Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (x 2023), pp. 1–x
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On the Diophantine equation $a\left(\frac{b^k-1}{b-1}\right) = U_n - U_m$

Pagdame Tiebekabe, Amadou Tall, and Ismaïla Diouf

Received 14 December 2023; Revised 2 January 2024; Accepted 05 February 2024

Abstract. Let b be an integer such that $b \geqslant 2$. In this paper, we show that there are only finitely many repdigits in base b which can be written as difference of two generalized Lucas numbers. In addition, we completely solve the considered Diophantine equation with the Pell sequence using the decimal base.

2010 AMS Classification: 11B39, 11J86, 11D61, 11Y50

Keywords: Linear forms in logarithms, Diophantine equations, Linear recurrent sequences, Lucas number, Pell number, Baker's theory, Reduction procedure.

Corresponding Author: Pagdame Tiebekabe (pagdame.tiebekabe@ucad.edu.sn)

1. Introduction

Numerical patterns and relationships have intrigued mathematicians for centuries. From prime numbers to Fibonacci sequences, these patterns often have a captivating appeal. In this article, we examine an intriguing connection between b-repdigits and generalized Lucas numbers, discovering an elegant representation of b-repdigits as the difference between two generalized Lucas numbers. A b-repdigit is a number composed of a repeated digit in base b. On the other hand, generalized Lucas numbers, denoted U_n, are a sequence of numbers that exhibit a recursive pattern, making them a fascinating subject of study in their own right. Recent papers have made significant contributions to the understanding of repdigits, exploring various aspects of these intriguing numerical patterns. Investigation of the integer sequences that are repdigits or the difference of two repdigits, or repdigits that are the difference between two integers sequences has been of interest to investigators (See [1, 2, 3, 4, 5, 6, 7, 8] for more details). They contribute to the field of number theory, inspiring new research in the exploration of repdigits and their complex links with other mathematical entities. With this in mind, we generalized Ray and Bhoi's work in [8]. We worked on b-repdigits, which are the difference between two

generalized Lucas numbers, and gave an application to the case of the Pell sequence in the decimal base.
This paper is organized as follows. In Section 2, we recall some useful results, Section 3 is devoted to the statement of our main results, Section 4 to the proof of the main results, and Section 5 to the application of the fundamental theorem to the special case of Pell numbers in decimal base.

2. Preliminaries

To make our results comprehensive, we have defined the concepts and presented preliminary results before stating our main findings.

2.1. **Some definitions and properties.**

This section is devoted to defining concepts.

Definition 2.1 (Generalized Lucas sequence)**.** The *generalized Lucas sequence* $(U_n)_{n\geqslant 0}$ is defined with initial values $U_0 = 0,\ U_1 = 1$ and the linear recurrence,

$$U_n = rU_{n-1} + sU_{n-2},$$

where r and s are integers such that $\Delta = r^2 + 4s > 0$.

The *Binet's formula* of the generalized Lucas sequence is given by

$$U_n = \frac{\delta^n - \gamma^n}{\delta - \gamma},$$

where $\delta = \dfrac{r + \sqrt{\Delta}}{2}$ and $\gamma = \dfrac{r - \sqrt{\Delta}}{2}$.

For more information about this sequence, the reader can refer to the book of Ribenboim (My Numbers, my friends)[9].

Recently, the following result was proved in [10].

Lemma 2.2. *The n-th term of the generalized Lucas sequence $(U_n)_{n\geqslant 0}$, with $r \geq 1$ and $s \in \{-1, 1\}$, satisfies the inequalities*

$$\delta^{n-2} \leqslant U_n < \delta^n$$

for $n \geqslant 2$.

Definition 2.3 (Pell sequence)**.** The *Pell sequence* $(P_n)_{n\geqslant 0}$ is a particular case of the generalized Lucas sequence with $r = 2$ and $s = 1$. In fact, we have $P_0 = 0, P_1 = 1$ and

$$P_n = 2P_{n-1} + P_{n-2}.$$

Definition 2.4 (Repdigit in base b)**.** Let $b \geqslant 2$ be an integer. A positive integer n is called a *repdigit* or simply a *b-repdigit*, if all of the digits in its base b expansion are equal. Indeed, n is of the form $a\left(\dfrac{b^k - 1}{b - 1}\right)$, where $1 \leqslant a \leqslant b - 1$ and $k \geqslant 1$.

2.2. **A lower bound for linear forms in logarithms.** The next tools are related to the transcendental approach to solving Diophantine equations. Let η be an algebraic number of degree d, let $a_0 > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \ldots, \eta^{(d)}$ denote its conjugates. The quantity defined by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right)$$

is called the *logarithmic height* of η. Some properties of height are as follows. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leqslant h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1\eta_2^{\pm 1}) &\leqslant h(\eta_1) + h(\eta_2), \\ h(\eta_1^m) &= |m| h(\eta_1). \end{aligned}$$

If $\eta = \dfrac{p}{q} \in \mathbb{Q}$ is a rational number in reduced form with $q > 0$, then the above definition reduces to $h(\eta) = \log(\max\{|p|, q\})$. We can now present the famous Matveev result used in this study. Thus, let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\eta_1, \ldots, \eta_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_s|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, s.$$

With the above notations, Matveev proved the following result.

Lemma 2.5 (Matveev [11])**.** *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

2.3. **Reduction methods.** Our next tool is a version of the reduction method of Baker and Davenport [12]. We use a slight variant of the version given by Dujella and Pethő [13] due to Bravo, Gomez and Luca [14].

Lemma 2.6 (Bravo-Gomez-Luca)**.** *Assume that τ and μ are real numbers and M is a positive integer. Let p/q be the convergent of the continued fraction of the irrational τ such that $q > 6M$, and let $A,\ B, \mu$ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \leqslant M \quad \text{and} \quad k \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

We also need the following result from Sanchez and Luca [15].

Lemma 2.7 (Sánchez-Luca)**.** *Let $r \geqslant 1$ and $H > 0$ be such that $H > (4r^2)^r$ and $H > L/(\log L)^r$. Then*

$$L < 2^r H (\log H)^r.$$

3. Statement of main results

The main results of this paper are the following.

Theorem 3.1. *Let b be a positive integer such that $b \geqslant 2$. If k, m and n are positive integers that satisfy the Diophantine equation*

$$a\left(\frac{b^k-1}{b-1}\right) = U_n - U_m \tag{3.1}$$

with $n > m$ and $1 \leqslant a \leqslant b-1$, then

$$k < 2.5n\log\delta$$

and

$$n\log\delta - \log(8.1\sqrt{\Delta}) < 2\cdot 10^{12}(1+\log D)\log\delta\log b\cdot\xi,$$

where

$$\xi = \log(4b^2\Delta(1+3\sqrt{\Delta}) + 2\cdot 10^{12}(1+\log D)\cdot\log\delta\cdot\log b\cdot(2\log b + \log\Delta)$$

with

$$D = 2.5n\log\delta.$$

Moreover, the above result implies the following corollary and theorem.

Corollary 3.2. *The Diophantine equation (3.1) has only finitely many solutions in positive integers k, m, n, b and a.*

By considering the case $b = 10$ and the particular case of Pell numbers, we get the following result.

Theorem 3.3. *The only repdigits that are differences between two Pell numbers are*

$$1, 3, 4, 7, 11 \text{ and } 99.$$

Moreover, we have

n	m	$P_n - P_m$	(a,k)
2	1	1	(1,1)
3	2	3	(3,1)
3	1	4	(4,1)
4	3	7	(7,1)
4	1	11	(1,2)
7	6	99	(9,2)

Table 1. Repdigits which are differences between two Pell numbers

where P_l is l-th term of Pell sequence.

4. Proof of main results

In this study, we reconsider Diophantine equation (3.1)

$$a\left(\frac{b^k-1}{b-1}\right)=U_n-U_m$$

with $n>m$ and $1\leqslant a\leqslant b-1$.
From (3.1), we deduce that $b^{k-1}<U_n\leqslant\delta^n$, where we used Lemma 2.2. Then we get $(k-1)\log b<n\log\delta$ which leads to

$$k<1+n\frac{\log\delta}{\log b}. \tag{4.1}$$

Using now Binet's formula for $(U_n)_{n\geqslant 0}$, Diophantine equation (3.1) becomes:

$$\frac{\delta^n-\gamma^n}{\delta-\gamma}-\frac{\delta^m-\gamma^m}{\delta-\gamma}=a\left(\frac{b^k-1}{b-1}\right),$$

which implies that

$$\frac{\delta^n}{\delta-\gamma}-\frac{ab^k}{b-1}=\frac{\gamma^n}{\delta-\gamma}+\frac{\delta^m}{\delta-\gamma}-\frac{\gamma^m}{\delta-\gamma}-\frac{a}{b-1}.$$

Taking absolute values on both sides we get:

$$\left|\frac{\delta^n}{\delta-\gamma}-\frac{ab^k}{b-1}\right|\leqslant\frac{|\gamma|^n}{\sqrt{\Delta}}+\frac{\delta^m}{\sqrt{\Delta}}+\frac{|\gamma|^m}{\sqrt{\Delta}}+\frac{a}{b-1}. \tag{4.2}$$

Note that $|\gamma|=\delta^{-1}$. Thus (4.2) becomes :

$$\begin{aligned}\left|\frac{\delta^n}{\delta-\gamma}-\frac{ab^k}{b-1}\right| &\leqslant\frac{1}{\delta^n\sqrt{\Delta}}+\frac{\delta^m}{\sqrt{\Delta}}+\frac{1}{\delta^m\sqrt{\Delta}}+\frac{a}{b-1}\\ &<3+\frac{\delta^m}{\sqrt{\Delta}}=\frac{3\sqrt{\Delta}+\delta^m}{\sqrt{\Delta}}.\end{aligned}$$

Since $s\in\{-1,1\}$, $\quad\delta\geqslant\dfrac{1+\sqrt{5}}{2}$,

$$\left|\frac{\delta^n}{\delta-\gamma}-\frac{ab^k}{b-1}\right|<3+\frac{\delta^m}{\sqrt{\Delta}}<\frac{1+3\sqrt{\Delta}}{\sqrt{\Delta}}\delta^m. \tag{4.3}$$

By dividing both sides of (4.3) by $\dfrac{\delta^n}{\sqrt{\Delta}}$, we get

$$\begin{aligned}\left|1-\delta^{-n}b^k\frac{a\sqrt{\Delta}}{b-1}\right| &<\frac{1+3\sqrt{\Delta}}{\sqrt{\Delta}}\cdot\frac{\sqrt{\Delta}}{\delta^n}\cdot\delta^m\\ &=\frac{1+3\sqrt{\Delta}}{\delta^{n-m}}.\end{aligned}$$

So we have

$$|\Gamma| := \left|1 - \delta^{-n} b^k \frac{a\sqrt{\Delta}}{b-1}\right| < \frac{1+3\sqrt{\Delta}}{\delta^{n-m}}. \tag{4.4}$$

Next, we have to show that $\Gamma \neq 0$.
If $\Gamma = 0$, then we get

$$\delta^n = b^k \frac{a\sqrt{\Delta}}{b-1}$$

which leads to

$$\delta^{2n} = b^{2k} \frac{a^2 \Delta}{(b-1)^2} = x + y\sqrt{\Delta},$$

where x and y are rational numbers. This is a contradiction since $n \geqslant 1$. Thus $\Gamma \neq 0$ and we can apply Matveev result to Γ.
Now we put

$$\eta_1 = \delta, \quad \eta_2 = b, \quad \eta_3 = \frac{a\sqrt{\Delta}}{b-1},$$

$$b_1 = -n, \quad b_2 = k, \quad b_3 = 1 \text{ and } s = 3.$$

Let $L := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\sqrt{\Delta})$. Then

$$d_L = [\mathbb{Q}(\eta_1, \eta_2, \eta_3) : \mathbb{Q}] = 2.$$

For the logarithm heights of η_1, η_2 and η_3, we have

$$h(\eta_1) = \frac{1}{2}\log\delta, \ \ h(\eta_2) = \log b$$

and

$$\begin{aligned} h(\eta_3) = h\left(\frac{a\sqrt{\Delta}}{b-1}\right) &\leqslant h\left(\frac{a}{b-1}\right) + h\left(\sqrt{\Delta}\right) \\ &\leqslant \log(b-1) + \frac{1}{2}\log\Delta \\ &< \log b + \frac{1}{2}\log\Delta. \end{aligned}$$

Thus we can take $A_1 = \log\delta, \quad A_2 = 2\log b$ and $A_3 = 2\log b + \log\Delta$.
Applying Lemma 2.5, we have

$$\log|\Gamma| > -1.4\cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2)\cdot(1+\log D)\cdot \log\delta \cdot 2\log b \cdot (2\log b + \log\Delta), \tag{4.5}$$

where, $D = \max\{|b_1|, |b_2|, |b_3|\} = \{1, n, k\}$.

Note that $k < 1 + n\dfrac{\log\delta}{\log b} = n\log\delta(\dfrac{1}{n\log\delta} + \dfrac{1}{\log b})$ for $b \geqslant 2$.

Since $b \geqslant 2$, $n \geqslant 2$ and $\delta \geqslant \dfrac{1+\sqrt{5}}{2}$,

$$k < n \log \delta \left(\frac{1}{2 \log \left(\dfrac{1+\sqrt{5}}{2} \right)} + \frac{1}{\log 2} \right) < 2.5 n \log \delta$$

we can take

$$D = 2.5 n \log \delta.$$

Combining (4.4) and (4.5), we get

$$\begin{aligned}(n-m)\log\delta - \log(1+3\sqrt{\Delta}) &< 1.4 \cdot 30^6 \cdot 3^{4.5} 2^2 (1+\log 2)(1+\log D) \\ &\quad \log\delta \cdot 2\log b \cdot (2\log b + \log\Delta) \\ &< 2 \cdot 10^{12}(1+\log D)\log\delta 2\log b(2\log b + \log\Delta).\end{aligned}$$

We rewrite Diophantine equation (3.1) to obtain that

$$\frac{\delta^n}{\sqrt{\Delta}} - \frac{\delta^m}{\sqrt{\Delta}} - \frac{ab^k}{b-1} = \frac{\gamma^n}{\sqrt{\Delta}} - \frac{\gamma^m}{\sqrt{\Delta}} - \frac{a}{b-1}.$$

After taking absolute values on both sides, we have

$$\left| \frac{\delta^n}{\sqrt{\Delta}} - \frac{\delta^m}{\sqrt{\Delta}} - \frac{ab^k}{b-1} \right| \leqslant \frac{1}{\delta^n\sqrt{\Delta}} + \frac{1}{\delta^m\sqrt{\Delta}} + \frac{a}{b-1} < 3.$$

So we have

$$\left| \frac{\delta^n}{\sqrt{\Delta}}(1-\delta^{m-n}) - \frac{ab^k}{b-1} \right| < 3.$$

Dividing both sides by $\dfrac{\delta^n}{\sqrt{\Delta}}(1-\delta^{m-n})$, we get that

$$\begin{aligned}\left| 1 - \delta^{-n} \cdot b^k \frac{a\sqrt{\Delta}}{(b-1)(1-\delta^{m-n})} \right| &< \frac{3\sqrt{\Delta}}{\delta^n(1-\delta^{m-n})} \\ &= \frac{3\sqrt{\Delta}\cdot\delta^{n-m}}{\delta^n(\delta^{n-m}-1)}.\end{aligned} \tag{4.6}$$

Moreover, $n - m \geqslant 1$. Let us show it.
From equation (3.1), we have $U_n - U_m > 0$. So

$$\delta^{m-2} \leqslant U_m < U_n < \delta^n.$$

Hence $m - 2 < n$ which implies that $n - m \geqslant -1$.
Note that $n - m$ cannot be equal to -1 or 0. Therefore we have to consider

$$n - m \geqslant 1.$$

Since $n-m \geqslant 1$, then $\delta^{n-m} \geqslant \delta \geqslant \alpha = \dfrac{1+\sqrt{5}}{2}$.

Using now the fact that the numerical function $f(x) = \dfrac{x}{x-1}$ is decreasing for $x \geqslant \dfrac{1+\sqrt{5}}{2}$, we have

$$\frac{\delta^{n-m}}{\delta^{n-m}-1} \leqslant \frac{\alpha}{\alpha-1} < 2.7.$$

Hence (4.6) becomes

$$\left|1-\delta^{-n}\cdot b^k\cdot\frac{a\sqrt{\Delta}}{(b-1)(1-\delta^{m-n})}\right| < \frac{8.1\cdot\sqrt{\Delta}}{\delta^n}. \tag{4.7}$$

Now set $\left|\Gamma^{'}\right| := \left|1-\delta^{-n}\cdot b^k\dfrac{a\sqrt{\Delta}}{(b-1)(1-\delta^{m-n})}\right|$.

Similarly, we can show that $\Gamma^{'} \neq 0$.

Put

$$\eta_1 = \delta, \quad ,\eta_2 = b, \quad \eta_3 = \frac{a\sqrt{\Delta}}{(b-1)(1-\delta^{m-n})}$$
$$b_1 = -n, \quad b_2 = k, \quad b_3 = 1.$$

Note that

$$\begin{aligned} h(\eta_3) &= h\left(\frac{a\sqrt{\Delta}}{(b-1)(1-\delta^{m-n})}\right) \\ &\leqslant h\left(\frac{a}{b-1}\right) + h\left(\sqrt{\Delta}\right) + h\left(\frac{1}{1-\delta^{m-n}}\right) \\ &< \log b + \frac{1}{2}\log\Delta + (n-m)\cdot\frac{\log\delta}{2} + \log 2 \\ &= \log\left(2b\sqrt{\Delta}\right) + \frac{n-m}{2}\log\delta. \end{aligned}$$

$$h(\eta_3) < \log\left(2b\sqrt{\Delta}\right) + 10^{12}(1+\log D)\cdot\log\delta\cdot\log b\cdot(2\log b+\log\Delta) + \frac{\log\left(1+3\sqrt{\Delta}\right)}{2}.$$

Then we can take

$$\begin{aligned} A_3 &= 2\log\left(2b\sqrt{\Delta}\right) + 2\cdot 10^{12}\left(1+\log D\right)\cdot\log\delta\cdot\log b\cdot\left(2\log b+\log\Delta\right) + \log(1+3\sqrt{\Delta}) \\ &= \log(4b^2\Delta(1+3\sqrt{\Delta}) + 2\cdot 10^{12}(1+\log D)\cdot\log\delta\cdot\log b\cdot(2\log b+\log\Delta). \end{aligned}$$

By Lemma 2.5, we get that

$$\log\left|\Gamma^{'}\right| > -1.4\cdot 30^6\cdot 3^{4.5}(1+\log 2)(1+\log D)\cdot\log\delta\cdot(2\log b)\cdot A_3.$$

Combining this with (4.7), we have

$$n\log\delta - \log(8.1\cdot\sqrt{\Delta}) < 2\cdot 10^{12}(1+\log D)\cdot\log\delta\cdot\log b\cdot A_3. \tag{4.8}$$

From (4.1) and (4.8), we have the proof of Theorem 3.1.

5. Application: Pell numbers in decimal base

In this section, we explicitly determine all repdigits which can be written as difference of two Pell numbers. So our result in this case is Theorem 3.3. In this case, U_n is Pell number. We have

$$(r,s)=(2,1), \quad \Delta=8, \quad \text{and } \delta=1+\sqrt{2}.$$

By the main theorem 3.1, we have

$$n\log(1+\sqrt{2})-\log(8.1\cdot\sqrt{8})<2\cdot 10^{12}(1+\log 8)\cdot\log(1+\sqrt{2})\cdot\log 10\cdot\xi$$

with

$$\xi=\log(4\times 10^2\cdot 8(1+3\sqrt{8})+2\cdot 10^{12}(1+\log D)\cdot\log(1+\sqrt{2})\cdot\log 10\cdot(2\log 10+\log 8)$$

and

$$D=2.5n\log(1+\sqrt{2})<2.21n.$$

First,

$$\begin{aligned}\xi &< 10.4+2.8\cdot 10^{13}(1+\log(2.21n))\\ &< 3\cdot 10^{13}(1+\log(2.21n)) \quad \text{for } n\geqslant 2.\end{aligned}$$

Then we get

$$\begin{aligned}n &< 1.4\cdot 10^{26}(1+\log(2.21n))^2\\ &= 1.4\cdot 10^{26}(1+\log 2.21+\log n)^2.\end{aligned}$$

Since $n\geqslant 2$, we obtain

$$n<1.82\cdot 10^{27}\cdot\log^2 n.$$

Now, we can apply the Lemma 2.7 by putting

$$l=2, \quad L=n, \quad \text{and } H=1.82\cdot 10^{27}.$$

Thus we have $n<2^2\cdot 1.82\cdot 10^{27}\cdot\left(\log(1.82\cdot 10^{27})\right)^2$, so

$$n<2.87\cdot 10^{31}.$$

Next, we need to reduce the bound on n by using the Lemma 2.6
Let

$$\Lambda_1:=-n\log\delta+k\log 10+\log\left(\frac{a\sqrt{8}}{9}\right).$$

The inequality (4.4) can be written as

$$\left|\mathrm{e}^{\Lambda_1}-1\right|<\frac{1+3\sqrt{8}}{\delta^{n-m}}.$$

Observe that $\Lambda_1\neq 0$ as $e^{\Lambda_1}-1=\Gamma\neq 0$.
Assume that $n-m\geqslant 4$. Then

$$\left|\mathrm{e}^{\Lambda_1}-1\right|<\frac{1+3\sqrt{8}}{\delta^{n-m}}<\frac{1}{2}.$$

This implies that:

$$|\Lambda_1|<2\frac{1+3\sqrt{8}}{\delta^{n-m}}$$

since $|x| < 2\,|e^x - 1|$ for every real x with $|x| < \dfrac{1}{2}$.
Dividing both sides by $\log\delta$, we get that:

$$\left| k\frac{\log 10}{\log\delta} - n + \frac{\log(a\sqrt{8}/9)}{\log\delta} \right| < \frac{21.6}{\delta^{n-m}}.$$

Thus we can take:

$$\tau = \frac{\log 10}{\log\delta}, \quad \mu = \frac{\log(a\sqrt{8}/9)}{\log\delta}, \quad A = 21.6, \quad B = \delta = 1+\sqrt{2} \quad \omega = n-m.$$

Let's show that $\tau = \dfrac{\log 10}{\log\delta}$ is irrational. Assume that τ is rational. Then, there exist two positive integers p and q with $\gcd(p,q) = 1$ such that $\tau = \dfrac{p}{q}$. This implies that $10^q = (1+\sqrt{2})^p$. This is impossible because we cannot find two positive integers p and q with $\gcd(p,q) = 1$ satisfying $10^q = (1+\sqrt{2})^p$. So τ is irrational. Moreover $k < 2.21n < 6.35 \cdot 10^{31}$. Then we take $M := 6.35 \cdot 10^{31}$. With *Mathematica*, we have

$$q_{73} = 1189285833530929228438091844076539, \epsilon = 0.108608, \quad \text{and } n-m \leqslant 93.$$

Put now

$$\Lambda_2 := -n\log\delta + k\log 10 + \log\left(\frac{a\sqrt{8}}{9\,(1-\delta^{m-n})}\right).$$

So the inequality (4.7) can be written as

$$\left|e^{\Lambda_2} - 1\right| < \frac{8.1\sqrt{8}}{\delta^n}.$$

Note also that $\Lambda_2 \neq 0$ as $e^{\Lambda_2} - 1 = \Gamma' \neq 0$.
Assuming $n \geqslant 5$, we get

$$\left|e^{\Lambda_2} - 1\right| < \frac{8.1\sqrt{8}}{\delta^n} < \frac{1}{2}$$

and then

$$|\Lambda_2| < \frac{16.2\sqrt{8}}{\delta^n}.$$

By dividing both sides by $\log\delta$, we get that:

$$\left| k\frac{\log 10}{\log\delta} - n + \frac{\log(a\sqrt{8}/(9(1-\delta^{m-n})))}{\log\delta} \right| < \frac{52}{\delta^n}.$$

To apply the Lemma 2.6, we can set

$$\tau = \frac{\log 10}{\log\delta}, \quad \mu = \frac{\log\left(\dfrac{a\sqrt{8}}{9(1-\delta^{m-n})}\right)}{\log\delta}, \quad A = 52, \quad B = \delta = 1+\sqrt{2} \quad \omega = n.$$

Since $k < 2.5n\log\delta < 2.21n < 6.35{\cdot}10^{31}$. Thus we take $M := 6.35{\cdot}10^{31}$. With *Mathematica*, we get $q_{73} = 1189285833530929228438091844076539$, $\epsilon = 0.423322$, and $n \leqslant 92$. So we have proved the Theorem 3.3.

Acknowledgements. The authors would like to thank the anonymous referees for their instructive suggestions. The first author is partially supported by Université de Kara (Togo). The authors thank K. N Adédji of the Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi (IMSP) for his remarks and suggestions, which considerably improved the quality of this paper.

References

[1] M. G. Duman, Padovan numbers as difference of two repdigits, Indian, Journal of Pure and Applied Mathematics (2023) https://doi.org/10.1007/s13226-023-00526-8.
[2] B. Faye and F. Luca, Pell and Pell-Lucas numbers with only one distinct digit, Ann. Math. Inform. 45 (2015) 55–60.
[3] R. Keskin and F. Erduvan, Repdigits as the difference of two Pell or Pell-Lucas numbers Korean J. Math. 31 (1) (2023) 63–73.
[4] B. Edjeou and B. Faye, Pell and Pell-Lucas numbers as the difference of two repdigits. Afr. Mat. 34(4) (2023).
[5] Z. Siar, F. Erduvan and R. Keskin, Repdigits base b as the difference of two Fibonacci numbers, J. Math. Study 55 (1) (2022) 84–94.
[6] F. Luca, Fibonacci and Lucas numbers with only one distinct digit. Port. Math. 57 (02) (2000) 243–254.
[7] F. Erduvan, R. Keskin and F. Luca, Fibonacci and Lucas numbers as the difference of two repdigits, Rend. Circ. Mat. Palermo II. 71 (2021) 575–589.
[8] P. Ray and K. Bhoi, Repdigits as difference of two Fibonacci or Lucas numbers, Mat. Stud. 56 (2021) 124–132.
[9] P. Ribenboim, My Numbers, My Friends: Popular Lectures on Number Theory, Springer, 2000.
[10] K. N. Adédji, J. Odjoumani and A. Togbé, Padovan and Perrin Numbers as Products of Two Generalized Lucas Numbers, Arch. Math. (Brno) 59 (2023) 351–373.
[11] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II, Izv. Math 64. (6) (2000) 1217–1269.
[12] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, Quart.J of Math. Ser. 20 (2) (1969) 129–137.
[13] A. Dujella and A. Petho, A generalization of a theorem of Baker and Davenport, Quart. J. Math. Oxford Ser. 49 (3) (1998) 291–306.
[14] J.J. Bravo, C. A. Gomez and F. Luca, Powers of two as sums of two k-Fibonacci numbers, Miskolc Math. Notes 17 (1) (2016) 85–100.
[15] S. G. Sanchez and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequence, Ann. Math. Qué. 38 (2014) 169–188.

Pagdame Tiebekabe (pagdame.tiebekabe@ucad.edu.sn)
Département de Mathématiques, Faculté des Sciences et Techniques (FST), Université de Kara, B.P. 404 Kara, Togo
Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algébrique et Applications (LACGAA), Université Cheikh Anta Diop de Dakar (UCAD), B.P. 5005 Dakar-Fann, Sénégal

Amadou Tall (amadou7.tall@ucad.edu.sn)
Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algébrique et Applications (LACGAA), Université Cheikh Anta Diop de Dakar (UCAD), B.P. 5005 Dakar-Fann, Sénégal

Ismaïla Diouf (shehab_ali@cic-cairo.com)
Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algébrique et Applications

(LACGAA), Université Cheikh Anta Diop de Dakar (UCAD), B.P. 5005 Dakar-Fann, Sénégal