X, then X is a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a fuzzy set in a fuzzy D-Baire space X and suppose λ is a fuzzy pseudo-open set in X. Since X is a fuzzy D-Baire space, by Proposition 4.9, λ is a fuzzy simply* open set in X. Then by Proposition 3.2, X is a fuzzy open hereditarily irresolvable space. □

Proposition 4.11. *If (X, T) is a fuzzy weakly Baire and weak fuzzy O_z-space, then X is not a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a fuzzy σ-boundary set in X. Since X is a weak fuzzy O_z-space, by Theorem 2.19, λ is a fuzzy somewhere dense set in X. Also, since (X, T) is a fuzzy weakly Baire sace, by Theorem 2.18, $\lambda, int(\lambda) = 0$. Then for the fuzzy somewhere dense set $\lambda, int(\lambda) = 0$. Thus X is not a fuzzy open hereditarily irresolvable space. □

The following proposition gives a condition under which fuzzy open hereditarily irresolvable spaces are becoming fuzzy almost P-spaces.

Proposition 4.12. *If each fuzzy G_δ-set is a fuzzy somewhere dense set in a fuzzy open hereditarily irresolvable space (X, T), then X is a fuzzy almost P-space.*

Proof. Let λ be a fuzzy G_δ-set in X and suppose λ is a fuzzy somewhere dense set in X. Since X is a fuzzy open hereditarily irresolvable space, $int(\lambda) \neq 0$. Then X is a fuzzy almost P-space. □

Proposition 4.13. *If each fuzzy dense set is a fuzzy G_δ-set in a fuzzy semi-P space (X, T), then X is a fuzzy open hereditarily irresolvable space.*

Proof. Let λ be a fuzzy somewhere dense set in X. Then. $intcl(\lambda) \neq 0$. Assume that $int(\lambda) = 0$. Then $cl(1 - \lambda) = 1 - int(\lambda) = 1$. Thus $1 - \lambda$ is a fuzzy dense set in X. By the hypothesis, $1 - \lambda$ is a fuzzy G_δ-set in the fuzzy semi-P space X. So $1 - \lambda$ is a fuzzy semi-open set in X. This implies that λ is a fuzzy semi-closed set in X and $intcl(\lambda) \leq \lambda$. Hence $int(intcl(\lambda)) \leq int(\lambda)$ and $intcl(\lambda) \leq 0$, i.e., $intcl(\lambda) = 0$. This is a contradiction. Therefore it must be that $int(\lambda) \neq 0$. It follows that (X, T) is a fuzzy open hereditarily irresolvable space. □

Proposition 4.14. *If (X, T) is a fuzzy open hereditarily irresolvable space, then X is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in X. Since X is a fuzzy open hereditarily irresolvable space, by Proposition 3.10, λ is a fuzzy residual and fuzzy α-open set in X. Then λ is a fuzzy semi-open set in X. Thus by Theorem 2.20, X is a fuzzy GID space. □

Proposition 4.15. *If each fuzzy set in a fts (X, T) is a fuzzy simply*-open set in X, then X is a fuzzy open hereditarily irresolvable space but not a fuzzy strongly irresolvable space.*

Proof. Let λ be a fuzzy set in X and suppose λ is a fuzzy simply*-open set in X. Then by Proposition 3.2, X is a fuzzy open hereditarily irresolvable space. Thus by Theorem 2.14, λ is not a fuzzy simply open set in X. So by Theorem 2.21, X is a not a fuzzy strongly irresolvable space. □

The following proposition gives a condition under which fuzzy open hereditarily irresolvability coincides with fuzzy strong irresolvability of fuzzy topological spaces.

Proposition 4.16. *If each fuzzy set in a fts (X, T) is a fuzzy simply*-open set in X such that $clint(\lambda) = 1$, then X is a fuzzy open hereditarily irresolvable and fuzzy strongly irresolvable space.*

Proof. Let λ be a fuzzy set in X and suppose λ is a fuzzy simply*-open set in X such that $clint(\lambda) = 1$. Then by Proposition 3.2, X is a fuzzy open hereditarily irresolvable space. Thus by Theorem 2.22, λ is a fuzzy simply open set in X. So by Theorem 2.21, X is a fuzzy strongly irresolvable space. □

It is established in [37] that fuzzy ultraconnected(but not fuzzy hyperconnected) spaces are fuzzy open hereditarily irresolvable spaces and also shown by an example that a fuzzy open openhereditarily irresolvable space need not be a fuzzy ultraconnected space. In this regard, an open question arises: Under what conditions does a fuzzy open hereditarily irresolvable space be a fuzzy ultraconnected space?

5. Conclusion

In this paper, the notion of fuzzy open hereditarily irresolvability of fuzzy topological spaces is characterized by means of fuzzy simply* open sets, fuzzy B^*-sets and fuzzy pre-open sets possessing fuzzy Baire property. It is found that fuzzy dense sets are fuzzy α-open sets and fuzzy nowhere dense sets are fuzzy α-closed sets in fuzzy open hereditarily irresolvable spaces. A condition under which fuzzy open hereditarily irresolvable spaces become fuzzy submaximal spaces is obtained. It is established that fuzzy open hereditarily irresolvable and fuzzy nodec spaces, are fuzzy quasi-submaximal spaces and fuzzy open hereditarily irresolvable and fuzzy nodef spaces, are fuzzy DG_δ-spaces. The conditions under which fuzzy D-Baire spaces and fuzzy semi-P-spaces become fuzzy open hereditarily irresolvable spaces are also obtained. It is established that fuzzy weakly Baire and weak fuzzy O_z-spaces are not fuzzy open hereditarily irresolvable spaces. It is also obtained that fuzzy open hereditarily irresolvable spaces are fuzzy GID spaces. A condition under which fuzzy open hereditarily irresolvability coincides with fuzzy strong irresolvability of fuzzy topological spaces is also obtained. New notions of fuzzy Brown spaces and fuzzy ultraconnected spaces have to be studied by incorporating the notion of fuzzy open hereditarily irresolvable spaces for future research.

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