

Power aggregation operators on trapezoidal fuzzy multi-numbers and their applications to a zero-waste problem

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ABSTRACT. In daily life, when conventional sets are insufficient in handling problems with multi-criterion, we use trapezoidal fuzzy multi-sets to solve decision-making problems in a lot of areas. Therefore, we introduce power aggregation operators on trapezoidal fuzzy multi-numbers as a novel way. By doing this, we aimed to introduce new operators on trapezoidal fuzzy multi-numbers which allow argument values to support each other in the aggregation process. Then, we described the properties of the operators and gave a formulation for the support function that is used in the operators. Moreover, we introduced two algorithms to solve multiple attribute group decision-making problems given with trapezoidal fuzzy multi-numbers. Lastly, we applied the introduced algorithms to a zero-waste problem to show the usage of the operators.

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1. INTRODUCTION

In 1965, the theory of fuzzy set was introduced by Zadeh [1] as an enlargement of the classical concept of a set of ambiguous information. Following of introduction of the theory, it was implemented in many areas. For example, Sahin et al. [2, 3] used fuzzy logic to conduct a study on education in 2021 and 2022. In time, a new concept of fuzzy set was introduced by Yager [4] which is called *fuzzy multi-sets* (fuzzy bags). The notion presents a new generalization of fuzzy sets. In addition, it gives complete information for some problems including situations in which each element has different membership values. Miyamoto [5] and Sebastian and Ramakrishnan [6, 7] expanded and studied in detailly the Yager’s multi-sets and fuzzy multi-sets. Since

some situations have multiple possibility of the same or different membership values, Uluçay et al. [8] developed TFM-numbers on the real number set \mathbb{R} . They are expansion of both fuzzy multi-sets and trapezoidal fuzzy numbers enabling the recurrent occurrences of any element. Later, many studies have been conducted by scientists. For example, Şahin et al. [9, 10] developed dice vector similarity measures and improved hybrid vector similarity measures of TFM-numbers in 2019. Then, Uluçay [11] introduced a new similarity function of TFM-numbers in 2020. In 2021, Deli and Keles [12] proposed distance measures on TFM-numbers and their application to multi-criteria decision-making problems. Then, Kesen and Deli [13] introduced weighted Bonferroni harmonic mean operator on TFM-numbers in 2022. In 2023, Şahin et al. [14] proposed a method by centroid point of TFM-numbers for multi-criteria decision-making problems. Deli and Kesen [15] introduced the Bonferroni geometric mean operator of TFM-numbers and they applied the operator to a multi-criteria decision-making problem. Then, Bakbak and Uluçay [16] proposed a paper related to the harmonic mean operator on intuitionistic TFM-numbers and they gave its application to architecture. In addition, Uluçay and Şahin [17] introduced some harmonic aggregation operators with TFM-numbers and their application of the law. Further, Kesen and Deli [18] conducted a study on Archimedean norms on TFM-numbers for multi-criteria decision-making problems. Lastly, Deli and Kesen [19] proposed a paper on Bonferroni arithmetic mean operator of TFM-numbers and gave an application to a multi-criteria decision-making problem.

By inspired Yager [20], in this work two aggregation operators are introduced: TFMPWA operator and TFMPWG operator. In addition, we mention their ordered types which are called TFMPOWA operator and TFMPOWG operator. They mainly cope with the information given with TFM-numbers. The main characteristic of these operators is that weight vectors depend on the arguments and that allows the values being aggregated to support and reinforce each other. In opposition to most aggregation operators, given operators incorporate information regarding the relationship between the values being combined. This is our main motivation for preparing the paper. Readers can find many papers about prominent aggregation operators in [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Technological developments resulting from the Industrial Revolution increasing population and urbanization with innovations lead to differentiation of standards and consumption habits. This situation causes depletion of natural resources and climate change led to air, water, and soil pollution which threaten bios. Creating waste, consuming natural resources, putting pressure on the soil, and using energy and water, mean polluting the environment and creating additional costs for waste management. To prevent these negative results, the material should not be accepted as waste and It is compulsory to transition to an understanding that argues that it should be transformed into useful products as much as possible. This understanding is expressed as zero-waste. Zero-waste accepts the zero as a good potential resource to benefit rather than a problem. This acceptance is influenced by the notion of waste as a valuable resource requiring solutions for collection, separation, management, and recovery which were seen as a burden for industries and societies because of its effect on global warming and resource depletion. What is targeted with Zero-waste are preventing waste, more efficient resource usage, reducing the amount of

waste, the establishment of an effective collection system, and the recycling of waste as a waste prevention approach that includes recycling. In the recycling and recovery process of waste disposal without recycling causes serious resource losses both material and energy. To leave a livable world to the next generations, we should adopt the zero-waste principle within the framework of sustainable development principles. Due to the importance of the topic, there are so many studies on zero-waste. For instance in 2018, Durgun and Durgun [33] reviewed the relationship between renewable energy consumption and economic development. Then, Onder [34] conducted a study related to circular economy as a new concept in the sustainable development approach. Okorie et al. [35] proposed a review of current research and future trends under digitization and the circular economy. Reike et al. [36], by focusing on history and resource value retention options, studied some debates on circular economy. In 2022, Uluçay and Okumus [37] proposed a study on combined fuzzy mathematical modeling and circular economy.

The organization of the paper is as follows: In the second section, we give some essential terms which are necessary for the next sections. In the third section, we introduce two aggregation techniques called TFMPWA operator and TFMPWG operator. Then, we investigate their properties and give their some special cases. In the section fourth, we introduce two algorithms to solve multiple attribute group decision-making problems given with TFM-numbers. In section five, we give a selection problem about to set a zero-waste factory which Turkiye government faced. The problem is containing five alternatives (plastic waste factory, paper waste factory, battery waste factory, organic waste factory, and glass waste factory) and four attributes (setup cost, human resource, adaptation period of public, and amount of waste) to be considered by decision-makers. We give this problem to see how effective introduced methods are. Then, in section six, we give a comparison table to show the compatibility of existing methods.

The following is the list of abbreviations.

TABLE 1. List of Abbreviations

PA	power average
PG	power geometric
TFM	trapezoidal fuzzy multi
GTF	generalized trapezoidal fuzzy
FWA	fuzzy weighted average
FWG	fuzzy weighted geometric
TFMG	trapezoidal fuzzy multi geometric
TFMPG	trapezoidal fuzzy multi power geometric
TFMPA	trapezoidal fuzzy multi power average
TFMPWA	trapezoidal fuzzy multi power weighted average
TFMPWG	trapezoidal fuzzy multi power weighted geometric
TFMBHM	trapezoidal fuzzy multi Bonferroni harmonic mean
TFMPOWA	trapezoidal fuzzy multi power ordered weighted average
TFMPOWG	trapezoidal fuzzy multi power ordered weighted geometric

2. PRELIMINARIES

Here, we give some basic notions connected to fuzzy set, fuzzy number and fuzzy multi-set which are needful for the rest of the paper.

Definition 2.1 ([1]). Let A be a non-empty set. A *fuzzy set* F on A is defined as:

$$F = \{ \langle x, \mu_F(x) \rangle : x \in A \},$$

where $\mu_F : A \rightarrow [0, 1]$ for $x \in A$.

In the context of this definition, the following two definitions are given:

Definition 2.2 ([38]). *t-norms* are monotonic, commutative, and associative functions t with two-valued mapping from $[0, 1] \times [0, 1]$ into $[0, 1]$ and satisfying following conditions: for each $x \in [0, 1]$,

- (i) $t(0, 0) = 0$, $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x)$,
- (ii) if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}(x), \mu_{X_4}(x))$,
- (iii) $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$,
- (iv) $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2}(x)), \mu_{X_3}(x))$.

Definition 2.3 ([38]). *s-norms* are monotonic, commutative, and associative functions s with two-placed mapping from $[0, 1] \times [0, 1]$ into $[0, 1]$ and satisfying following conditions: for each $x \in [0, 1]$,

- (i) $s(1, 1) = 1$, $s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x)$,
- (ii) if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$,
- (iii) $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$,
- (iv) $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2}(x)), \mu_{X_3}(x))$.

For example; $t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x)\mu_{X_2}(x)$ is a t-norm and $s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x)\mu_{X_2}(x)$ is a s-norm.

Definition 2.4 ([39]). Let $\eta_T \in [0, 1]$ and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in R$ such that $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4$. Then a *GTF-number* $T = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_T \rangle$ is a special fuzzy set on the real number set \mathbb{R} . Its membership functions are defined as follows:

$$\mu_T(x) = \begin{cases} (x - \epsilon_1)\eta_T / (\epsilon_2 - \epsilon_1) & \epsilon_1 \leq x < \epsilon_2 \\ \eta_T & \epsilon_2 \leq x \leq \epsilon_3 \\ (\epsilon_4 - x)\eta_T / (\epsilon_4 - \epsilon_3) & \epsilon_3 < x \leq \epsilon_4 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5 ([6]). Let $X \neq \emptyset$. A *fuzzy multi-set* T on X is defined as

$$T = \{ \langle x, \mu_T^1(x), \mu_T^2(x), \dots, \mu_T^i(x), \dots \rangle : x \in X \},$$

where $x \in X$ and $\mu_T^i : X \rightarrow [0, 1]$ for all $i \in \{1, 2, \dots, p\}$.

In the paper, I_m, I_n, I_p and I_t will be used instead of $\{1, 2, \dots, m\}, \{1, 2, \dots, n\}, \{1, 2, \dots, p\}$ and $\{1, 2, \dots, t\}$ respectively.

Definition 2.6 ([8]). Let $\eta_T^i \in [0, 1]$ ($i \in I_P$) and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in R$ such that $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4$. Then a *TFM-number* $T = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_T^1, \eta_T^2, \dots, \eta_T^P \rangle$ is a special fuzzy multi-set on the real number set \mathbb{R} . Its membership functions are defined as follows:

$$\mu_T^i(x) = \begin{cases} (x - \epsilon_1)\eta_T^i / (\epsilon_2 - \epsilon_1) & \epsilon_1 \leq x < \epsilon_2 \\ \eta_T^i & \epsilon_2 \leq x \leq \epsilon_3 \\ (\epsilon_4 - x)\eta_T^i / (\epsilon_4 - \epsilon_3) & \epsilon_3 < x \leq \epsilon_4 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.7 ([8]). Let $T_1 = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_{T_1}^1, \eta_{T_1}^2, \dots, \eta_{T_1}^P \rangle$ and $T_2 = \langle (\rho_1, \rho_2, \rho_3, \rho_4); \eta_{T_2}^1, \eta_{T_2}^2, \dots, \eta_{T_2}^P \rangle$ be two TFM-numbers and $\gamma \neq 0$ be any real number. Then

- (i) $T_1 \oplus T_2 = (\epsilon_1 + \rho_1, \epsilon_2 + \rho_2, \epsilon_3 + \rho_3, \epsilon_4 + \rho_4); s(\eta_{T_1}^1, \eta_{T_2}^1), s(\eta_{T_1}^2, \eta_{T_2}^2), \dots, s(\eta_{T_1}^P, \eta_{T_2}^P)$,
- (ii) $T_1 \otimes T_2 = \begin{cases} \langle (\epsilon_1 \rho_1, \epsilon_2 \rho_2, \epsilon_3 \rho_3, \epsilon_4 \rho_4); t(\eta_{T_1}^1, \eta_{T_2}^1), t(\eta_{T_1}^2, \eta_{T_2}^2), \dots, t(\eta_{T_1}^P, \eta_{T_2}^P) \rangle & (\epsilon_4 > 0, \rho_4 > 0) \\ \langle (\epsilon_1 \rho_4, \epsilon_2 \rho_3, \epsilon_3 \rho_2, \epsilon_4 \rho_1); t(\eta_{T_1}^1, \eta_{T_2}^1), t(\eta_{T_1}^2, \eta_{T_2}^2), \dots, t(\eta_{T_1}^P, \eta_{T_2}^P) \rangle & (\epsilon_4 < 0, \rho_4 > 0) \\ \langle (\epsilon_4 \rho_4, \epsilon_3 \rho_3, \epsilon_2 \rho_2, \epsilon_1 \rho_1); t(\eta_{T_1}^1, \eta_{T_2}^1), t(\eta_{T_1}^2, \eta_{T_2}^2), \dots, t(\eta_{T_1}^P, \eta_{T_2}^P) \rangle & (\epsilon_4 < 0, \rho_4 < 0) \end{cases}$
- (iii) $\gamma T_1 = \langle (\gamma \epsilon_1, \gamma \epsilon_2, \gamma \epsilon_3, \gamma \epsilon_4); 1 - (1 - \eta_{T_1}^1)^\gamma, 1 - (1 - \eta_{T_1}^2)^\gamma, \dots, 1 - (1 - \eta_{T_1}^P)^\gamma \rangle (\gamma \geq 0)$,
- (iv) $T_1^\gamma = \langle (\epsilon_1^\gamma, \epsilon_2^\gamma, \epsilon_3^\gamma, \epsilon_4^\gamma); (\eta_{T_1}^1)^\gamma, (\eta_{T_1}^2)^\gamma, \dots, (\eta_{T_1}^P)^\gamma \rangle (\gamma \geq 0)$.

Here, t-norms and s-norms are mappings that are given in Definitions 2.2 and 2.3, respectively.

Based on the complement of a fuzzy set given by Zimmermann [38], we can give following property for TFM-numbers.

Definition 2.8. Let $T = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_T^1, \eta_T^2, \dots, \eta_T^P \rangle$ be a TFM-number. Then the *complement* of T , denoted by T^c , is defined as follows:

$$T^c = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); 1 - \eta_T^1, 1 - \eta_T^2, \dots, 1 - \eta_T^P \rangle$$

Note that if $\epsilon_1 \leq \rho_1, \epsilon_2 \leq \rho_2, \epsilon_3 \leq \rho_3, \epsilon_4 \leq \rho_4, \eta_{T_1}^1 \leq \eta_{T_2}^1, \eta_{T_1}^2 \leq \eta_{T_2}^2, \dots, \eta_{T_1}^P \leq \eta_{T_2}^P$ then, we said $T_1 \leq T_2$.

Definition 2.9 ([8]). Let $T_1 = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_{T_1}^1, \eta_{T_1}^2, \dots, \eta_{T_1}^P \rangle$ and $T_2 = \langle (\rho_1, \rho_2, \rho_3, \rho_4); \eta_{T_2}^1, \eta_{T_2}^2, \dots, \eta_{T_2}^P \rangle$ be two TFM-numbers. Then the *Hamming distance* between T_1 and T_2 is defined as follows:

$$d(T_1, T_2) = \frac{1}{8p} \sum_{i=1}^P (|(1 + \eta_{T_1}^i) \epsilon_1 - (1 + \eta_{T_2}^i) \rho_1| + |(1 + \eta_{T_1}^i) \epsilon_2 - (1 + \eta_{T_2}^i) \rho_2| + |(1 + \eta_{T_1}^i) \epsilon_3 - (1 + \eta_{T_2}^i) \rho_3| + |(1 + \eta_{T_1}^i) \epsilon_4 - (1 + \eta_{T_2}^i) \rho_4|).$$

Definition 2.10 ([8]). $T_1 = \langle (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_{T_1}^1, \eta_{T_1}^2, \dots, \eta_{T_1}^P \rangle$ and $T_2 = \langle (\rho_1, \rho_2, \rho_3, \rho_4); \eta_{T_2}^1, \eta_{T_2}^2, \dots, \eta_{T_2}^P \rangle$ be two TFM-numbers and d be Hamming distance between two TFM-numbers. Then the comparison of T_1 and T_2 defined as follows:

- (i) if $d(T_1, r^+) < d(T_2, r^+)$, then T_1 is *bigger than* T_2 , symbolised by $T_1 \succ T_2$,
- (ii) if $d(T_1, r^+) = d(T_2, r^+)$ and $d(T_1, r^-) < d(T_2, r^-)$, then T_2 is *bigger than* T_1 , symbolised by $T_1 \prec T_2$,
- (iii) if $d(T_1, r^+) = d(T_2, r^+)$ and $d(T_1, r^-) = d(T_2, r^-)$, then T_1 is *similar to* T_2 , symbolised by $T_1 \simeq T_2$,

where r_A^+ is positive ideal and r_A^- is negative ideal and defined as follows, respectively:

$$r_A^+ = \langle (a^+, b^+, c^+, d^+); (\eta_A^1)^+, (\eta_A^2)^+, \dots, (\eta_A^P)^+ \rangle = \langle (1, 1, 1, 1); 1, 1, \dots, 1 \rangle,$$

$$r_A^- = \langle (a^-, b^-, c^-, d^-); (\eta_A^1)^-, (\eta_A^2)^-, \dots, (\eta_A^P)^- \rangle = \langle (0, 0, 0, 0); 0, 0, \dots, 0 \rangle.$$

Definition 2.11 ([20]). Let R_i ($i \in I_n$) be a collection of non-negative real numbers. To aggregate this collection, the *PA operator* is defined as follows:

$$PA(R_1, R_2, \dots, R_n) = \frac{\sum_{i=1}^n (1 + T(R_i))R_i}{\sum_{i=1}^n (1 + T(R_i))},$$

where $T(R_i) = \sum_{j=1, i \neq j}^n Sup(R_i, R_j)$.

$Sup(R_i, R_j)$ is denoted as support from R_i to R_j and satisfies following properties:

- (1) $Sup(R_i, R_j) \in [0, 1]$,
- (2) $Sup(R_i, R_j) = Sup(R_j, R_i)$,
- (3) $Sup(R_i, R_j) \geq Sup(R_x, R_y)$, if $|R_i - R_j| < |R_x - R_y|$.

Definition 2.12 ([26]). Let R_i ($i \in I_n$) be a collection of non-negative real numbers. To aggregate this collection, the *PG operator* is defined as follows:

$$PG(R_1, R_2, \dots, R_n) = \prod_{i=1}^n R_i^{\frac{1+T(R_i)}{\sum_{i=1}^n (1+T(R_i))}},$$

where $T(R_i) = \sum_{j=1, i \neq j}^n Sup(R_i, R_j)$.

$Sup(R_i, R_j)$ is denoted as support from R_i to R_j and satisfies following properties:

- (1) $Sup(R_i, R_j) \in [0, 1]$,
- (2) $Sup(R_i, R_j) = Sup(R_j, R_i)$,
- (3) $Sup(R_i, R_j) \geq Sup(R_x, R_y)$, if $|R_i - R_j| < |R_x - R_y|$.

3. POWER AGGREGATION OPERATORS ON TFM-NUMBERS

In this section, we developed two TFM-Aggregation operators based on power aggregation operators for multi-criteria decision-making problems given with TFM information.

Definition 3.1. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the *TFMPWA operator* is defined as follows:

$$(3.1) \quad TFMPWA(F_1, F_2, \dots, F_n) = \frac{\bigoplus_{i=1}^n v_i(1 + T(F_i))F_i}{\sum_{i=1}^n v_i(1 + T(F_i))},$$

where $T(F_i) = \sum_{j=1, i \neq j}^n Sup(F_i, F_j)$, $Sup(F_i, F_j) = 1 - d(F_i, F_j)$ and $d(F_i, F_j)$ is Hamming distance [8] between F_i and F_j .

Theorem 3.2. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the aggregated value by TFMPWA operator is a TFM-number and computed as follows:

For simplicity, let $\frac{v_i(1+T(F_i))}{\sum_{i=1}^n v_i(1+T(F_i))} = \varsigma_i$.

(3.2)

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_n) &= \bigoplus_{i=1}^n \varsigma_i F_i \\ &= \varsigma_1 F_1 \oplus \varsigma_2 F_2 \oplus \dots \oplus \varsigma_n F_n \\ &= \langle (\bigoplus_{i=1}^n \varsigma_i \epsilon_i, \bigoplus_{i=1}^n \varsigma_i \rho_i, \bigoplus_{i=1}^n \varsigma_i \delta_i, \bigoplus_{i=1}^n \varsigma_i \gamma_i); \\ &\quad 1 - \prod_{i=1}^n (1 - \eta_{F_i}^1)^{\varsigma_i}, 1 - \prod_{i=1}^n (1 - \eta_{F_i}^2)^{\varsigma_i}, \dots, 1 - \prod_{i=1}^n (1 - \eta_{F_i}^P)^{\varsigma_i} \rangle, \end{aligned}$$

where $T(F_i) = \sum_{j=1, i \neq j}^n Sup(F_i, F_j)$.

Proof. By using the mathematical induction on n and considering operational laws in Definition 2.7, we can prove the theorem. We need to show that:

(3.3)

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_n) &= \bigoplus_{i=1}^n \varsigma_i F_i \\ &= \varsigma_1 F_1 \oplus \varsigma_2 F_2 \oplus \dots \oplus \varsigma_n F_n \\ &= \langle (\bigoplus_{i=1}^n \varsigma_i \epsilon_i, \bigoplus_{i=1}^n \varsigma_i \rho_i, \bigoplus_{i=1}^n \varsigma_i \delta_i, \bigoplus_{i=1}^n \varsigma_i \gamma_i); \\ &\quad 1 - \prod_{i=1}^n (1 - \eta_{F_i}^1)^{\varsigma_i}, 1 - \prod_{i=1}^n (1 - \eta_{F_i}^2)^{\varsigma_i}, \dots, 1 - \prod_{i=1}^n (1 - \eta_{F_i}^P)^{\varsigma_i} \rangle. \end{aligned}$$

(i) Suppose $n = 2$. Then we have

$$\begin{aligned} TFMPWA(F_1, F_2) &= \bigoplus_{i=1}^2 \varsigma_i F_i \\ &= \varsigma_1 F_1 \oplus \varsigma_2 F_2 \\ &= \langle (\varsigma_1 \epsilon_1 + \varsigma_2 \epsilon_2, \varsigma_1 \rho_1 + \varsigma_2 \rho_2, \varsigma_1 \delta_1 + \varsigma_2 \delta_2, \varsigma_1 \gamma_1 + \varsigma_2 \gamma_2); \\ &\quad s(\varsigma_1 \eta_{F_1}^1, \varsigma_2 \eta_{F_2}^1), s(\varsigma_1 \eta_{F_1}^2, \varsigma_2 \eta_{F_2}^2) \rangle \\ &= \langle (\bigoplus_{i=1}^2 \varsigma_i \epsilon_i, \bigoplus_{i=1}^2 \varsigma_i \rho_i, \bigoplus_{i=1}^2 \varsigma_i \delta_i, \bigoplus_{i=1}^2 \varsigma_i \gamma_i); \\ &\quad 1 - \prod_{i=1}^2 (1 - \eta_{F_i}^1)^{\varsigma_i}, 1 - \prod_{i=1}^2 (1 - \eta_{F_i}^2)^{\varsigma_i}, \dots, 1 - \prod_{i=1}^2 (1 - \eta_{F_i}^P)^{\varsigma_i} \rangle. \end{aligned}$$

Thus the Equation (3.3) is right.

(ii) Suppose the Equation (3.3) is right for $n = k$, i.e.,

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_n) &= \bigoplus_{i=1}^k \varsigma_i F_i \\ &= \varsigma_1 F_1 \oplus \varsigma_2 F_2 \oplus \dots \oplus \varsigma_k F_k \\ &= \left\langle \left(\bigoplus_{i=1}^k \varsigma_i \epsilon_i, \bigoplus_{i=1}^k \varsigma_i \rho_i, \bigoplus_{i=1}^k \varsigma_i \delta_i, \bigoplus_{i=1}^k \varsigma_i \gamma_i \right); \right. \\ &\quad \left. 1 - \prod_{i=1}^k (1 - \eta_{F_i}^1)^{\varsigma_i}, 1 - \prod_{i=1}^k (1 - \eta_{F_i}^2)^{\varsigma_i}, \dots, 1 - \prod_{i=1}^k (1 - \eta_{F_i}^P)^{\varsigma_i} \right\rangle. \end{aligned}$$

Now, we need to prove it is true for $n = k + 1$ as well. Then from the Equation (3.2) for $n = k + 1$, we obtain the following:

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_{k+1}) &= \bigoplus_{i=1}^k \varsigma_i F_i + \varsigma_{k+1} F_{k+1} \\ &= \varsigma_1 F_1 \oplus \varsigma_2 F_2 \oplus \dots \oplus \varsigma_n F_n \oplus \varsigma_{k+1} F_{k+1} \\ &= \left\langle \left(\bigoplus_{i=1}^k \varsigma_i \epsilon_i + \varsigma_{k+1} \epsilon_{k+1}, \bigoplus_{i=1}^k \varsigma_i \rho_i + \varsigma_{k+1} \rho_{k+1}, \right. \right. \\ &\quad \left. \bigoplus_{i=1}^k \varsigma_i \delta_i + \varsigma_{k+1} \delta_{k+1}, \bigoplus_{i=1}^k \varsigma_i \gamma_i + \varsigma_{k+1} \gamma_{k+1} \right); \\ &\quad s(1 - \prod_{i=1}^k (1 - \eta_{F_i}^1)^{\varsigma_i}, \varsigma_{k+1} \eta_{F_{k+1}}^1), \\ &\quad s(1 - \prod_{i=1}^k (1 - \eta_{F_i}^2)^{\varsigma_i}, \varsigma_{k+1} \eta_{F_{k+1}}^2), \dots, \\ &\quad \left. s(1 - \prod_{i=1}^k (1 - \eta_{F_i}^P)^{\varsigma_i}, \varsigma_{k+1} \eta_{F_{k+1}}^P) \right\rangle \\ &= \bigoplus_{i=1}^{k+1} \varsigma_i F_i. \end{aligned}$$

Thus the Equation (3.2) holds for $n = k + 1$. So the Equation (3.2) holds for all n . □

Note 3.3. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers. If $v = (1/n, 1/n, \dots, 1/n)^T$, then TFMPWA operator (3.1) converted into a TFMPA operator:

$$(3.4) \quad TFMPA(F_1, F_2, \dots, F_n) = \frac{((1 + T(F_1))F_1) \oplus ((1 + T(F_2))F_2) \oplus \dots \oplus ((1 + T(F_n))F_n)}{\sum_{i=1}^n (1 + T(F_i))},$$

where $T(F_i) = \sum_{j=1, i \neq j}^n Sup(F_i, F_j)$.

Proposition 3.4. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers. TFMPWA operator has the following desirable properties:

(1) **(Commutativity)** if $(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n)$ is any permutation of (F_1, F_2, \dots, F_n) , then $TFMPWA(F_1, F_2, \dots, F_n) = TFMPWA(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n)$,

(2) **(Idempotency)** if $F_i = F$ for all i , then $TFMPWA(F_1, F_2, \dots, F_n) = F$,

(3) **(Boundedness)** $F^- \leq TFMPWA(F_1, F_2, \dots, F_n) \leq F^+$,

where $F^+ = \langle (\max\{\epsilon_i\}_{i \in I_n}, \max\{\rho_i\}_{i \in I_n}, \max\{\delta_i\}_{i \in I_n}, \max\{\gamma_i\}_{i \in I_n}); \max\{\eta_{F_i}^1\}_{i \in I_n}, \max\{\eta_{F_i}^2\}_{i \in I_n}, \dots, \max\{\eta_{F_i}^P\}_{i \in I_n} \rangle$

and

$$F^- = \langle (\min\{\epsilon_i\}_{i \in I_n}, \min\{\rho_i\}_{i \in I_n}, \min\{\delta_i\}_{i \in I_n}, \min\{\gamma_i\}_{i \in I_n}); \min\{\eta_{F_i}^1\}_{i \in I_n}, \min\{\eta_{F_i}^2\}_{i \in I_n}, \dots, \min\{\eta_{F_i}^P\}_{i \in I_n} \rangle.$$

Proof. (1) Suppose $(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n)$ is any permutation of (F_1, F_2, \dots, F_n) . Then we get

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_n) &= \frac{\bigoplus_{i=1}^n v_i(1+T(F_i))F_i}{\sum_{i=1}^n v_i(1+T(F_i))} \\ &= \frac{\bigoplus_{i=1}^n v_i(1+T(\dot{F}_i))\dot{F}_i}{\sum_{i=1}^n v_i(1+T(\dot{F}_i))} \\ &= TFMPWA(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n). \end{aligned}$$

(2) Suppose $F_i = F$ for all i . Then we get

$$\begin{aligned} TFMPWA(F_1, F_2, \dots, F_n) &= \frac{\bigoplus_{i=1}^n v_i(1+T(F_i))F_i}{\sum_{i=1}^n v_i(1+T(F_i))} \\ &= \frac{\bigoplus_{i=1}^n v_i(1+T(F))F}{\sum_{i=1}^n v_i(1+T(F))} \\ &= \frac{(1+T(F))F \sum_{i=1}^n v_i}{(1+T(F)) \sum_{i=1}^n v_i} \\ &= \frac{(1+T(F))F}{(1+T(F))} \\ &= TFMPWA(F, F, \dots, F) \\ &= F. \end{aligned}$$

(3) Let's denote

$$\begin{aligned} \epsilon_b &= \max\{\epsilon_i\}_{i \in I_n}, \quad \epsilon_l = \min\{\epsilon_i\}_{i \in I_n}, \\ \rho_b &= \max\{\rho_i\}_{i \in I_n}, \quad \rho_l = \min\{\rho_i\}_{i \in I_n}, \\ \delta_b &= \max\{\delta_i\}_{i \in I_n}, \quad \delta_l = \min\{\delta_i\}_{i \in I_n}, \\ \gamma_b &= \max\{\gamma_i\}_{i \in I_n}, \quad \gamma_l = \min\{\gamma_i\}_{i \in I_n}. \end{aligned}$$

Similarly,

$$\begin{aligned} \eta_{F_b}^1 &= \max\{\eta_{F_i}^1\}_{i \in I_n}, \quad \eta_{F_l}^1 = \min\{\eta_{F_i}^1\}_{i \in I_n}, \\ \eta_{F_b}^2 &= \max\{\eta_{F_i}^2\}_{i \in I_n}, \quad \eta_{F_l}^2 = \min\{\eta_{F_i}^2\}_{i \in I_n}, \end{aligned}$$

⋮

$$\eta_{F_b}^P = \max\{\eta_{F_i}^P\}_{i \in I_n}, \eta_{F_l}^P = \min\{\eta_{F_i}^P\}_{i \in I_n}.$$

and let

$$F^+ = \langle (\epsilon_b, \rho_b, \delta_b, \gamma_b); \eta_{F_b}^1, \eta_{F_b}^2, \dots, \eta_{F_b}^P \rangle$$

and

$$F^- = \langle (\epsilon_l, \rho_l, \delta_l, \gamma_l); \eta_{F_l}^1, \eta_{F_l}^2, \dots, \eta_{F_l}^P \rangle.$$

Since $\epsilon_l \leq \epsilon_i \leq \epsilon_b, \rho_l \leq \rho_i \leq \rho_b, \delta_l \leq \delta_i \leq \delta_b$ and $\gamma_l \leq \gamma_i \leq \gamma_b$,

$$\eta_{F_l}^1 \leq \eta_{F_i}^1 \leq \eta_{F_b}^1, \eta_{F_l}^2 \leq \eta_{F_i}^2 \leq \eta_{F_b}^2, \dots, \eta_{F_l}^P \leq \eta_{F_i}^P \leq \eta_{F_b}^P,$$

we get

$$(3.5) \quad \begin{aligned} \bigoplus_{i=1}^n v_i \epsilon_l &\leq \bigoplus_{i=1}^n v_i \epsilon_i \leq \bigoplus_{i=1}^n v_i \epsilon_b, \bigoplus_{i=1}^n v_i \rho_l \leq \bigoplus_{i=1}^n v_i \rho_i \leq \bigoplus_{i=1}^n v_i \rho_b, \\ \bigoplus_{i=1}^n v_i \delta_l &\leq \bigoplus_{i=1}^n v_i \delta_i \leq \bigoplus_{i=1}^n v_i \delta_b \quad \text{and} \quad \bigoplus_{i=1}^n v_i \gamma_l \leq \bigoplus_{i=1}^n v_i \gamma_i \leq \bigoplus_{i=1}^n v_i \gamma_b. \end{aligned}$$

On the other hand, we get

$$1 - \eta_{F_b}^1 \leq 1 - \eta_{F_i}^1 \leq 1 - \eta_{F_l}^1, 1 - \eta_{F_b}^2 \leq 1 - \eta_{F_i}^2 \leq 1 - \eta_{F_l}^2, \dots, 1 - \eta_{F_b}^P \leq 1 - \eta_{F_i}^P \leq 1 - \eta_{F_l}^P.$$

Then we have

$$(3.6) \quad \begin{aligned} 1 - \prod_{i=1}^n (1 - \eta_{F_l}^1)^{v_i} &\leq 1 - \prod_{i=1}^n (1 - \eta_{F_i}^1)^{v_i} \leq 1 - \prod_{i=1}^n (1 - \eta_{F_b}^1)^{v_i}, \\ 1 - \prod_{i=1}^n (1 - \eta_{F_l}^2)^{v_i} &\leq 1 - \prod_{i=1}^n (1 - \eta_{F_i}^2)^{v_i} \leq 1 - \prod_{i=1}^n (1 - \eta_{F_b}^2)^{v_i}, \dots, \\ 1 - \prod_{i=1}^n (1 - \eta_{F_l}^P)^{v_i} &\leq 1 - \prod_{i=1}^n (1 - \eta_{F_i}^P)^{v_i} \leq 1 - \prod_{i=1}^n (1 - \eta_{F_b}^P)^{v_i}. \end{aligned}$$

Thus from inequalities in (3.5) and (3.6), we get

$$F^- \leq TFMPWA(F_1, F_2, \dots, F_n) \leq F^+.$$

□

Definition 3.5. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the *TFMPOWA operator* is defined as follows:

$$TFMPOWA(F_1, F_2, \dots, F_n) = \frac{\sum_{i=1}^n v_i (1 + T(F_{\sigma(i)})) F_{\sigma(i)}}{\sum_{i=1}^n v_i (1 + T(F_{\sigma(i)}))},$$

where $F_{\sigma(i)}$ is the i^{th} largest of the trapezoidal fuzzy sets (F_1, F_2, \dots, F_n) and v_i is the collection of weights such that

$$(3.7) \quad v_i = f\left(\frac{R_i}{TV}\right) - f\left(\frac{R_{i-1}}{TV}\right), R_i = \sum_{j=1}^i V_{\sigma(j)}, TV = \sum_{i=1}^n V_{\sigma(i)}, V_{\sigma(i)} = 1 + T(F_{\sigma(i)})$$

and $T(F_{\sigma(i)})$ denotes the support of the i^{th} largest trapezoidal fuzzy variable $F_{\sigma(i)}$ by all the other trapezoidal fuzzy variables, i.e.,

$$(3.8) \quad T(F_{\sigma(i)}) = \sum_{j=1, i \neq j}^n Sup(T(F_{\sigma(i)}), T(F_{\sigma(j)})),$$

where $Sup(T(F_{\sigma(i)}), T(F_{\sigma(j)}))$ represents the support of j^{th} largest trapezoidal fuzzy variable $F_{\sigma(j)}$ for the i^{th} largest trapezoidal fuzzy variable $F_{\sigma(i)}$ and $f : [0, 1] \rightarrow [0, 1]$ is a basic unit interval monotonic function which satisfies the following properties:

- (i) $f(0) = 0$,
- (ii) $f(1) = 1$,
- (iii) $f(x) \geq f(y)$, if $x > y$.

Definition 3.6. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the aggregated value by the TFMPWG operator is defined as follows:

$$(3.9) \quad TFMPWG(F_1, F_2, \dots, F_n) = \bigotimes_{i=1}^n (F_i)^{\frac{1}{n-1} \left(1 - \frac{v_i(1+T(F_i))}{\sum_{i=1}^n v_i(1+T(F_i))} \right)}$$

$$= F_1^{\frac{1}{n-1} \left(1 - \frac{v_1(1+T(F_1))}{\sum_{i=1}^n v_i(1+T(F_i))} \right)} \otimes F_2^{\frac{1}{n-1} \left(1 - \frac{v_2(1+T(F_2))}{\sum_{i=1}^n v_i(1+T(F_i))} \right)} \otimes \dots \otimes F_n^{\frac{1}{n-1} \left(1 - \frac{v_n(1+T(F_n))}{\sum_{i=1}^n v_i(1+T(F_i))} \right)},$$

where $T(F_i) = \sum_{j=1, i \neq j}^n Sup(F_i, F_j)$ and $Sup(F_i, F_j) = 1 - d(F_i, F_j)$.

Theorem 3.7. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the aggregated value by the TFMPWG is a TFM-number and computed as follows:

For simplicity, let $\frac{1}{n-1} \left(1 - \frac{v_i(1+T(F_i))}{\sum_{i=1}^n v_i(1+T(F_i))} \right) = \zeta_i$

$$\begin{aligned} TFMPWG(F_1, F_2, \dots, F_n) &= \bigotimes_{i=1}^n F_i^{\zeta_i} \\ &= \langle (\bigotimes_{i=1}^n \epsilon_i^{\zeta_i}, \bigotimes_{i=1}^n \rho_i^{\zeta_i}, \bigotimes_{i=1}^n \delta_i^{\zeta_i}, \bigotimes_{i=1}^n \gamma_i^{\zeta_i}); \\ &\quad \bigotimes_{i=1}^n \eta_{F_i}^1 \zeta_i, \bigotimes_{i=1}^n \eta_{F_i}^2 \zeta_i, \dots, \bigotimes_{i=1}^n \eta_{F_i}^P \zeta_i \rangle. \end{aligned}$$

Proof. The proof can be done similar to Theorem 3.2. □

Note 3.8. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers. If $v = (1/n, 1/n, \dots, 1/n)^T$, then the TFMPWG operator (3.9) converted into TFMPG operator:

$$\begin{aligned} TFMPG(F_1, F_2, \dots, F_n) &= F_1^{\frac{1}{n-1} \left(1 - \frac{1+T(F_1)}{\sum_{i=1}^n (1+T(F_i))} \right)} \\ &\quad \otimes F_2^{\frac{1}{n-1} \left(1 - \frac{1+T(F_2)}{\sum_{i=1}^n (1+T(F_i))} \right)} \otimes \dots \otimes F_n^{\frac{1}{n-1} \left(1 - \frac{1+T(F_n)}{\sum_{i=1}^n (1+T(F_i))} \right)} \end{aligned} \tag{3.10}$$

Proposition 3.9. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers. TFMPWG operator has the following desirable properties:

(1) **(Commutativity)** if $(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n)$ is any permutation of (F_1, F_2, \dots, F_n) , then $TFMPWG(F_1, F_2, \dots, F_n) = TFMPWG(\dot{F}_1, \dot{F}_2, \dots, \dot{F}_n)$,

(2) **(Idempotency)** if $F_i = F$ for all i , then $TFMPWG(F_1, F_2, \dots, F_n) = F$,

(3) **(Boundedness)** $F^- \leq TFMPWG(F_1, F_2, \dots, F_n) \leq F^+$,

where $F^+ = \langle (\max\{\epsilon_i\}_{i \in I_n}, \max\{\rho_i\}_{i \in I_n}, \max\{\delta_i\}_{i \in I_n}, \max\{\gamma_i\}_{i \in I_n}); \max\{\eta_{F_i}^1\}_{i \in I_n}, \max\{\eta_{F_i}^2\}_{i \in I_n}, \dots, \max\{\eta_{F_i}^P\}_{i \in I_n} \rangle$

and

$$F^- = \langle (\min\{\epsilon_i\}_{i \in I_n}, \min\{\rho_i\}_{i \in I_n}, \min\{\delta_i\}_{i \in I_n}, \min\{\gamma_i\}_{i \in I_n}); \min\{\eta_{F_i}^1\}_{i \in I_n}, \min\{\eta_{F_i}^2\}_{i \in I_n}, \dots, \min\{\eta_{F_i}^P\}_{i \in I_n} \rangle.$$

Proof. The proof can be done similar to Proposition 3.4. □

Definition 3.10. Let $F_i = \langle (\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{F_i}^1, \eta_{F_i}^2, \dots, \eta_{F_i}^P \rangle$ ($i \in I_n$) be a collection of TFM-numbers and $v = (v_1, v_2, \dots, v_n)^T$ weight vector of F_i ($i \in I_n$), where $v_i \geq 0$ ($i \in I_n$) and $\sum_{i=1}^n v_i = 1$. Then the TFMPPOWG operator is defined as follows:

$$TFMPPOWG(F_1, F_2, \dots, F_n) = \prod_{i=1}^n (F_{\sigma(i)})^{\frac{v_i(1+T(F_{\sigma(i)}))}{\sum_{i=1}^n v_i(1+T(F_{\sigma(i)}))}}, \tag{3.11}$$

where v_i ($i \in I_n$) is the collection of weights which satisfies conditions (3.7) and (3.8).

Especially, if $f(x) = x$, then the TFMPWA operator (3.11) converted into the TFMPA operator (3.10).

4. TWO APPROACHES TO MULTI-ATTRIBUTE GROUP DECISION-MAKING WITH TRAPEZOIDAL FUZZY MULTI INFORMATION

In this section, by inspiring Xu [40], two approaches to solve multi-attribute group decision-making problem introduced.

Let alternative's set be $Z = \{Z_i | i \in I_m\}$ and attributes' set be $G = \{G_j | j \in I_n\}$. The weight vector of attributes is $v = (v_1, v_2, \dots, v_n)$ satisfying $v_j \geq 0$ ($j \in I_n$), $\sum_{j=1}^n v_j = 1$. Moreover, let decision maker's set be $D = \{D_k | (k \in I_t)\}$ whose weight

vector $\tau = (\tau_1, \tau_2, \dots, \tau_t)$ with $\tau_k \geq 0$ ($k \in I_t$), $\sum_{k=1}^t \tau_k = 1$.

Approach I

Step 1. Construct the decision matrix of each expert.

Step 2. Normalise each attribute value $F_{ij}^{(k)}$ in the matrix F_k into a corresponding element in the matrix $\tilde{R}_k = (\tilde{r}_{ij}^{(k)})_{m \times n}$ such that

$$(4.1) \quad \tilde{r}_{ij}^{(k)} = \begin{cases} \langle (\epsilon_{ij}, \rho_{ij}, \delta_{ij}, \gamma_{ij}); \eta_{F_{ij}}^1, \eta_{F_{ij}}^2, \dots, \eta_{F_{ij}}^k \rangle & \text{for benefit attribute } G_j \\ \langle (\epsilon_{ij}, \rho_{ij}, \delta_{ij}, \gamma_{ij}); 1 - \eta_{F_{ij}}^1, 1 - \eta_{F_{ij}}^2, \dots, 1 - \eta_{F_{ij}}^k \rangle & \text{for cost attribute } G_j. \end{cases}$$

Step 3. Support measure is calculated as follows:

$$(4.2) \quad Sup(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}) = 1 - d(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}), (l \in I_t).$$

Here $d(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)})$ is calculated with Hamming distance in Definition 2.9 and utilize weights $\tau = (\tau_1, \tau_2, \dots, \tau_t)$ of the decision makers $D_k (k \in I_t)$ to find the weighted support $T(\tilde{r}_{ij}^{(k)})$ of the trapezoidal fuzzy preference value $(\tilde{r}_{ij}^{(k)})$ by other trapezoidal fuzzy preference value $(\tilde{r}_{ij}^{(l)})$ ($l \in I_t$ and $l \neq k$)

$$T(\tilde{r}_{ij}^{(k)}) = \sum_{l=1, l \neq k}^t \tau_l Sup(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)})$$

and weights $\eta_{ij}^{(k)}$ ($k \in I_t$) of the trapezoidal fuzzy preference value $\tilde{r}_{ij}^{(k)}$ ($k \in I_t$) is calculated as follows:

$$\eta_{ij}^{(k)} = \frac{\tau_k(1 + T(\tilde{r}_{ij}^{(k)}))}{\sum_{k=1}^t \tau_k(1 + T(\tilde{r}_{ij}^{(k)}))} \quad (k \in I_t),$$

where $\eta_{ij}^{(k)} \geq 0$ ($k \in I_t$) and $\sum_{k=1}^t \eta_{ij}^{(k)} = 1$.

Step 4. Use the matrix \tilde{R}_k and find the TFMPWA operator:

$$\sum_{k=1}^t \frac{\tau_k(1 + T(\tilde{r}_{ij}^{(k)}))\tilde{r}_{ij}^{(k)}}{\sum_{k=1}^t \tau_k(1 + T(\tilde{r}_{ij}^{(k)}))} = \sum_{k=1}^t \eta_{ij}^{(k)} \tilde{r}_{ij}^{(k)} \quad (i \in I_m, j \in I_n)$$

or the TFMPWG operator:

$$\prod_{k=1}^t (\tilde{r}_{ij}^{(k)})^{\frac{\tau_k(1+T(\tau_{ij}^{(k)}))}{\sum_{k=1}^t \tau_k(1+T(\tau_{ij}^{(k)}))}} = \prod_{k=1}^t (\tilde{r}_{ij}^{(k)})^{\eta_{ij}^{(k)}} \quad (i \in I_m, j \in I_n).$$

Step 5. By using the FWA operator, find the aggregation of all trapezoidal fuzzy preference value \tilde{r}_{ij} ($j \in I_n$):

$$\tilde{r}_i = FWA_v = \sum_{j=1}^n v_j \tilde{r}_{ij} \quad (i \in I_m)$$

or FWG operator:

$$\tilde{r}_i = FWG_v = \prod_{j=1}^n \tilde{r}_{ij}^{v_j} \quad (i \in I_m)$$

Step 6. Find the Hamming distance between \tilde{r}_i and positive ideal (or negative ideal) given in Definition 2.10.

Step 7. Rank all the alternatives. If Hamming distance between \tilde{r}_i and positive ideal taken consideration, then the bigger result, the better alternative. If Hamming distance between \tilde{r}_i and negative ideal taken consideration, then the smaller result, the better alternative.

Approach II

Step 1. Same as **Approach I**.

Step 2. Same as **Approach I**.

Step 3. Support measure is calculated as follows:

$$Sup(\tilde{r}_{ij}^{\sigma(k)}, \tilde{r}_{ij}^{\sigma(l)}) = 1 - d(\tilde{r}_{ij}^{\sigma(k)}, \tilde{r}_{ij}^{\sigma(l)})$$

which remarks the support of l^{th} largest trapezoidal fuzzy preference value r_{ij}^l for the k^{th} largest trapezoidal fuzzy preference value \tilde{r}_{ij}^k of \tilde{r}_{ij}^k .

Step 4. Find the value of the support $T(\tilde{r}_{ij}^{(k)})$ of the k^{th} largest trapezoidal fuzzy preference value $\tilde{r}_{ij}^{(k)}$ ($l \in I_t$ and $l \neq k$)

$$T(\tilde{r}_{ij}^{\sigma(k)}) = \sum_{l=1, l \neq k}^t Sup(\tilde{r}_{ij}^{\sigma(k)}, \tilde{r}_{ij}^{\sigma(l)})$$

and use equalities in (3.7) to find the weights $v_{ij}^{(k)}$ ($k \in I_t$) related to the k^{th} largest trapezoidal fuzzy preference value $\tilde{r}_{ij}^{(k)}$, where

$$v_{ij}^{(k)} = f\left(\frac{Q_{ij}^{(k)}}{TV_{ij}}\right) - f\left(\frac{Q_{ij}^{(k-1)}}{TV_{ij}}\right),$$

$$Q_{ij}^{(k)} = \sum_{l=1}^k \eta_{ij}^{\sigma(l)}, TV_{ij} = \sum_{l=1}^t \eta_{ij}^{\sigma(l)}, \eta_{ij}^{\sigma(l)} = 1 + T(\tilde{r}_{ij}^{\sigma(l)}),$$

where $v_{ij}^{(k)} \geq 0$ ($k \in I_t$) and $\sum_{k=1}^t v_{ij}^{(k)} = 1$.

Step 5. To convert all the individual decision matrices \tilde{R}_k ($k \in I_t$) into the collective decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m \times n}$,

Utilise TFMPWA operator:

$$TFMPWA = \sum_{k=1}^t v_{ij}^{(k)} \tilde{r}_{ij}^{(k)} \quad (i \in I_m \quad \text{and} \quad j \in I_n)$$

or the TFMPOWG operator:

$$TFMPOWG = \prod_{k=1}^t (\bar{r}_{ij}^{(k)})^{v_{ij}^{(k)}} \quad (i \in I_m \quad \text{and} \quad j \in I_n).$$

Step 6. Same as **Approach I**.

Step 7. Same as **Approach I**.

5. APPLICATION

Here, we give an illustrative example to show the efficiency of the proposed operators.

TABLE 2. TFM-numbers response to linguistic terms

Linguistic terms	TFM-numbers
Absolutely high(AH)	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$
Very Very High(VVH)	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$
Very High(VH)	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$
High(H)	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$
Fairly high(FH)	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
Medium(M)	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$
Fairly low(FL)	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$
Low(L)	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$
Very Low(VL)	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$
Very Very Low(VVL)	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$
Absolutely low(AL)	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$

5.1. An Illustrative Example.

Example 5.1. In Turkiye, awareness of zero-waste has been growing recently as in other developing countries. As a result of this growth, wastes created by people have been collected more than ever. This issue triggers Turkish government to take action for recycling the collected wastes instead of releasing them to nature. As the first step of the action, the Government wants to set a zero-waste factory in the biggest city of the country, Istanbul. The government has five possible zero-waste factories to be chosen and four attributes to consider. List of five possible alternatives as follows:

- (1) Z_1 (plastic waste),
- (2) Z_2 (paper waste),
- (3) Z_3 (battery waste),
- (4) Z_4 (organic waste),
- (5) Z_5 (glass waste).

The government will decide by considering the attributes given below:

- (1) G_1 is setup cost,
- (2) G_2 is human resource,
- (3) G_3 is adaptation period of public,
- (4) G_4 is amount of waste.

Their weight vector is $(0.3, 0.1, 0.2, 0.4)$.

Three decision makers D_k ($k \in I_3$) will take part in decision process. Their weight vector is $\tau = (0.4, 0.3, 0.3)$. Trapezoidal fuzzy decision matrices are shown in Tables 6, 7 and 8.

Step 1. We constructed decision matrices of each decision maker according to Table 2 [13] linguistically (in Tables 3, 4 and 5) and numerically (in Tables 6, 7 and 8):

TABLE 3. Decision Matrix F_1

	G_1	G_2	G_3	G_4
Z_1	VL	FL	FH	VH
Z_2	VVL	L	M	H
Z_3	AH	VVH	L	AL
Z_4	M	VVL	VVH	M
Z_5	VVH	AH	H	L

TABLE 4. Decision Matrix F_2

	G_1	G_2	G_3	G_4
Z_1	VVL	FH	AL	VVL
Z_2	VVH	L	FH	VVH
Z_3	AL	AH	VVL	M
Z_4	VH	VL	H	L
Z_5	FH	FL	M	AH

TABLE 5. Decision Matrix F_3

	G_1	G_2	G_3	G_4
Z_1	VVL	FL	L	VL
Z_2	H	M	M	FH
Z_3	AH	VVL	H	AH
Z_4	FH	VL	VVH	L
Z_5	VH	AL	AL	M

TABLE 6. Decision Matrix F_1

	G_1	G_2	G_3	G_4
Z_1	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3
Z_2	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6
Z_3	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4
Z_4	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8
Z_5	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1

TABLE 7. Decision Matrix F_2

	G_1	G_2	G_3	G_4
Z_1	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1
Z_2	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9
Z_3	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8
Z_4	(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1
Z_5	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2

TABLE 8. Decision Matrix F_3

	G_1	G_2	G_3	G_4
Z_1	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3
Z_2	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4
Z_3	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2
Z_4	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1
Z_5	(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8

Step 2. Since G_1 and G_3 are cost attributes, normalized decision matrix \tilde{R}_k is constructed. The results are shown in Tables 9, 10 and 11:

TABLE 9. Decision Matrix \tilde{R}_1

	G_1	G_2	G_3	G_4
Z_1	(0.10, 0.15, 0.15, 0.20); 0.8, 0.6, 0.5, 0.7	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.30, 0.35, 0.40, 0.45); 0.4, 0.9, 0.2, 0.6	(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3
Z_2	(0.05, 0.10, 0.15, 0.20); 0.8, 0.7, 0.6, 0.9	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.25, 0.30, 0.35, 0.40); 0.6, 0.5, 0.4, 0.2	(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6
Z_3	(0.70, 0.80, 0.90, 1.00); 0.3, 0.2, 0.1, 0.8	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9	(0.10, 0.20, 0.20, 0.30); 0.7, 0.6, 0.2, 0.9	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4
Z_4	(0.45, 0.55, 0.65, 0.75); 0.3, 0.2, 0.4, 0.7	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.50, 0.60, 0.70, 0.80); 0.9, 0.3, 0.2, 0.1	(0.25, 0.30, 0.35, 0.40); 0.3, 0.4, 0.8, 0.1
Z_5	(0.50, 0.60, 0.70, 0.80); 0.9, 0.3, 0.2, 0.1	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.40, 0.45, 0.50, 0.55); 0.2, 0.1, 0.7, 0.4	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1

TABLE 10. Decision Matrix \tilde{R}_2

	G_1	G_2	G_3	G_4
Z_1	(0.05, 0.10, 0.15, 0.20); 0.8, 0.7, 0.6, 0.9	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4	(0.01, 0.05, 0.10, 0.15); 0.9, 0.8, 0.7, 0.6	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1
Z_2	(0.50, 0.60, 0.70, 0.80); 0.9, 0.3, 0.2, 0.1	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1	(0.30, 0.35, 0.40, 0.45); 0.4, 0.9, 0.2, 0.6	(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9
Z_3	(0.01, 0.05, 0.10, 0.15); 0.9, 0.8, 0.7, 0.6	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2	(0.05, 0.10, 0.15, 0.20); 0.8, 0.7, 0.6, 0.9	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8
Z_4	(0.45, 0.55, 0.65, 0.75); 0.3, 0.2, 0.4, 0.7	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3	(0.40, 0.45, 0.50, 0.55); 0.2, 0.1, 0.7, 0.4	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1
Z_5	(0.30, 0.35, 0.40, 0.45); 0.4, 0.9, 0.2, 0.6	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.25, 0.30, 0.35, 0.40); 0.6, 0.5, 0.4, 0.2	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2

TABLE 11. Decision Matrix \tilde{R}_3

	G_1	G_2	G_3	G_4
Z_1	(0.05, 0.10, 0.15, 0.20); 0.8, 0.7, 0.6, 0.9	(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5	(0.10, 0.20, 0.20, 0.30); 0.7, 0.8, 0.2, 0.9	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3
Z_2	(0.40, 0.45, 0.50, 0.55); 0.2, 0.1, 0.7, 0.4	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8	(0.25, 0.30, 0.35, 0.40); 0.6, 0.5, 0.4, 0.2	(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4
Z_3	(0.70, 0.80, 0.90, 1.00); 0.3, 0.2, 0.1, 0.8	(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1	(0.40, 0.45, 0.50, 0.55); 0.2, 0.1, 0.7, 0.4	(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2
Z_4	(0.30, 0.35, 0.40, 0.45); 0.4, 0.9, 0.2, 0.6	(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3	(0.50, 0.60, 0.70, 0.80); 0.9, 0.3, 0.2, 0.1	(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1
Z_5	(0.45, 0.55, 0.65, 0.75); 0.3, 0.2, 0.4, 0.7	(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4	(0.01, 0.05, 0.10, 0.15); 0.9, 0.8, 0.7, 0.6	(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8

Step 3. Use equations (4.1) and (4.2) to find weight $\eta_{ij}^{(k)}$ ($i \in I_5, j \in I_4, k \in I_3$) related to attribute values $\tilde{r}_{ij}^{(k)}$ ($i \in I_5, j \in I_4, k \in I_3$) which are given in the matrices $\eta^{(k)} = (\eta_{ij}^{(k)})_{5 \times 4}$ ($k \in I_3$) and given in Tables 12, 13 and 14 respectively.

TABLE 12. Weight Matrix $\eta^{(1)}$

	G_1	G_2	G_3	G_4
Z_1	0.385	0.389	0.384	0.372
Z_2	0.383	0.389	0.387	0.392
Z_3	0.399	0.403	0.391	0.391
Z_4	0.388	0.385	0.389	0.382
Z_5	0.388	0.367	0.387	0.394

TABLE 13. Weight Matrix $\eta^{(2)}$

	G_1	G_2	G_3	G_4
Z_1	0.307	0.301	0.305	0.313
Z_2	0.304	0.310	0.304	0.303
Z_3	0.279	0.308	0.309	0.324
Z_4	0.301	0.308	0.300	0.309
Z_5	0.304	0.321	0.313	0.284

TABLE 14. Weight Matrix $\eta^{(3)}$

	G_1	G_2	G_3	G_4
Z_1	0.307	0.310	0.312	0.315
Z_2	0.314	0.301	0.309	0.306
Z_3	0.321	0.289	0.300	0.285
Z_4	0.311	0.308	0.311	0.309
Z_5	0.308	0.312	0.300	0.322

Step 4. Use the TFMPWA and TFMPWG operators to find the aggregation of all individual decision matrices into the collective decision matrix in Tables 15 and 16.

TABLE 15. Decision Matrix \tilde{R} (TFMPWA)

	G_1	G_2	G_3	G_4
Z_1	(0.069, 0.119, 0.150, 0.200); 0.800, 0.664, 0.564, 0.817	(0.195, 0.215, 0.295, 0.345); 0.469, 0.489, 0.473, 0.472	(0.149, 0.212, 0.246, 0.312); 0.720, 0.810, 0.407, 0.740	(0.214, 0.283, 0.336, 0.405); 0.445, 0.582, 0.513, 0.243
Z_2	(0.148, 0.362, 0.427, 0.492); 0.750, 0.452, 0.549, 0.658	(0.331, 0.290, 0.345, 0.330); 0.332, 0.432, 0.754, 0.428	(0.189, 0.315, 0.365, 0.415); 0.547, 0.694, 0.345, 0.352	(0.324, 0.465, 0.520, 0.593); 0.610, 0.727, 0.673, 0.702
Z_3	(0.507, 0.590, 0.676, 0.762); 0.594, 0.457, 0.338, 0.757	(0.432, 0.517, 0.603, 0.688); 0.370, 0.662, 0.778, 0.642	(0.175, 0.244, 0.275, 0.344); 0.645, 0.533, 0.519, 0.829	(0.285, 0.345, 0.409, 0.473); 0.423, 0.537, 0.665, 0.544
Z_4	(0.526, 0.391, 0.456, 0.521); 0.463, 0.651, 0.544, 0.520	(0.081, 0.131, 0.150, 0.200); 0.290, 0.363, 0.464, 0.229	(0.469, 0.555, 0.630, 0.725); 0.813, 0.245, 0.404, 0.203	(0.157, 0.238, 0.257, 0.338); 0.330, 0.440, 0.739, 0.493
Z_5	(0.424, 0.509, 0.593, 0.678); 0.686, 0.597, 0.268, 0.499	(0.308, 0.374, 0.442, 0.510); 0.472, 0.615, 0.642, 0.371	(0.236, 0.283, 0.333, 0.383); 0.655, 0.523, 0.627, 0.419	(0.319, 0.403, 0.447, 0.531); 0.476, 0.586, 0.795, 0.464

TABLE 16. Decision Matrix \tilde{R} (TFMPWG)

	G_1	G_2	G_3	G_4
Z_1	(0.065, 0.117, 0.156, 0.200); 0.800, 0.660, 0.559, 0.817	(0.185, 0.219, 0.288, 0.339); 0.452, 0.458, 0.304, 0.469	(0.076, 0.162, 0.211, 0.283); 0.610, 0.705, 0.283, 0.681	(0.141, 0.214, 0.259, 0.327); 0.319, 0.473, 0.499, 0.213
Z_2	(0.193, 0.276, 0.349, 0.418); 0.537, 0.294, 0.451, 0.358	(0.132, 0.226, 0.237, 0.327); 0.327, 0.428, 0.734, 0.187	(0.264, 0.314, 0.365, 0.415); 0.530, 0.598, 0.324, 0.279	(0.392, 0.455, 0.517, 0.579); 0.390, 0.426, 0.545, 0.599
Z_3	(0.213, 0.309, 0.487, 0.588); 0.408, 0.295, 0.172, 0.738	(0.285, 0.391, 0.485, 0.574); 0.222, 0.371, 0.679, 0.300	(0.122, 0.206, 0.241, 0.318); 0.501, 0.367, 0.409, 0.706	(0.095, 0.197, 0.281, 0.354); 0.275, 0.400, 0.514, 0.411
Z_4	(0.316, 0.378, 0.439, 0.501); 0.429, 0.456, 0.322, 0.410	(0.077, 0.129, 0.150, 0.200); 0.290, 0.358, 0.459, 0.197	(0.468, 0.559, 0.633, 0.713); 0.575, 0.216, 0.291, 0.152	(0.142, 0.233, 0.248, 0.335); 0.335, 0.436, 0.717, 0.221
Z_5	(0.414, 0.496, 0.577, 0.658); 0.501, 0.370, 0.248, 0.314	(0.114, 0.216, 0.301, 0.376); 0.319, 0.474, 0.394, 0.333	(0.114, 0.205, 0.276, 0.337); 0.443, 0.309, 0.588, 0.364	(0.233, 0.338, 0.367, 0.463); 0.419, 0.523, 0.754, 0.238

Step 5. By using the decision information in Tables 15 and 16, TFMPWA-TFMPWG operators and weight vector of the attributes $v=(0.3,0.1,0.2,0.4)$, we access to the overall preference values of the alternatives. The aggregating results are presented in Tables 17 and 18.

TABLE 17. The overall preference values of the alternatives (TFMPWA)

Z_1	((0.155, 0.216, 0.258, 0.319); 0.645, 0.645, 0.506, 0.635)
Z_2	((0.245, 0.381, 0.438, 0.502); 0.629, 0.630, 0.598, 0.613)
Z_3	((0.344, 0.416, 0.482, 0.555); 0.525, 0.529, 0.576, 0.697)
Z_4	((0.263, 0.337, 0.383, 0.457); 0.509, 0.477, 0.564, 0.431)
Z_5	((0.333, 0.408, 0.468, 0.543); 0.586, 0.581, 0.642, 0.458)

TABLE 18. The overall preference values of the alternatives (TFMPWG)

Z_1	((0.101, 0.171, 0.213, 0.275); 0.495, 0.558, 0.442, 0.435)
Z_2	((0.264, 0.339, 0.396, 0.464); 0.448, 0.408, 0.478, 0.392)
Z_3	((0.142, 0.257, 0.339, 0.423); 0.341, 0.372, 0.364, 0.529)
Z_4	((0.216, 0.301, 0.338, 0.418); 0.381, 0.376, 0.450, 0.244)
Z_5	((0.223, 0.328, 0.389, 0.473); 0.435, 0.420, 0.482, 0.291)

Step 6. According to the results shown in Tables 17 and 18, Definitions 3.1 and 3.6, the ordering of the alternatives is shown below:

for TFMPWA

- $$\begin{aligned}
 d(Z_1, r^+) &= \frac{1}{32} (|(1 + 0.645)0.155 - (1 + 1)1| + |(1 + 0.645)0.155 - (1 + 1)1| \\
 &\quad + |(1 + 0.506)0.155 - (1 + 1)1| + |(1 + 0.635)0.155 - (1 + 1)1| \\
 &\quad + |(1 + 0.645)0.216 - (1 + 1)1| + |(1 + 0.645)0.216 - (1 + 1)1| \\
 &\quad + |(1 + 0.506)0.216 - (1 + 1)1| + |(1 + 0.635)0.216 - (1 + 1)1| \\
 &\quad + |(1 + 0.645)0.258 - (1 + 1)1| + |(1 + 0.645)0.258 - (1 + 1)1| \\
 &\quad + |(1 + 0.506)0.258 - (1 + 1)1| + |(1 + 0.635)0.258 - (1 + 1)1| \\
 &\quad + |(1 + 0.645)0.319 - (1 + 1)1| + |(1 + 0.645)0.319 - (1 + 1)1| \\
 &\quad + |(1 + 0.506)0.319 - (1 + 1)1| + |(1 + 0.635)0.319 - (1 + 1)1|) \\
 &= 0.807
 \end{aligned}$$
 - $d(Z_2, r^+) = 0.683$
 - $d(Z_3, r^+) = 0.645$
 - $d(Z_4, r^+) = 0.731$
 - $d(Z_5, r^+) = 0.657$

Then we get $Z_1 > Z_4 > Z_2 > Z_5 > Z_3$ and the best alternative is Z_1 that is, the government should set plastic waste factory.

for **TFMPWG**

- $d(Z_1, r^+) = 0.859$
- $d(Z_2, r^+) = 0.738$
- $d(Z_3, r^+) = 0.797$
- $d(Z_4, r^+) = 0.783$
- $d(Z_5, r^+) = 0.751$

Therefore we get $Z_1 > Z_3 > Z_4 > Z_5 > Z_2$ and the best alternative is Z_1 , i.e., the government should set plastic waste factory.

6. COMPARISON TABLE

TABLE 19. Some rankings in terms of different methods and proposed methods of Example 5.1

Methods	Operator	Ranking
Proposed Method 1	<i>TFMPWA</i>	$Z_1 > Z_4 > Z_2 > Z_5 > Z_3$
Proposed Method 2	<i>TFMPWG</i>	$Z_1 > Z_3 > Z_4 > Z_5 > Z_2$
Method of Uluçay et al. [8]	<i>TFMG_v</i>	$Z_5 > Z_3 > Z_4 > Z_1 > Z_2$
Method of Şahin et al. [9]	<i>D_v</i>	$Z_3 > Z_5 > Z_1 > Z_4 > Z_2$
Method of Uluçay [11]	<i>S_v</i>	$Z_4 > Z_3 > Z_1 > Z_5 > Z_2$
Method of Deli and Keleş [12]	<i>Sⁱ(Z_i)</i>	$Z_5 > Z_3 > Z_4 > Z_1 > Z_2$
Method of Kesen and Deli [13]	<i>TFMBHM_v^(1,1)</i>	$Z_1 > Z_3 > Z_5 > Z_2 > Z_4$

In the Table 19, we gave a brief comparison of introduced operators with some existing operators such as the weighted Bonferroni harmonic mean operator given by Kesen and Deli [13], distance measure operator proposed by Deli and Keleş [12], TFM weighted geometric operator introduced by Uluçay et al. [8], weighted dice vector similarity operator submitted by Şahin et al. [9] and vector similarity operator given by Uluçay [11] based on Example 5.1 including a zero-waste problem having five alternatives and four attributes and given under trapezoidal fuzzy multi-environment. Zero-waste has been drawing attention all around the world for saving nature and decreasing the destruction of the natural surroundings. This is the reason why we chose the zero-waste problem. As for the operators we introduced, we used them efficiently on the problem. If the comparison table

is analyzed, results of the proposed aggregation methods present a new perspective to decision-making process and are compatible with the existing methods. Therefore, decision-makers can easily use the proposed methods to solve decision-making problems with multiple criteria.

7. CONCLUSION AND FUTURE SCOPE OF STUDIES

In this article, we proposed TFMPWA operator and TFMPWG operator to find the solution of a zero-waste decision-making problem. By using these operators, this article aimed to investigate how to solve zero-waste group decision-making problem with multi-criterion by using trapezoidal fuzzy multi-numbers and to access a solution of the problem. In the paper, a zero-waste problem including five alternatives and four attributes was handled. In consideration of the given data in the problem, introduced operators were efficiently utilized to have a solution. The results we obtained may help to decision-makers to find the best alternative in a decision-making problem. In addition, they are consistent with the existing methods. As seen in the application, the main advantage of the operators is allowing the argument values to support each other in the aggregation process. As for limitation, since the operators don't contain parameters, decision-makers lack flexibility during the decision-making process.

In the future, the operators will be applied to other zero-waste problems given with triangular intuitionistic fuzzy multi-numbers, trapezoidal intuitionistic fuzzy multi-numbers, and trapezoidal neutrosophic fuzzy multi-numbers. In addition, the operators will be extended to bipolar soft sets and bipolar complex fuzzy sets and applications. In the daily life that are given under the bipolar soft sets and the bipolar complex fuzzy sets will be given.

REFERENCES

- [1] L. A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965) 338–353.
- [2] S. Sahin, B. Bozkurt and A. Kargin, Comparing the social justice leadership behaviors of school administrators according to teacher perceptions using classical and fuzzy Logic, *NeuroAlgebra Theory I* (2021) 145–160.
- [3] S. Sahin, M. Kisaoglu and Kargin, In determining the level of teachers' commitment to the teaching profession using classical and fuzzy logic, *Neutrosophic Algebraic Structures and Their Applications* (2022) 183–200.
- [4] R. R. Yager, On the theory of bags, *International Journal of General Systems* 13 (1986) 23-37.
- [5] S. Miyamoto, Fuzzy multi sets and their generalizations, *Lecture Notes in Computer Science* 2235 (2001) 225–235.
- [6] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy sets, *International Mathematical Forum* 5 (50) (2010) 2471–2476.
- [7] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy extensions of functions, *Advances in Adaptive Data Analysis* 3 (3) (2011) 339–350.
- [8] V. Uluçay, I. Deli and M. Şahin, Trapezoidal fuzzy multi-number and its application to multi-criteria decision-making problems, *Neural Computing and Applications* 30 (5) (2018) 1469–1478.
- [9] M. Şahin, V. Uluçay and F. S. Yilmaz, Dice vector similarity measure of trapezoidal fuzzy multi-numbers based on multi-criteria decision making, *Neutrosophic Triplet Structures* (1) (2019) 185–197.
- [10] M. Şahin, V. Uluçay and F. S. Yilmaz, Improved hybrid vector similarity measures and their applications on trapezoidal fuzzy multi numbers, *Neutrosophic Triplet Structures* (1) (2019) 158–184.
- [11] V. Uluçay, A new similarity function of trapezoidal fuzzy multi-numbers based on multi-criteria decision making, *Journal of the Institute of Science and Technology* 10 (2) (2020) 1233–1246.
- [12] I. Deli and M. A. Keleş, Distance measures on trapezoidal fuzzy multi-numbers and application to multi-criteria decision-making problems, *Soft Computing* 25 (8) (2021) 5979–5992.
- [13] D. Kesen and I. Deli, A novel operator to solve decision-making problems under trapezoidal fuzzy multi numbers and its application, *Journal of New Theory* (40) (2022) 60–73.

- [14] M. Şahin, I. Deli and D. Kesen, A decision-making method under trapezoidal fuzzy multi-numbers based on centroid point and circumcenter of centroids, *Neutrosophic SuperHyperAlgebra And New Types of Topologies* 2023 148–171.
- [15] I. Deli and D. Kesen, Bonferroni geometric mean operator of trapezoidal fuzzy multi numbers and its application to multiple attribute decision making problems, *Neutrosophic Super Hype Algebra And New Types of Topologies* 2023 237–252.
- [16] D. Bakbak and V. Uluçay, Multi-criteria decision-making method based on intuitionistic trapezoidal fuzzy multi-numbers and some harmonic aggregation operators: Application of architecture, *Neutrosophic Super Supe Hype Algebra And New Types of Topologies* 2023 172–192.
- [17] V. Uluçay and N. M. Şahin, Some harmonic aggregation operators with trapezoidal fuzzy multi-numbers: Application of law, *Neutrosophic SuperHyperAlgebra And New Types of Topologies* 2023 202–218.
- [18] D. Kesen and I. Deli, Trapezoidal fuzzy multi aggregation operator based on archimedean norms and their application to multi attribute decision-making problems, *Data-Driven Modelling with Fuzzy Sets: Embracing Uncertainty* CRC Press/Taylor & Francis Group (In Press).
- [19] D. Kesen and I. Deli, Bonferroni arithmetic mean operator of trapezoidal fuzzy multi numbers and its decision-making application to crafting the ideal student dormitory, *Journal of New Results in Science* 12 (3) (2023) 166–187.
- [20] R. R. Yager, The power average operator, *IEEE Transactions on Systems, Man, and Cybernetics-Part A* 31 (6) (2001) 724–731.
- [21] N. Jamil, M. Riaz, Cubic bipolar fuzzy aggregation operator with priority degree with multi-criteria decision-making, *Scientia Iranica* (In Press) 2023 1–43.
- [22] T. Mahmood, U. U. Rehman, W. Emam, Z. Ali and W. Haolun, Partitioned dual Maclaurin symmetric mean operators based on picture fuzzy sets and their applications in multi-attribute decision-making problems, *Scientific Reports* 13 (2023) 20834.
- [23] Z. Ali, Decision-making techniques based on complex intuitionistic fuzzy power interaction aggregation operators and their applications, *Journal of Innovative Research in Mathematical and Computational Sciences* 1 (1) (2022) 107–125.
- [24] A. Hussain and D. Pamucar, Multi-attribute group decision-making based on Pythagorean fuzzy rough set and novel Schweizer-Sklar t-norm and t-conorm, *Journal of Innovative Research in Mathematical and Computational Sciences* 1 (2) (2022) 1–17.
- [25] A. Hussain, S. Ali and K. Ullah, A novel approach of picture fuzzy sets with unknown degree of weights based on Schweizer-Sklar aggregation operators, *Journal of Innovative Research in Mathematical and Computational Sciences* 1 (2) (2022) 18–39.
- [26] Z. Xu and R. R. Yager, Power geometric operators and their use in group decision making, *IEEE Transactions on Fuzzy Systems* 18 (2010) 94–105.
- [27] Z. Xu and X. Cai, Uncertain power average operators for aggregating interval fuzzy preference relations, *Group Decision and Negotiation* (21) (2010) 381–397
- [28] Z. Xu, Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators, *Knowledge-Based Systems* 24 (2011) 749–760.
- [29] H. Garg, J. Gwak, T. Mahmood and Z. Ali, Power aggregation operators and VIKOR methods for complex q-rung orthopair fuzzy sets and their applications, *Mathematics* 8 (4) (2020) 538.
- [30] T. Mahmood, W. Ali, Z. Ali and R. Chinram, Power aggregation operators and similarity measures based on improved intuitionistic hesitant fuzzy sets and their applications to multiple attribute decision making, *Computer Modeling in Engineering and Sciences* 126 (3) (2020) 1165–1187.
- [31] H. Garg, K. Ullah, T. Mahmood, N. Hassan and N. Jan, T-spherical fuzzy power aggregation operators and their applications in multi-attribute decision making, *Journal of Ambient Intelligence and Humanized Computing* (12) (2021) 9067–9080.
- [32] T. Mahmood, Z. Ali and M. Aslam, Applications of complex picture fuzzy soft power aggregation operators in multi-attribute decision making, *Scientific Reports* (12) (2022) 16449.
- [33] B. Durgun and F. Durgun, Yenilebilir enerji tüketimi ile ekonomik buyume arasinda nedensellik iliskisi Turkiye ornegi, *International Review Of Economics and Management* 6 (1) (2018) 1–27.
- [34] H. Onder, Sürdürülebilir kalkinma anlayisinda yeni bir kavram: Dongusel ekonomi, *Dumlupinar Universitesi Sosyal Bilimler Dergisi* (57) (2018) 196–204.

- [35] O. Okorie, K. Salonitis, F. Charnley, M. Moreno, C. Turner and A. Tiwari, Digitisation and the circular economy: A review of current research and future trends, *Energies* 11 (2018) 1–31.
- [36] D. Reike, J. V. V. Walter and W. Sjors, The circular economy: New or refurbished As CE 3.0? exploring controversies in the conceptualization of the circular economy through a focus on history and resource value retention options, *Resources, Conservation ve Recycling* 135 (2018) 246–264.
- [37] V. Uluçay and N. Okumus, A comparative analysis for multi-criteria decision-making methods: TOPSIS and VIKOR methods using NVTN-numbers for application of circular economy, *Neutrosophic Algebraic Structures and Their Applications* 2022 201–220.
- [38] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*, Fourth Edition, Kluwer Academic Publishers 2001.
- [39] A. Kaufmann and M. M. Gupta, *Fuzzy mathematical models in engineering and management science*, Elsevier 1988.
- [40] Z. Xu, *Intuitionistic Fuzzy Aggregation and Clustering*, *Studies in Fuzziness and Soft Computing*, Springer, 2012.

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