Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (x 2023), pp. 1–x
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

New separation axioms in soft bitopological ordered spaces

S. H. Shalil, S. A. El-Sheikh, S. A. Kandil

Received 8 December 2023; Revised 28 January 2024; Accepted 29 February 2024

Abstract. This paper focuses on the topic of ordered soft separation axioms, which involve using soft points on soft bitopological ordered space. The main objective is to examine the properties and characterizations of these axioms, and establish some important results linking them to other concepts such as soft topological and soft hereditary properties. Furthermore, the paper presents examples and properties to highlight the distinctions between the separation axioms introduced in this study and those in [1]. Notably, the separation axioms proposed in this paper are more robust than other separation axioms.

2020 AMS Classification: 54A10, 54B05, 54D10, 54D15, 54D30

Keywords: Soft set, Soft point, *Bi*-ordered soft separation axioms.

Corresponding Author: Salama Hussien Ali Shalil (slamma_elarabi@yahoo.com)

1. Introduction

By adding partial order relations to topological structures, Nachbin [2] introduced the concept of topological ordered spaces as a generalization of topological spaces in 1965. McCartan [3] went on to utilize monotone neighborhoods in order to study ordered separation axioms. In order to deal with the vagueness and uncertainty of real-life problems, various mathematical tools have been developed such as fuzzy sets, intuitionistic fuzzy sets, rough sets, and vague sets. One such tool, soft sets, was introduced by Molodtsov [4] in 1999 and has since been developed and applied to decision-making problems, algebraic structures, and topological spaces. Soft separation axioms have been defined and investigated for both crisp and soft points, and four types of separation axioms have been identified and characterized. In 2020, Solai and Maltepe [5] introduced soft separation axioms using soft points on soft topological spaces and discussed their properties and characterizations.

El-Shafei et al. [6, 7] developed a new framework for soft topological ordered spaces by introducing two novel soft relations, namely partial belong and total non-belong. They also introduced the concept of ordered soft separation axioms, denoted as P-soft T_i-ordered, where i ranges from 0 to 4. Şenel [8] presented the soft topology generated by L-soft sets. Additionally, in a 2016, Şenel [9] proposed a new approach to hausdorff space theory via the soft sets. Şenel and Çağman [10] introduced soft topological subspaces in 2015. Furthermore, Şenel and Çağman [11] explored soft closed sets on soft bitopological space in 2014. In 2020, Şenel et al. [12] studied distance and similarity measures for octahedron sets introduced by Lee et al. [13]. El-Sheikh et al. [14] established the concept of a soft bitopological ordered space, which comprises a soft bitopological space with a partial order relation. They introduced and studied various concepts related to increasing and decreasing pairwise open and closed soft sets, increasing and decreasing total and partial pairwise soft neighborhoods, and increasing, decreasing, and balancing pairwise open soft neighborhoods. They also defined the concept of increasing and decreasing pairwise soft closure and interior.

The paper is divided into three sections. The preliminary section, Section 2, provides an overview of soft sets and soft topologies, including their definitions and properties. Section 3 is dedicated to introducing new soft separation axioms, called $PSST_i$-spaces ($i = 0, 1, 2, 3, 4$), and exploring their relationships and various properties. One of the key aspects of this section is the introduction of a new concept called PSS-regular spaces, which is weaker than P^*-soft regular space, as studied in [1]. The properties of PSS-regular spaces are analyzed in detail in this section.

2. Preliminaries

In this section, we briefly review some concepts and some related results of soft set, soft point, soft topological space and soft bitopological ordered space which are needed to use in current paper. For more details about these concepts you can see in [1, 15, 16]. Henceforth, X denotes the universe set, E denotes the fixed set of parameters and 2^X denotes the power set of X.

Definition 2.1 ([4])**.** A pair (G, E) is said to be a *soft set over* X, where $G : E \longrightarrow 2^X$. The family of all soft sets over X denoted by $P(X)^E$.

Remark 2.2. (1) For short, we use the notation G_E instead of (G, E).

(2) A soft set can be defined as a set of ordered pairs:

$$G_E = \{(\alpha, G(\alpha)) : \alpha \in E \text{ and } G(\alpha) \in 2^X\}.$$

Definition 2.3 ([17])**.** Let $G_E \in P(X)^E$. Then G_E is called:

(i) a *null soft set*, denoted by $\widehat{\phi}$, if $G(\alpha) = \varnothing \ \forall \alpha \in E$,

(ii) an *absolute soft set*, denoted by X_E, if $G(\alpha) = X \ \forall \alpha \in E$.

Definition 2.4 ([18])**.** Let $G_E,\ N_E \in P(X)^E$.

(i) N_E is called a *soft subset* of G_E, denoted by $N_E \sqsubseteq G_E$, if $N(\alpha) \subseteq G(\alpha) \ \forall \alpha \in E$.

(ii) N_E and G_E are said to be *equal*, denoted by $N_E = G_E$, if $N_E \sqsubseteq G_E$ and $G_E \sqsubseteq N_E$.

(iii) The *union* of N_E and G_E, denoted by $N_E \sqcup G_E$, is a soft set H_E over X defined by $H(\alpha) = N(\alpha) \cup G(\alpha) \ \forall \alpha \in E$.

(iv) The *intersection* of N_E and G_E, denoted by $N_E \sqcap G_E$, is a soft set H_E over X defined by $H(\alpha) = N(\alpha) \cap G(\alpha)\ \forall \alpha \in E$.

Definition 2.5 ([16]). Let $G_E,\ N_E \in P(X)^E$.

(i) The *difference* of N_E and G_E, denoted by $H_E = N_E - G_E$ is a soft set H_E over X defined by $H(\alpha) = N(\alpha) - G(\alpha)\ \forall \alpha \in E$.

(ii) The *complement* of N_E, denoted by N_E^c, is a soft set over X defined by $N^c(\alpha) = (N(\alpha))^c\ \forall \alpha \in E$.

Definition 2.6 ([19]). The soft set $N_E : E \to 2^X$ given by:

$$N(e) = \begin{cases} \{x\} & \text{if } e = \alpha \\ \phi & \text{if } e \in E - \{\alpha\} \end{cases}$$

is called a *soft point* over X and denoted by x^α. The family of all soft points over X denoted by $Sp(X)_E$.

Definition 2.7 ([20]). A soft point x^α is said to *belong to* a soft set N_E, denoted by $x^\alpha \widehat{\in} N_E$, if $x^\alpha(\alpha) \subseteq N(\alpha)$ for each $\alpha \in E$.

Definition 2.8 ([4, 6]). For a soft set N_E over X and $a \in X$, we say:

(i) $a \in N_E$, if $a \in N(\alpha)$ for each $\alpha \in E$ and $a \notin N_E$, if $a \notin N(\alpha)$ for some $\alpha \in$,

(ii) $a \Subset N_E$, if $a \in N(\alpha)$, for some $\alpha \in E$ and $a \not\Subset N_E$, if $a \notin N(\alpha)$ for each $\alpha \in E$.

The notations $\in, \notin, \Subset$ and $\not\Subset$ are respectively read belong, non-belong, partial belong and total non-belong relations.

Definition 2.9 ([16]). A *soft topology* on X is a collection τ of soft sets over X under E satisfying the following axioms:

(i) $\widehat{\phi}$ and X_E belong to τ,

(ii) the union of any member of soft sets in τ belongs to τ,

(iii) the intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is said to be a *soft topological space* over X. Every member of τ is called a *soft open set* and its relative complement is called a *soft closed set*.

Definition 2.10 ([21]). A soft subset W_E of a soft topological space (X, τ, E) is called a *soft neighbourhood* of $x^e \in Sp(X)_E$, if there exists $N_E \in \tau$ such that $x^e \widehat{\in} N_E \sqsubseteq W_E$.

Definition 2.11 ([22, 23]). A *soft bitopological space* is a quadrable system (X, τ_1, τ_2, E), where τ_1 and τ_2 are any soft topologies on X with a fixed set of parameters E.

Given a soft bitopological space (X, τ_1, τ_2, E), a soft set N_E in X is called a *pairwise open soft* (briefly, *PO*-soft) *set*, if there exist a τ_1-open soft set N_E^1 and a τ_2-open soft set N_E^2 such that $N_E = N_E^1 \sqcup N_E^2$. Similarly, a soft set N_E in X is called a *pairwise closed soft* (briefly, *PC*-soft) *set*, if the complement of N_E is a *PO*-soft set. Furthermore, the family of all *PO*-soft sets in a soft bitopological space (X, τ_1, τ_2, E) forms a supra soft topology on X, denoted by τ_{12}, given by $\tau_{12} = \{N_E : N_E = G_E^1 \sqcup G_E^2, G_E^j \in \tau_j, j = 1, 2\}$. Finally, we note that the supra soft topological space associated with the soft bitopological space (X, τ_1, τ_2, E) is the triple (X, τ_{12}, E).

Definition 2.12 ([24])**.** Let (X, τ_1, τ_2, E) be a soft bitopological space, G_E be a soft set over X and $x^e \in Sp(X)_E$. Then G_E is said to be a *pairwise soft neighbourhood* (briefly, P-soft nbd) of x^e, if there exists a PO-soft set N_E such that $x^e \widehat{\in} N_E \sqsubseteq G_E$.

Definition 2.13 ([23])**.** The following concepts are defined for a subset G_E of (X, τ_1, τ_2, E).

(i) The *pairwise soft closure* of G_E, denoted by $cl^s_{12}(G_E)$, is the intersection of all PC-soft sets containing G_E.

(ii) The *pairwise soft interior* of G_E, denoted by $int^s_{12}(G_E)$, is the union of all $PO-$soft sets which are contained in G_E.

Definition 2.14 ([15])**.** A binary relation $\leq$ on X is said to be a *partial order relation*, if $\leq$ is reflexive, anti-symmetric and transitive.

$\{(a, a) : \text{for every } a \in X\}$ is the equality relation on X and it is denoting by $\blacktriangle$.

Definition 2.15 ([2])**.** A triple $(X, \tau, \leq)$ is said to be a *topological ordered space*, if (X, τ) is a topological space and $(X, \leq)$ is a partially ordered set.

Definition 2.16 ([6])**.** A triple $(X, E, \leq)$ is said to be a *partially ordered soft space*, where $\leq$ is a partial order relation on X.

Definition 2.17 ([6])**.** Let $(X, E, \leq)$ be a partially ordered soft space. Then an increasing soft operator $i : (P(X)^E, \leq) \to (P(X)^E, \leq)$ and a decreasing soft operator $d : (P(X)^E, \leq) \to (P(X)^E, \leq)$ are defined, respectively as follows: for each soft set N_E in $P(X)^E$,

(i) $i(N_E) = (iN)_E$, where iN is a mapping of E into X given by: for each $\alpha \in E$,

$$iN(\alpha) = i(N(\alpha)) = \{a \in X : b \leq a \text{ for } \text{ some } b \in N(\alpha)\},$$

(ii) $d(N_E) = (dN)_E$, where dN is a mapping of E into X given by: for each $\alpha \in E$,

$$dN(\alpha) = d(N(\alpha)) = \{a \in X : a \text{ for } \text{ some } b \in N(\alpha)\}.$$

Definition 2.18 ([6])**.** A soft subset N_E of a partially ordered soft space $(X, E, \leq)$ is *increasing* (resp. *decreasing*), if $N_E = i(N_E)$ (resp. $N_E = d(N_E)$).

Definition 2.19 ([6])**.** A quadrable system $(X, \tau, E, \leq)$ is said to be a *soft topological ordered space* (briefly, $STOS$), if (X, τ, E) is a soft topological space and $(X, E, \leq)$ is a partially ordered soft space.

Definition 2.20 ([6])**.** Let $Y \subseteq X$ and $(X, \tau, E, \leq)$ be an $STOS$. Then $(Y, \tau_Y, E, \leq_Y$) is called a *soft ordered subspace* of $(X, \tau, E, \leq)$, provided that (Y, τ_Y, E) is soft subspace of (X, τ, E), where $\leq_Y$ is a partially ordered relation on Y.

Lemma 2.21 ([6])**.** *If W_E is an increasing (resp. a decreasing) soft subset of an $STOS$ $(X, \tau, E, \leq)$, then $W_E \sqcap Y_E$ is an increasing (resp. a decreasing) soft subset of a soft ordered subspace $(Y, \tau_Y, E, \leq_Y)$.*

Definition 2.22 ([14])**.** The system $(X, \tau_1, \tau_2, E, \leq)$ is said to be a *soft bitopological ordered space* (briefly, $SBTOS$), if (X, τ_1, τ_2, E) is a soft bitopological space and $(X, E, \leq)$ is a partially ordered soft space.

Definition 2.23 ([14])**.** Let $(X, \tau_1, \tau_2, E, \leq)$ be a *SBTOS*. A soft set M_E over X is said to be:

(i) an *increasing pairwise open soft* (briefly, *IPO*-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2,\ M_E^\beta \in \tau_\beta$ and increasing for $\beta = 1,\ 2$,

(ii) a *decreasing pairwise open soft* (briefly, *DPO*-soft) *set*, if $M_E = M_E^1 \sqcup M_E^2,\ M_E^\beta \in \tau_\beta$ and decreasing for $\beta = 1,\ 2$,

(iii) an *increasing pairwise closed soft* (briefly, *IPC*-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2,\ M_E^\beta \in \tau_\beta^c$ and increasing for $\beta = 1,\ 2$,

(iv) a *decreasing pairwise closed soft* (briefly, *DPC*-soft) *set*, if $M_E = M_E^1 \sqcap M_E^2,\ M_E^\beta \in \tau_\beta^c$ and decreasing for $\beta = 1,\ 2$.

Definition 2.24 ([14])**.** A soft set W_E in an *SBTOS* $(X, \tau_1, \tau_2, E, \leq)$ is called an *increasing* (resp. a *decreasing*) *pairwise soft neighborhood* (briefly, *IPS* (resp. *DPS*)-nbd) of $x^e \widehat{\in} X_E$, if there exists a *PO*-soft set H_E such that $x^e \widehat{\in} H_E \sqsubseteq W_E$ and W_E is increasing (resp. decreasing).

Definition 2.25 ([1])**.** Let x^{e_1} and y^{e_2} be two soft points in a partially ordered space $(X, E, \leq)$. We say that $x^{e_1} \nleq y^{e_2}$ if and only if $x \nleq y$ or $e_1 \neq e_2$.

Definition 2.26 ([1])**.** A soft subset W_E of an *STOS* $(X, \tau, E, \leq)$ is called an *increasing* (resp. a *decreasing*) *soft neighbourhood* of $x^e \in Sp(X)_E$, if W_E is soft neighbourhood of x^e and increasing (resp. decreasing).

Definition 2.27 ([1])**.** An *STOS* $(X, \tau, E, \leq)$ is said to be:

a *lower* P^**-soft* T_1*-ordered space*, if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1} \nleq y^{e_2}$, there exists an increasing soft neighbourhood W_E of x^{e_1} such that $y^{e_2} \widehat{\notin} W_E$,

(ii) an *upper* P^**-soft* T_1*-ordered space*, if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1} \nleq y^{e_2}$ there exists a decreasing soft neighbourhood W_E of y^{e_2} such that $x^{e_1} \widehat{\notin} W_E$,

(iii) a P^**-soft* T_0*-ordered* (briefly, P^*ST_0-ordered) *space*, if it is lower P^*-soft T_1-ordered or upper P^*-soft T_1-ordered,

(iv) a P^**-soft* T_1*-ordered* (briefly, P^*ST_1-ordered) *space*, if it is lower P^*-soft T_1-ordered and upper P^*-soft T_1-ordered,

(v) a P^**-soft* T_2*-ordered space*, if for every pair of soft points $x^{e_1},\ y^{e_2}$ such that $x^{e_1} \nleq y^{e_2}$ there exist disjoint soft neighbourhoods W_E and V_E of x^{e_1} and y^{e_2}, respectively such that W_E is increasing and V_E is decreasing.

Definition 2.28 ([1])**.** A soft point a^e in a partially ordered space $(X, E, \leq)$ is called the *smallest* (resp. *largest*) *soft element* of X_E if $a^e \leq x^e$ ($x^e \leq a^e$) for all $x^e \in X_E$.

3. New $Bi-$ordered soft separation axioms

In this section, we introduce Bi-ordered soft separation axioms or $PSST_i$-ordered spaces. We explore their main properties and the relationships between different separation axioms. We provide various examples to illustrate the results obtained in this section.

Definition 3.1. An SBTOS $(X, \tau_1, \tau_2, E, \leq)$ is said to be:

(i) a *lower pairwise soft ST_1-ordered* (briefly, $LPSST_1$-ordered) *space*, if for every pair of soft points x^{e_1}, y^{e_2} such that $x^{e_1} \nleq y^{e_2}$ there exists an IPS-nbd W_E of x^{e_1} such that $y^{e_2} \widehat{\notin} W_E$,

(ii) an *upper pairwise soft ST_1-ordered* (briefly, $UPSST_1$-ordered) *space*, if for every pair of soft points x^{e_1}, y^{e_2} such that $x^{e_1} \nleq y^{e_2}$ there exists a DPS-nbd W_E of y^{e_2} such that $x^{e_1} \widehat{\notin} W_E$,

(iii) a *$PSST_0$-ordered space*, if it is $LPSST_1$-ordered or $UPSST_1$-ordered,

(iv) a *$PSST_1$-ordered space*, if it is $LPSST_1$-ordered and $UPSST_1$-ordered,

(v) a *$PSST_2$-ordered space*, if for every pair of soft points x^{e_1}, y^{e_2} such that $x^{e_1} \nleq y^{e_2}$ there exist disjoint P-soft nbds W_E and V_E of x^{e_1} and y^{e_2}, respectively such that W_E is increasing and V_E is decreasing.

Proposition 3.2. *Every $PSST_i$-ordered space $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_{i-1}$-ordered, for $i = 1,\ 2$.*

Proof. The proof comes immediately from Definition 3.1 □

We present two examples to illustrate that the converse of the above proposition fails.

Example 3.3. Let $E = \{e_1, e_2\}, \leq = \blacktriangle \cup \{(x, y)\}$ be a partial order relation on $X = \{x, y\}$. Define $\tau_1 = \{X_E, \widehat{\phi}, G^1_E, G^2_E, G^3_E\}$ and $\tau_2 = \{X_E, \widehat{\phi}, F^1_E, F^2_E\}$, where

$$G^1_E = \{(e_1, \varnothing), (e_2, \{y\})\}\ G^2_E = \{(e_1, \{y\}), (e_2, \varnothing)\},\ G^3_E = \{(e_1, \{y\}), (e_2, \{y\})\},$$
$$F^1_E = \{(e_1, X), (e_2, \varnothing)\},\ F^2_E = \{(e_1, \varnothing), (e_2, X)\}.$$

Then $\tau_{12} = \{X_E, \widehat{\phi}, G^1_E, G^2_E, G^3_E, F^1_E, F^2_E, H^1_E, H^2_E\}$,
where $H^1_E = \{(e_1, \{y\}), (e_2, X)\}$, $H^2_E = \{(e_1, X), (e_2, \{y\})\}$.

For the soft points x^{e_1}, y^{e_2} with $x^{e_1} \nleq y^{e_2}$, there exists an IPS-nbd F^1_E of x^{e_1} such that $y^{e_2} \widehat{\notin} F^1_E$.

For the soft points x^{e_1}, x^{e_2} with $x^{e_1} \nleq x^{e_2}$, there exists an IPS-nbd F^1_E of x^{e_1} such that $x^{e_2} \widehat{\notin} F^1_E$.

For the soft points x^{e_2}, x^{e_1} with $x^{e_2} \nleq x^{e_1}$, there exists an IPS-nbd F^2_E of x^{e_2} such that $x^{e_1} \widehat{\notin} F^2_E$.

For the soft points x^{e_2}, y^{e_1} with $x^{e_2} \nleq y^{e_1}$, there exists an IPS-nbd F^2_E of x^{e_2} such that $y^{e_1} \widehat{\notin} F^2_E$.

For the soft points y^{e_1}, x^{e_2} with $y^{e_1} \nleq x^{e_2}$, there exists an IPS-nbd G^2_E of y^{e_1} such that $x^{e_2}\ \widehat{\notin} G^2_E$.

For the soft points y^{e_1}, y^{e_2} with $y^{e_1} \nleq y^{e_2}$, there exists an IPS-nbd G^2_E of y^{e_1} such that $y^{e_2} \widehat{\notin} G^2_E$.

For the soft points y^{e_2}, x^{e_1} with $y^{e_2} \nleq x^{e_1}$, there exists an IPS-nbd G^3_E of y^{e_2} such that $x^{e_1} \widehat{\notin} G^3_E$.

For the soft points y^{e_2}, y^{e_1} with $y^{e_2} \nleq y^{e_1}$, there exists an IPS-nbd G^1_E of y^{e_2} such that $y^{e_1} \widehat{\notin} G^1_E$.

For the soft points y^{e_1}, x^{e_1} with $y^{e_1} \nleq x^{e_1}$, there exists an IPS-nbd G^2_E of y^{e_1} such that $x^{e_1} \widehat{\notin} G^2_E$.

For the soft points y^{e_2}, x^{e_2} with $y^{e_2} \nleq x^{e_2}$, there exists an IPS-nbd G_E^3 of y^{e_2} such that $x^{e_2} \widehat{\notin} G_E^3$.
Thus $(X, \tau_1, \tau_2, E, \leq)$ is $LPSST_1$-ordered. So $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_0$-ordered. On the other hand, we have $y^{e_1} \nleq x^{e_1}$ but there does not exist a DPS- nbd contain x^{e_1} and does not contain y^{e_1}. Hence $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_1$-ordered.

Example 3.4. By taking $\tau_1 = \tau_2 = \tau$. The example is referring to an Example 4.7 in aprevious work [6]. It is stated that this example is $PSST_1$-ordered, but not $PSST_2$-ordered.

Theorem 3.5. *Let $(X, \tau_1, \tau_2, E, \leq)$ be an SBTOS. Then the following three statements are equivalent:*
(1) $(X, \tau_1, \tau_2, E, \leq)$ *is* $UPSST_1$ (resp. $LPSST_1$)*-ordered,*
(2) *for any two soft points x^{e_1}, $y^{e_2} \in Sp(X)_E$ such that $x^{e_1} \nleq y^{e_2}$, there is a PO-soft set G_E containing y^{e_2}* (resp. x^{e_1}) *in which $x^{e_1} \nleq z^{e_3}$* (resp. $z^{e_3} \nleq y^{e_2}$) *for every $z^{e_3} \widehat{\in} G_E$,*
(3) *for any soft point x^e, $i(x^e)$* (resp. $d(x^e)$) *is a PC-soft set.*

Proof. (1)⇒(2): Suppose $(X, \tau_1, \tau_2, E, \leq)$ is a $UPSST_1$-ordered space and let x^{e_1}, y^{e_2} be two soft points such that $x^{e_1} \nleq y^{e_2}$. Then there exists a DPS- nbd W_E of y^{e_2} such that $x^{e_1} \widehat{\notin} W_E$. Putting $G_E = sint_{12}^s(W_E)$. Assume that $G_E \not\sqsubseteq (i(x^{e_1}))^c$. Then there exists $z^{e_3} \widehat{\in} G_E$ and $z^{e_3} \widehat{\notin} (i(x^{e_1}))^c$. Thus $z^{e_3} \widehat{\in} i(x^{e_1})$ and this implies that $x^{e_1} \leq z^{e_3}$. Now, $z^{e_3} \widehat{\in} G_E \sqsubseteq W_E$ implies that $x^{e_1} \widehat{\in} W_E$. This contradicts that $x^{e_1} \widehat{\notin} W_E$. So $G_E \sqsubseteq (i(x^{e_1}))^c$. Hence $x^{e_1} \nleq z^{e_3}$, for every $z^{e_3} \widehat{\in} G_E$.

(2)⇒(3): Suppose the condition (2) holds and let x^{e_1}, $y^{e_2} \in Sp(X)_E$ such that $y^{e_2} \widehat{\in} (i(x^{e_1}))^c$. Then $x^{e_1} \nleq y^{e_2}$. Thus there exists a PO-soft set G_E containing y^{e_2} such that $G_E \sqsubseteq (i(x^{e_1}))^c$. Since x^{e_1} and y^{e_2} are chosen arbitrary, a soft set $(i(x^{e_1}))^c$ is PO-soft open for any soft point x^{e_1}. So $i(x^{e_1})$ is PC-soft for any $x^{e_1} \in Sp(X)_E$.

(3)⇒(1): Suppose the condition (3) holds and let x^{e_1} and y^{e_2} be two soft points such that $x^{e_1} \nleq y^{e_2}$. Obviously, $i(x^{e_1})$ is increasing. By the hypothesis, $i(x^{e_1})$ is a PC-soft set. Then $(i(x^{e_1}))^c$ is a DPO-soft set satisfies that $y^{e_2} \widehat{\in} (i(x^{e_1}))^c$ and $x^{e_1} \widehat{\notin} (i(x^{e_1}))^c$. Thus the proof is completed.

A similar proof can be given for the case between parentheses. □

Proposition 3.6. *Let $(X, \tau_1, \tau_2, E, \leq)$ be an SBTOS with $\tau_1 = \tau_2 = \tau$. If $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_i$-ordered, then $(X, \tau, E, \leq)$ is always P^*ST_i-ordered for $i = 0,\ 1,\ 2$.*

Proof. We have shown the proposition when $i = 1$, and the other instance can be shown similarly. Let x^{e_1}, y^{e_2} be two soft points in $(X, \tau, E, \leq)$ such that $x^{e_1} \nleq y^{e_2}$. As $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_1$, there exist an IPS-nbd W_E of x^{e_1} such that $y^{e_2} \widehat{\notin} W_E$ and a DPS-nbd F_E of y^{e_2} such that $x^{e_1} \widehat{\notin} F_E$. Since $\tau_1 = \tau_2 = \tau$, W_E is an increasing soft neighborhood of x^{e_1} such that $y^{e_2} \widehat{\notin} W_E$ and F_E is a decreasing soft neighborhood of y^{e_2} such that $x^{e_1} \widehat{\notin} F_E$. Then $(X, \tau, E, \leq)$ is P^*ST_1-ordered. □

Proposition 3.7. *If a^e is one of the smallest soft element of a finite $LPSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$, then a^e is a DPC-soft point.*

Proof. Suppose a^e is one of the smallest soft element of a finite $LPSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$. Then $a^e \leq x^e$ for all $x^e \widehat{\in} X_E$. Thus a^e is a DPC- soft point. □

Proposition 3.8. *If a^e is one of the largest soft element of a finite $UPSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$, then a^e is a DPC-soft pint.*

Proof. Suppose a^e is one of the largest soft element of a finite $UPSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$. Then $x^e \leq a^e$ for all $x^e \widehat{\in} X^E$. Thus a^e is a DPC- soft point. □

Proposition 3.9. *If a^e is a smallest* (resp. a largest) *soft element of a finite $PSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$, then a^e is a DPO* (resp. an IPO)*-soft point.*

Proof. Suppose a^e is a smallest soft element of a finite $PSST_1$-ordered space $(X, \tau_1, \tau_2, E, \leq)$. Then $a^e \leq x^e$ for all $x^e \widehat{\in} X_E$. By the anti-symmetric of $\leq$, we have $x^e \nleq a^e$ for all $x^e \widehat{\in} X_E$. Thus by the hypothesis, there is a DPS- nbd W_E of a^e such that $x^e \widehat{\notin} W_E$. It follows that $a^e = \sqcap W_E$. Since X is finite, a^e is a DPO-soft point.

A similar proof can be given for the case between parentheses. □

Proposition 3.10. *A finite $SBTOS$ $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_1$-ordered if and only if it is $PSST_2$-ordered.*

Proof. Necessity: For each $y^{e'} \widehat{\in} (i(x^e))^c$, we have $d(y^{e'})$ is PC-soft. Since X is finite, $\sqcup_{y^{e'} \widehat{\in} (i(x^e))^c} d(y^{e'})$ is PC-soft. Then $(\sqcup_{y^{e'} \widehat{\in} (i(x^e))^c} d(y^{e'}))^c$ is a PO-soft set. Thus $(X, \tau_1, \tau_2, E, \leq)$ is a $PSST_2$-ordered space.

Sufficiency: It follows immediately from Proposition 3.2. □

Theorem 3.11. *The property of being a $PSST_i$-ordered space is hereditary for* $i = 0,\ 1,\ 2$.

Proof. Let $(Y, \tau_Y, \eta_Y, E, \leq_Y)$ be a soft bitopological ordered subspace of a $PSST_2$-ordered space $(X, \tau_1, \tau_2, E, \leq)$. Let $x^e,\ y^{e'} \widehat{\in} Y_E$ such that $x^e \nleq_Y y^{e'}$. Then $x^e \nleq y^{e'}$. Thus by the hypothesis, there exist disjoint P- soft nbds W_E and V_E of x^e and $y^{e'}$, respectively such that W_E is increasing and V_E is decreasing. Setting $U_E = Y_E \sqcap W_E$ and $G_E = Y_E \sqcap V_E$, from Lemma 2.21, we obtain that U_E is an IPS-nbd of x^e and G_E is a DPS-nbd of $y^{e'}$. Since the P-soft nbds U_E and G_E are disjoint, it follows that $(Y, \tau_Y, \eta_Y, E, \leq_Y)$ is $PSST_2$-ordered.

The theorem can be proven similarly in case of $i = 0,\ 1$. □

Definition 3.12. An $SBTOS$ $(X, \tau_1, \tau_2, E, \leq)$ is said to be:

(i) a *lower* (resp. an *upper*) *PSS-regularly ordered space*, if for each DPC (resp. IPC)-soft set H_E and $x^e \in Sp(X)_E$ such that $x^e \widehat{\notin} H_E$, there exist disjoint P-soft nbds W_E of H_E and V_E of x^e such that W_E is decreasing (resp. increasing) and V_E is increasing (resp. decreasing),

(ii) a *PSS-regularly ordered space*, if it is both lower PSS-regularly ordered and upper PSS-regularly ordered,

(iii) a *lower* (resp. an *upper*) *$PSST_3$-ordered space*, if it is both $LPSST_1$ (resp. $UPSST_1$)-ordered and lower (resp. upper) PSS-soft regularly ordered,

(iv) a *$PSST_3$-ordered space*, if it is both lower $PSST_3$-ordered and upper $PSST_3$-ordered.

Theorem 3.13. *An $SBTOS$ $(X,\tau_1,\tau_2,E,\leq)$ is lower* (resp. *upper*) *PSS-regularly ordered if and only if for all $x^e\in Sp(X)_E$ and every IPO (resp. DPO)-soft set U_E containing x^e, there is an IPS* (resp. *DPS*)*-nbd V_E of x^e satisfies that $scl^s_{12}(V_E)\sqsubseteq U_E$.*

Proof. Necessity: Suppose $(X,\tau_1,\tau_2,E,\leq)$ is lower PSS-regularly ordered and let $x^e\in Sp(X)_E$ and U_E be an IPO-soft set containing x^e. Then U^c_E is DPC-soft such that $x^e\widehat{\notin}U^c_E$. Thus by the hypothesis, there exist disjoint P-soft nbds V_E of x^e and W_E of U^c_E such that V_E is increasing and W_E is decreasing. So there is a PO-soft set G_E such that $U^c_E\sqsubseteq G_E\sqsubseteq W_E$. Since $V_E\sqsubseteq W^c_E$, $V_E\sqsubseteq W^c_E\sqsubseteq G^c_E\sqsubseteq U_E$. Since G^c_E is PC-soft, $scl^s_{12}(V_E)\sqsubseteq G^c_E\sqsubseteq U_E$.

Sufficiency: Suppose the sufficient condition holds and let $x^e\in Sp(X)_E$ and H_E be a DPC-soft set such that $x^e\widehat{\notin}H_E$. Then H^c_E is an IPO-soft set containing x^e. Thus by the hypothesis, there is an IPS-nbd V_E of x^e such that $scl^s_{12}(V_E)\sqsubseteq H^c_E$. So $(scl^s_{12}(V_E))^c$ is a PO-soft set containing H_E. Assume that $V_E\sqcap d((scl^s_{12}(V_E))^c)\neq\widehat{\phi}$. Then there exists $y^{e'}\in Sp(X)_E$ such that $y^{e'}\widehat{\in}V_E$ and $y^{e'}\widehat{\in}d((scl^s_{12}(V_E))^c)$. Thus there exists $z^{e''}\widehat{\in}(cl^s_{12}(V_E))^c$ satisfies that $y^{e'}\leq z^{e''}$. This means that $z^{e''}\widehat{\in}V_E$. This contradicts the disjointedness between V_E and $(scl^s_{12}(V_E))^c$. So $V_E\sqcap d((scl^s_{12}(V_E))^c)=\widehat{\phi}$. This completes the proof.

A similar proof can be given for the case between parentheses. □

Proposition 3.14. *The following three properties are equivalent if $(X,\tau_1,\tau_2,E,\leq)$ is $PSS-$regularly ordered:*

(1) *$(X,\tau_1,\tau_2,E,\leq)$ is $PSST_2$-ordered,*
(2) *$(X,\tau_1,\tau_2,E,\leq)$ is $PSST_1$-ordered,*
(3) *$(X,\tau_1,\tau_2,E,\leq)$ is $PSST_0$-ordered.*

Proof. The proofs of (1)⇒(2)⇒(3) are obvious.

(3)⇒(1): Suppose the condition (3) holds and let x^e, $y^{e'}\in Sp(X)_E$ such that $x^e\not\leq y^{e'}$. Since $(X,\tau_1,\tau_2,E,\leq)$ is $PSST_0$-ordered, it is $LPSST_1$-ordered or $UPSST_1$-ordered, say it is $UPSST_1$-ordered. Then from Theorem 3.5, we have $i(x^e)$ is PC-soft. Obviously, $i(x^e)$ is increasing and $y^{e'}\widehat{\notin}i(x^e)$. Since $(X,\tau_1,\tau_2,E,\leq)$ is PSS-regularly ordered, there exist disjoint P-soft nbds W_E and V_E of $y^{e'}$ and $i(x^e)$, respectively such that W_E is decreasing and V_E is increasing. Thus $(X,\tau_1,\tau_2,E,\leq)$ is $PSST_2$-ordered. □

Corollary 3.15. *The following three properties are equivalent if $(X,\tau_1,\tau_2,E,\leq)$ is lower (resp. upper) $PSS-$regularly ordered:*

(1) *$(X,\tau_1,\tau_2,E,\leq)$ is $PSST_2$-ordered,*
(2) *$(X,\tau_1,\tau_2,E,\leq)$ is $PSST_1$-ordered,*
(3) *$(X,\tau_1,\tau_2,E,\leq)$ is $LPSST_1$* (resp. *$UPSST_1$*)*-ordered.*

Definition 3.16. An $SBTOS$ $(X,\tau_1,\tau_2,E,\leq)$ is said to be:

(i) a Bi-Soft normally ordered space, if for each disjoint PC-soft sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint P-soft nbds W_E of F_E and V_E of H_E such that W_E is increasing and V_E is decreasing,

(ii) a $PSST_4$-ordered space, if it is $Bi-$soft normally ordered and $PSST_1$-ordered.

Theorem 3.17. *An SBTOS* $(X, \tau_1, \tau_2, E, \leq)$ *is Bi-soft normally ordered if and only if for every DPC* (resp. *IPC*)*-soft set* F_E *and every DPS* (resp. *IPS*)*- nbd* U_E *of* F_E*, there is a DPS* (resp. an *IPS*)*-nbd* V_E *of* F_E *satisfies that* $scl_{12}^s(V_E) \sqsubseteq U_E$.

Proof. Necessity: Suppose $(X, \tau_1, \tau_2, E, \leq)$ is Bi-soft normally ordered and let F_E be a DPC-soft set and U_E be a DPS-nbd of F_E. Then U_E^c is an IPC-soft set and $F_E \sqcap U_E^c = \widehat{\phi}$. Since $(X, \tau_1, \tau_2, E, \leq)$ is Bi-soft normally ordered, there exist disjoint DPS-nbd V_E of F_E and IPS-nbd W_E of U_E^c. Since W_E is P-soft nbd of U_E^c, there exists a PO-soft set H_E such that $U_E^c \sqsubseteq H_E \sqsubseteq W_E$. Thus $W_E^c \sqsubseteq H_E^c \sqsubseteq U_E$ and $V_E \sqsubseteq W_E^c$. So it follows that $scl_{12}^s(V_E) \sqsubseteq scl_{12}^s(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E$. Hence we have

$$F_E \sqsubseteq scl_{12}^s(V_E) \sqsubseteq scl_{12}^s(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E.$$

Therefore the necessary part holds.

Sufficiency: Suppose the necessary condition holds and let F_E^1 and F_E^2 be two disjoint PC-soft sets such that F_E^1 is decreasing and F_E^2 is increasing. Then F_E^{2c} is a DPO-soft set containing F_E^1. By the hypothesis, there exists a DPS-nbd V_E of F_E^1 such that $scl_{12}^s(V_E) \sqsubseteq F_E^{2c}$. Setting $H_E = X_E - scl_{12}^s(V_E)$. This means that H_E is a PO-soft set containing F_E^2. Obviously, $F_E^2 \sqsubseteq H_E,\ F_E^1 \sqsubseteq V_E$ and $H_E \sqcap V_E = \widehat{\phi}$. Now, $i(H_E)$ is an IPS-nbd of F_E^2. Assume that $i(H_E) \sqcap V_E \neq \widehat{\phi}$. Then there exists $x^e \in Sp(X)_E$ such that $x^e \widehat{\in} i(H_E)$ and $x^e \widehat{\in} V_E = d(V_E)$. This implies that there exist $a^{e'} \widehat{\in} H_E$ and $b^{e''} \widehat{\in} V_E$ such that $a^{e'} \leq x^e$ and $x^e \leq b^{e''}$. As $\leq$ is transitive, $a^{e'} \leq b^{e''}$. Thus $b^{e''} \widehat{\in} H_E \sqcap V_E$. This contradicts the disjointedness between H_E and V_E. So $i(H_E) \sqcap V_E = \widehat{\phi}$. Hence the proof is completed.

A similar proof can be given for the case between parentheses. □

Proposition 3.18. *Every* $PSST_i$*-ordered space* $(X, \tau_1, \tau_2, E, \leq)$ *is* $PSST_{i-1}$*-ordered for* $i = 3,\ 4$.

Proof. From Proposition 3.14, we obtain that every $PSST_3$-ordered space is $PSST_2$-ordered. To prove the proposition in case of $i = 4$, let $x^e \in Sp(X)_E$ and F_E be a DPC-soft set such that $x^e \widehat{\notin} F_E$. Since $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_1$-ordered, $i(x^e)$ is an IPC-soft set. Since $(X, \tau_1, \tau_2, E, \leq)$ is Bi-soft normally ordered, there exist disjoint P-soft nbds W_E and V_E of $i(x^e)$ and F_E, respectively such that W_E is increasing and V_E is decreasing. Then $(X, \tau_1, \tau_2, E, \leq)$ is lower $PSS-$regularly ordered. If F_E is an IPC-soft set, then we prove similarly that $(X, \tau_1, \tau_2, E, \leq)$ is upper PSS-regularly ordered. Thus $(X, \tau_1, \tau_2, E, \leq)$ is PSS-regularly ordered. So $(X, \tau_1, \tau_2, E, \leq)$ is $PSST_3$-ordered. □

The converse of the above proposition is not always true as illustrated in the following two examples.

Example 3.19. Let $E = \{e_\alpha, e_\beta\}$ be a set of parameters, $\leq = \blacktriangle \cup \{(1, 2)\}$ be a partial order relation on the set of natural numbers $\aleph$. Define $\tau_1 = \{G_E \sqsubseteq \aleph_E$ such that $1 \in G_E$ and G_E^c is infinite $\}$ and $\tau_2 = \{F_E \sqsubseteq \aleph_E$ such that $1 \in F_E^c\} \cup \aleph_E$. Then $(\aleph, \tau_1, \tau_2, E, \leq)$ is a soft bitopological ordered space. In the following, we illustrate that $(\aleph, \tau_1, \tau_2, E, \leq)$ is $PSST_2$-ordered. We have the following 10 cases. Let $i \in \{\alpha, \beta\}$ and $x,\ y \in \aleph - \{1, 2\}$ with $x \neq y$.

Case 1. Suppose $1^{e_i} \nleq 1^{e_j},\ i \neq j$. Then we define two soft sets W_E, V_E as follows:

$$W(e_i) = \{1,2\},\ W(e_j) = \varnothing \text{ and } V(e_j) = \{1\},\ V(e_i) = \varnothing,\ i \neq j.$$

Thus W_E is an IPO-soft set containing 1^{e_i}, V_E is a DPO-soft set containing 1^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 2. Suppose $1^{e_i} \nleq 2^{e_j},\ i \neq j$. Then we define two soft sets W_E, V_E as follows:

$$W(e_i) = \{1,2\},\ W(e_j) = \varnothing \text{ and } V(e_j) = \{1,2\},\ V(e_i) = \varnothing,\ i \neq j.$$

Thus W_E is an IPO-soft set containing 1^{e_i}, V_E is a DPO-soft set containing 2^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 3. Suppose $1^{e_i} \nleq x^{e_j}\ \forall i,\ j$. Then we define two soft sets W_E, V_E as follows:

$$W(e_i) = \{1,2\},\ W(e_j) = \varnothing \text{ and } V(e) = \{x\} \forall e \in E.$$

Thus W_E is an IPO-soft set containing 1^{e_i}, V_E is a DPO-soft set containing x^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 4. $2^{e_i} \nleq 1^{e_j}\ \forall i,\ j$. Then we define two soft sets W_E, V_E as follows:

$$W(e) = \{2\}\ \forall e \in E \text{ and } V(e_j) = \{1\},\ V(e_i) = \varnothing.$$

Thus W_E is an IPO-soft set containing 2^{e_i}, V_E is a DPO-soft set containing 1^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 5. Suppose $2^{e_i} \nleq 2^{e_j},\ i \neq j$. Then we define two soft sets W_E, V_E as follows:

$$W(e_i) = \{1,2\},\ W(e_j) = \varnothing \text{ and } V(e_j) = \{1,2\},\ V(e_i) = \varnothing,\ i \neq j.$$

Thus W_E is an IPO-soft set containing 2^{e_i}, V_E is a DPO-soft set containing 2^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 6. Suppose $2^{e_i} \nleq x^{e_j} \forall i,\ j$. Then we define two soft sets W_E, V_E as follows:

$$W(e_i) = \{1,2\},\ W(e_j) = \varnothing \text{ and } V(e) = \{x\} \forall e \in E.$$

Thus W_E is an IPO-soft set containing 2^{e_i}, V_E is a DPO-soft set containing x^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 7. Suppose $x^{e_i} \nleq 1^{e_j}\ \forall i,\ j$. Then we define two soft sets W_E, V_E as follows:

$$W(e) = \{x\}\ \forall e \in E \text{ and } V(e_j) = \{1\},\ V(e_i) = \varnothing.$$

Thus W_E is an IPO-soft set containing x^{e_i}, V_E is a DPO-soft set containing 1^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 8. Suppose $x^{e_i} \nleq 2^{e_j}\ \forall i,\ j$. Then we define two soft sets $W_E,\ V_E$ as follows:

$$W(e) = \{x\}\ \forall e \in E \text{ and } V(e_j) = \{1,2\},\ V(e_i) = \varnothing.$$

Thus W_E is an IPO-soft set containing x^{e_i}, V_E is a DPO-soft set containing 2^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 9. Suppose $x^{e_i} \nleq x^{e_j},\ i \neq j$. Then we define two soft sets $W_E,\ V_E$ as follows:

$$W(e_i) = \{1,2,x\},\ W(e_j) = \varnothing \text{ and } V(e_j) = \{1,x\},\ V(e_i) = \varnothing,\ i \neq j.$$

Thus W_E is an IPO-soft set containing x^{e_i}, V_E is a DPO-soft set containing x^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

Case 10. Suppose $x^{e_i} \nleq y^{e_j}\ \forall i,\ j$. Then we define two soft sets W_E, V_E as follows:

$$W(e) = \{x\}\ \forall e \in E \text{ and } V(e) = \{y\}\ \forall e \in E.$$

Thus W_E is an IPO-soft set containing x^{e_i}, V_E is a DPO-soft set containing y^{e_j} and $W_E \sqcap V_E = \widehat{\phi}$.

To illustrate that $(\aleph, \tau_1, \tau_2, E, \leq)$ is not lower PSS-regularly ordered, we define a decreasing soft closed set H_E as follows:

$$H_E = \{(e_\alpha, \{1, 2, 4, 5, \cdots\}),\ (e_\beta, \{1, 2, 4, 5, \cdots\})\}.$$

Since $3^{e_\alpha} \widehat{\notin} H_E$ and there do not exist disjoint P-soft nbds W_E and V_E containing H_E and 3^{e_α}, respectively, $(\aleph, \tau_1, \tau_2, E, \leq)$ is not lower PSS-regularly ordered. Then $(\aleph, \tau_1, \tau_2, E, \leq)$ is not $PSST_3$-ordered.

Example 3.20. Let $E = \{e_1, e_2, e_3\}$ be a set of parameters, $\leq = \blacktriangle \cup \{(1, 2)\}$ be a partial order relation on the set of natural numbers $\aleph$. Define $\tau_1 = \{G_E \sqsubseteq \aleph_E$ such that $1 \in G^c_E\} \cup \aleph_E$ and $\tau_2 = \{F_E \sqsubseteq \aleph_E$ such that $1 \in F(e_2)$ and F^c_E is finite$\}$. Then $(\aleph, \tau_1, \tau_2, E, \leq)$ is a $SBTOS$. In the following, we illustrate that $(\aleph, \tau_1, \tau_2, E, \leq)$ is PSS-regularly ordered.

A soft subset H_E of $(\aleph, \tau_1, \tau_2, E, \leq)$ is PC-soft, if $1 \in H_E$ or $1 \notin H(e_2)$ and H_E is finite.

On the one hand, consider $\widehat{\phi} \neq H_E \neq \aleph_E$ is a DPO-soft set. Then we have the following two cases.

Case 1. Suppose $1 \in H_E$. Then for each $x^e \in Sp(X)_E$ such that $x^e \widehat{\notin} H_E$, we define a soft set G_E by: $G_E = x^e$. Thus G_E is an IPO-soft set containing x^e and its relative complement is a DPO-soft set containing H_E.

Case 2. Suppose $1 \notin H(e_2)$, H_E is finite and $x^e \widehat{\notin} H_E$. Then we have the following two cases.

(1) If $x^e = 1^{e_2}$, then $2^{e_2} \widehat{\notin} H_E$. Thus we define a soft set G_E by: $G(e) = \aleph - H(e)$ for each $e \in E$. So G_E is an IPO-soft set containing 1^{e_2} and its relative complement is a DPO-soft set containing H_E.

(2) If $x^e \neq 1^{e_2}$, then we define a soft set G_E by: $G(e) = \aleph - H(e)$ for each $e \in E$. Thus G_E is an IPO-soft set containing x^e and its relative complement is a DPO-soft set containing H_E. So $(\aleph, \tau_1, \tau_2, E, \leq)$ is lower PSS-regularly ordered.

On the other hand, consider $\widehat{\phi} \neq H_E \neq \aleph_E$ is an IPC-soft set. Then we have the following two cases.

Case 1. Suppose $1 \in H_E$. Then $2 \in H_E$. Thus for each $x^e \in Sp(X)_E$ such that $x^e \widehat{\notin} H_E$, we define a soft set G_E by: $G_E = x^e$. So G_E is a DPO-soft set containing x^e and its relative complement is an IPO-soft set containing H_E.

Case 1. Suppose [$1 \notin H(e_2)$, H_E is finite and $x^e \widehat{\notin} H_E$. Then we have the following two cases.

(1) If $x^e = 1^{e_2}$, then we define a soft set G_E by: $G(e) = \aleph - H(e)$ for each $e \in E$. Thus G_E is a DPO-soft set containing 1^{e_2} and its relative complement is an IPO-soft set containing H_E.

(2) If $x^e \neq 1^{e_2}$ and $x^e = 2^{e_2}$, then $1 \notin H_E$. Thus by the definition of PO-soft sets, we obtain that H_E is an IPO- soft set. Obviously, its relative complement is a DPO-soft set containing x^e. If $x^e \neq 1^{e_2} \neq 2^{e_2}$, then we define a soft set G_E by: $G(e) = \aleph - H(e)$ for each $e \in E$. Thus G_E is a

DPO-soft set containing x^e and its relative complement is an IPO-soft set containing H_E. So $(\aleph, \tau_1, \tau_2, E, \leq)$ is upper PSS-regularly ordered.

From the above discussion, we conclude that $(\aleph, \tau_1, \tau_2, E, \leq)$ is PSS-regularly ordered. Hence $(\aleph, \tau_1, \tau_2, E, \leq)$ is $PSST_3$-ordered.

To illustrate that $(\aleph, \tau_1, \tau_2, E, \leq)$ is not Bi-soft normally ordered, we define an IPC-soft set H_E and a DPC-soft soft set F_E as follows:

$$H_E = \{(e_1, \{1,2\}), (e_2, \{3\}), (e_3, \{4\})\},\ F_E = \{(e_1, \{3\}), (e_2, \{4\}), (e_3, \{1,5\})\}.$$

Since the two $PC-$soft sets are disjoint and there do not exist disjoint P-soft nbds W_E and V_E containing H_E and F_E, respectively, $(\aleph, \tau_1, \tau_2, E, \leq)$ is not Bi-soft normally ordered. Then $(\aleph, \tau_1, \tau_2, E, \leq)$ is not $PSST_4$-ordered.

4. Conclusion

In summary, we have introduced a new class of ordered soft separation axioms, called $PSST_i$-ordered spaces, and investigated their interrelations. This provides a useful framework for studying soft topology. We have also given illustrative examples to clarify the obtained results. As future work, we plan to extend these concepts to supra soft topological spaces. We hope that our work will be beneficial for researchers and scholars in advancing the study of soft topology.

Acknowledgements. I would like to express my heartfelt appreciation to everyone who supported me during the writing of this paper. Firstly, I am deeply grateful to my supervisor, Prof. Dr. S. A. El-Sheikh, for his invaluable guidance and insightful feedback throughout the writing process. Secondly, I extend my thanks to my classmates who provided me with timely references and information, without which completing this paper would have been much more challenging. I am grateful for the support and encouragement of all those who helped me along the way.

References

[1] S. A. El-Sheikh, S. A. Kandil and S. H. Shalil, New separation axioms in soft topological ordered spaces, International Journal of Innovative Science, Engineering and Technology 10 (6) (2023) 41–53.

[2] L. Nachbin, Topology and ordered, D. Van Nostrand Inc., Princeton, New Jersey 1965.

[3] S. D. McCartan, Separation axioms for topological ordered spaces, Math. Proc. Cambridge Philos Soc. 64 (1968) 965–973.

[4] D. A. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.

[5] R. Solai and V. Subbiah Maltepe, Soft separation axioms and soft product of soft topological spaces, Journal of Mathematics II (2) (2020) Pages 61–75.

[6] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, On soft topological ordered spaces, Journal of King Saud University-Science 31 (4) (2018) 556–566.

[7] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, Partial soft separation axioms and soft compact spaces, Filomat 32 (13) (2018) 4755–4771.

[8] G. Şenel, Soft topology generated by L-soft sets, Journal of New Theory 4 (24) (2018) 88–100.

[9] G. Şenel, A new approach to hausdorff space theory via the soft sets, Mathematical Problems in Engineering 9 (2016) 1–6.

[10] G. Şenel and N. Çağman, Soft topological subspaces, Ann. Fuzzy Math. Inform. 10 (4) (2015) 525–535.

[11] G. Şenel and N. Çağman, Soft closed sets on soft bitopological space, Journal of New Results in Science 3 (5) (2014) 57–66.

[12] G. Şenel, J. Lee and K. Hur, Distance and similarity measures for octahedron sets and their application to MCGDM Problems, Mathematics 8 (2020) 1–16.
[13] J. G. Lee, G. Şenel, P. K. Lim, J. Kim and K. Hur, Octahedron sets, Ann. Fuzzy Math. Inform. 19 (3) (2020) 211–238.
[14] S. A. El-Sheikh, S. A. Kandil and S. H. Shalil, On soft bitopological ordered spaces, Malaysian Journal of Mathematical Sciences, accepted.
[15] J. C. Kelly, General Topology, Springer Verlag 1975.
[16] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[17] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[18] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[19] S. Nazmul and S. K. Samanta, Neighbourhood properties of soft topological spaces, Ann. Fuzzy Math. Inform. 6 (1) (2013) 1–15.
[20] S. Nazmul and S. K. Samanta, Some properties of soft topologies and group soft topologies, Ann. Fuzzy Math. Inform. 8 (4) (2014) 645–661.
[21] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 171–185.
[22] B. M. Ittanagi, Soft bitopological spaces, International Journal of Computer Applications 107 (7) (2014) 1–4.
[23] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and S. A. Hazza, Pairwise open (closed) soft sets in soft bitopological spaces, Ann. Fuzzy Math. Inform. 11 (4) (2016) 571–588.
[24] T. Y. Ozturk and C. G. Aras, Soft pairwise contiuity in soft bitopological spaces, CBU J. of Sci. 13 (2) (2017) 413–422.

S. H. Shalil (slamma_elarabi@yahoo.com)
Mathematics Department, Faculty of science, Helwan University, Cairo, Egypt

S. A. El-Sheikh (elsheikh33@hotmail.com)
Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt

S. A. Kandil (shehab_ali@cic-cairo.com)
Mathematics Department, Canadian International college, Cairo, Egypt