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On L-almost separation axioms in L-fuzzifying bitopological spaces

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Abstract. Khalaf and Abd El-Latif [1] introduced and investigated the concept of L-almost separation axioms in L-fuzzifying bitopological spaces, where L is a complete residuated lattice, but we note that some of their results are incorrect (See Theorems 6, 7, 8, 18 in [1]). Firstly in this paper we give some counterexamples to show that these results generally need not be true. Secondly, we introduced the notions of L-almost continuity, L-almost open function and L-completely continuous function in L-fuzzifying bitopological spaces with studying some important results. Finally, under these types of L-fuzzy mappings we study the image of these kinds of L-fuzzifying bitopological spaces.

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1. Introduction

In 1963, Kelley [2] defined the concept of bitopological spaces by spaces equipped with its two (arbitrary) topologies. In 1993, Shen [3] is considered the first to study the separation axioms in fuzzifying topologies.

In 2000, Zahran [4] introduced the notion of regular open sets in I-fuzzifying topological spaces but some of his results were incorrect so, in 2004, Sayed and Zahran [5] gave corrections of them. Also, Sayed and et al. [6, 7, 8, 9, 10, 11, 12, 13] studied many deferent separation axioms in fuzzifying topology and (L,M)-fuzzy convexity spaces. Allam et al. [14, 15] studied separation axioms and semi-separation axioms in fuzzifying bitopological spaces. In 2023, Binshahnah et al. [16] studied strongly separation axioms in fuzzifying bitopological spaces

In 2019, Khalaf and Abd El-Latif [1] deal with regular open sets similar to open sets in their study of L-almost separation axioms. So, they introduced their results as the same as the results on open sets but this is a big mistake because regular open sets and open sets are independent in I-fuzzifying topology (See [5]) and thus in L-fuzzifying bitopology as we illustrate in this paper.

The contents of this paper are arranged as follows: In Section 3, we prove that some results obtained in [1] need not be true, by giving some counterexamples. In Section 4, we introduced the notions of L-almost continuity, L-almost open function and L-completely continuous function in L-fuzzifying bitopological spaces with studying some important results. Finally, in Section 5, under these types of L-fuzzy mappings we study the image of these kinds of L-fuzzifying bitopological spaces

2. Preliminaries

Definition 2.1 ([17, 18])**.** A structure $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is called a *complete residuated lattice*, if the following conditions are satisfied:

(i) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice whose greatest and least element are $\top, \perp$ respectively,

(ii) $(L, *, \top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L and

(b) for every $a \in L, a * \top = a$,

(iii) $\longrightarrow$ is related with $*$ as: $a * b \leq c$ if and only if $a \leq b \longrightarrow c \ \ \forall a, b, c \in L$, where " $\longrightarrow$ " is defined by: $\alpha \longrightarrow \beta = \bigvee\{\lambda \in L : \alpha * \lambda \leq \beta\}\ \forall \alpha, \beta \in L$.

In each statement in the rest of this paper, L is assumed to be a complete residuated lattice. Sometimes we need to add more conditions on L such as the completely distributive law (briefly, CDL) or the double negation law (briefly, DNL).

Definition 2.2 ([19])**.** *We say that L satisfies CDL*, if the following law is satisfied:

$$\bigwedge_{j \in J} \bigvee A_j = \bigvee_{f \in \prod\limits_{j \in J} A_j} (\bigwedge_{j \in J} f(j))\ \forall \{A_j | j \in J\} \subseteq 2^L,$$

where 2^L is the power subset of L.

Definition 2.3 ([20])**.** We say that L *satisfies DNL*, if the following law is satisfied:

$$(a \longrightarrow \perp) \longrightarrow \perp = a\ \forall\ a \in L.$$

Definition 2.4 ([20])**.** Let $f,\ g \in L^X$. Then the *L-equality* between f and g, denoted by $[[f, g]]$, is defined as follows: $[[f, g]] = \bigwedge\limits_{x \in X} \big((f(x) \longrightarrow g(x)) \wedge (g(x) \longrightarrow f(x))\big)$.

Definition 2.5 ([20, 21, 22, 23])**.** The *L-fuzzifying topology* is a mapping $\varrho : 2^X \longrightarrow L$ satisfying the following conditions:

(i) $\varrho(X) = \varrho(\varnothing) = \top$,

(ii) $\varrho(\bigcup\limits_{\gamma \in \Upsilon} \mathcal{O}_\gamma) \geq \bigwedge\limits_{\gamma \in \Upsilon} \varrho(\mathcal{O}_\gamma)\ \forall \{\mathcal{O}_\gamma | \gamma \in \Upsilon\} \subseteq 2^X$,

(iii) $\varrho(\mathcal{O} \cap \mathcal{G}) \geq \varrho(\mathcal{O}) \wedge \varrho(\mathcal{G})\ \forall \mathcal{O},\ \mathcal{G} \in 2^X$.

A pair (X, ϱ) is called an *L-fuzzifying topological space*.

Definition 2.6 ([24])**.** Let (X, ϱ_1) and (X, ϱ_2) be two L-fuzzifying topological spaces. Then a system $(X, \varrho_1, \varrho_2)$ is called an *L-fuzzifying bitopological space* (briefly, L-fbs).

Definition 2.7 ([1])**.** Let $(X, \varrho_1, \varrho_2)$ be an L-fbs.

(i) The set of all L-fuzzifying (s, k)-regular open sets is denoted by $R\varrho_{(s,k)} \in L^{2^X}$ and defined as follows:

$$R\varrho_{(s,k)}(\mathcal{O}) = \min\Big(\bigwedge_{x\in\mathcal{O}} I_k(C_s(\mathcal{O}))(x), \bigwedge_{x\in X-\mathcal{O}} (I_k(C_s(\mathcal{O}))(x) \longrightarrow \bot)\Big),$$

where $s, k = 1, 2$ and $s \neq k$, $I_k(A)$ is means the interior of a set A with respect to ϱ_k and $C_s(A)$ is means the closure of a set A with respect to ϱ_s, $\forall A \in 2^X$.

(ii) The set of all L-fuzzifying (s, k)-regular closed sets is denoted by $R\mathcal{F}_{(s,k)} \in L^{2^X}$ and defined as follows: $R\mathcal{F}_{(s,k)}(\mathcal{O}) = R\varrho_{(s,k)}(X - \mathcal{O})$.

Definition 2.8 ([1])**.** Let $(X, \varrho_1, \varrho_2)$ be an L-fbs and $x \in \mathcal{O}$. Then

(i) an *(s, k)-regular neighborhood system of x*, denoted by $RN_x^{(s,k)} \in L^{2^X}$, is defined as follows: $RN_x^{(s,k)}(\mathcal{O}) = \bigvee_{x\in B\subseteq\mathcal{O}} R\varrho_{(s,k)}(B)$,

(ii) an (s, k)-*regular closure operator*, denoted by $RC_{(s,k)} \in (L^X)^{2^X}$, is defined as follows:

$$RC_{(s,k)}(\mathcal{O})(x) = RN_x^{(s,k)}(\mathcal{O}) \longrightarrow \bot.$$

For simplicity, we take:

$K_R^{(s,k)}(x, y) = (\bigvee_{y\notin A} RN_x^{(s,k)}(A)) \vee (\bigvee_{x\notin A} RN_y^{(s,k)}(A))$,

$H_R^{(s,k)}(x, y) = (\bigvee_{y\notin B} RN_x^{(s,k)}(B)) \wedge (\bigvee_{x\notin C} RN_y^{(s,k)}(C))$,

$M_R^{(s,k)}(x, y) = \bigvee_{C\cap B=\phi} \big(RN_x^{(s,k)}(B) \wedge RN_y^{(s,k)}(A)\big)$,

$V_R^{(s,k)}(x, D) = \bigvee_{A\cap B=\phi, D\subseteq B} (RN_x^{(s,k)}(A) \wedge R\varrho_{(s,k)}(B))$,

where $x, y \in X$ and $A, B, D \in 2^X$.

Definition 2.9 ([1])**.** Let Ω be the class of all L-fbss. Then the *unary L-predicates L-almost–$T_n^{(s,k)}$* $\in L^\Omega$, denoted by $RT_n^{(s,k)}$, $n = 0, 1, 2, 3$ are defined as follows:

(i) $RT_0^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} K_R^{(s,k)}(x, y)$,

(ii) $RT_1^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} H_R^{(s,k)}(x, y)$,

(iii) $RT_2^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\neq y} M_R^{(s,k)}(x, y)$,

(iv) $RT_3^{(s,k)}(X, \varrho_1, \varrho_2) = \bigwedge_{x\notin D} (R\mathcal{F}_{(s,k)}(D) \longrightarrow V_R^{(s,k)}(x, D))$.

Definition 2.10 ([1])**.** Let $(X, \varrho_1, \varrho_2)$ be an L-fbs. The *L-fuzzifying derived set* of $\mathcal{O} \subseteq X$, denoted by $Rd_{(s,k)} \in (L^X)^{2^X}$, is defined by: for each $x \in X$,

$$Rd_{(s,k)}(\mathcal{O})(x) = RN_x^{(s,k)}\big((X - \mathcal{O}) \cup \{x\}\big) \longrightarrow \bot.$$

Definition 2.11 ([1])**.** Let Ω be the class of all L-fbss. Then the *unary L-predicate* ${}^1RT_3^{(s,k)} \in L^\Omega$ are defined as follows:

${}^1RT_3^{(s,k)}(X, \varrho_1, \varrho_2)$

$$= \bigwedge_{x\notin D}\Big(R\mathcal{F}_s(D) \longrightarrow \bigvee_{A\in 2^X}\big(RN_x^{(s,k)}(A) \wedge (\bigwedge_{y\in D} RC_{(s,k)}(A)(y) \longrightarrow \bot)\big)\Big).$$

3. L-almost separation axioms in L-fbss

First, the following example shows that

(1) For any $\{\mathcal{O}_\gamma : \gamma \in \Upsilon\}$, $R\varrho_{(s,k)}(\bigcup_{\gamma\in\Upsilon}\mathcal{O}_\gamma) \geq \bigwedge_{\gamma\in\Upsilon} R\varrho_{(s,k)}(\mathcal{O}_\gamma)$ and

(2) For any $\mathcal{O},\ \mathcal{G} \subseteq X$, $R\varrho_{(s,k)}(\mathcal{O}\cap\mathcal{G}) \leq R\varrho_{(s,k)}(\mathcal{O}) \wedge R\varrho_{(s,k)}(\mathcal{G})$, generally need not be true.

Example 3.1. Let $S = \{l,t,m\}$ and $L = [0,1]$, ϱ_1, ϱ_2 be two fuzzifying topologies defined on S as follows:

$$\varrho_1(B) = \begin{cases} 1 & \text{if} \quad B \in \{\varnothing, S, \{l\}, \{l,m\}\}, \\ 1/8 & \text{if} \quad B \in \{\{m\},\{t,m\}\}, \\ 0 & \text{if} \quad B \in \{\{t\},\{l,t\}\}, \end{cases}$$

$$\varrho_2(B) = \begin{cases} 1 & \text{if} \quad B \in \{\varnothing, S, \{l\}, \{l,m\}\}, \\ 1/4 & \text{if} \quad B \in \{\{m\},\{t,m\}\}, \\ 0 & \text{if} \quad B \in \{\{t\},\{l,t\}\}. \end{cases}$$

Note that

$$R\varrho_{(1,2)}(B) = \begin{cases} 1 & \text{if} \quad B \in \{\varnothing, S\}, \\ 1/4 & \text{if} \quad B \in \{\{m\},\{t,m\}\}, \\ 1/8 & \text{if} \quad B \in \{\{l\},\{l,t\}\}, \\ 0 & \text{if} \quad B \in \{\{t\},\{l,m\}\}, \end{cases}$$

$$R\mathcal{F}_{(1,2)}(B) = \begin{cases} 1 & \text{if} \quad B \in \{\varnothing, S\}, \\ 1/4 & \text{if} \quad B \in \{\{l\},\{l,t\}\}, \\ 1/8 & \text{if} \quad B \in \{\{m\},\{t,m\}\}, \\ 0 & \text{if} \quad B \in \{\{t\},\{l,m\}\}. \end{cases}$$

We note that $R\varrho_{(1,2)}(\{l,t\}) \wedge R\varrho_{(1,2)}(\{t,m\}) = 1/4 \wedge 1/8 = 1/8$ and $R\varrho_{(1,2)}(\{l,t\} \cap \{t,m\}) = R\varrho_{(1,2)}(\{t\}) = 0$. Then $R\varrho_{(1,2)}(\{l,t\}\cap\{t,m\}) \ngeq R\varrho_{(1,2)}(\{l,t\})\wedge\varrho_{(1,2)}(\{t,m\})$. Also, $R\varrho_{(1,2)}(\{l\} \cup \{m\}) = R\varrho_{(1,2)}(\{l,m\}) = 0 \ngeq 1/8 = R\varrho_{(1,2)}(\{l\}) \wedge R\varrho_{(1,2)}(\{m\})$.

In Definition 6 (1) [1], Khalaf and Abd El-Latif said $R\varrho_{(s,k)}(\mathcal{O}) = \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$, but generally, this is not true as shown by the following theorem.

Theorem 3.2. *Let $(X, \varrho_1, \varrho_2)$ be an L-fbs. Then* $R\varrho_{(s,k)}(\mathcal{O}) \leq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$.

Proof. Note that $\bigwedge_{y\in\mathcal{O}}\bigvee_{y\in B\subseteq\mathcal{O}} R\varrho_{(s,k)}(B) \geq R\varrho_{(s,k)}(\mathcal{O})$. Then we have

$$R\varrho_{(s,k)}(\mathcal{O}) \leq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O}).$$

□

The following example illustrates that $R\varrho_{(s,k)}(\mathcal{O}) \neq \bigwedge_{y\in\mathcal{O}} RN_y^{(s,k)}(\mathcal{O})$ in general.

Example 3.3. From Example 3.1, we have

$$R\varrho_{(1,2)}(\{l,m\}) = 0 \neq 1/8 = \bigwedge_{x\in\{l,m\}} RN_x^{(1,2)}(\{l,m\}).$$

Theorem 3.4 (Theorem 6, [1])**.** *Let* (X,ϱ_1,ϱ_2) *be an* L*-fbs. If* L *satisfies CDL, then* $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) = \bigwedge_{x\in X} R\mathcal{F}_{(s,k)}(\{x\})$.

The following example shows that $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) > \bigwedge_{x\in X} R\mathcal{F}_{(s,k)}(\{x\})$. Then the above theorem is not true in general.

Example 3.5. From Example 3.1, we have

$$RT_1^{(1,2)}(S,\varrho_1,\varrho_2) = 1/8 > 0 = \bigwedge_{x\in S} R\mathcal{F}_{(1,2)}(\{x\}).$$

Theorem 3.6 (Theorem 7, [1])**.** *Let* (X,ϱ_1,ϱ_2) *be an* L*-fbs and* $\mathcal{O}\subseteq X$*. If* L *satisfies CDL, then* $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) \leq \bigwedge_{x\in X} RN_x^{(s,k)}\big((X-\mathcal{O})\cup\{x\}\big)$.

Theorem 3.7 (Theorem 8, [1])**.** *Let* (X,ϱ_1,ϱ_2) *be an* L*-fbs and* $\mathcal{O}\subseteq X$*. If* L *satisfies CDL, then* $RT_1^{(s,k)}(X,\varrho_1,\varrho_2) \leq [[Rd_{(s,k)}(\mathcal{O}), 1_\varnothing]]$.

The following example shows that Theorem 3.6 and Theorem 3.7 are incorrect in general.

Example 3.8. From Example 3.1, take $\mathcal{O}=\{l,m\}$, we have
(1) $\bigwedge_{x\in S} RN_t^{(1,2)}(\{t\}\cup\{x\}) = 0 \ngeq 1/8 = RT_1^{(1,2)}(S,\varrho_1,\varrho_2)$,
(2) $[[Rd_{(1,2)}(\mathcal{O}), 1_\varnothing]] = 0 \ngeq 1/8 = RT_1^{(1,2)}(S,\varrho_1,\varrho_2)$.

Theorem 3.9 (Theorem 18, [1])**.** *Let* (X,ϱ_1,ϱ_2) *be an* L*-fbs. If* L *satisfies CDL and DNL, then* $RT_3^{(s,k)}(X,\varrho_1,\varrho_2) = {}^1RT_3^{(s,k)}(X,\varrho_1,\varrho_2)$.

The following example shows that Theorem 3.9 need not be true in general.

Example 3.10. From Example 3.1, we have

$$R\varrho_1(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/8 & \text{if} \quad B\in\{\{l\},\{l,t\},\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\},\end{cases}$$

$$R\mathcal{F}_1(B) = \begin{cases} 1 & \text{if} \quad B\in\{\varnothing, S\},\\ 1/8 & \text{if} \quad B\in\{\{l\},\{l,t\},\{m\},\{t,m\}\},\\ 0 & \text{if} \quad B\in\{\{t\},\{l,m\}\}.\end{cases}$$

Then $RT_3^{(1,2)}(S,\varrho_1,\varrho_2) = 7/8 \neq 1 = {}^1RT_3^{(1,2)}(S,\varrho_1,\varrho_2)$.

4. L-almost continuity, L-almost opennes And L-completely continuity in L-fbss

Definition 4.1. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $AC_{(s,k)}\in L^{(Y^X)}$ is called *L-almost continuous*, provide that

$$AC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\big).$$

Theorem 4.2. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. Then* $AC_{(s,k)}(g)\le\bigwedge_{x\in X}\bigwedge_{W\in 2^Y}\big(RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W))\big)$.

Proof. We have $AC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\big)$. Then

$$AC_{(s,k)}(g)\le R\xi_{(s,k)}(W)\longrightarrow\varrho_s(g^{-1}(W))\ \forall x\in X,\ \forall W\in 2^Y.$$

Thus $AC_{(s,k)}(g)*R\xi_{(s,k)}(W)\le\varrho_s(g^{-1}(W))$ implies

$$AC_{(s,k)}(g)*R\xi_{(s,k)}(W)\le\bigvee_{x\in H\subseteq g^{-1}(W)}\varrho_s(H)=N^s_x(g^{-1}(W)).$$

So $\bigvee_{g(x)\in H\subseteq W}\big(AC_{(s,k)}(g)*R\xi_{(s,k)}(H)\big)\le\bigvee_{g(x)\in H\subseteq W}N^s_x(g^{-1}(H))$ implies

$$AC_{(s,k)}(g)*\bigvee_{g(x)\in H\subseteq W}R\xi_{(s,k)}(H)\le\bigvee_{x\in g^{-1}(H)\subseteq f^{-1}(W)}N^s_x(g^{-1}(H)).$$

Hence $AC_{(s,k)}(g)*RN^{(s,k)}_{g(x)}(W)\le N^s_x(g^{-1}(W))$ implies

$$AC_{(s,k)}(g)\le RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W)).$$

Therefore $AC_{(s,k)}(g)\le\bigwedge_{x\in X}\bigwedge_{W\in\ 2^Y}\big(RN^{(s,k)}_{g(x)}(W)\longrightarrow N^s_x(g^{-1}(W))\big)$. □

Definition 4.3. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $AO_{(s,k)}\in L^{(Y^X)}$ is called *L-almost open*, provide that

$$AO_{(s,k)}(g)=\bigwedge_{U\in 2^X}\big(R\varrho_{(s,k)}(U)\longrightarrow\xi_s(g(U))\big).$$

Theorem 4.4. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. Then* $AO_{(s,k)}(g)\le\bigwedge_{x\in X}\bigwedge_{W\in 2^Y}\big(RN^{(s,k)}_x(g^{-1}(W))\longrightarrow N^s_{g(x)}(W)\big)$.

Proof. We can prove this theorem in the same way as the proof of Theorem 4.2. □

Definition 4.5. Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. A unary L-predicate $RC_{(s,k)}\in L^{(Y^X)}$ is called *L-completely continuous*, provide that

$$RC_{(s,k)}(g)=\bigwedge_{W\in 2^Y}\big(\xi_s(W)\longrightarrow R\varrho_{(s,k)}(g^{-1}(W))\big).$$

Theorem 4.6. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss and $g:(X,\varrho_1,\varrho_2)\to(Y,\xi_1,\xi_2)$ be a mapping. Then* $RC_{(s,k)}(g)\leq\bigwedge_{x\in X}\bigwedge_{W\in 2^Y}\big(N^s_{g(x)}(W)\longrightarrow RN^{(s,k)}_x(g^{-1}(W))\big)$.

Proof. We can prove this theorem in the same way as the proof of Theorem 4.2. □

5. L-almost separation axioms and L-fuzzy mappings In L-fbss

Theorem 5.1. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss. If $g:(X,\varrho_1,\varrho_2)\longrightarrow(Y,\xi_1,\xi_2)$ is a bijective mapping and $[AC_{(s,k)}(g)]=\top$, then*

(1) $RT^{(s,k)}_0(Y,\xi_1,\xi_2)\leq T_0(X,\varrho_s)$,
(2) $RT^{(s,k)}_1(Y,\xi_1,\xi_2)\leq T_1(X,\varrho_s)$,
(3) $RT^{(s,k)}_2(Y,\xi_1,\xi_2)\leq T_2(X,\varrho_s)$.

Proof. (1) From Theorem 4.2 and $[AC_{(s,k)}(g)]=\top$, we have

$$\begin{aligned}RT^{(s,k)}_0(Y,\xi_1,\xi_2)&=\bigwedge_{z\neq w}\Big(\bigvee_{w\notin W}RN^{(s,k)}_z(W)\ \vee\ \bigvee_{z\notin W}RN^{(s,k)}_w(W)\Big)\\&=\bigwedge_{gg^{-1}(z)\neq gg^{-1}(w)}\Big(\bigvee_{gg^{-1}(w)\notin W}RN^{(s,k)}_{gg^{-1}(z)}(W)\\&\qquad\vee\bigvee_{gg^{-1}(z)\notin W}RN^{(s,k)}_{gg^{-1}(w)}(W)\Big)\\&\leq\bigwedge_{g^{-1}(z)\neq g^{-1}(w)}\Big(\bigvee_{g^{-1}(w)\notin g^{-1}(W)}N^s_{g^{-1}(z)}(g^{-1}(W))\\&\qquad\vee\bigvee_{g^{-1}(z)\notin g^{-1}(W)}N^s_{g^{-1}(w)}(g^{-1}(W))\Big)\end{aligned}$$

$$=\bigwedge_{x\neq y}\Big(\bigvee_{y\notin U}N^s_x(U)\ \vee\ \bigvee_{x\notin U}N^s_y(U)\Big)=T_0(X,\varrho_s).$$

We can prove (2) and (3) in the same way as the proof of (1) above. □

Theorem 5.2. *Let $(X,\varrho_1,\varrho_2),(Y,\xi_1,\xi_2)$ be two L-fbss. If $g:(X,\varrho_1,\varrho_2)\longrightarrow(Y,\xi_1,\xi_2)$ is a bijective mapping and $[AO_{(s,k)}(g)]=\top$, then*

(1) $RT^{(s,k)}_0(X,\varrho_1,\varrho_2)\leq T_0(Y,\xi_s)$,
(2) $RT^{(s,k)}_1(X,\varrho_1,\varrho_2)\leq T_1(Y,\xi_s)$,
(3) $RT^{(s,k)}_2(X,\varrho_1,\varrho_2)\leq T_2(Y,\xi_s)$.

Proof. (1) From Theorem 4.4 and $[AO_{(s,k)}(g)] = \top$, we have

$$\begin{aligned}
RT_0^{(s,k)}(X, \varrho_1, \varrho_2) &= \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} RN_x^{(s,k)}(\mathcal{O}) \ \vee \ \bigvee_{x \notin \mathcal{O}} RN_y^{(s,k)}(\mathcal{O}) \Big) \\
&= \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} RN_x^{(s,k)}(g^{-1}g(\mathcal{O})) \ \vee \ \bigvee_{x \notin \mathcal{O}} RN_y^{(s,k)}(g^{-1}g(\mathcal{O})) \Big) \\
&\leq \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} N^s_{g(x)}(g(\mathcal{O})) \ \vee \ \bigvee_{x \notin \mathcal{O}} N^s_{g(y)}(g(\mathcal{O})) \Big) \\
&= \bigwedge_{g(x) \neq g(y)} \Big(\bigvee_{g(y) \notin g(\mathcal{O})} N^s_{g(x)}(g(\mathcal{O})) \ \vee \ \bigvee_{g(x) \notin g(\mathcal{O})} N^s_{g(y)}(g(\mathcal{O})) \Big) \\
&= \bigwedge_{z \neq w} \Big(\bigvee_{w \notin H} N^s_z(H) \ \vee \ \bigvee_{z \notin H} N^s_w(H) \Big) \\
&= T_0(Y, \xi_s).
\end{aligned}$$

We can prove (2) and (3) in the same way as the proof of (1) above. □

Theorem 5.3. *Let $(X, \varrho_1, \varrho_2), (Y, \xi_1, \xi_2)$ be two L-fbss. If $g : (X, \varrho_1, \varrho_2) \longrightarrow (Y, \xi_1, \xi_2)$ is an injective mapping and $[RC_{(s,k)}(g)] = \top$, then*

(1) $T_0(Y, \xi_s) \leq RT_0^{(s,k)}(X, \varrho_1, \varrho_2)$,
(2) $T_1(Y, \xi_s) \leq RT_1^{(s,k)}(X, \varrho_1, \varrho_2)$,
(3) $T_2(Y, \xi_s) \leq RT_2^{(s,k)}(X, \varrho_1, \varrho_2)$.

Proof. (1) From Theorem 4.2 and $[RC_{(s,k)}(g)] = \top$, we have for every $W \in 2^Y$ and $x \in X$, $N^s_{g(x)}(W) \leq RN_x^{(s,k)}(g^{-1}(W))$. Therefore

$$\begin{aligned}
RT_0^{(s,k)}(X, \varrho_1, \varrho_2) &= \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} RN_x^{(s,k)}(\mathcal{O}) \ \vee \ \bigvee_{x \notin \mathcal{O}} RN_y^{(s,k)}(\mathcal{O}) \Big) \\
&= \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} RN_x^{(s,k)}(g^{-1}g(\mathcal{O})) \ \vee \ \bigvee_{x \notin \mathcal{O}} RN_y^{(s,k)}(g^{-1}g(\mathcal{O})) \Big) \\
&\geq \bigwedge_{x \neq y} \Big(\bigvee_{y \notin \mathcal{O}} N^s_{g(x)}(g(\mathcal{O})) \ \vee \ \bigvee_{x \notin \mathcal{O}} N^s_{g(y)}(g(\mathcal{O})) \Big) \\
&= \bigwedge_{g(x) \neq g(y)} \Big(\bigvee_{g(y) \notin g(\mathcal{O})} N^s_{g(x)}(g(\mathcal{O})) \ \vee \ \bigvee_{g(x) \notin g(\mathcal{O})} N^s_{g(y)}(g(\mathcal{O})) \Big) \\
&= \bigwedge_{z \neq w} \Big(\bigvee_{w \notin H} N^s_z(H) \ \vee \ \bigvee_{z \notin H} N^s_w(H) \Big) = T_0(Y, \xi_s).
\end{aligned}$$

The proof of (2) and (3) is similar to (1) above. □

6. Conclusions

In the present paper, we prove that some results obtained in [1] need not be true, by giving some counterexamples. Also, we study some types of L-fuzzy mappings and the image of these kinds of L-fuzzifying bitopological spaces. In the future, we can take these properties in applications of L-fuzzifying bitopological spaces.

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