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Some mixed neutrosophic sets

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Abstract. In this paper, we introduce and study some subsets in mixed neutrosophic topological spaces and obtain some of their basic properties. Moreover, we introduce and investigate not only some mixed generalized open sets but also the features of extremally disconnectedness in the context of mixed neutrosophic topological spaces.

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1. Notations and terminology

The impact of fuzzy set theory and its applications have been great in almost all aspects of mathematics since its advent and introduction by Zadeh [1]. The theory of fuzzy topological space was introduced and developed by Chang [2] and since then various notions in classical topology have been extended into the context of fuzzy topological space. The idea of "intuitionistic fuzzy set" was first published by Atanassov [3] and some research in this respect have been done by him and his colleagues [4, 5, 6]. Later, this concept was generalized to "intuitionistic L - fuzzy sets" by Atanassov and Stoeva [7]. Smarandache introduced the important and useful concepts of neutrosophy and neutrosophic set [8, 9]. The concepts of neutrosophic crisp set and neutrosophic crisp topological space were introduced by Salama and Alblowi [10]. The rudimentary notions and basic results related to neutrosophic topological spaces were introduced and discussed by Dhavaseelan et al. [11].

In this paper, after introducing mixed neutrosophic topological spaces, we present some of their properties. Then, we offer some new notions of mixed generalized open and closed sets and discuss some of their features. Moreover, we obtain some results related to the extremally disconnectedness in the context of mixed neutrosophic topological spaces. Here we begin to mention some well-known notions.

Definition 1.1. Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$ with $sup_T = t_{sup}$, $inf_T = t_{inf}$, $sup_I = i_{sup}$, $inf_I = i_{inf}$, $sup_F = f_{sup}, inf_F = f_{inf}$,

$$n - sup = t_{sup} + i_{sup} + f_{sup},\ n - inf = t_{inf} + i_{inf} + f_{inf}.$$

Then T, I, F are called *neutrosophic components*.

Definition 1.2. Let X be a nonempty fixed set. A *neutrosophic set* A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.3. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.4. Let X be a nonempty set and the neutrosophic sets A and B in the form

$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(iii) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$ [The complement of A],
(iv) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$,
(v) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$,
(vi) $[\ ]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$,
(vii) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.5. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(i) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$,
(ii) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in X as follows.

Definition 1.6. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.7. [10]A *neutrosophic topology* on a nonempty set X is a family τ of neutrosophic subsets of X which satisfies the following three conditions:

(i) $0_N, 1_N \in \tau$,
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \tau$.

The pair (X, τ) is called a neutrosophic topological space.

Definition 1.8. Members of τ are called *neutrosophic open sets* and the complement of neutrosophic open sets are called *neutrosophic closed sets*, where the complement of a neutrosophic set A, denoted by A^c, is $1 - A$.

2. Some mixed neutrosophic sets

Definition 2.1. Let (X, τ_1) and (X, τ_2) be two neutrosophic topological spaces. Then the system (X, τ_1, τ_2) is called a *mixed neutrosophic topological space.*

Remark 2.2. Here we denote the interior and the closure operators by Int and Cl respectively. If $A \in \tau_1$ or $A \in \tau_2$, this means that $A = Int_1(A)$ (A is open with respect to τ_1) or $A = Int_2(A)$ (A is open with respect to τ_2). A is closed with respect to τ_1 iff $A = Cl_1(A)$, and also A is closed with respect to τ_2 iff $A = Cl_2(A)$.

Definition 2.3. A subset A of a mixed neutrosophic topological space (X, τ_1, τ_2) is said to be:
(i) (τ_i, τ_j)-*regular open*, if $A = \mathrm{Int}_i(\mathrm{Cl}_j(A))$,
(ii) (τ_i, τ_j)-*semiopen*, if $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$,
(iii) (τ_i, τ_j)-*preopen*, if $A \subset \mathrm{Int}_i(\mathrm{Cl}_j(A))$,
(iv) (τ_i, τ_j)-α-*open*, if $A \subset \mathrm{Int}_i(\mathrm{Cl}_j(\mathrm{Int}_i(A)))$,
(v) (τ_i, τ_j)-b-*open*, if $A \subset \mathrm{Int}_i(\mathrm{Cl}_j(A)) \cup \mathrm{Cl}_j(\mathrm{Int}_i(A))$,
(vi) (τ_i, τ_j)-β-*open*, if $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(\mathrm{Cl}_j(A)))$,
(vii) (τ_i, τ_j)-δ-*open*, if $\mathrm{Int}_i(\mathrm{Cl}_j(A)) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$,
where $i,\ j = 1,\ 2$ and $i \neq j$.

The complement of an (i, j)-semiopen (resp. (i, j)-preopen, (i, j)-b-open, (i, j)-β-open, (i, j)-regular open) set is called an (i, j)-*semiclosed* (resp. (i, j)-*preclosed*, (i, j)-b-*closed*, (i, j)-β-*closed*, (i, j)-*regular closed*) *set*.

The family of all (i, j)-regular open (resp. (i, j)-preopen, (i, j)-semiopen, (i, j)-b-open, (i, j)-β-open, (i, j)-regular closed, (i, j)-preclosed, (i, j)-semiclosed, (i, j)-b-closed, (i, j)-β-closed) subsets of (X, τ_1, τ_2) is denoted by (i, j)-$RO(X)$ (resp. (i, j)-$PO(X)$, (i, j)-$SO(X)$, (i, j)-$BO(X)$, (i, j)-$\beta O(X)$, (i, j)-$RC(X)$, (i, j)-$PC(X)$, (i, j)-$SC(X)$, (i, j)-$BC(X)$, (i, j)-$\beta C(X)$).

Theorem 2.4. *Let A and B be neutrosophic subsets of (X, τ_1, τ_2).*
(1) *A is (τ_1, τ_2)-semiopen if and only if $\mathrm{Cl}_2(A) = \mathrm{Cl}_2(\mathrm{Int}_1(A))$.*
(2) *A is (τ_2, τ_1)-semiopen if and only if $\mathrm{Cl}_1(A) = \mathrm{Cl}_1(\mathrm{Int}_2(A))$.*
(3) *If $A \in \tau_1$ and B is (τ_1, τ_2)-preopen, then $A \cap B$ is (τ_1, τ_2)-preopen.*
(4) *If $A \in \tau_2$ and B is (τ_2, τ_1)-preopen, then $A \cap B$ is (τ_2, τ_1)-preopen.*

Proof. We prove only (1) since they follow from definition 2.3 and Remark 2.2. Since A is (τ_1, τ_2)-semiopen, then we have $A \subset \mathrm{Cl}_2(\mathrm{Int}_1(A))$. If we impose Cl_2 on both sides, then we get $\mathrm{Cl}_2(A) = \mathrm{Cl}_2(\mathrm{Int}_1(A))$. Conversely if $\mathrm{Cl}_2(A) = \mathrm{Cl}_2(\mathrm{Int}_1(A))$, then it is clear that $A \subset \mathrm{Cl}_2(\mathrm{Int}_1(A))$. □

Theorem 2.5. *Let A and B be any two neutrosophic subsets of a mixed neutrosophic topological space (X, τ_1, τ_2).*
(1) *If A is a (τ_1, τ_2)-semiopen set or B is a (τ_1, τ_2)-semiopen set, then*

$$\mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)) = \mathrm{Int}_1(\mathrm{Cl}_2(A)) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B)).$$

(2) *If A is a (τ_2, τ_1)-semiopen set or B is a (τ_2, τ_1)-semiopen set, then*

$$\mathrm{Int}_2(\mathrm{Cl}_1(A \cap B)) = \mathrm{Int}_2(\mathrm{Cl}_1(A)) \cap \mathrm{Int}_2(\mathrm{Cl}_1(B)).$$

Proof. (1) Suppose A is a (τ_1, τ_2)-semiopen set. Then $\mathrm{Cl}_2(A) = \mathrm{Cl}_2(\mathrm{Int}_1(A))$. Note that $\mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)) \subset \mathrm{Int}_1(\mathrm{Cl}_2(A)) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))$. Thus we have

$$\begin{aligned} \mathrm{Int}_1(\mathrm{Cl}_2(A)) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B)) &= \mathrm{Int}_1(\mathrm{Cl}_2(A) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))) \\ &= \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A)) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))) \\ &\subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))) \\ &= \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A) \cap \mathrm{Cl}_2(B)))) \\ &\subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A \cap B))))) \\ &\subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)))) \\ &= \mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)). \end{aligned}$$

(2) The proof is analogous. □

Theorem 2.6. *Let A and B be any two neutrosophic subsets of a mixed neutrosophic topological space (X, τ_1, τ_2).*

(1) *If B is a (τ_1, τ_2)-α-open set if and only if there exists $B \in \tau_1$ such that $A \subset B \subset \mathrm{Int}_1(\mathrm{Cl}_2(A))$.*

(2) *If A is a (τ_1, τ_2)-α-open set and $A \subset B \subset \mathrm{Int}_1(\mathrm{Cl}_2(A))$, then A is (τ_1, τ_2)-α-open set.*

(3) *If B is a (τ_2, τ_1)-α-open set if and only if there exists $B \in \tau_2$ such that $A \subset B \subset \mathrm{Int}_2(\mathrm{Cl}_1(A))$.*

(4) *If A is a (τ_2, τ_1)-α-open set and $A \subset B \subset \mathrm{Int}_2(\mathrm{Cl}_1(A))$, then A is (τ_2, τ_1)-α-open set.*

Proof. (1) Suppose B is a (τ_1, τ_2)-α-open set and let $\mathrm{Int}_1(A) = B$. Then clearly, $B \in \tau_1$ and $B \subset A \subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A))) = \mathrm{Int}_1(\mathrm{Cl}_2(A))$.

Conversely, suppose the necessary condition holds. Then $\mathrm{Int}_1(B) = B \subset \mathrm{Int}_1(A))$. Thus $A \subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(B))) \subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A)))$. So B is a (τ_1, τ_2)-α-open set.

The other proofs can be carried on by the same token. □

Theorem 2.7. *Let A and B be any two neutrosophic subsets of a mixed neutrosophic topological space (X, τ_1, τ_2).*

(1) *If A is a (τ_1, τ_2)-α-open set and B is a (τ_1, τ_2)-β-open set, then $A \cap B$ is a (τ_1, τ_2)-β-open set.*

(2) *If A is a (τ_2, τ_1)-α-open set and B is a (τ_2, τ_1)-β-open set, then $A \cap B$ is a (τ_2, τ_1)-β-open set.*

(3) *If A is a (τ_1, τ_2)-α-open set and B is a (τ_1, τ_2)-semiopen set, then $A \cap B$ is a (τ_1, τ_2)-semiopen set.*

(4) *If A is a (τ_2, τ_1)-α-open set and B is a (τ_2, τ_1)-semiopen set, then $A \cap B$ is a (τ_2, τ_1)-semiopen set.*

Proof. (1) Suppose A is a (τ_1, τ_2)-α-open set and B is a (τ_1, τ_2)-β-open set. Then we have

$$\begin{aligned} A \cap B &\subset \mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A))) \cap \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(B))) \\ &\subset \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A))) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))) \\ &= \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A)) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B)))) \\ &\subset \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A) \cap \mathrm{Int}_1(\mathrm{Cl}_2(B))))) \\ &= \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Int}_1(A) \cap \mathrm{Cl}_2(B))))) \\ &\subset \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(\mathrm{Int}_1(A) \cap B))))) \\ &\subset \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2\,\mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)))) \end{aligned}$$

$\subset \mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(A \cap B)))$.

Thus $A \cap B$ is a (τ_1, τ_2)-β-open set.

The other proofs are analogous. □

Theorem 2.8. *Let A be a neutrosophic subset of a mixed neutrosophic topological space (X, τ_1, τ_2). Then*

(1) *A is (τ_1, τ_2)-semiclosed if and only if* $\mathrm{Int}_2(\mathrm{Cl}_1(A)) \subset A$,
(2) *A is (τ_2, τ_1)-semiclosed if and only if* $\mathrm{Int}_1(\mathrm{Cl}_2(A)) \subset A$,
(3) *A is (τ_1, τ_2)-preclosed if and only if* $\mathrm{Cl}_1(\mathrm{Int}_2(A)) \subset A$,
(4) *A is (τ_2, τ_1)-preclosed if and only if* $\mathrm{Cl}_1(\mathrm{Int}_2(A)) \subset A$,
(5) *A is (τ_1, τ_2)-α-closed if and only if* $\mathrm{Cl}_2(\mathrm{Int}_1(\mathrm{Cl}_2(A))) \subset A$,
(6) *A is (τ_2, τ_1)-α-closed if and only if* $\mathrm{Cl}_1(\mathrm{Int}_2(\mathrm{Cl}_1(A))) \subset A$,
(7) *A is (τ_1, τ_2)-β-closed if and only if* $\mathrm{Int}_2(\mathrm{Cl}_1(\mathrm{Int}_2(A))) \subset A$,
(8) *A is (τ_2, τ_1)-β-closed if and only if* $\mathrm{Cl}_1(\mathrm{Int}_2(\mathrm{Cl}_1(A))) \subset A$.

Proof. The proofs follow from the respective definitions. □

Lemma 2.9. *Let A be a neutrosophic subset of a mixed neutrosophic topological space (X, τ_1, τ_2). Then*

(1) $\mathrm{Cl}_i(\mathrm{Int}_j(A)) = \mathrm{Cl}_i(\mathrm{Int}_j(\mathrm{Cl}_i(\mathrm{Int}_j(A))))$,
(2) $\mathrm{Int}_i(\mathrm{Cl}_j(A)) = \mathrm{Int}_i(\mathrm{Cl}_j(\mathrm{Int}_i(\mathrm{Cl}_j(A))))$.

Proof. (1) Clearly, the following holds $\mathrm{Int}_j(A) \subset \mathrm{Cl}_i(\mathrm{Int}_j(A))$. Then we get

$$\mathrm{Int}_j(\mathrm{Int}_j(A)) = \mathrm{Int}_j(A) \subset \mathrm{Int}_j(\mathrm{Cl}_i(\mathrm{Int}_j(A))).$$

Thus $\mathrm{Cl}_i(\mathrm{Int}_j(A)) \subset \mathrm{Cl}_i(\mathrm{Int}_j(\mathrm{Cl}_i(\mathrm{Int}_j(A))))$.

Conversely, one has that $\mathrm{Int}_j(\mathrm{Cl}_i(\mathrm{Int}_j(A))) \subset \mathrm{Cl}_i(\mathrm{Int}_j(A))$. Then we have

$$\mathrm{Cl}_i(\mathrm{Int}_j(\mathrm{Cl}_i(\mathrm{Int}_j(A)))) \subset \mathrm{Cl}_i(\mathrm{Cl}_i(\mathrm{Int}_j(A))) = \mathrm{Cl}_i(\mathrm{Int}_j(A)).$$

So the proof is complete.

(2) The proof is dual to (1). □

Proposition 2.10. (1) *Every (τ_i, τ_j)-α-open set is (τ_i, τ_j)-semiopen.*
(2) *Every (τ_i, τ_j)-semiopen set is (τ_i, τ_j)-b-open.*

Proof. The proof follows from the definitions. □

Corollary 2.11. (1) *Every (τ_i, τ_j)-semiopen set is (τ_i, τ_j)-δ-open.*
(2) *Every (τ_i, τ_j)-semiopen set is (τ_i, τ_j)-b-open.*

Remark 2.12. It is clear that (τ_i, τ_j)-semiopenness and (τ_i, τ_j)-preopenness are independent notions.

Theorem 2.13. *If $\{A_\alpha\}_{\alpha \in \Delta}$ is the collection of (τ_i, τ_j)-semiopen sets of (X, τ_1, τ_2), then $\bigcup_{\alpha \in \Delta} A_\alpha$ is also a (τ_i, τ_j)-semiopen set.*

Proof. Since each A_α is (τ_i, τ_j)-semiopen and $A_\alpha \subset A_\alpha$, $\underset{\alpha \in \Delta}{\cup} A_\alpha \subset \mathrm{Cl}_j(\mathrm{Int}_i(\underset{\alpha \in \Delta}{\cup} A_\alpha))$. Then $\bigcup_{\alpha \in \Delta} A_\alpha$ is also a (τ_i, τ_j)-semiopen set in (X, τ_1, τ_2). □

Proposition 2.14. *A subset A of X is (τ_i, τ_j)-semiopen if and only if* $\mathrm{Cl}_j(A) = \mathrm{Cl}_j(\mathrm{Int}_i(A))$.

Proof. Suppose $A \in (\tau_i, \tau_j)$-$SO(X)$. Then we have $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Thus $\mathrm{Cl}_j(A) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. So $\mathrm{Cl}_j(A) = \mathrm{Cl}_j(\mathrm{Int}_i(A))$.

The converse is obvious. □

Corollary 2.15. *If A is a nonempty (τ_i, τ_j)-semiopen set, then $\mathrm{Int}_i(A) \neq \varnothing$.*

Proof. Since A is (τ_i, τ_j)-semiopen, by Proposition 2.14, we have $\mathrm{Cl}_j(A) = \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Assume that $\mathrm{Int}_i(A) = \varnothing$. Then we have $\mathrm{Cl}_j(A) = \varnothing$. Thus $A = \varnothing$. This is contrary to the hypothesis. So $\mathrm{Int}_i(A) \neq \varnothing$. □

Proposition 2.16. *A subset A is (τ_i, τ_j)-semiopen if and only if there exists $U \in \tau_i$ such that $U \subset A \subset \mathrm{Cl}_j(U)$.*

Proof. Suppose $A \in (\tau_i, \tau_j)$-$SO(X)$. Then we have $A \subset \mathrm{Cl}_j(\mathrm{Int}(A))$. Take $\mathrm{Int}_i(A) = U$. Then $U \subset A \subset \mathrm{Cl}_j(U)$.

Conversely, suppose the necessary condition holds. Since $U \subset A$, $U \subset \mathrm{Int}_i(A)$. Then $\mathrm{Cl}_j(U) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Thus $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. □

Proposition 2.17. *If A is a (τ_i, τ_j)-semiopen set in a mixed neutrosophic topological space (X, τ_1, τ_2) and $A \subset B \subset \mathrm{Cl}_j(A)$, then B is a (τ_i, τ_j)-semiopen set in (X, τ_1, τ_2).*

Proof. Suppose A is a (τ_i, τ_j)-semiopen set and $A \subset B \subset \mathrm{Cl}_j(A)$. Since A is (τ_i, τ_j)-semiopen, there exists a τ_i-open set U such that $U \subset A \subset \mathrm{Cl}_j(U)$. Since $A \subset B \subset \mathrm{Cl}_j(A)$, we have $U \subset A \subset B \subset \mathrm{Cl}_j(A) \subset \mathrm{Cl}_j(\mathrm{Cl}_j(U)) = \mathrm{Cl}_j(U)$. Then $U \subset B \subset \mathrm{Cl}_j(U)$. Thus by Proposition 2.16, $B \in (\tau_i, \tau_j)$-$SO(X)$. □

Theorem 2.18. *A subset A of X is (τ_i, τ_j)-semiopen if and only if it is both (τ_i, τ_j)-δ-open and (τ_i, τ_j)-β-preopen.*

Proof. Suppose A is a (τ_i, τ_j)-semiopen set. Then $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(A)) \subset \mathrm{Cl}_j(\mathrm{Int}_i(\mathrm{Cl}_j(A)))$. This shows that A is (τ_i, τ_j)-β-open. Moreover, $\mathrm{Int}_i(\mathrm{Cl}_j(A)) \subset \mathrm{Cl}_j(A) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Thus A is (τ_i, τ_j)-δ-open.

Conversely, suppose A is (τ_i, τ_j)-δ-open and (τ_i, τ_j)-β-open set. Then we have $\mathrm{Int}_i(\mathrm{Cl}_j(A)) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Thus $\mathrm{Cl}_j(\mathrm{Int}_i(\mathrm{Cl}_j(A))) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. Since A is (τ_i, τ_j)-β-open, we have $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(\mathrm{Cl}_j(A))) \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$ and $A \subset \mathrm{Cl}_j(\mathrm{Int}_i(A))$. So A is a (τ_i, τ_j)-semiopen set. □

Theorem 2.19. *A subset A of X is (τ_i, τ_j)-semiclosed if and only if there exists a τ_j-closed set F such that $\mathrm{Int}_i(F) \subset A \subset F$.*

Proof. Suppose A is (τ_i, τ_j)-semiclosed. Then $\mathrm{Int}_i(\mathrm{Cl}_j(A)) \subset A$. Let $F = \mathrm{Cl}_j(A)$. Then F is τ_j-closed set such that $\mathrm{Int}_i(F) \subset A \subset F$.

Conversely, let F be a τ_j-closed set such that $\mathrm{Int}_i(F) \subset A \subset F$. But $F \supset \mathrm{Cl}_j(A)$. Then $\mathrm{Int}_i(F) \supset \mathrm{Int}_i(\mathrm{Cl}_j(A))$. Thus $\mathrm{Int}_i(\mathrm{Cl}_j(A)) \subset A$. So A is (τ_i, τ_j)-semiclosed. □

Proposition 2.20. *A subset A of X is (τ_i, τ_j)-β-closed and (τ_i, τ_j)-δ-open, then it is (τ_i, τ_j)-semiclosed.*

Proof. The proof follows from the definitions. □

Theorem 2.21. *Arbitrary intersection of (τ_i, τ_j)-semiclosed sets is always (τ_i, τ_j)-semiclosed.*

Proof. Follows from Theorem 2.13. □

Definition 2.22. Let A be subset of a mixed neutrosophic topological space (X, τ_1, τ_2). Then

(i) the (τ_i, τ_j)-*semiclosure* of A, denoted by (τ_i, τ_j)-$s\,\mathrm{Cl}(A)$, is defined as the intersection of all (τ_i, τ_j)-semiclosed sets containing A, i.e.,

$$(\tau_i, \tau_j)\text{-}s\,\mathrm{Cl}(A) = \bigcap\{F : F \text{ is } (\tau_i, \tau_j)\text{-semiclosed and } A \subset F\},$$

(ii) the (τ_i, τ_j)-*semiinterior* of A, denoted by (τ_i, τ_j)-$s\,\mathrm{Int}(A)$, is defined as the union of all (τ_i, τ_j)-semiopen sets contained in A, i.e.,

$$(\tau_i, \tau_j)\text{-}s\,\mathrm{Int}(A) = \bigcup\{U : U \text{ is } (\tau_i, \tau_j)\text{-semiopen and } U \subset A\}.$$

Theorem 2.23. *For a subset A of X, the following hold:*

(1) (τ_i, τ_j)-$s\,\mathrm{Cl}(A) = A \cup \mathrm{Int}_i(\mathrm{Cl}_j(A))$,
(2) (τ_i, τ_j)-$s\,\mathrm{Int}(A) = A \cap \mathrm{Cl}_i(\mathrm{Int}_j(A))$.

Proof. The proof follows from the definitions. □

3. Extremally disconnected mixed neutrosophic topological spaces

Definition 3.1. A mixed neutrosophic topological space (X, τ_1, τ_2) is said to be:

(i) (τ_i, τ_j)-*extremally disconnected*, if τ_j-closure of every τ_i-open set is τ_i-open in X,

(ii) *pairwise extremally disconnetcted*, if (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected and (τ_2, τ_1)-extremally disconnected.

Theorem 3.2. *A mixed neutrosophic topological space (X, τ_1, τ_2) is pairwise extremally disconnected if and only if for each τ_i-open set A and each τ_j-open set B such that $A \cap B = \varnothing$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$.*

Proof. Suppose (X, τ_1, τ_2) is pairwise extremally disconnected. Let A and B, respectively, be τ_1-open and τ_2-open sets such that $A \cap B = \varnothing$. Then τ_j-$\mathrm{Cl}(A) \in \tau_i$. Thus τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$.

Conversely, suppose the necessary conditions hold and let U be a τ_i-open set in X. Then $X \backslash \tau_j$-$\mathrm{Cl}(U)$ is τ_j-open in X. Now, we have

$$\begin{aligned} & U \cap (X \backslash \tau_j\text{-}\mathrm{Cl}(U)) = \varnothing \\ \Rightarrow\ & \tau_j\text{-}\mathrm{Cl}(U) \cap \tau_i\text{-}\mathrm{Cl}(X \backslash \tau_j\text{-}\mathrm{Cl}(U)) \\ \Rightarrow\ & \tau_i\text{-}\mathrm{Cl}(X \backslash \tau_j\text{-}\mathrm{Cl}(U)) \subset X \backslash \tau_j\text{-}\mathrm{Cl}(U) \\ \Rightarrow\ & \tau_i\text{-}\mathrm{Cl}(X \backslash \tau_j\text{-}\mathrm{Cl}(U)) = X \backslash \tau_j\text{-}\mathrm{Cl}(U) \\ \Rightarrow\ & (X \backslash \tau_j\text{-}\mathrm{Cl}(U)) \text{ is } \tau_i\text{-closed} \\ \Rightarrow\ & \tau_j\text{-}\mathrm{Cl}(U) \text{ is } \tau_i\text{-open.} \end{aligned}$$

Thus (X, τ_1, τ_2) is (τ_i, τ_j)-extremally disconnected. Similarly, (X, τ_1, τ_2) is (τ_j, τ_i)-extremally disconnected. So (X, τ_1, τ_2) is pairwise extremally disconnected. □

Theorem 3.3. *The following are equivalent for a mixed neutrosophic topological space (X, τ_1, τ_2):*

(1) *(X, τ_1, τ_2) is pairwise extremally disconnected,*
(2) *for each (τ_j, τ_i)-semiopen set A in X, τ_j-$\mathrm{Cl}(A)$ is τ_i-open set,*
(3) *for each (τ_i, τ_j)-semiopen set A in X, (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_i-open set,*
(4) *for each (τ_i, τ_j)-semiopen set A and each (τ_j, τ_i)-semiopen set B with $A \cap B = \varnothing$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$,*

(5) *for each* (τ_j, τ_i)*-semiopen set* A *in* X*,* τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$*,*
(6) *for each* (τ_i, τ_j)*-semiopen set* A *in* X*,* (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ *is* τ_j*-closed set,*
(7) *for each* (τ_i, τ_j)*-semiclosed set* A *in* X*,* τ_j-$\mathrm{Int}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A)$*,*
(8) *for each* (τ_i, τ_j)*-semiclosed set* A *in* X*,* (τ_j, τ_i)-$s\,\mathrm{Int}(A)$ *is* τ_j*-open set.*

Proof. (1) ⇒ (2): Clear.

(1) ⇒ (5): Since (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$ for any set A of X, it is sufficient to show that (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \supset \tau_j$-$\mathrm{Cl}(A)$ for any (τ_i, τ_j)-semiopen set A of X. Let $x \notin (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j, τ_i)-semiopen set W with $x \in W$ such that $W \cap A = \varnothing$. Thus τ_j-$\mathrm{Int}(W)$ and τ_i-$\mathrm{Int}(A)$ are, respectively, τ_j-open and τ_i-open such that τ_j-$\mathrm{Int}(X) \cap \tau_i$-$\mathrm{Int}(A) = \varnothing$. By Theorem 3.2, we get

$$\tau_i\text{-}\mathrm{Cl}(\tau_j\text{-}\mathrm{Int}(W)) \cap \tau_j\text{-}\mathrm{Cl}(\tau_i\text{-}\mathrm{Int}(A)) = \varnothing.$$

So $x \notin \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)) = \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$.

(5) ⇒ (6): Obvious.

(6) ⇒ (5): For any set A in X, $A \subset (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Then we have

$$\tau_j\text{-}\mathrm{Cl}(A) = \tau_j\text{-}\mathrm{Cl}((\tau_j, \tau_i)\text{-}s\,\mathrm{Cl}(A)).$$

Since A is (τ_i, τ_j)-semiopen, by (6), (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_j-closed. Thus τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$.

(6) ⇔ (8): Clear.

(7) ⇒ (8): Obvious.

(8) ⇒ (7): For any subset A of X, τ_j-$\mathrm{Int}(A) \subset (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A) \subset A$. Then

$$\tau_j\text{-}\mathrm{Int}(A) = \tau_j\text{-}\mathrm{Int}((\tau_j, \tau_i)\text{-}s\,\mathrm{Int}(A)).$$

Since A is (τ_i, τ_j)-semiclosed, by (8), (τ_j, τ_i)-$s\,\mathrm{Int}(A)$ is τ_j-open. Thus τ_j-$\mathrm{Int}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A)$.

(1) ⇒ (4): Let A be a (τ_i, τ_j)-open set and B a (τ_j, τ_i)-semiopen set such that $A \cap B = \varnothing$. Then τ_i-$\mathrm{Int}(A) \cap \tau_j$-$\mathrm{Int}(B) = \varnothing$. Thus by Theorem 3.2,

$$\tau_j\text{-}\mathrm{Cl}(\tau_j\text{-}\mathrm{Int}(A)) \cap \tau_i\text{-}\mathrm{Cl}(\tau_j\text{-}\mathrm{Int}(B)) = \varnothing.$$

So τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$.

(4) ⇒ (2): Let A be a (τ_i, τ_j)-semiopen subset of X. Then $X \backslash \tau_j$-$\mathrm{Cl}(A)$ is (τ_j, τ_i)-semiopen and $A \cap (X \backslash \tau_j$-$\mathrm{Cl}(A))$. Thus by (4), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(X \backslash \tau_j$-$\mathrm{Cl}(A)) = \varnothing$ which implies τ_j-$\mathrm{Cl}(A) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$. So τ_j-$\mathrm{Cl}(A) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$. Hence τ_j-$\mathrm{Cl}(A)$ is τ_i-open in X.

(5) ⇒ (4): Let A be a (τ_i, τ_j)-semiopen set and B be a (τ_j, τ_i)-semiopen set such that $A \cap B = \varnothing$. Then (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is (τ_i, τ_j)-semiopen and (τ_i, τ_j)-$s\,\mathrm{Cl}(B)$ is (τ_j, τ_i)-semiopen in X. Thus (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \cap (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(B) = \varnothing$. So By (5), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$.

(1) ⇒ (3): Clear.

(3) ⇒ (1): Let A be a τ_i-open set in (X, τ_1, τ_2). It is sufficient to prove that τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Obviously, (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Let $x \notin (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j, τ_i)-semiopen set U with $x \in U$ such that $A \cap U = \varnothing$. Thus (τ_i, τ_j)-$s\,\mathrm{Cl}(U) \subset (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(X \backslash A) = X \backslash A$. So (τ_i, τ_j)-$s\,\mathrm{Cl}(U) \cap A = \varnothing$. Since (τ_i, τ_j)-$s\,\mathrm{Cl}(U)$ is a τ_j-open set with $x \in (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(U)$, $x \notin \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j, \tau_i)$-$\mathrm{Cl}(A)$. □

Definition 3.4. A point x in a mixed neutrosophic toplogical space (X, τ_1, τ_2) is said to be a (τ_i, τ_j)*-*θ*-cluster point* of a set A, if for every τ_i-open, say, U containing

x, τ_j-Cl$(U) \cap A \neq \varnothing$. The set of all (τ_i, τ_j)-θ-closure of A and will be denoted by (τ_i, τ_j)-Cl$_\theta(A)$. A set A is called (τ_i, τ_j)-*θ-closed*, if $A = (\tau_i, \tau_j)$-Cl$_\theta(A)$.

Lemma 3.5. *For any (τ_j, τ_i)-preopen set A in a mixed neutrosophic topological space (X, τ_1, τ_2), τ_i-Cl$(A) = (\tau_i, \tau_j)$-Cl$_\theta(A)$.*

Proof. It is obvious that τ_i-Cl$(A) \subset (\tau_i, \tau_j)$-Cl$_\theta(A)$ for any subset A of (X, τ_1, τ_2). Then it remains to be shown that (τ_i, τ_j)-Cl$_\theta(A) \subset \tau_i$-Cl$(A)$. If $x \notin \tau_i$-Cl(A), then there exists a τ_i-open set U containing x such that $U \cap A = \varnothing$. Thus $U \cap \tau_i$-Cl$(A) = \varnothing$. But $U \cap \tau_j$-Int$(\tau_i$-Cl$(A)) = \varnothing$ which implies τ_j-Cl$(U) \cap \tau_j$-Int$(\tau_i$-Cl$(A)) = \varnothing$ and so τ_j-Cl$(U) \cap A = \varnothing$ since A is (τ_j, τ_i)-preopen. Hence $x \notin (\tau_j, \tau_i)$-Cl$_\theta(A)$ and consequently (τ_j, τ_i)-Cl$_\theta(A) \subset \tau_i$-Cl$(A)$. □

Theorem 3.6. *The following are equivalent for a mixed neutrosophic topological space (X, τ_1, τ_2):*

1. *(X, τ_1, τ_2) is pairwise extremally disconnected,*
2. *the τ_j-closure of every (τ_i, τ_j)-β-open set of X is τ_i-open set,*
3. *the (τ_j, τ_i)-θ-closure of every (τ_i, τ_j)-preopen set of X is τ_i-open set,*
4. *the τ_j-closure of every (τ_i, τ_j)-preopen set of X is τ_i-open set.*

Proof. (1) $\Rightarrow$ (2): Let A be a (τ_i, τ_j)-β-open set. Then we have

$$\tau_j\text{-Cl}(A) = \tau_j\text{-Cl}(\tau_i\text{-Int}(\tau_j\text{-Cl}(A))).$$

Since (X, τ_1, τ_2) is pairwise extremally disconnected, τ_j-Cl(A) is a τ_i-open set.

(2) $\Rightarrow$ (4): Follows from the fact that every (τ_i, τ_j)-preopen set is (τ_i, τ_j)-β-open.
(4) $\Rightarrow$ (1): Clear.
(3) $\Leftrightarrow$ (4): Follows from Lemma 3.5. □

Theorem 3.7. *A mixed neutrosophic topological space (X, τ_1, τ_2) is pairwise extremally disconnected if and only if every (τ_i, τ_j)-semiopen set is a (τ_i, τ_j)-preopen set.*

Proof. Suppose (X, τ_1, τ_2) is pairwise extremally disconnected and let A be a (τ_i, τ_j)-semiopen set. Then $A \subset \tau_j$-Cl$(\tau_i$-Int$(A))$. Since X is pairwise extremally disconnected, τ_j-Cl$(\tau_i$-Int$(A))$ is a τ_i-open set. Thus we have

$$A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(A)) = \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(A))) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(A)).$$

So A is a (τ_i, τ_j)-preopen set.

Conversely, Suppose the necessary condition holds and let A be a τ_i-open set. Since τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(A))$, we have τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Then τ_j-Cl(A) is (τ_j, τ_i)-regular closed. Thus A is (τ_i, τ_j)-semiopen. By the hypothesis, A is (τ_i, τ_j)-propen. So τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence τ_j-Cl(A) is τ_i-open in X. Therefore X is pairwise extremally disconnected. □

Lemma 3.8. *For a subset A of a mixed neutrosophic topological space (X, τ_1, τ_2),*

1. *τ_j-Int$(\tau_i$-Cl$(A)) \subset (\tau_i, \tau_j)$-$s\,Cl(A)$,*
2. *τ_j-Int$((\tau_i, \tau_j)$-$s\,$Cl$(A)) = \tau_j$-Int$(\tau_i$-Cl$(A))$.*

Proof. (1) Since (τ_i, τ_j)-$s\,$Cl(A) is (τ_i, τ_j)-semiclosed, there exists a τ_i-closed set U in X such that τ_j-Int$(U) \subset (\tau_i, \tau_j)$-$s\,Cl(A) \subset U$. Then we have

$$\tau_j\text{-Int}(U) \subset (\tau_i,\tau_j)\text{-}s\,\text{Cl}(A) \subset \tau_i\text{-Cl}(A) \subset U.$$

Thus τ_j-Int$(U) \subset \tau_j$-Int$(\tau_i$-Cl$(A)) \subset \tau_j$-Int(U). So τ_j-Int$(\tau_i$-Cl$(A)) \subset (\tau_i,\tau_j)$-$s\,Cl(A)$.

(2) Follows easily from (1). □

Theorem 3.9. *Let A be a subset of a mixed neutrosophic topological space (X,τ_1,τ_2). Then A is (τ_i,τ_j)-regular open if and only if A is τ_i-open and τ_j-closed.*

Proof. Suppose A is a (τ_i,τ_j)-regular open set of a bitoplogical space (X,τ_1,τ_2). Then τ_i-Int$(\tau_j$-Cl$(A)) = A$. Now, $X\backslash\tau_j$-Cl(A) and A are, respectively, τ_j-open and τ_i-open such that $(X\backslash\tau_j$-Cl$(A)) \cap A = \varnothing$. Since (X,τ_1,τ_2) is pairwise extremally disconnected, by Theorem 3.2, τ_i-Cl$(X\backslash\tau_j$-Cl$(A)) \cap \tau_j$-Cl$(A) = \varnothing$. Thus τ_i-Cl$(X\backslash\tau_j$-Cl$(A)) = X\backslash\tau_j$-Cl(A) and $X\backslash\tau_j$-Cl(A) is τ_i-closed. So τ_j-Cl(A) is τ_i-open. Hence τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A)) = A$ is τ_i-open and τ_j-closed.

The converse is clear. □

Lemma 3.10. *Let A be a subset of a mixed neutrosophic topological space (X,τ_1,τ_2). Then we have*

(1) A is (τ_i,τ_j)-preopen if and only if (τ_j,τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$,

(2) A is (τ_i,τ_j)-preopen if and only if (τ_j,τ_i)-$s\,$Cl(A) is (τ_i,τ_j)-regular open,

(3) A is (τ_i,τ_j)-regular open if and only if A is (τ_i,τ_j)-preopen and (τ_j,τ_i)-semiclosed.

Proof. (1) Suppose A is a (τ_i,τ_j)-preopen set. Then we have

$$(\tau_j,\tau_i)\text{-}s\,\text{Cl}(A) \subset (\tau_j,\tau_i)\text{-}s\,\text{Cl}(\tau_i\text{-Int}(\tau_j\text{-Cl}(A))).$$

Since τ_i-Int$(\tau_j$-Cl$(A))$ is (τ_j,τ_i)-semiclosed, (τ_j,τ_i)-$s\,$Cl$(A) \subset \tau_i$-Int$(\tau_j$-Cl$(A))$. Then by Lemma 3.8 (1), (τ_j,τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$.

The converse is obvious.

(2) Suppose (τ_j,τ_i)-$s\,$Cl(A) is a (τ_i,τ_j)-regular open set. Then we have

$$(\tau_j,\tau_i)\text{-}s\,\text{Cl}(A) = \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_j,\tau_i)\text{-}s\,\text{Cl}(A)).$$

Thus (τ_j,τ_i)-$s\,$Cl$(A) \subset \tau_i$-Int$(\tau_j$-Cl$(\tau_j$-Cl$(A))) = \tau_i$- Int$(\tau_j$-Cl$(A))$. So by Lemma 3.8 (1), we have (τ_j,τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence A is a (τ_i,τ_j)-preopen set from (1).

The converse follows from (1).

(3) Suppose A is a (τ_i,τ_j)-preopen and a (τ_j,τ_i)-semiclosed set. Then by (2), A is (τ_i,τ_j)-regular open in X.

Conversely, suppose A is a (τ_i,τ_j)-regular open set. Then $A = \tau_i$-Int$(\tau_j$-Cl$(A))$. Thus τ_i-Int$(\tau_j$-Cl$(A)) = (\tau_j,\tau_i)$-$s\,$Cl$(A) = A$. So A is (τ_i,τ_j)-preopen and (τ_j,τ_i)-semiclosed. □

Theorem 3.11. *In a mixed neutrosophic topological space (X,τ_1,τ_2), the following are equivalent:*

(1) X is pairwise extremally disconnected,

(2) (τ_j,τ_i)-$s\,$Cl$(A) = (\tau_j,\tau_i)$-Cl$_\theta(A)$ for every (τ_i,τ_j)-preopen (or *(τ_i,τ_j)-semiopen*) *set A in X,*

(3) (τ_j,τ_i)-$s\,$Cl$(A) = \tau_j$-Cl(A) for every (τ_i,τ_j)-β-open set A in X.

Proof. (1) ⇒ (2): Since (τ_j,τ_i)-$s\,$Cl$(A) \subset (\tau_j,\tau_i)$-Cl$_\theta(A)$ for any subset A of X, it is sufficient to show that (τ_j,τ_i)-Cl$_\theta(A) \subset (\tau_j,\tau_i)$-$s\,Cl(A)$ for any (τ_i,τ_j)-preopen or

(τ_i,τ_j)-semiopen set A of X. Let $x \notin (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j,τ_i)-semiopen set U with $x \in U$ such that $U \cap A = \varnothing$. Thus there exists a τ_j-open set V such that $V \subset U \subset \tau_j$-$\mathrm{Cl}(V)$ with $V \cap A = \varnothing$ which implies $V \cap \tau_j$-$\mathrm{Cl}(A) = \varnothing$. This means $V \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A)) = \varnothing$. So τ_i-$\mathrm{Cl}(V) \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A)) = \varnothing$. Now, if A is (τ_i,τ_j)-preopen, then $A \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$ and thus τ_i-$\mathrm{Cl}(V) \cap A = \varnothing$. If A is (τ_i,τ_j)-semiopen, since X is pairwise extremally disconnected, τ_i-$\mathrm{Cl}(V)$ is τ_j-open and thus τ_i-$\mathrm{Cl}(V) \cap \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))) = \varnothing$ which implies τ_i-$\mathrm{Cl}(V) \cap A = \varnothing$. So in any case, $x \notin (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$.

(2) ⇒ (1): First let A be a (τ_i,τ_j)-preopen set in X. By Lemmas 3.10 and 3.5, we have τ_i-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A)) = (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A) = \tau_j$-$\mathrm{Cl}(A)$. Then τ_j-$\mathrm{Cl}(A)$ is τ_i-open. Thus by Theorem 3.6, X is pairwise extremally disconnected. Next, let A be a (τ_i,τ_j)-semiopen set in X. Then we have

$$(\tau_j,\tau_i)\text{-}\mathrm{Cl}(A) \subset \tau_j\text{-}\mathrm{Cl}(A) \subset (\tau_j,\tau_i)\text{-}\mathrm{Cl}_\theta(A) = (\tau_j,\tau_i)\text{-}s\,\mathrm{Cl}(A).$$

Thus (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}(A)$. So X is pairwise extremally disconnected from Theorem 3.6.

(1) ⇒ (3): Let A be a (τ_i,τ_j)-β-open set in X. Since X is pairwise extremally disconnected, by Theorem 3.6, τ_j-$\mathrm{Cl}(A)$ is τ_i-open in X. Then by Lemma 3.10, (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}(A)$.

(3) ⇒(1): Let U and V, respectively, be τ_i-open and τ_j-open sets such that $U \cap V = \varnothing$. Then $U \subset X \backslash V$ which implies (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(X\backslash V) = X\backslash V$. Thus (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap V = \varnothing$. Since (τ_j,τ_i)-$s\,\mathrm{Cl}(U)$ is (τ_i,τ_j)-semiopen in X, (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap (\tau_i,\tau_j)$-$s\,\mathrm{Cl}(V) = \varnothing$. So by (3), τ_j-$\mathrm{Cl}(U) \cap \tau_i$-$\mathrm{Cl}(V) = \varnothing$. Hence by Theorem 3.2, X is pairwise extremally disconnected. □

Theorem 3.12. *In a mixed neutrosophic topological space* (X,τ_1,τ_2), *the following are equivalent:*

(1) *X is pairwise extremally disconnected,*

(2) *for each (τ_i,τ_j)-β-open set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \varnothing$, τ_i-$\mathrm{Cl}(A) \cap \tau_j$-$\mathrm{Cl}(B) = \varnothing$,*

(3) *for each (τ_i,τ_j)-preopen set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \varnothing$, τ_i-$\mathrm{Cl}(A) \cap \tau - j$-$\mathrm{Cl}(B) = \varnothing$.*

Proof. (1)⇒ (2): Let A be a (τ_i,τ_j)-β-open set and B a (τ_j,τ_i)-semiopen set such that $A \cap B = \varnothing$. Then $A \cap \tau_j$-$\mathrm{Int}(B) = \varnothing$. Thus τ_j-$\mathrm{Cl}(A) \cap \tau_j$-$\mathrm{Int}(B) = \varnothing$. By Theorem 3.6, τ_j-$\mathrm{Cl}(A)$ is a τ_i-open set in X. So τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(B)) = \varnothing$. Since B is (τ_j,τ_i)-semiopen in X, τ_i-$\mathrm{Cl}(B) = \tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(B))$. Hence τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$.

(2) ⇒ (3): Straightforward.

(3) ⇒ (1): Let A be a τ_i-open set and B a τ_j-open set such that $A \cap B = \varnothing$. Since every τ_i-open set is a (τ_i,τ_j)-semiopen set and every τ_j-open set is a (τ_i,τ_j)-semiopen set and every τ_j-open set is a (τ_j,τ_i)-preopen set, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \varnothing$. Then by Theorem 3.2, X is pairwise extremally disconnected. □

4. Conclusion

We have introduced not only the notion of mixed neutrosophic topological space but also several generalized open sets in the context of such spaces. We obtained many new, useful and important properties. The notion of extremally disconnectedness is also introduced, characterized and discussed with respect to some generalized open sets. The fertile ground of Mixed neutrosophic topological spaces demands more research for example with respect to separation axioms, compactness, multifunctions, different types of continuities and decision-making problems, to name a few.

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