**A**nnals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (x 2023), pp. 1–x
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr



# Pair-wise continuity in soft bitopological ordered spaces

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Received 8 December 2023; Revised 3 January 2024; Accepted 23 February 2024

Abstract. This paper extends soft bitopological spaces to soft bitopological ordered spaces by introducing and analyzing properties of increasing (decreasing, balancing) soft pairwise continuous (open, closed, homeomorphism) maps, denoted as $xSP$-continuous (open, closed, homeomorphism) maps, where $x$ can take values $I$, $D$, or $B$. The investigation explores the interrelationships among these concepts, establishing equivalent conditions for each. Additionally, the study offers a comparative analysis of these notions. This research aims to contribute to the advancement of soft bitopological ordered spaces by providing a comprehensive understanding of their fundamental properties and relationships, paving the way for further exploration and applications in the field.

2020 AMS Classification: 54A40, 54C55, 06D72, 54E55

Keywords: $ISP(DSP, BSP)$ continuous, $ISP$ ($DSP$, $BSP$)-open, $ISP$ ($DSP$, $BSP$)-closed, $ISP$ ($DSP$, $BSP$)-homeomorphism.

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## 1. Introduction

In 1965, Nachbin [1] introduced the concept of a topological ordered space as a triple $(X, \tau, \leq)$, where $\tau$ is a topological structure and $\leq$ is a partial order relation on a non-empty set $X$. He established several topological ordered notions based on this structure. In 1999, Molotdsov [2] proposed the idea of soft sets as a means of dealing with uncertainties and vagueness in real-world situations and phenomena. He highlighted the advantages of soft set theory over fuzzy theory and probability theory and explored its applications in various fields. The use of soft sets in overcoming incomplete data has motivated researchers to introduce soft operations between soft sets and apply them in decision-making, information science, mathematics, and related disciplines. To incorporate soft sets into topology studies, Shabir and Naz [3] formulated the notion of soft topological spaces in 2011 by drawing an analogy with the definition of topology. El-Shafei et al. [4] introduced the concept of soft topological ordered spaces as a generalization of topological ordered spaces, along with the notions of increasing and decreasing soft sets and maps. They further extended this concept by introducing the notions of soft $x\alpha$-continuous, soft $x\alpha$-open, soft $x\alpha$-closed, and soft $x\alpha$-homeomorphism maps for $x = I, D$, and $B$ via soft topological ordered spaces. El-Sheikh et al. [5] generalized the notion of soft topological spaces by introducing the concept of supra soft topological spaces, while Ittanagi [6] introduced the notion of soft bitopological space defined over an initial universal set with a fixed set of parameters. Şenel [7] presented the soft topology generated by $L$-soft sets. Additionally, in a 2016, Şenel [8] proposed a new approach to hausdorff space theory via the soft sets. Şenel et al. [9] introduced soft topological subspaces in 2015. Furthermore, Şenel et al. [10] explored soft closed sets on soft bitopological space in 2014. In 2020, Şenel et al. [11] investigated distance and similarity measures for octahedron sets proposed by Lee et al. [12]. Kandil et al. [13] provided some structures on soft bitopological spaces and defined some basic notions such as pairwise open and closed soft sets, pairwise soft closure, interior, kernel operators, and related topics. They also studied pairwise soft continuous mappings and open and closed soft mappings between two soft bitopological spaces. El-Sheikh et al. [14] established the concept of a soft bitopological ordered space, which comprises a soft bitopological space with a partial order relation. They introduced and studied various concepts related to increasing and decreasing pairwise open and closed soft sets, increasing and decreasing total and partial pairwise soft neighborhoods, and increasing, decreasing, and balancing pairwise open soft neighborhoods. They also defined the concept of increasing and decreasing pairwise soft closure and interior.

The main objective of this research is to introduce and examine the notions of $xSP$-continuous, $xSP$-open, $xSP$-closed mappings and $xSP$-homeomorphism for $x = I; D; B$ in soft bitopological ordered spaces. The study aims to provide a comprehensive understanding of these concepts by exploring their equivalent conditions and establishing their relationships. Several examples are presented to demonstrate the connections between these maps and to show that they are more powerful than their $P$-soft counterparts. Moreover, the research characterizes each of these maps and emphasizes the crucial role of extended soft topologies in studying the links between these maps and their corresponding maps in soft bitopological ordered spaces.

## 2. Preliminaries

In the remaining part of this section, we will introduce some important definitions and results that will be necessary in the following sections.

**Definition 2.1** ([2])**.** Let $X$ be a universe set and let $E$ be a fixed set of parameters. If $G_E : E \to 2^X$ is a function, then an ordered pair $(G, E)$ is is called a *soft set*, where $2^X$ is the power set of $X$. The set of all soft sets over $X$ is denoted by $P(X)^E$.

**Definition 2.2** ([15])**.** Let $F_E,\ G_E \in P(X)^E$. Then:
(i) $F_E$ is called a *null soft set*, denoted by $\Phi$, if $F(e) = \varnothing\ \forall e \in E$,
(ii) $F_E$ is called an *absolute soft set*, denoted by $X_E$, if $F(e) = X\ \forall e \in E$,
(iii) $F_E$ is called a *soft subset* of $G_E$, denoted by $F_E \sqsubseteq G_E$, if $F(e) \subseteq G(e)\ \forall e \in E$,

(iv) $F_E$ and $G_E$ are said to *be equal*, denoted by $F_E = G_E$, if $F_E \sqsubseteq G_E$ and $G_E \sqsubseteq F_E$,

(v) The *union* of $F_E$ and $G_E$, denoted by $F_E \sqcup G_E$, is a soft set $H_E$ over $X$ defined as: $H(e) = F(e) \cup G(e)\ \forall e \in E$,

(vi) The *intersection* of $F_E$ and $G_E$, denoted by $F_E \sqcap G_E$, is a soft set $H_E$ over $X$ defined as: $H(e) = F(e) \cap G(e)\ \forall e \in E$,

(vii) The *difference* of $F_E$ and $G_E$, denoted by $F_E - G_E$, is a soft set $H_E$ over $X$ defined as: $H(e) = F(e) - G(e)\ \forall e \in E$,

(viii) The *complement* of $F_E$, denoted by $F_E^c$, is a soft set over $X$ defined as: $F^c(e) = (F(e))^c\ \forall e \in E$.

**Definition 2.3** ([16])**.** Let $\phi : X \to Y$ and $\psi : E \to K$ be two mappings. Then the mapping $\phi_\psi$ is called a *soft mapping from $X$ to $Y$*, denoted by $\phi_\psi : P(X)^E \to P(Y)^K$. The mapping $\phi_\psi$ itself is defined as follows: for any soft set $G_E \in P(X)^E$, $\phi_\psi$ maps it to a soft set $F_K \in P(Y)^K$ such that:

$$F_K(k) = \{y \in Y : y = \phi(x),\ x \in X,\ \text{and } k = \psi(e)\ e \in E\}.$$

Let $G_E \in P(X)^E$ and let $F_K \in P(Y)^K$. Then

(i) the *image of $G_E$ under $\phi_\psi$*, denoted by $\phi_\psi(G_E)$, is the soft set over $Y$ defined as follows: for each $k \in K$,

$$\phi_\psi(G_E)(k) = \begin{cases} \bigcup_{e \in \psi^{-1}(k)} G_E(e) & \text{if } \psi^{-1}(k) \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

(ii) the *inverse image of a soft set $F_K$ under $\phi_\psi$*, denoted by $\phi_\psi^{-1}(F_K)$, is the soft set over $X$ defined as follows: for each $e \in E$,

$$\phi_\psi^{-1}(F_K)(e) = \phi^{-1}(F_K(\psi(e))).$$

From Definition 2.4, we can easily see that $F_K(k) = \phi_\psi(G_E)(k)$ for each $k \in K$ if and only if for each $y \in Y$, the following conditions hold:

(1) there exist $x \in X$, $e \in E$ such that $y = \phi(x)$ and $k = \psi(e)$, $x \in G_E(e)$,

(2) there exist $e \in E$, $x \in G_E(e)$ such that $k = \psi(e)$ and $y = \phi(x)$.

**Definition 2.4** ([17])**.** Let $P(X)^E$ and $P(Y)^K$ be two families of soft sets over $X$ and $Y$, respectively. A soft mapping $\phi_\psi : P(X)^E \to P(Y)^K$ is called *soft surjective* (*injective*) *mapping*, if $\phi, \psi$ are surjective (injective) mappings, respectively. A soft mapping which is a soft surjective and soft injective mapping is called a *soft bijection mapping*.

**Proposition 2.5** ([16])**.** *Consider $\phi_\psi : P(X)^E \to P(Y)^K$ is a soft map and let $G_E$ and $H_K$ be two soft subsets of $P(X)^E$ and $P(Y)^K$, respectively. Then we have the following results:*

(1) *$G_E \sqsubseteq \phi_\psi^{-1}(\phi_\psi(G_E))$ and the equality relation holds if $\phi_\psi$ is injective,*

(2) *$\phi_\psi(\phi_\psi^{-1}(H_K)) \sqsubseteq H_K$ and the equality relation holds if $\phi_\psi$ is surjective.*

**Definition 2.6** ([3])**.** A *soft topology* on a set $X$ is a collection $\tau$ of soft sets over $X$ that satisfies the following conditions:

(i) both the universal set $X_E$ and the empty set $\Phi$ belong to $\tau$,

(ii) the union of any collection of soft sets in $\tau$ belongs to $\tau$,

(iii) the intersection of any two soft sets in $\tau$ belongs to $\tau$.
The elements of $\tau$ are called *open soft sets* in $X$ and the complement of an open soft set is called a *closed soft set* in $X$.

**Definition 2.7** ([18])**.** A *partial order relation* $\leq$ on a set $X$ satisfies three properties: reflexivity, antisymmetry, and transitivity. The equality relation on $X$ is a special case of a partial order relation, denoted by $\blacktriangle$, where $\blacktriangle = \{(a,a) : a \in X\}$ and it satisfies all three properties of a partial order relation.

**Definition 2.8** ([1])**.** A triple $(X, \tau, \leq)$ is said to be a *topological ordered space*, if $(X, \tau)$ is a topological space and $(X, \leq)$ is a partially ordered set.

**Definition 2.9** ([4])**.** A triple $(X, E, \leq)$ is said to be a *partially ordered soft space*, where $\leq$ is a partial order relation on $X$.

**Definition 2.10** ([4])**.** An *increasing soft operator* $i$ and a *decreasing soft operator* $d$ can be defined on a partially ordered soft space $(X, E, \leq)$ as follows: for each $F_E \in P(X)^E$,
(i) $i(F_E) = (iF)_E$, where $iF$ is a mapping of $E$ into $X$ given by

$$iF(\alpha) = i(F(\alpha)) = \{a \in X : b \leq a \text{ for some } b \in F(\alpha)\}.$$

(ii) $d(F_E) = (dF)_E$, where $dF$ is a mapping of $E$ into $X$ given by

$$dF(\alpha) = d(F(\alpha)) = \{a \in X : a \leq b \text{ for some } b \in F(\alpha)\}.$$

**Definition 2.11** ([4])**.** Let $(X, E, \leq)$ be a partially ordered soft space and let $F_E$ be a soft set over $X$. Then $F_E$ is said to be:
(i) *increasing*, if $F_E$ is equal to its own image under $i$,
(ii) *decreasing*, if $F_E$ is equal to its own image under $d$,
(iii) *balancing*, if it is both increasing and decreasing.

**Proposition 2.12** ([4])**.** *Let $(X, E, \leq)$ be a partially ordered soft space, and let $\{F_E^\beta : \beta \in \Omega\}$ be a collection of increasing* (resp. decreasing) *soft sets in $(X, E, \leq)$. Then*
(1) $\sqcup_{\beta\in\Omega} F_E^\beta$ *is increasing* (resp. decreasing),
(2) $\sqcap_{\beta\in\Omega} F_E^\beta$ *is increasing* (resp. decreasing).

**Definition 2.13** ([4])**.** A quadrable system $(X, \tau, E, \leq)$ is said to be a *soft topological ordered space*, if $(X, \tau, E)$ is a soft topological space and $(X, \leq)$ is a partially ordered set.

**Definition 2.14** ([4])**.** Let $(X, \tau, E, \leq)$ be a soft topological ordered space. A soft set $G_E$ over $X$ is called:
(i) an *increasing open soft* (briefly, *IO*-soft), if $G_E$ is open and increasing,
(ii) a *decreasing open soft set* (briefly, *DO*-soft set), if $G_E$ is open and decreasing,
(iii) a *balancing open soft set* (briefly, *BO*-soft set), if $G_E$ is an *IO*-soft and *DO*-soft set.

**Definition 2.15** ([19])**.** A soft map $\phi_\psi : (X, \tau, E, \leq_1) \to (Y, \eta, K, \leq_2)$ is said to be $IS$ (resp. $DS$, $BS$)-*continuous*, if f the inverse image of each open soft subset of $Y$ is a soft $IO$ (resp. $DO$, $BO$)-soft subset of $X$.

**Definition 2.16** ([6]). A quadruple $(X, \tau_1, \tau_2, E)$ is said to be a *soft bitopological space*, where $\tau_1, \tau_2$ are arbitrary soft topologies on $X$ with a fixed set of parameter $E$.

**Definition 2.17** ([13]). A soft set $F_E$ in a soft bitopological space $(X, \tau_1, \tau_2, E)$ is said to be a pairwise open soft (briefly, $PO$-soft) *set*, if it can be expressed as the union of a $\tau_1$-open soft set $F_E^1$ and a $\tau_2$-open soft set $F_E^2$, i.e., $F_E = F_E^1 \sqcup F_E^2$. On the other hand, $F_E$ is said to be a *pairwise closed soft* (briefly, $PC$-soft) *set*, if its complement in $X$, denoted by $X - F_E$, is a $PO-$soft set.

**Definition 2.18** ([13]). The following concepts are defined for a subset $G_E$ of $(X, \tau_1, \tau_2, E)$.

(i) The *pairwise soft closure* of $G_E$, denoted by $cl_{12}^s(G_E)$, is the intersection of all $PC$-soft sets containing $G_E$.

(ii) The *pairwise soft interior* of $G_E$, denoted by $int_{12}^s(G_E)$, is the union of all $PO$-soft sets which are contained in $G_E$.

**Definition 2.19** ([20]). A soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E) \to (Y, \eta_1, \eta_2, K)$ is said to be:

(i) *pairwise soft continuous* (briefly, $P$-soft continuous), if the inverse image of any $PO-$soft set in $(Y, \eta_1, \eta_2, K)$ is a $PO$-soft set in $(X, \tau_1, \tau_2, E)$,

(ii) *pairwise soft open* (briefly, $P$-soft open), if the image of any $PO-$soft set in $(X, \tau_1, \tau_2, E)$ is a $PO$-soft set in $(Y, \eta_1, \eta_2, K)$,

(iii) *pairwise soft closed* (briefly, $P$-soft closed), if the image of any $PC-$soft set in $(X, \tau_1, \tau_2, E)$ is a $PC$-soft set in $(Y, \eta_1, \eta_2, K)$,

(iv) *pairwise soft homeomorphism* (briefly, $P$-soft homeomorphism), if it is bijective, $P$-soft continuous and $P$-soft open.

**Definition 2.20** ([14]). A system $(X, \tau_1, \tau_2, E, \leq)$ is said to be a *soft bitopological ordered space*, if $(X, \tau_1, \tau_2, E)$ is a soft bitopological space and $(X, \leq)$ is a partially ordered set.

**Definition 2.21** ([14]). Let $(X, \tau_1, \tau_2, E, \leq)$ be a soft bitopological ordered space and let $G_E$ be a soft set over $X$. Then $G_E$ is said to be:

(i) an *increasing* (resp. a *decreasing*) *pairwise open* (briefly, $IPO$) (resp. briefly, $DPO$)-*soft set*, if it can be expressed as $G_E = G_E^1 \sqcup G_E^2$, where $G_E^\beta \in \tau_\beta$ and is increasing ( decreasing ) for $\beta = 1, 2$.

(ii) an *increasing* (resp. *decreasing*) *pairwise closed* (briefly, $IPC$) (resp. briefly, $DPC$)-*soft set*, if it can be expressed as $G_E = G_E^1 \sqcap G_E^2$, where $G_E^\beta \in \tau_\beta^c$ and is increasing ( decreasing ) for $\beta = 1, 2$.

(iii) a *balancing pairwise open* (resp. *closed*) (briefly, $BPO$) (resp. briefly, $BPC$)-*soft set*, if it is both increasing pairwise open (resp. closed) and decreasing pairwise open (resp. closed).

**Definition 2.22** ([14]). Let $(X, \tau_1, \tau_2, E, \leq)$ be a soft bitopological ordered space and let $G_E$ be a soft set over $X$.

(i) The *increasing* (resp. *decreasing*, *balancing*) *pairwise soft closure* of $G_E$, denoted by $Icl_{12}^s(G_E)$ (resp. $Dcl_{12}^s(G_E)$, $Bcl_{12}^s(G_E)$), is the intersection of all $IPO$ (resp. $DPO$, $BPO$)-soft sets containing $G_E$.

(ii) The *increasing* (resp. *decreasing*, *balancing*) *pairwise soft interior* of $G_E$, denoted by $Iint^s_{12}(G_E)$ (resp. $Dint^s_{12}(G_E)$, $Bint^s_{12}(G_E)$), is the union of all $IPO$ ($DPO$, $BPO$)-soft sets contained in $G_E$.

## 3. $ISP$ ($DSP$, $BSP$)- continuous mappings

In this section, we introduce a different kind of a definition of soft continuity in a soft bitopological ordered space.

**Definition 3.1.** A soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ is said to be $ISP$ (resp. $DSP$, $BSP$)-*continuous*, if $\phi_\psi^{-1}(G_E)$ is an $IPO$ (resp. $DPO$, $BPO$)-soft set in $X$, whenever $G_E$ is a $PO$-soft set in $Y$.

**Example 3.2.** Let $E = \{\alpha_1, \alpha_2\}$ and $\leq = \blacktriangle \cup \{(a, b), (b, c), (a, c)\}$ be a partial order relation on $X = \{a, b, c\}$ and let $\tau_1 = \{X_E, \Phi, G_E\}$ and $\tau_2 = \{X_E, \Phi, F_E\}$, where $G_E = \{(\alpha_1, \{b\}), (\alpha_2, \{a\})\}$, $F_E = \{(\alpha_1, \{a, c\}), (\alpha_2, \varnothing)\}$.
Then clearly, $(X, \tau_1, \tau_2, E, \leq)$ is a soft bitopological ordered space. Consider $\phi$ and $\psi$ are identity mappings. Since $G_E$ is a $PO$-soft set and $\phi_\psi^{-1}(G)(\alpha_1) = \phi^{-1}(G(\psi(\alpha_1))) = \phi^{-1}(G(\alpha_1)) = \phi^{-1}(\{b\}) = \{b\}$, $\phi_\psi^{-1}(G_E)$ is not an element in $xPOS(X, \tau_1, \tau_2)_E, x = I; D; B$. Then $\phi_\psi$ is $P$-soft continuous but it is not $xSP$-continuous for $x = I; D; B$.

**Example 3.3.** Let $E = \{\alpha_1, \alpha_2\} = K$ and $X = \{a, b, c\} = Y$, $\leq_1 = \blacktriangle \cup \{(a, c)\}$, $\leq_2 = \blacktriangle \cup \{(a, b), (b, c), (a, c)\}$ and let $\tau_1 = \{X_E, \Phi, G_E\} = \eta_1$ and $\tau_2 = \{X_E, \Phi, F_E\} = \eta_2$, where $G_E = \{(\alpha_1, \{c\}), (\alpha_2, \{a, c\})\}$, $F_E = \{(\alpha_1, \{c\}), (\alpha_2, \{b, c\})\}$.
Then clearly, $(X, \tau_1, \tau_2, E, \leq_1)$ and $(Y, \eta_1, \eta_2, K, \leq_2)$ are soft bitopological ordered spaces and $\tau_{12} = \eta_{12} = \{X_E, \Phi, G_E, F_E, H_E\}$, where $H_E = \{(\alpha_1, \{c\}), (\alpha_2, X)\}$. Consider $\phi$ and $\psi$ are identity mappings. Then we have

$$\phi_\psi^{-1}(G)(\alpha_1) = \phi^{-1}(G(\psi(\alpha_1))) = \phi^{-1}(G(\alpha_1)) = \phi^{-1}(\{c\}) = \{c\},$$
$$\phi_\psi^{-1}(G)(\alpha_2) = \phi^{-1}(G(\psi(\alpha_2))) = \phi^{-1}(G(\alpha_2)) = \phi^{-1}(\{a, c\}) = \{a, c\},$$
$$\phi_\psi^{-1}(F)(\alpha_1) = \phi^{-1}(F(\psi(\alpha_1))) = \phi^{-1}(F(\alpha_1)) = \phi^{-1}(\{c\}) = \{c\},$$
$$\phi_\psi^{-1}(F)(\alpha_2) = \phi^{-1}(F(\psi(\alpha_2))) = \phi^{-1}(F(\alpha_2)) = \phi^{-1}(\{b, c\}) = \{b, c\},$$
$$\phi_\psi^{-1}(H)(\alpha_1) = \phi^{-1}(H(\psi(\alpha_1))) = \phi^{-1}(H(\alpha_1)) = \phi^{-1}(\{c\}) = \{c\},$$
$$\phi_\psi^{-1}(H)(\alpha_2) = \phi^{-1}(H(\psi(\alpha_2))) = \phi^{-1}(H(\alpha_2)) = \phi^{-1}(X) = X.$$

Thus $\phi_\psi^{-1}(G_E) = G_E$, $\phi_\psi^{-1}(F_E) = F_E$ and $\phi_\psi^{-1}(H_E) = H_E$ are $IPO$-soft sets in $X$. So $\phi_\psi$ is $ISP$-continuous but it is not $DSP$-continuous. However $\phi_\psi$ is not $BSP$-continuous.

**Example 3.4.** Let $E = \{\alpha_1, \alpha_2\} = K$ and $X = \{a, b, c\} = Y$, $\leq_1 = \leq_2 = \blacktriangle \cup \{(a, b), (b, c), (a, c)\}$ and let $\tau_1 = \{X_E, \Phi, G^1_E\}, \tau_2 = \{X_E, \Phi, G^2_E\}$, and $\eta_1 = \{Y_K, \Phi, F^1_K\}, \eta_2 = \{Y_K, \Phi, F^2_K\}$, where $G^1_E = \{(\alpha_1, \{a\}), (\alpha_2, \varnothing)\}$, $G^2_E = \{(\alpha_1, \varnothing), (\alpha_2, \{a, b\})\}$, $F^1_K = \{(\alpha_1, \{a, b\}), (\alpha_2, \{b\})\}, F^2_K = \{(\alpha_1, \varnothing), (\alpha_2, \{b\})\}$.
Then clearly, $(X, \tau_1, \tau_2, E, \leq_1)$ and $(Y, \eta_1, \eta_2, K, \leq_2)$ are soft bitopological ordered spaces and $\tau_{12} = \{X_E, \Phi, G^1_E, G^2_E, G^3_E\}$ and $\eta_{12} = \{Y_K, \Phi, F^1_K, F^2_K\}$, where $G^3_E = \{(\alpha_1, \{a\}), (\alpha_2, \{a, b\})\}$.
Define: $\phi(a) = b, \phi(b) = a, \phi(c) = c$ and $\psi(\alpha_1) = \alpha_2, \psi(\alpha_2) = \alpha_1$. Then we have

$$\phi_\psi^{-1}(F^1)(\alpha_1) = \phi^{-1}(F^1(\psi(\alpha_1))) = \phi^{-1}(F^1(\alpha_2)) = \phi^{-1}(\{b\}) = \{a\},$$

$$\phi_\psi^{-1}(F^1)(\alpha_2) = \phi^{-1}(F^1(\psi(\alpha_2))) = \phi^{-1}(F^1(\alpha_1)) = \phi^{-1}(\{a,b\}) = \{a,b\},$$

Then, $\phi_\psi^{-1}(F_K^1) = G_E^3$ is a $DPO-$soft set in $X$.

$$\phi_\psi^{-1}(F^2)(\alpha_1) = \phi^{-1}(F^2(\psi(\alpha_1))) = \phi^{-1}(F^2(\alpha_2)) = \phi^{-1}(\{b\}) = \{a\},$$

$$\phi_\psi^{-1}(F^2)(\alpha_2) = \phi^{-1}(F^2(\psi(\alpha_2))) = \phi^{-1}(F^2(\alpha_1)) = \phi^{-1}(\varnothing) = \varnothing.$$

Thus $\phi_\psi^{-1}(F_K^1) = G_E^3$ and $\phi_\psi^{-1}(F_K^2) = G_E^1$ are $DPO$-soft sets in $X$. So $\phi_\psi$ is $DSP$-continuous, but it is not $ISP$-continuous map. However $\phi_\psi$ is not $BSP$-continuous.

**Remark 3.5.** The following diagram shows the relation between $ISP$-continuous, $DSP$-continuous and $BSP$-continuous mappings. For a soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ where $\phi : X \to Y, \psi : E \to K$, we have the diagram:

$\phi_\psi$ is $ISP$-continuous $\nleftrightarrow$ $\phi_\psi$ is $DSP$-continuous

$\searrow \quad \swarrow$

$\phi_\psi$ is $P$-soft continuous

$\nwarrow \quad \uparrow \quad \nearrow$

$\phi_\psi$ is $BSP$-continuous

**Theorem 3.6.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. The following statements are equivalent:*

(1) *$\phi_\psi$ is $ISP$-continuous,*
(2) *$\phi_\psi(Icl_{12}^s(G_E)) \sqsubseteq cl_{12}^s(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$,*
(3) *$Icl_{12}^s(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl_{12}^s(F_K))$ for any $F_K \in P(Y)^K$,*
(4) *for any $PC$-soft subset $M_K$ of $(Y, \eta_1, \eta_2, K, \leq_2), \phi_\psi^{-1}(M_K)$ is $DPC$-soft subset of $(X, \tau_1, \tau_2, E, \leq_1)$.*

*Proof.* (1)⇒ (2): Suppose (1) holds and let $G_E \in P(X)^E$. Since $Y - cl_{12}^s(\phi_\psi(G_E))$ is $PO$-soft set in $Y$ and $\phi_\psi$ is $ISP$-continuous, $\phi_\psi^{-1}(Y - cl_{12}^s(\phi_\psi(G_E)))$ is $IPO$-soft set in $X$. Then clearly, $X - \phi_\psi^{-1}(Y - cl_{12}^s(\phi_\psi(G_E)))$ is $DPC$-soft set in $X$. Since $X - \phi_\psi^{-1}(Y - cl_{12}^s(\phi_\psi(G_E))) = \phi_\psi^{-1}(cl_{12}^s(\phi_\psi(G_E)))$, $\phi_\psi^{-1}(cl_{12}^s(\phi_\psi(G_E)))$ is $DPC$-soft set in $X$. Since $G_E \sqsubseteq \phi_\psi^{-1}(cl_{12}^s(\phi_\psi(G_E)))$ and $Icl_{12}^s(G_E)$ is the smallest $DPC$-soft set containing $G_E$ in $X$, we have

$$Icl_{12}^s(G_E) \sqsubseteq \phi_\psi^{-1}(cl_{12}^s((\phi_\psi(G_E))),\ \phi_\psi(Icl_{12}^s(G_E)) \sqsubseteq \phi_\psi(\phi_\psi^{-1}(cl_{12}^s(\phi_\psi(G_E)))).$$

Thus $\phi_\psi(Icl_{12}^s(G_E)) \sqsubseteq cl_{12}^s(\phi_\psi(G_E))$.

(2)⇒ (3): Suppose (2) holds, and let $F_K \in P(Y)^K$ and let $H_E = \phi_\psi^{-1}(F_K)$. Then clearly, $\phi_\psi(H_E) = \phi_\psi(\phi_\psi^{-1}(F_K)) \sqsubseteq F_K$. Thus $cl_{12}^s(\phi_\psi(H_E)) \sqsubseteq cl_{12}^s(F_K)$. By the condition (2), we have

$$Icl_{12}^s(\phi_\psi^{-1}(F_K)) = Icl_{12}^s(H_E) \sqsubseteq \phi_\psi^{-1}(\phi_\psi(Icl_{12}^s(H_E))) \sqsubseteq \phi_\psi^{-1}(cl_{12}^s(\phi_\psi(H_E))).$$

But $\phi_\psi^{-1}(cl^s_{12}(\phi_\psi(H_E))) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(F_K))$. So $Icl^s_{12}(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(F_K))$.

(3)⇒ (4): Suppose (3) holds and let $M_K$ be any $PC$-soft subset of $Y$. Then clearly, $Icl^s_{12}(\phi_\psi^{-1}(M_K)) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(M_K))$. Thus $\phi_\psi^{-1}(M_K)$ is a $DPC$-soft set of $X$.

(4)⇒ (1): Suppose (4) holds and let $N_K$ be a $PO$-soft subset of $Y$. Then $\phi_\psi^{-1}(Y - N_K)$ is a $DPC$-soft set of $X$, since $Y - N_K$ is a $PC$-soft subset of $Y$. But $X - \phi_\psi^{-1}(N_K) = \phi_\psi^{-1}(Y - N_K)$. Thus $X - \phi_\psi^{-1}(N_K)$ is a $DPC$-soft set of $X$. So $\phi_\psi^{-1}(N_K)$ is an $IPO$-soft set of $X$. Hence $\phi_\psi$ is $ISP$-continuous. □

The following two theorems characterized $DSP$-continuous and $BSP$-continuous mappings, whose proofs are similar to as that of the above Theorem 3.6.

**Theorem 3.7.** *For a soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$, the following statements are equivalent:*

1. *$\phi_\psi$ is $DSP$-continuous,*
2. *$\phi_\psi(Dcl^s_{12}(G_E)) \sqsubseteq cl^s_{12}(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$,*
3. *$Dcl^s_{12}(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(F_K))$ for any $F_K \in P(Y)^K$,*
4. *for any $PC-$soft subset $M_K$ of $(Y, \eta_1, \eta_2, K, \leq_2), \phi_\psi^{-1}(M_K)$ is $IPC$-soft subset of $(X, \tau_1, \tau_2, E, \leq_1)$.*

**Theorem 3.8.** *For a soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$, the following statements are equivalent:*

1. *$\phi_\psi$ is $BSP$-continuous,*
2. *$\phi_\psi(Bcl^s_{12}(G_E)) \sqsubseteq cl^s_{12}(\phi_\psi(G_E))$ for any $G_E \in S(X)^E$,*
3. *$Bcl^s_{12}(\phi_\psi^{-1}(F_K)) \sqsubseteq \phi_\psi^{-1}(cl^s_{12}(F_K))$ for any $F_K \in S(Y)^K$.*
4. *for any $P-$closed soft subset $M_K$ of $(Y, \eta_1, \eta_2, K, \leq_2), \phi_\psi^{-1}(M_K)$ is $BPC$-soft subset of $(X, \tau_1, \tau_2, E, \leq_1)$.*

**Theorem 3.9.** *Let $(X, \tau_1, \tau_2, E, \leq_1)$ and $(Y, \eta_1, \eta_2, K, \leq_2)$ are soft bitopological ordered spaces and $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. Then $\phi_\psi$ is an $ISP$-continuous if and only if $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ and $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ are $IS$-continuous mappings.*

*Proof.* Suppose $\phi_\psi$ is an $ISP$-continuous mapping and let $G_E$ be a $PO$-soft set over $Y$. Then there exist $G^1_E \in \eta_1$ and $G^2_E \in \eta_2$ such that $G_E = G^1_E \sqcup G^2_E$. Since $\phi_\psi$ is an $ISP$-continuous mapping, $\phi_\psi^{-1}(G_E) = \phi_\psi^{-1}(G^1_E \sqcup G^2_E) = \phi_\psi^{-1}(G^1_E) \sqcup \phi_\psi^{-1}(G^2_E)$. In this case, $\phi_\psi^{-1}(G^1_E)$ is $\tau_1$-increasing and $\phi_\psi^{-1}(G^2_E)$ is $\tau_2$-increasing. Thus $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ and $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ are $IS$-continuous mappings.

Conversely, suppose $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ and $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ are $IS$-continuous mappings and let $G^1_E \in \eta_1, G^2_E \in \eta_2$. Then there exists a $PO$-soft set $G_E$ such that $G_E = G^1_E \sqcup G^2_E$. Since $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ and $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ are $IS$-continuous mappings, $\phi_\psi^{-1}(G^1_E)$ is $IO$-soft set in $\tau_1$ and $\phi_\psi^{-1}(G^2_E)$ is $IO$-soft set in $\tau_2$. Thus $\phi_\psi^{-1}(G^1_E) \sqcup \phi_\psi^{-1}(G^2_E) = \phi_\psi^{-1}(G^1_E \sqcup G^2_E) = \phi_\psi^{-1}(G_E)$ is an $IPO$-soft set. So $\phi_\psi$ is an $ISP$-continuous mapping. □

The following two theorems characterized $DSP$-continuous and $BSP$-continuous mappings, whose proofs are similar to as that of the above Theorem 3.9.

**Theorem 3.10.** *Let* $(X, \tau_1, \tau_2, E, \leq_1)$ *and* $(Y, \eta_1, \eta_2, K, \leq_2)$ *are soft bitopological ordered spaces and* $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ *be a soft mapping. Then* $\phi_\psi$ *is an* $DSP$*-continuous if and only if* $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ *and* $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ *are* $DS$*-continuous mappings.*

**Theorem 3.11.** *Let* $(X, \tau_1, \tau_2, E, \leq_1)$ *and* $(Y, \eta_1, \eta_2, K, \leq_2)$ *are soft bitopological ordered spaces and* $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ *be a soft mapping. Then* $\phi_\psi$ *is an* $BSP$*-continuous if and only if* $\phi_\psi : (X, \tau_1, E, \leq_1) \to (Y, \eta_1, K, \leq_2)$ *and* $\phi_\psi : (X, \tau_2, E, \leq_1) \to (Y, \eta_2, K, \leq_2)$ *are* $BS$*-continuous mappings.*

**Theorem 3.12.** *Let* $(X, \tau_1, \tau_2, E, \leq_1), (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $(Z, \delta_1, \delta_2, L, \leq_3)$ *are soft bitopological ordered spaces. If* $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *are* $ISP$*-continuous mappings, then* $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *is* $ISP$*-continuous.*

*Proof.* Let $W_L \in \delta_{12}$ be a $PO$-soft set over $Z$ and let us show that $(\Delta_\beta \circ \phi_\psi)^{-1}(W_L)$ is an $IPO$-soft set in $X$. Since $(\Delta_\beta \circ \phi_\psi)^{-1}(W_L) = \phi_\psi^{-1}(\Delta_\beta^{-1}(W_L))$ and $\Delta_\beta$ is $ISP$-continuous, $\Delta_\beta^{-1}(W_L)$ is an $IPO$-soft set in $Y$. On the other hand, since $\phi_\psi$ is $ISP$-continuous mapping, $\phi_\psi^{-1}(\Delta_\beta^{-1}(W_L))$ is an $IPO$-soft set in $X$. Then $\Delta_\beta \circ \phi_\psi$ is $ISP$-continuous mapping. □

The following two theorems characterized $DSP$-continuous and $BSP$-continuous mappings, whose proofs are similar to as that of the above Theorem 3.12.

**Theorem 3.13.** *Let* $(X, \tau_1, \tau_2, E, \leq_1), (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $(Z, \delta_1, \delta_2, L, \leq_3)$ *are soft bitopological ordered spaces. If* $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *are* $DSP$*-continuous mappings, then* $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *is* $DSP$*-continuous.*

**Theorem 3.14.** *Let* $(X, \tau_1, \tau_2, E, \leq_1), (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $(Z, \delta_1, \delta_2, L, \leq_3)$ *are soft bitopological ordered spaces. If* $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ *and* $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *are* $BSP$*-continuous mappings, then* $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ *is* $BSP$*-continuous.*

## 4. $ISP$ ($DSP$, $BSP$)-open and $ISP$ ($DSP$, $BSP$)-closed mappings

In this section, we introduce the concepts of $ISP$-open and $ISP$-closed mappings, as well as $DSP$-open and $DSP$-closed maps, and $BSP$-open and $BSP$-closed maps. We then establish the relationships among these concepts and provide equivalent conditions for each type of soft map. Finally, we investigate the interrelations between these soft maps and their counterparts on bitopological ordered spaces.

**Definition 4.1.** A soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ is called $ISP$ (resp. $SP$, $BSP$)-*open*, if $\phi_\psi(G_E)$ is $IPO$ (resp. $DPO$, $BPO$)-soft set in $Y$, whenever $G_E$ is a $PO$-soft set in $X$.

**Example 4.2.** From Example 3.2, $\tau_{12} = \{X_E, \Phi, G_E, F_E, H_E\}$, where $H_E = \{(\alpha_1, X), (\alpha_2, \{a\})\}$. Then clearly, $G_E$, $F_E$ and $H_E$ are $PO$-soft sets in $X$. Moreover, we have

$$\phi_\psi(G)(\alpha_1) = \cup_{e \in \psi^{-1}(\alpha_1)} \phi(G(e)) = \phi(G(\alpha_1)) = \phi(\{b\}) = \{b\},$$
$$\phi_\psi(G)(\alpha_2) = \cup_{e \in \psi^{-1}(\alpha_2)} \phi(G(e)) = \phi(G(\alpha_2)) = \phi(\{a\}) = \{a\},$$

$$\phi_\psi(F)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(F(e)) = \phi(F(\alpha_1)) = \phi(\{a, c\}) = \{a, c\},$$
$$\phi_\psi(F)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(F(e)) = \phi(F(\alpha_2)) = \phi(\varnothing) = \varnothing,$$
$$\phi_\psi(H)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(H(e)) = \phi(H(\alpha_1)) = \phi(\{c\}) = \{c\},$$
$$\phi_\psi(H)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(H(e)) = \phi(H(\alpha_2)) = \phi(X) = X.$$

Thus $\phi_\psi$ is $P$-soft open mapping but it is not $xSP$-open for $x = I; D; B$.

**Example 4.3.** Let $X, Y, \tau_1, \tau_2, \eta_1, \eta_2, E, K, \phi, \psi$ as in Example 3.3. Consider $\leq_1 = \blacktriangle \cup \{(a, b), (a, c)\}$ and $\leq_2 = \blacktriangle \cup \{(a, c)\}$. Then we get

$$\phi_\psi(G)(\alpha_1) = \phi(G(\alpha_1)) = \phi(\{c\}) = \{c\},$$
$$\phi_\psi(G)(\alpha_2) = \phi(G(\alpha_2)) = \phi(\{a, c\}) = \{a, c\},$$
$$\phi_\psi(F)(\alpha_1) = \phi(F(\alpha_1)) = \phi(\{c\}) = \{c\},$$
$$\phi_\psi(F)(\alpha_2) = \phi(F(\alpha_2)) = \phi(\{b, c\}) = \{b, c\},$$
$$\phi_\psi(M)(\alpha_1) = \phi(M(\alpha_1)) = \phi(\{c\}) = \{c\},$$
$$\phi_\psi(M)(\alpha_2) = \phi(M(\alpha_2)) = \phi(X) = Y.$$

Thus $\phi_\psi(G_E) = G_E$, $\phi_\psi(G_E) = G_E$ and $\phi_\psi(M_E) = H_E$ are $IPO$-soft sets in $Y$. So $\phi_\psi$ is an $ISP$-open mapping, but it is not a $DSP$-open mapping. However $\phi_\psi$ is not $BSP$-open.

**Example 4.4.** In Example 3.4, consider $\leq_1 = \leq_2 = \blacktriangle \cup \{(b, a), (a, c), (b, c)\}, \eta_1 = \{Y_K, \Phi, F^1_K, F^3_K\}$, where $F^3_K = \{(\alpha_1, \{a, b\}), (\alpha_2, \varnothing)\}$ and $\eta_{12} = \{Y_K, \Phi, F^1_K, F^2_K, F^3_K\}$ Then we have

$$\phi_\psi(G^1)(\alpha_1) = \phi(G^1(\alpha_2)) = \phi(\varnothing) = \varnothing,$$
$$\phi_\psi(G^1)(\alpha_2) = \phi(G(\alpha_1)) = \phi(\{a\}) = \{b\},$$
$$\phi_\psi(G^2)(\alpha_1) = \phi(G^2(\alpha_2)) = \phi(\{a, b\}) = \{a, b\},$$
$$\phi_\psi(G^2)(\alpha_2) = \phi(G^2(\alpha_1)) = \phi(\varnothing) = \varnothing,$$
$$\phi_\psi(G^3)(\alpha_1) = \phi(G^3(\alpha_2)) = \phi(\{a, b\}) = \{a, b\},$$
$$\phi_\psi(G^3)(\alpha_3) = \phi(G^2(\alpha_1)) = \phi(\{a\}) = \{b\}.$$

Thus $\phi_\psi(G^1_E) = F^2_K$, $\phi_\psi(G^2_E) = F^3_K$ and $\phi_\psi(G^3_E) = F^1_K$ are $DPO$-soft sets in $Y$. So $\phi_\psi$ is a $DSP$-open mapping, but it is not an $ISP$-open mapping. However $\phi_\psi$ is not $BSP$-open.

**Remark 4.5.** For a soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$, We have the following diagram:

$\phi_\psi$ is $ISP$-open $\not\leftrightarrow$ $\phi_\psi$ is $DSP$-open

$\searrow \quad \swarrow$

$\phi_\psi$ is $P$-soft open

$\nwarrow \quad \uparrow \quad \nearrow$

$$\phi_\psi \text{ is } BSP\text{-open.}$$

**Theorem 4.6.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. The following statements are equivalent:*

(1) *$\phi_\psi$ is $ISP$-open,*
(2) *$\phi_\psi(int^s_{12}(G_E)) \sqsubseteq Iint^s_{12}(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$,*
(3) *$int^s_{12}(\phi^{-1}_\psi(F_K)) \sqsubseteq \phi^{-1}_\psi(Iint^s_{12}(F_K))$ for any $F_K \in P(Y)^K$.*

*Proof.* (1)⇒ (3): Suppose (1) holds and let $F_K \in P(Y)^K$. Since $int^s_{12}(\phi^{-1}_\psi(F_K))$ is a $PO$-soft set in $X$ and $\phi_\psi$ is $ISP$-open map, $\phi_\psi(int^s_{12}(\phi^{-1}_\psi(F_K)))$ is an $IPO$-soft set in $Y$. Also, $\phi_\psi(int^s_{12}(\phi^{-1}_\psi(F_K))) \sqsubseteq \phi_\psi(\phi^{-1}_\psi(F_K)) \sqsubseteq F_K$. Then $\phi_\psi(int^s_{12}(\phi^{-1}_\psi(F_K))) \sqsubseteq Iint^s_{12}(F_K)$, since $Iint^s_{12}(F_K)$ is the largest $IPO$-soft set contained in $F_K$. Thus $int^s_{12}(\phi^{-1}_\psi(F_K)) \sqsubseteq \phi^{-1}_\psi(Iint^s_{12}(F_K))$.

(3)⇒ (2): Suppose (3) holds and let $G_E \in P(X)^E$. Replacing $F_K$ by $\phi_\psi(G_E)$ in (3), we get $int^s_{12}(\phi^{-1}_\psi(\phi_\psi(G_E))) \sqsubseteq \phi^{-1}_\psi(Iint^s_{12}(\phi_\psi(G_E)))$. Since $int^s_{12}(G_E)) \sqsubseteq int^s_{12}(\phi^{-1}_\psi(\phi_\psi(G_E)))$, $int^s_{12}(G_E)) \sqsubseteq \phi^{-1}_\psi(Iint^s_{12}(\phi_\psi(G_E)))$. Thus we have

$$\phi_\psi(int^s_{12}(G_E)) \sqsubseteq \phi_\psi(\phi^{-1}_\psi(Iint^s_{12}(\phi_\psi(G_E)))) \sqsubseteq Iint^s_{12}(\phi_\psi(G_E)).$$

So $\phi_\psi(int^s_{12}(G_E)) \sqsubseteq Iint^s_{12}(\phi_\psi(G_E))$.

(2)⇒ (1): Suppose (2) holds and let $G_E$ be any $PO$-soft set in $X$. Then we have

$$\phi_\psi(G_E) = \phi_\psi(int^s_{12}(G_E)) \sqsubseteq Iint^s_{12}(\phi_\psi(G_E)).$$

Thus $\phi_\psi(G_E)$ is an $IPO$-soft set in $Y$. So $\phi_\psi$ is $ISP$-open mapping. □

The following two theorems characterized $DSP$-open and $BSP$-open mappings, whose proofs are similar to as that of the above Theorem 4.6.

**Theorem 4.7.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. The following statements are equivalent:*

(1) *$\phi_\psi$ is $DSP$-open,*
(2) *$\phi_\psi(int^s_{12}(G_E)) \sqsubseteq Dint^s_{12}(\phi_\psi(G_E))$ for any $G_E \in P(X)^E$.*
(3) *$int^s_{12}(\phi^{-1}_\psi(F_K)) \sqsubseteq \phi^{-1}_\psi(Dint^s_{12}(F_K))$ for any $F_K \in P(Y)^K$.*

**Theorem 4.8.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. The following statements are equivalent:*

(1) *$\phi_\psi$ is a $BSP$-open mapping,*
(2) *$\phi_\psi(int^s_{12}(G_E)) \sqsubseteq Bint^s_{12}(\phi_\psi(G_E))$ for any $G_E \in S(X)^E$,*
(3) *$int^s_{12}(\phi^{-1}_\psi(F_K)) \sqsubseteq \phi^{-1}_\psi(Bint^s_{12}(F_K))$ for any $F_K \in S(Y)^K$.*

**Theorem 4.9.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ and $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ be two soft mappings. If $\phi_\psi$ and $\Delta_\beta$ are $ISP$-open, then $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ is $ISP$-open mapping.*

*Proof.* Suppose $\phi_\psi$ and $\Delta_\beta$ are $ISP$-open. Let $G_E$ be a $PO-$soft set over $X$ and let us show that $(\Delta_\beta \circ \phi_\psi)(G_E)$ is an $IPO$-soft set in $Z$. Since $(\Delta_\beta \circ \phi_\psi)(G_E) = \Delta_\beta(\phi_\psi(G_E))$ and $\phi_\psi$ is $ISP$-open, $\phi_\psi(G_E)$ is an $IPO$-soft set in $Y$. On the other hand, since $\Delta_\beta$ is $ISP$-open, $\Delta_\beta(\phi_\psi(G_E))$ is an $IPO$-soft set in $Z$. Thus $\Delta_\beta \circ \phi_\psi$ is $ISP$-open mapping. □

The following two theorems characterized $DSP$-open and $BSP$-open mapping, whose proofs are similar to as that of the above Theorem 4.9.

**Theorem 4.10.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ and $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ be two soft mappings. If $\phi_\psi$ and $\Delta_\beta$ are $DSP$-open, then $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ is $DSP$-open mapping.*

**Theorem 4.11.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ and $\Delta_\beta : (Y, \eta_1, \eta_2, K, \leq_2) \to (Z, \delta_1, \delta_2, L, \leq_3)$ be two soft mappings. If $\phi_\psi$ and $\Delta_\beta$ are $BSP$-open, then $\Delta_\beta \circ \phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Z, \delta_1, \delta_2, L, \leq_3)$ is $BSP$-open mapping.*

**Definition 4.12.** A soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ is called $ISP$ (resp. $DSP$, $BSP$)-*closed*, if $\phi_\psi(G_E)$ is $IPC$ (resp. $DPC$, $BPC$)-soft set in $Y$, whenever $G_E$ is a $PC$-soft set in $X$.

**Example 4.13.** From Example 4.2, we have $G^c_E = \{(\alpha_1, \{a, c\}), (\alpha_2, \{b, c\})\}, F^c_E = \{(\alpha_1, \{b\}), (\alpha_2, X)\}, H^c_E = \{(\alpha_1, \varnothing), (\alpha_2, \{b, c\})\}$. Since $G^c_E$, $F^c_E$ and $H^c_E$ are $PC$-soft set in $X$, we have

$\phi_\psi(G^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(G^c(e)) = \phi(G^c(\alpha_1)) = \phi(\{a, c\}) = \{a, c\}$,
$\phi_\psi(G^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(G^c(e)) = \phi(G^c(\alpha_2)) = \phi(\{b, c\}) = \{b, c\}$,
$\phi_\psi(F^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(F^c(e)) = \phi(F^c(\alpha_1)) = \phi(\{b\}) = \{b\}$,
$\phi_\psi(F^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(F^c(e)) = \phi(F^c(\alpha_2)) = \phi(X) = X$,
then $\phi_\psi(H^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(H^c(e)) = \phi(H^c(\alpha_1)) = \phi(\varnothing) = \varnothing$,
$\phi_\psi(H^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(H^c(e)) = \phi(H^c(\alpha_2)) = \phi(\{b, c\}) = \{b, c\}$.

Then clearly, $\phi_\psi$ is $P$-soft closed map, but it is not $xSP$-closed for $x = I; D; B$.

**Example 4.14.** From Example 4.3, we have $G^c_E = \{(\alpha_1, \{a, b\}), (\alpha_2, \{b\})\}, F^c_E = \{(\alpha_1, \{a, b\}), (\alpha_2, \{a\})\}, H^c_E = \{(\alpha_1, \{a, b\}), (\alpha_2, \varnothing)\}$. Then we have

$\phi_\psi(G^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(G^c(e)) = \phi(G^c(\alpha_1)) = \phi(\{a, b\}) = \{a, b\}$,
$\phi_\psi(G^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(G^c(e)) = \phi(G^c(\alpha_2)) = \phi(\{b\}) = \{b\}$,
$\phi_\psi(F^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(F^c(e)) = \phi(F^c(\alpha_1)) = \phi(\{a, b\}) = \{a, b\}$,
$\phi_\psi(F^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(F^c(e)) = \phi(F^c(\alpha_2)) = \phi(\{a\}) = \{a\}$,
$\phi_\psi(H^c)(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(H^c(e)) = \phi(H^c(\alpha_1)) = \phi(\{a, b\}) = \{a, b\}$,
$\phi_\psi(H^c)(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(H^c(e)) = \phi(H^c(\alpha_2)) = \phi(\varnothing) = \varnothing$.

Thus $\phi_\psi(G^c_E) = G^c_E$, $\phi_\psi(F^c_E) = F^c_E$ and $\phi_\psi(H^c_E) = H^c_E$ are an $IPC$-soft sets in $Y$. So $\phi_\psi$ is $ISP$-closed, but it is not a $DSP$-closed mapping. However $\phi_\psi$ is not $BSP$-closed.

**Example 4.15.** From Example 4.4 consider $\leq_2 = \blacktriangle \cup \{(a, b)\}$, we have $G^{1c}_E = \{(\alpha_1, \{b, c\}), (\alpha_2, X)\}, G^{2c}_E = \{(\alpha_1, X), (\alpha_2, \{c\})\}, G^{c3}_E = \{(\alpha_1, \{b, c\}), (\alpha_2, \{c\})\}, F^{1c}_K = \{(\alpha_1, \{c\}), (\alpha_2, \{a, c\})\}, F^{2c}_K = \{(\alpha_1, X), (\alpha_2, \{a, c\})\}, F^{c3}_K = \{(\alpha_1, \{c\}), (\alpha_2, X)\}$. Then

$\phi_\psi(G^{1c})(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(G^{1c}(e)) = \phi(G^{1c}(\alpha_2)) = \phi(X) = X$,
$\phi_\psi(G^{1c})(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(G^{1c}(e)) = \phi(G^c(\alpha_1)) = \phi(\{b, c\}) = \{a, c\}$,
$\phi_\psi(G^{2c})(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(G^{2c}(e)) = \phi(G^{2c}(\alpha_2)) = \phi(\{c\}) = \{c\}$,
$\phi_\psi(G^{2c})(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(G^{2c}(e)) = \phi(G^{2c}(\alpha_1)) = \phi(X) = X$,
$\phi_\psi(G^{3c})(\alpha_1) = \cup_{e\in\psi^{-1}(\alpha_1)}\phi(G^{3c}(e)) = \phi(G^{3c}(\alpha_2)) = \phi(\{c\}) = \{c\}$,
$\phi_\psi(G^{3c})(\alpha_2) = \cup_{e\in\psi^{-1}(\alpha_2)}\phi(G^{3c}(e)) = \phi(G^{3c}(\alpha_1)) = \phi(\{b, c\}) = \{a, c\}$.

Thus $\phi_\psi(G^{1c}_E) = F^{2c}_K$, $\phi_\psi(G^c_E) = F^{3c}_K$ and $\phi_\psi(G^{3c}_E) = F^{1c}_K$ are $DPC$-soft set in $Y$. So $\phi_\psi$ is $DSP$-closed, but it is not an $ISP$-closed mapping. However $\phi_\psi$ is not $BSP$-closed.

**Remark 4.16.** For a soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$, we have the following diagram:

$\phi_\psi$ is $ISP-$closed $\not\leftrightarrow$ $\phi_\psi$ is $DSP$-closed

$\searrow \quad \swarrow$

$\phi_\psi$ is $P$-soft closed

$\nwarrow \quad \uparrow \quad \nearrow$

$\phi_\psi$ is $BSP$-closed.

**Theorem 4.17.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. Then $\phi_\psi$ is an $ISP$-closed mapping if and only if $Icl^s_{12}(\phi_\psi(G_E)) \sqsubseteq \phi_\psi(cl^s_{12}(G_E))$ for any $G_E \in P(X)^E$.*

*Proof.* Necessity: Suppose $\phi_\psi$ is an $ISP$-closed mapping and let $G_E \in P(X)^E$. Then by the hypothesis, $\phi_\psi(cl^s_{12}(G_E))$ is an $IPC$-soft set in $Y$ and $\phi_\psi(G_E) \sqsubseteq \phi_\psi(cl^s_{12}(G_E))$. Thus $Icl^s_{12}(\phi_\psi(G_E)) \sqsubseteq \phi_\psi(cl^s_{12}(G_E))$, since $Icl^s_{12}(\phi_\psi(G_E))$ is the smallest $IPC$-soft set containing $\phi_\psi(G_E)$ in $Y$.

Sufficiency: Suppose the necessary condition holds and et $F_E$ be any $PC$-soft set in $X$. Then $\phi_\psi(F_E) \sqsubseteq Icl^s_{12}(\phi_\psi(F_E)) \sqsubseteq \phi_\psi(cl^s_{12}(F_E)) = \phi_\psi(F_E)$. Thus $\phi_\psi(F_E) = Icl^s_{12}(\phi_\psi(F_E))$. So $\phi_\psi(F_E)$ is an $IPC$-soft set in $Y$. So $\phi_\psi$ is an $ISP$-closed mapping. □

The following two theorems characterized $DSP$-closed and $BSP$-closed mapping, whose proofs are similar to as that of the above Theorem 4.17.

**Theorem 4.18.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. Then $\phi_\psi$ is a $DSP$-closed mapping if and only if $Dcl^s_{12}(\phi_\psi(G_E)) \sqsubseteq \phi_\psi(cl^s_{12}(G_E))$ for any $G_E \in P(X)^E$.*

**Theorem 4.19.** *Let $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ be a soft mapping. Then $\phi_\psi$ is a $BSP$-closed mapping if and only if $Bcl^s_{12}(\phi_\psi(G_E)) \sqsubseteq \phi_\psi(cl^s_{12}(G_E))$ for any $G_E \in P(X)^E$.*

**Theorem 4.20.** *The following three statements hold for a bijective soft map $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ :*

(1) *$\phi_\psi$ is $ISP$* (resp. $DSP$, $BSP$)*-open if and only if $\phi_\psi$ is $DSP$* (resp. $ISP$, $BSP$)*-closed.*
(2) *$\phi_\psi$ is $ISP$* (resp. $DSP$, $BSP$)*-open if and only if $\phi_\psi^{-1}$ is $ISP$* (resp. $DSP$, $BSP$)*-continuous.*
(3) *$\phi_\psi$ is $ISP$* (resp. $DSP$, $BSP$)*-closed if and only if $\phi_\psi^{-1}$ is $ISP$* (resp. $DSP$, $BSP$)*-continuous.*

*Proof.* To summarize, we only give proofs of cases outside the parenthesis for the three statements above and the cases between parenthesis can be made similarly.

(1) Suppose $\phi_\psi$ is an $ISP$-open mapping and let $G_E$ be a $PC$-soft set in $X$. Then $G_E^c$ be a $PO$-soft set and $\phi_\psi(G_E^c)$ is an $IPO$-soft set in $Y$. Since $\phi_\psi$ is bijective, $\phi_\psi(G_E^c) = (\phi_\psi(G_E))^c$. Thus $\phi_\psi(G_E)$ is a $DPC$-soft set in $Y$. So $\phi_\psi$ is an $DSP$-closed mapping. The sufficiency condition can be proved in a similar manner.

(2) Suppose $\phi_\psi$ is an $ISP$-open mapping and let $F_E$ be a $PC$-soft set in $X$. Then $\phi_\psi(F_E)$ is a $IPO$-soft set in $Y$. Since $\phi_\psi$ is bijective, $\phi_\psi(F_E) = (\phi_\psi^{-1})^{-1}(F_E)$. Thus $(\phi_\psi^{-1})^{-1}(F_E)$ is an $IPO$-soft set in $Y$. So $\phi_\psi^{-1}$ is an $ISP$-continuous mapping. The sufficiency condition can be proved in a similar manner.

(3) The proof of this statement comes immediately from (1) and (2). □

## 5. $ISP$ ($DSP$, $BSP$)-Homeomorphisms

The concepts of $ISP$ (resp. $DSP$, $BSP$)-homeomorphisms are introduced and their main properties are discussed. Some examples are constructed to illustrate the relationships among the initiated soft mappings.

**Definition 5.1.** A soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ is called $ISP$ (resp. $DSP$, $BSP$)-*homeomorphism*, if it satisfies the following conditions:
(i) $\phi_\psi$ is $ISP$ (resp. $DSP$, $BSP$)-open,
(ii)$\phi_\psi$ is $ISP$ (resp. $DSP$, $BSP$)-closed,
(iii) $\phi_\psi$ is $ISP$ (resp. $DSP$, $BSP$)-continuous,
(iv $\phi_\psi^{-1}$ is $ISP$ (resp. $DSP$, $BSP$)- continuous.

**Example 5.2.** From Example 3.2, 4.2, 4.13, we obtain that $\phi_\psi$ is not $xSP$-homeomorphism for $x = I; D; B$, but it is a $P$-soft homeomorphism.

**Example 5.3.** Let $E = \{\alpha_1, \alpha_2\} = K$ and $X = \{a, b, c, d\} = Y$, $\leq_1 = \blacktriangle \cup \{(c, b)\}, \leq_2 = \blacktriangle \cup \{(a, d)\}$ and let $\tau_1 = \eta_1 = \{X_E, \Phi, G_E\}$ and $\tau_2 = \eta_2 = \{X_E, \Phi, F_E\}$, where $G_E = \{(\alpha_1, \{a, d\}), (\alpha_2, \{b, c\})\}$, $F_E = \{(\alpha_1, \{a, b, d\}), (\alpha_2, \{b, c\})\}$. Then clearly, $(X, \tau_1, \tau_2, E, \leq_1)$ and $(Y, \eta_1, \eta_2, K, \leq_2)$ are soft bitopological ordered spaces and $\tau_{12} = \eta_{12} = \{X_E, \Phi, G_E, F_E\}$. Let us consider identity mappings $\phi$ and $\psi$.

(i) It is obvious that

$$\phi_\psi(G_E)(\alpha_1) = \phi(\{a, d\}) = \{a, d\},\ \phi_\psi(G_E)(\alpha_2) = \phi(\{b, c\}) = \{b, c\},$$

$$\phi_\psi(F_E)(\alpha_1) = \phi(\{a, b, d\}) = \{a, b, d\},\ \phi_\psi(F_E)(\alpha_2) = \phi(\{b, c\}) = \{b, c\}.$$

Then $\phi_\psi(G_E) = G_E$ and $\phi_\psi(F_E) = F_E$ are $IPO$-soft sets in $Y$. Thus $\phi_\psi$ is $ISP$-open.

(ii) It is clear that

$$\phi_\psi(G_E^c)(\alpha_1) = \phi(\{b, c\}) = \{b, c\},\ \phi_\psi(G_E^c)(\alpha_2) = \phi(\{a, d\}) = \{a, d\},$$

$$\phi_\psi(F_E^c)(\alpha_1) = \phi(\{c\}) = \{c\},\ \phi_\psi(F_E^c)(\alpha_2) = \phi(\{a, d\}) = \{a, d\}.$$

Then $\phi_\psi(G_E^c) = G_E^c$ and $\phi_\psi(F_E^c) = F_E^c$ are $IPC$-soft sets in $Y$. Thus $\phi_\psi$ is $ISP$-closed.

(iii) We can easily prove that $\phi_\psi$ is $ISP$-continuous.

(iv) We obtain easily the followings:

$$\phi_\psi^{-1}(G_E)(\alpha_1) = \phi^{-1}(G_E(\psi(\alpha_1))) = \phi^{-1}(G_E(\alpha_1)) = \phi^{-1}(\{a, d\}) = \{a, d\},$$

$$\phi_\psi^{-1}(G_E)(\alpha_2) = \phi^{-1}(G_E(\psi(\alpha_2))) = \phi^{-1}(G_E(\alpha_2)) = \phi^{-1}(\{b,c\}) = \{b,c\},$$

$$\phi_\psi^{-1}(F_E)(\alpha_1) = \phi^{-1}(F_E(\psi(\alpha_1))) = \phi^{-1}(F_E(\alpha_1)) = \phi^{-1}(\{a,b,d\}) = \{a,b,d\},$$

$$\phi_\psi^{-1}(F_E)(\alpha_2) = \phi^{-1}(F_E(\psi(\alpha_2))) = \phi^{-1}(F_E(\alpha_2)) = \phi^{-1}(\{b,c\}) = \{b,c\}.$$

Then $\phi_\psi^{-1}(G_E) = G_E$ and $\phi_\psi^{-1}(F_E) = F_E$ are $IPO$-soft sets in $X$. Thus $\phi_\psi^{-1}$ is $ISP$-continuous. So $\phi_\psi$ is an $ISP$-homeomorphism, but it is not a $DSP$-homeomorphism. However $\phi_\psi$ is not a $BSP$-homeomorphism.

**Example 5.4.** In Example 5.3, consider $\leq_1 = \blacktriangle \cup \{(b,c)\}$ and $\leq_2 = \blacktriangle \cup \{(d,a)\}$.

(i) We obtain easily the followings:

$$\phi_\psi(G_E)(\alpha_1) = \phi(\{a,d\}) = \{a,d\},\ \phi_\psi(G_E)(\alpha_2) = \phi(\{b,c\}) = \{b,c\},$$

$$\phi_\psi(F_E)(\alpha_1) = \phi(\{a,b,d\}) = \{a,b,d\},\ \phi_\psi(F_E)(\alpha_2) = \phi(\{b,c\}) = \{b,c\}.$$

Then $\phi_\psi(G_E) = G_E$ and $\phi_\psi(F_E) = F_E$ are $DPO$-soft sets in $Y$. Thus $\phi_\psi$ is $DSP$-open.

(ii) It can be calculated:

$$\phi_\psi(G_E^c)(\alpha_1) = \phi(\{b,c\}) = \{b,c\},\ \phi_\psi(G_E^c)(\alpha_2) = \phi(\{a,d\}) = \{a,d\},$$

$$\phi_\psi(F_E^c)(\alpha_1) = \phi(\{c\}) = \{c\},\ \phi_\psi(F_E^c)(\alpha_2) = \phi(\{a,d\}) = \{a,d\}.$$

Then $\phi_\psi(G_E^c) = G_E^c$ and $\phi_\psi(F_E^c) = F_E^c$ are $DPC$-soft sets in $Y$. Thus $\phi_\psi$ is $DSP$-closed.

(iii) It is clear that $\phi_\psi$ is $DSP$-continuous.

(iv) We have

$$\phi_\psi^{-1}(G_E)(\alpha_1) = \phi^{-1}(G_E(\psi(\alpha_1))) = \phi^{-1}(G_E(\alpha_1)) = \phi^{-1}(\{a,d\}) = \{a,d\},$$

$$\phi_\psi^{-1}(G_E)(\alpha_2) = \phi^{-1}(G_E(\psi(\alpha_2))) = \phi^{-1}(G_E(\alpha_2)) = \phi^{-1}(\{b,c\}) = \{b,c\}.$$

Then $\phi_\psi^{-1}(G_E) = G_E$ and $\phi_\psi^{-1}(F_E) = F_E$ are $DPO$-soft sets in $X$. Thus $\phi_\psi$ is a $DSP$-homeomorphism, but it is not an $ISP$-homeomorphism. However $\phi_\psi$ is not a $BSP$-homeomorphism.

**Theorem 5.5.** *If a bijective soft mapping $\phi_\psi : (X, \tau_1, \tau_2, E, \leq_1) \to (Y, \eta_1, \eta_2, K, \leq_2)$ is $ISP$* (resp. *$DSP$, $BSP$*)*-continuous, then the following three statements are equivalent:*

(1) *$\phi_\psi$ is a $ISP$* (resp. *$DSP$, $BSP$*)*-homeomorphism,*
(2) *$\phi_\psi^{-1}$ is $ISP$* (resp. *$DSP$, $BSP$*)*-continuous,*
(3) *$\phi_\psi$ is $DSP$* (resp. *$ISP$, $BSP$*)*-closed.*

*Proof.* (1)⇒ (2): Suppose (1) holds. Then clearly, $\phi_\psi$ is $ISP$ (resp. $DSP$, $BSP$)-open. Thus by Theorem 4.20 (2), $\phi_\psi^{-1}$ is $ISP$ (resp. $DSP$, $BSP$)-continuous.

(2)⇒ (3): The proof follows from Theorem 4.20 (3).

(3)⇒ (1): It sufficient to prove that $\phi_\psi$ is $ISP$ (resp. $DSP$, $BSP$)-open. This follows from Theorem 4.20 (1). □

## 6. Conclusions

This paper introduces and defines the concepts of $xSP$-continuous, $xSP$-open, $xSP$-closed, and $xSP$-homeomorphism maps via soft bitopological ordered spaces. Relationships between the introduced soft maps and their counterparts in bitopological ordered spaces are established. These new soft ordered maps provide a basis for further developments in soft bitopological ordered spaces. Future research will introduce additional soft ordered bitopological concepts and explore their properties.

**Acknowledgements.** I would like to express my heartfelt appreciation to everyone who supported me during the writing of this paper. Firstly, I am deeply grateful to my supervisor, **Prof. Dr. S. A. El-Sheikh**, for his invaluable guidance and insightful feedback throughout the writing process. Secondly, I extend my thanks to my classmates who provided me with timely references and information, without which completing this paper would have been much more challenging. I am grateful for the support and encouragement of all those who helped me along the way.

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