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Interval type-2 fuzzy linear programming problem with vagueness parameters

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Abstract. Interval type-2 fuzzy linear programming models, which are extensions of type-2 fuzzy linear programming models, have been extensively considered and used in recent years, and numerous studies have been conducted to solve this type of problem. However, so far, not many studies have been conducted on the fuzzy linear programming problem with vagueness in the parameters with interval type-2 membership functions, and this is a necessity for conducting this research. In these types of problems, the input data are modeled using fuzzy preference-based membership functions. This study investigates interval type-2 fuzzy linear programming problems with vagueness parameters. In addition, we present the membership functions associated with each model and propose new modeling techniques. Depending on the position of vagueness in the problem, such as vagueness in the objective function vector, technological coefficients, resources vector, and any possible combination of them, various problems arise. Therefore, we introduce these types of problems, provide membership functions, and propose different solution methods. We evaluate the efficiency and performance of each of the proposed methods using an example.

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1. Introduction

Given that real-world data are often imprecision, employing fuzzy linear programming (FLP) problems is regarded as an ideal modeling technique. Zadeh [1] was the first to introduce fuzzy sets (FSs). As an extension of type-1 fuzzy sets (T1FSs), he introduced type-2 fuzzy sets (T2FSs) (See [2, 3, 4]). The latter one features degrees of fuzzy memberships that outperformed at decreasing the uncertainty impact and modeling problems. Many researchers have studied and introduced several methods for solving different types of FLP problems. One of the first studies was conducted by Tanaka et al. [5]. They considered the Bellman-Zadeh's decision-making principle (the max-min operator) [6]. To acquire further scholarly investigations and examinations that have been conducted recently regarding FLP models, the following can be mentioned: Karmakar et al. proposed type-2 intuitionistic fuzzy matrix games based on a new distance measure in a practical problem [7]. Also in [8], they proposed a novel and applicable defuzzification approach of type-2 fuzzy variables to solve matrix games. El Alaoui presented a fully FLP approach in which all parameters are represented by unrestricted interval type 2 fuzzy numbers (IT2FN) and variables [9]. Javanmard et al. introduced a method for solving the FLP problem with IT2FNs [10].

Fuzzy mathematical programming is classified into different categories according to how it models the fuzzy parameters and numbers in the problem. One of the most important categories in the fuzzy medium was presented by Lai and Hwang [11], who divided the problems into two categories: FLP problems and linear programming (LP) problems with fuzzy parameters. In the first category, the input data are modeled using fuzzy preference-based membership functions (MFs), and in the second category, they are modeled on the basis of possibility distributions, which are called flexible and possibilistic LP problems. This study reviews the modeling of vagueness data using MFs, i.e., flexible LP problems. Figueroa enhanced the Zimmerman's method [12] based on MFs to solve interval type-2 fuzzy linear programming (IT2FLP) problems with the resources vector (RsV) [13]. In addition, he introduced different methods for solving different problems that may arise because of the position of vagueness in a problem. Sargolzaei and Mishmast Nehi proposed three methods to solve the IT2FLP problem with vagueness in the RsV [14]. Furthermore, they proposed several methods to solve the multi-objective IT2FLP problems with vagueness in coefficients [15]. Also they proposed an algorithm for multi-objective IT2FLP problems with ambiguous parameters [16]. Golpayegani and Mishmastnehi proposed a new technique for solving IT2FLP problems [17]. To represent the uncertainty in the degree of compliance between the constraints and the objective function, this method also employed MFs. According to the position of the vagueness in the problem, there are different modes of interval type-2 fuzzy linear programming problems. That is, an IT2FLP problem with vagueness in the OFV, TCs, RsV, and any possible combination of them.

Few studies have been conducted on the IT2FLP problem with vagueness coefficients. In this study, we propose new methods to solve these problems. As an innovation, some ideas were used in this study, such as Gasimove and Yenilmez's method for solving the FLP problem with vagueness in technological coefficients (TCs) [18], two methods based on the idea presented by Chandra and Aggarwal in solving the FLP problems with vagueness in the objective function vector (OFV) and RsV [19], and a method based on Farhadinia's idea in solving the FLP problems with vagueness in the RsV and TCs [20]. New methods were proposed for solving flexible IT2FLP problems. As mentioned above, various problems were raised because of vagueness in the problem, such as vagueness in the OFV, TCs, RsVs, and any combination of these, all of which, except the IT2FLP problem with vagueness in the RsV (See [14]), will be reviewed. The novelty and need for this study arises from the fact that most studies conducted thus far have primarily focused on FLP problems with vagueness parameters. However, there is a significant lack of research on IT2FLP problems with vagueness parameters. In short, the main advantages and benefits of this study can be described as follows:

- Lack of sufficient studies regarding IT2FLP problems with the vagueness in the parameters;
- Using the ideas used to solve FLP problems with vagueness and expanding them to solve IT2FLP problems with the vagueness in the parameters;
- The proposed methods have simple solution steps and no computational complexity;
- Since the Bellman-Zadeh operator is used to find a crisp solution to the IT2FL problem, our proposed methods are flexible and interpretable. Hence, they are appropriate for numerous similar problems.
- One of the advantages of most of our proposed methods is that for different values of $\alpha \in [0, 1]$, the corresponding values for the optimal objective function and optimal solutions are obtained. Therefore, the decision-maker (DM) can choose one of the optimal solutions according to the real conditions of the problem and what he/she is considering.

The objective function of maximization and the constraints are considered to be less than or equal to in this study. In addition, It should be noted that, the TCs and values of the OFV can also have negative values. Also, we consider the TCs with triangular MFs. The structure of this study is as follows:

First, the preliminary and basic concepts are introduced in Section 2. In the Section 3, the IT2FLP problem with vagueness in the OFV is expressed, and a novel technique is presented to solve it. In the Section 4, the IT2FLP problem with vagueness in the TCs is introduced, and two new methods are presented to solve it. In the Section 5, the IT2FLP problem with vagueness in the OFV and RsV is stated, and two new solutions are proposed. The Section 6 presents the IT2FLP problem with vagueness in the TCs and RsV and proposed a new method to solve it. The Section 7 introduced the IT2FLP problem with vagueness in the OFV, TCs, and RsV and proposed a new method to solve it. Then, for a better understanding of the proposed method(s), examples are presented at the end of each section, and comparisons are provided if required. Finally, the Section 8 includes the conclusion, which states several limitations and potential future work. In the Table 1, you will find a list of the symbols used in this study.

2. Preliminaries

The interval linear programming (ILP) problem and basic T2FS terminologies are introduced in this section.

2.1. **The IT2FSs.** A T2FS gathers an unbounded quantity of FSs and is distinguished by two MFs. In this subsection, we introduce significant and indispensable explanations associated with interval type-2 fuzzy sets (IT2Fs). Please consult [21] and [22] for further details.

TABLE 1. Reference of the symbols used in this article

Symbols	Description
T2FS	$\tilde{\tilde{a}}$
Upper membership function (UMF)	$\overline{a}$
Lower membership function (LMF)	$\underline{a}$
Left MF	$a^{\vee}$
Right MF	$a^{\wedge}$
Interval number	$a^{\pm}$

Definition 2.1. $\tilde{\tilde{A}}$ is the T2FS and definition as follows:

$$\begin{aligned}\tilde{\tilde{A}} &= \int_{x\in X}\int_{u\in J_x} f_x(u)/(x,u)\\ &= \int_{x\in X}\left[\int_{u\in J_x} f_x(u)/u\right]\Big/ x,\end{aligned}$$

where $\int\int$ denotes union over all admissible x and u, $J_x \subseteq [0,1]$ is the primary membership of x, $x \in X$, $u \in [0,1]$, and $f_x(u) \in [0,1]$ and is the secondary grade.

Definition 2.2. $\tilde{\tilde{A}}$ is an IT2FS and we described it as:

$$\tilde{\tilde{A}} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} 1/u\right]\Big/ x. \tag{2.1}$$

The subsequent explanation enables us to depict a T2FS visually in two dimensions as opposed to three dimensions. Figure 1 showcases an illustrative depiction of IT2FS.

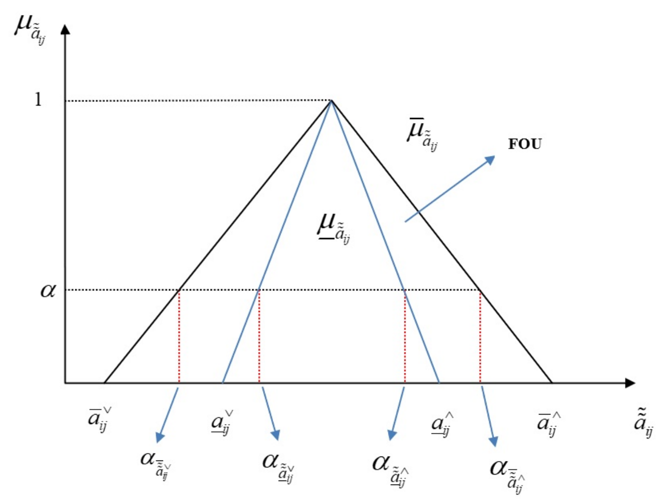

FIGURE 1. The IT2FS and its α-cut.

Definition 2.3. The footprint of uncertainty (FOU), which we refer to as the bounded area of uncertainty, is present in the initial memberships of a T2FS $\tilde{\tilde{A}}$. It consists of all initial memberships combined, i.e.

$$FOU(\tilde{\tilde{A}}) = \bigcup_{x\in X} J_x.$$

A distinction can be observed between the T1FS and T2FS in terms of the FOU of a T2FS that receives an infinite number of T1FSs.

Definition 2.4. An IT2FS's α-cut is

$$\alpha_{\tilde{\tilde{a}}_{ij}} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, J_{\tilde{\tilde{a}}_{ij}} \geq \alpha, u \in [0,1] \right\}.$$

It is displayed in the following two parts

$$\alpha_{\overline{\mu}_{\tilde{\tilde{a}}_{ij}}} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, \overline{\mu}_{\tilde{\tilde{a}}_{ij}} \geq \alpha \right\} \tag{2.2}$$

and

$$\alpha_{\underline{\mu}_{\tilde{\tilde{a}}_{ij}}} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, \underline{\mu}_{\tilde{\tilde{a}}_{ij}} \geq \alpha \right\}. \tag{2.3}$$

where UMF $\tilde{\tilde{a}}_{ij}$ corresponding $\alpha \in [0,1]$ is represented by relation (2.2) and its LMF is represented by relation (2.3).

The decomposition of any T2FS can be achieved by utilizing α-cuts, which are facilitated by the employment of $\overline{\mu}_{\tilde{\tilde{a}}_{ij}}$ and $\underline{\mu}_{\tilde{\tilde{a}}_{ij}}$.

Definition 2.5. The IT2FS is considered to be characterized by known UMF and LMF. Establish ${\alpha^I}_{\tilde{\tilde{a}}^{\vee}_{ij}}$ and ${\alpha^I}_{\tilde{\tilde{a}}^{\wedge}_{ij}}$ as the left and right interval-valued bound of relation (2.1). In addition, the boundaries of $\alpha_{\tilde{\tilde{a}}_{ij}}$ are defined in the following:

$${\alpha^I}_{\tilde{\tilde{a}}^{\vee}_{ij}} = \left[\inf_{\tilde{\tilde{a}}_{ij}} \alpha_{\overline{\mu}_{\tilde{\tilde{a}}_{ij}}}(\tilde{\tilde{a}}_{ij}, u); \inf_{\tilde{\tilde{a}}_{ij}} \alpha_{\underline{\mu}_{\tilde{\tilde{a}}_{ij}}}(\tilde{\tilde{a}}_{ij}, u) \right] = \left[\alpha_{\overline{\tilde{\tilde{a}}}^{\vee}_{ij}}, \alpha_{\underline{\tilde{\tilde{a}}}^{\vee}_{ij}} \right],$$

$${\alpha^I}_{\tilde{\tilde{a}}^{\wedge}_{ij}} = \left[\sup_{\tilde{\tilde{a}}_{ij}} \alpha_{\underline{\mu}_{\tilde{\tilde{a}}_{ij}}}(\tilde{\tilde{a}}_{ij}, u); \sup_{\tilde{\tilde{a}}_{ij}} \alpha_{\overline{\mu}_{\tilde{\tilde{a}}_{ij}}}(\tilde{\tilde{a}}_{ij}, u) \right] = \left[\alpha_{\underline{\tilde{\tilde{a}}}^{\wedge}_{ij}}, \alpha_{\overline{\tilde{\tilde{a}}}^{\wedge}_{ij}} \right].$$

2.2. **The ILP problem.** Here, we investigate the ILP problem and demonstrate two theorems regarding the determination of values for the optimal objective function of the ILP, [23].

Definition 2.6 ([24])**.** ILP problems are described as:

$$\begin{array}{ll} \max & z = \sum_{j=1}^{n} \mathbf{c}_j^{\pm} \mathbf{x}_j \\ s.t. & \sum_{j=1}^{n} a_{ij}^{\pm} \mathbf{x}_j \leq \mathbf{b}_i^{\pm}, \quad i = 1, 2, ..., m, \\ & \mathbf{x}_j \geq 0 \quad j = 1, 2, \cdots, n. \end{array} \tag{2.4}$$

ILP problems have been addressed using various strategies. Several methods have been introduced to solve the ILP problem in the [24],[25],[26],[27], and [28]. The best-worst case (BWC) method is one of the fundamental techniques [29]. By expressing two sub-models as the worst and the best sub-models, the BWC method is used to solve ILP problem (2.4). The best and worst values of the optimal objective function of the ILP problem can be found by applying the following theorems:

Theorem 2.7 ([23])**.** *The optimal values of the objective function of the ILP problem* (2.4) *can be determined by solving two separate problems, each yielding the best and worst sub-problems.*

$$\begin{array}{ll} \max & z^+ = \sum_{j=1}^{n} \mathbf{c}_j^+ \mathbf{x}_j \\ s.t. & \sum_{j=1}^{n} a_{ij}^- \mathbf{x}_j \le \mathbf{b}_i^+, \qquad i = 1, 2, \cdots, m, \\ & \mathbf{x}_j \ge 0 \qquad j = 1, 2, \cdots, n, \end{array}$$

and

$$\begin{array}{ll} \max & z^- = \sum_{j=1}^{n} \mathbf{c}_j^- \mathbf{x}_j \\ s.t. & \sum_{j=1}^{n} a_{ij}^+ \mathbf{x}_j \le \mathbf{b}_i^-, \qquad i = 1, 2, \cdots, m, \\ & \mathbf{x}_j \ge 0 \qquad j = 1, 2, \cdots, n. \end{array}$$

Theorem 2.8 ([23])**.** *The range of the objective function values in the BWC method encompasses the value of the objective function in any arbitrary characteristic model of ILP problem* (2.4)*.*

In the following sections, we presented the IT2FLP problems with vagueness in the coefficients, OFV, any combination. First, we provide the MF for each one, and then we proposed new method(s) to solve each one.

3. The IT2FLP problem with vagueness in OFV

In this specific section, the MF of the OFV is initially articulated, followed by the introduction of a novel methodology aimed at resolving the IT2FLP problem when confronted with vagueness in the OFV. The $\tilde{\tilde{c}}_j$, $j = 1, 2, \cdots, n$, are the IT2FSs, which are defined by the UMF, $\bar{\mu}_{\tilde{\tilde{c}}_j}$, and LMF, $\underline{\mu}_{\tilde{\tilde{c}}_j}$ (see the Figure 2).

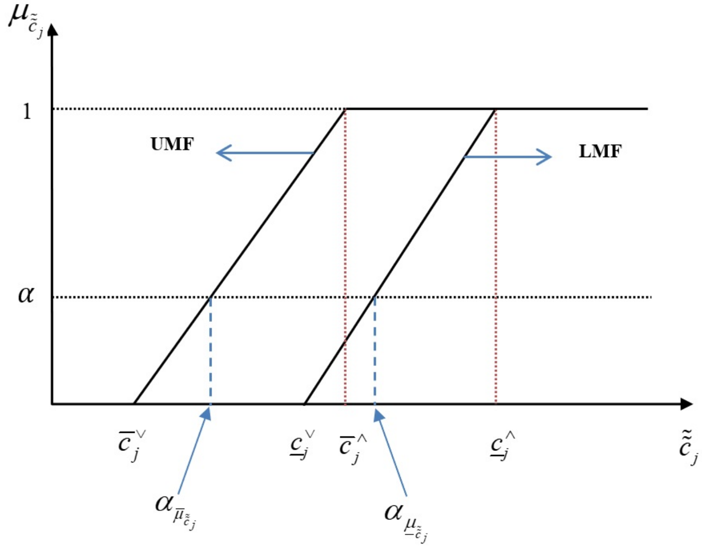

Figure 2. The MF of the interval type-2 OFV.

Since the objective function is a maximization, thus the UMF is equal to:

$$\bar{\mu}_{\tilde{\tilde{c}}_j}(c_j;\bar{c}_j^{\vee},\bar{c}_j^{\wedge})=\begin{cases}1, & c_j\geq \bar{c}_j^{\wedge},\\ \frac{c_j-\bar{c}_j^{\vee}}{\bar{c}_j^{\wedge}-\bar{c}_j^{\vee}}, & \bar{c}_j^{\vee}\leq c_j\leq \bar{c}_j^{\wedge},\\ 0, & c_j\leq \bar{c}_j^{\vee},\end{cases}$$

and its LMF is expressed as below:

$$\underline{\mu}_{\tilde{\tilde{c}}_j}(c_j;\underline{c}_j^{\vee},\underline{c}_j^{\wedge})=\begin{cases}1, & c_j\geq \underline{c}_j^{\wedge},\\ \frac{c_j-\underline{c}_j^{\vee}}{\underline{c}_j^{\wedge}-\underline{c}_j^{\vee}}, & \underline{c}_j^{\vee}\leq c_j\leq \underline{c}_j^{\wedge},\\ 0, & c_j\leq \underline{c}_j^{\vee}.\end{cases}$$

3.1. **The new solving method.** Here, we introduce a method for solving the IT2FLP problem with vagueness in the OFV. The IT2FLP problem with vagueness in the OFV is considered, which is a dual-mode IT2FLP problem with vagueness in the RsV, [14].

$$\begin{aligned} &\max \sum_{j=1}^{n}\tilde{\tilde{c}}_j x_j\\ &s.t.\ \sum_{j=1}^{n}a_{ij}x_j\leq b_i, \quad i=1,2,\cdots,m,\\ &\qquad x_j\geq 0, \qquad j=1,2,\cdots,n,\end{aligned}\tag{3.1}$$

which $\tilde{\tilde{c}}_j, j=1,\cdots,n$, are interval type-2 fuzzy OFVs. As shown, by applying the α-cut, the interval $\left[\alpha_{\bar{\mu}_{\tilde{\tilde{c}}_j}},\alpha_{\underline{\mu}_{\tilde{\tilde{c}}_j}}\right]$ is obtained. Considering Figure 2, we have $\left[\alpha_{\bar{\mu}_{\tilde{\tilde{c}}_j}},\alpha_{\underline{\mu}_{\tilde{\tilde{c}}_j}}\right]=\left[\frac{c_j-\bar{c}_j^{\vee}}{\bar{c}_j^{\wedge}-\bar{c}_j^{\vee}},\frac{c_j-\underline{c}_j^{\vee}}{\underline{c}_j{}^{\wedge}-\underline{c}_j^{\vee}}\right]$. Now, the problem (3.1) is rewritten as follows:

$$\begin{aligned} &\max \sum_{j=1}^{n}\left[\frac{c_j-\bar{c}_j^{\vee}}{\bar{c}_j^{\wedge}-\bar{c}_j^{\vee}},\frac{c_j-\underline{c}_j^{\vee}}{\underline{c}_j^{\wedge}-\underline{c}_j^{\vee}}\right]x_j\\ &s.t.\ \sum_{j=1}^{n}a_{ij}x_j\leq b_i, \qquad i=1,2,\cdots,m,\\ &\qquad x_j\geq 0,\ \alpha\in[0,1], \quad j=1,2,\cdots,n,\end{aligned}\tag{3.2}$$

which (3.2) is equivalent to

$$\begin{aligned} &\max \sum_{j=1}^{n}\left[\bar{c}_i^{\vee}+\alpha(\bar{c}_i^{\wedge}-\bar{c}_i^{\vee}),\underline{c}_i^{\vee}+\alpha(\underline{c}_i^{\wedge}-\underline{c}_i^{\vee})\right]x_j\\ &s.t.\ \sum_{j=1}^{n}a_{ij}x_j\leq b_i, \qquad i=1,\cdots,m,\\ &\qquad x_j\geq 0,\ \alpha\in[0,1], \quad j=1,\cdots,n,\end{aligned}\tag{3.3}$$

since the objective function is maximization, the optimal solution for the problem (3.3), given the interval programming, is as follows:

$$\begin{aligned} &\max \sum_{j=1}^{n}(\underline{c}_i^{\vee}+\alpha(\underline{c}_i^{\wedge}-\underline{c}_i^{\vee}))x_j\\ &s.t.\ \sum_{j=1}^{n}a_{ij}x_j\leq b_i, \qquad i=1,\cdots,m,\\ &\qquad x_j\geq 0,\ \alpha\in[0,1], \quad j=1,\cdots,n.\end{aligned}\tag{3.4}$$

The problem (3.4) is a nonlinear that is solved for different values of $\alpha \in [0,1]$. In this method, in addition to α values, the optimal value of the objective function and optimal solutions are also calculated, which is one of the advantages of this method. Therefore, for different values of $\alpha \in [0,1]$, a table of optimal solutions and the optimal value of the objective function corresponding to each α value are obtained, and the DM can choose a value from among the obtained solutions.

Example 3.1. Consider the following data for IT2FLP problem with vagueness in the OFV: $\underline{c}^{\vee} = \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$, $\underline{c}^{\wedge} = \begin{bmatrix} 14 \\ 20 \\ 12 \end{bmatrix}$, $\bar{c}^{\vee} = \begin{bmatrix} 8 \\ 13 \\ 5 \end{bmatrix}$, $\bar{c}^{\wedge} = \begin{bmatrix} 15 \\ 22 \\ 13 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 3 & 7 \\ 10 & 4 & 9 \\ 4 & 6 & 3 \\ 2 & 7 & 7 \\ 5 & 6 & 11 \end{bmatrix}$, $b = \begin{bmatrix} 66 \\ 92 \\ 60 \\ 85 \\ 68.5 \end{bmatrix}$.

According to these data and the problem (3.4), the non-LP problem is expressed as follows:

$$
\begin{aligned}
\max\ & (8+7\alpha)x_1 + (13+9\alpha)x_2 + (5+8\alpha)x_3 \\
s.t.\ & 5x_1 + 3x_2 + 7x_3 \leq 66, \\
& 10x_1 + 4x_2 + 9x_3 \leq 92, \\
& 4x_1 + 6x_2 + 3x_3 \leq 60, \\
& 2x_1 + 7x_2 + 7x_3 \leq 85, \\
& 5x_1 + 6x_2 + 11x_3 \leq 68.5, \\
& \alpha \in [0,1],\ x_j \geq 0,\ \ j = 1,2,3.
\end{aligned} \tag{3.5}
$$

Since the problem (3.5) is nonlinear, we solve that for different values of $\alpha \in [0,1]$. From Table 2, we can see for $\alpha = 1$, the maximum value of the optimal objective

TABLE 2. Solution of the problem (3.5)

$\alpha \in [0,1]$	$\mathbf{x}_1^*$	$\mathbf{x}_2^*$	$\mathbf{x}_3^*$	z^*
0	0	10.0000	0	130.0000
0.1	0	10.0000	0	139.0000
0.2	0	10.0000	0	148.0000
0.3	0	10.0000	0	157.0000
0.4	0	10.0000	0	166.0000
0.5	0	9.4687	1.0625	175.2656
0.6	0	9.4687	1.0625	184.6375
0.7	0	9.4687	1.0625	194.0094
0.8	0	9.4687	1.0625	203.3812
0.9	6.9000	5.3000	0.2000	212.9400
1	6.9000	5.3000	0.2000	222.7000

function is obtained as $z^* = 222.7000$ and the best optimal solution is $(x_1^*, x_2^*, x_3^*) = (6.9, 5.3, 0.2)$. As mentioned above, according to the real conditions of the problem, the DM can choose a value from among the solutions obtained in Table 1.

4. The IT2FLP problem with vagueness in the TCs

In this section, we review the MFs of interval type-2 fuzzy TCs and propose a method to solve such problems. In this study, the MFs of the TCs have a triangular shape. The main structure of an IT2FLP problem with vagueness in the TCs is as follows:

$$
\begin{aligned}
&\max \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \leq b_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \quad j = 1, \cdots, n,
\end{aligned} \tag{4.1}
$$

where (for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$), $c_j, x_j \in \mathbb{R}^n$, $b_i \in \mathbb{R}^m$, and $\tilde{\tilde{a}}_{ij} \in \mathbb{R}^{m\times n}$ are the IT2FSs. We examine the MF of interval type-2 fuzzy TCs with imprecision of the vagueness type. The MF representing this FS is shown in the Definition 2.5 (See Figure 1). Then for each α-cut of two ILP problems,

$$
\begin{aligned}
{}^{\alpha}z_1 : \quad &\max \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \left[\alpha_{\bar{\tilde{a}}^{\vee}_{ij}}, \alpha_{\underline{\tilde{a}}^{\vee}_{ij}}\right] x_j \leq b_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \qquad\qquad\qquad\qquad j = 1, \cdots, n,
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
{}^{\alpha}z_2 : \quad &\max \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \left[\alpha_{\underline{\tilde{a}}^{\wedge}_{ij}}, \alpha_{\bar{\tilde{a}}^{\wedge}_{ij}}\right] x_j \leq b_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \qquad\qquad\qquad\qquad j = 1, \cdots, n,
\end{aligned} \tag{4.3}
$$

to solve the programming problems (4.2) and (4.3), the best and worst objective function values are displayed with ${}^{\alpha}_{b}z_i$ and ${}^{\alpha}_{w}z_i$, respectively, where $i = 1, 2$.

$$
\begin{aligned}
{}^{\alpha}_{b}z_1 : \quad &\max \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \alpha_{\bar{\tilde{a}}^{\vee}_{ij}} x_j \leq b_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \qquad\qquad\quad j = 1, \cdots, n,
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
{}^{\alpha}_{w}z_1 : \quad &\max \sum_{j=1}^{n} c_j x_j \\
s.t. \quad &\sum_{j=1}^{n} \alpha_{\underline{\tilde{a}}^{\vee}_{ij}} x_j \leq b_i, \quad i = 1, \cdots, m, \\
&\qquad x_j \geq 0, \qquad\qquad\quad j = 1, \cdots, n,
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
&{}^{\alpha}_{b}z_2: \quad \max \sum_{j=1}^{n} c_j x_j \\
(4.6) \qquad & s.t. \quad \sum_{j=1}^{n} \alpha_{\underline{\tilde{\bar{a}}}_{ij}^{\wedge}} x_j \le b_i, \quad i=1,\cdots,m, \\
& \qquad x_j \ge 0, \qquad j=1,\cdots,n,
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{\alpha}_{w}z_2: \quad \max \sum_{j=1}^{n} c_j x_j \\
(4.7) \qquad & s.t. \quad \sum_{j=1}^{n} \alpha_{\tilde{\bar{a}}_{ij}^{\wedge}} x_j \le b_i, \quad i=1,\cdots,m, \\
& \qquad x_j \ge 0, \qquad j=1,\cdots,n.
\end{aligned}
$$

Theorem 4.1 ([30]). *Consider the interval inequalities* $\sum_{j=1}^{n} \left[\alpha_{\tilde{\bar{a}}_{ij}^{\vee}}, \alpha_{\underline{\tilde{\bar{a}}}_{ij}^{\vee}}\right] x_j \le b_i$ *and* $\sum_{j=1}^{n} \left[\alpha_{\underline{\tilde{\bar{a}}}_{ij}^{\wedge}}, \alpha_{\tilde{\bar{a}}_{ij}^{\wedge}}\right] x_j \le b_i$*. The biggest and smallest feasible regions are equal to* $\sum_{j=1}^{n} \alpha_{\tilde{\bar{a}}_{ij}^{\vee}} x_j \le b_i$ *and* $\sum_{j=1}^{n} \alpha_{\tilde{\bar{a}}_{ij}^{\wedge}} x_j \le b_i$*, respectively.*

Theorem 4.2 ([30]). *For each α-cut, the UMF of the objective function is obtained by solving* (4.4) *and* (4.7).

Given the theorems 4.1 and 4.2, the best solution is obtained by solving the problem (4.4).

Next, we propose two new approaches for solving the IT2FLP problem with vagueness in the TCs.

4.1. **The first new method.** In this subsection, a new solution method is suggested based on Gasimov's idea for solving the IT2FLP problem with vagueness in the TCs [18]. Consider the IT2FLP problem with vagueness in the TCs (problem (4.1)). As shown in Figure 1, four modes $(\bar{a}_{ij}^{\vee}, \bar{a}_{ij}^{\wedge}, \underline{a}_{ij}^{\vee}, \underline{a}_{ij}^{\wedge})$ exist for $\tilde{a}_{ij}$, which a FLP problem exists for each. Therefore, as these four problems are solved, four values are obtained for the fuzzy objective function, i.e., $(\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge})$. Because the problem (4.1) is an asymmetrical problem, we try to make it symmetrical. Therefore, it is assumed that

$$
\begin{aligned}
z_b &= \max\{\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge}\}, \\
z_w &= \min\{\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge}\},
\end{aligned}
$$

as a result, the MF of the fuzzy objective function is expressed as follows:

$$
(4.8) \qquad \mu_{\tilde{\bar{C}}_j}(x) = \begin{cases} 1, & \sum_{j=1}^{n} c_j x_j > z_b, \\ \dfrac{\sum_{j=1}^{n} c_j x_j - z_w}{z_b - z_w}, & z_w \le \sum_{j=1}^{n} c_j x_j \le z_b, \\ 0, & \sum_{j=1}^{n} c_j x_j < z_w. \end{cases}
$$

In addition, for $i = 1, 2, \cdots, m$, the MF of fuzzy constraints is defined as below:

$$(4.9) \qquad \mu_{\tilde{\tilde{G}}_i}(x) = \begin{cases} 1, & b_i > \sum\limits_{j=1}^{n} (a_{ij} + \Delta_{ij})x_j, \\ \dfrac{b_i - \sum\limits_{j=1}^{n} a_{ij}x_j}{\sum\limits_{j=1}^{n} \Delta_{ij}x_j}, & \sum\limits_{j=1}^{n} a_{ij}x_j \le b_i \le \sum\limits_{j=1}^{n} (a_{ij} + \Delta_{ij})x_j, \\ 0, & b_i < \sum\limits_{j=1}^{n} a_{ij}x_j. \end{cases}$$

Regarding the max-min operator [6], we have

$$(4.10) \qquad \begin{array}{ll} \max & \alpha \\ s.t. & \mu_{\tilde{\tilde{C}}_j}(x) \ge \alpha, \ \ i = 1, \cdots, n, \\ & \mu_{\tilde{\tilde{G}}_i}(x) \ge \alpha, \\ & \alpha \in [0, 1], \, x \ge 0. \end{array}$$

The problem (4.10) is rewritten on the basis of two defined MFs for the fuzzy objective (4.8) and fuzzy constraints (4.9) as follows:

$$(4.11) \qquad \begin{array}{ll} \max & \alpha \\ s.t. & \alpha(z_b - z_w) - \sum\limits_{j=1}^{n} c_j x_j + z_w \le 0, \\ & \sum\limits_{j=1}^{n} (a_{ij} + \Delta_{ij}\alpha)x_j - b_i \le 0, \quad i = 1, \cdots, m, \\ & \alpha \in [0, 1], \, x_j \ge 0, \qquad j = 1, \cdots, n. \end{array}$$

From now on, two modes can be considered. First, only the UMF is taken and the problem is solved on the basis of it. Then, the LMF is considered and the problem is solved. Thus, an interval is obtained for the α value, from which the mean α for obtaining a specified amount for α^* is obtained. Second, the mean of the left upper and lower MFs and the mean of the right upper and lower MFs are calculated and an α^* value is obtained. Then, the two proposed modes are studied in detail.
Mode 1: For the UMF, consider the intervals $\tilde{\tilde{a}}_{ij} \in \left[\underline{a}^{\wedge}_{ij}, \bar{a}^{\wedge}_{ij}\right]$ and $\Delta_{ij} \in \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right]$. Then the problem (4.11) is displayed as an interval non-LP problem:

$$\begin{array}{ll} \max & \alpha \\ s.t. & \alpha(z_b - z_w) - \sum\limits_{j=1}^{n} c_j x_j + z_w \le 0, \\ & \sum\limits_{j=1}^{n} (\left[\underline{a}^{\wedge}_{ij}, \bar{a}^{\wedge}_{ij}\right] + \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right] \alpha)x_j \le b_i, \quad i = 1, \cdots, m, \\ & \alpha \in [0, 1], \, x_j \ge 0, \qquad\qquad j = 1, \cdots, n, \end{array}$$

which, given interval programming, the best problem is expressed as follows:

$$
\begin{aligned}
&\max \quad \alpha \\
&s.t. \quad \alpha(z_b - z_w) - \sum_{j=1}^{n} c_j x_j + z_w \le 0, \\
&\qquad \sum_{j=1}^{n} (\underline{a}_{ij}^{\wedge} + \underline{\Delta}_{ij}\alpha) x_j \le b_i, \qquad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1]\,,\ x_j \ge 0, \qquad\qquad j = 1, \cdots, n.
\end{aligned} \tag{4.12}
$$

The problem (4.12) is a nonlinear. Thus, it for $\alpha \in [0,1]$ for solving the IT2FLP problem with vagueness in the TCs should be solved. Also, for the LMF, consider the intervals $\tilde{\tilde{a}}_{ij} \in \left[\bar{a}_{ij}^{\vee}, \underline{a}_{ij}^{\vee}\right]$ and $\Delta_{ij} \in \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right]$, we have

$$
\begin{aligned}
&\max \quad \alpha \\
&s.t. \quad \alpha(z_b - z_w) - \sum_{j=1}^{n} c_j x_j + z_w \le 0, \\
&\qquad \sum_{j=1}^{n} (\bar{a}_{ij}^{\vee} + \underline{\Delta}_{ij}\alpha) x_j \le b_i, \qquad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1]\,,\ x_j \ge 0, \qquad\qquad j = 1, \cdots, n.
\end{aligned} \tag{4.13}
$$

The problem (4.13) is also a nonlinear. Therefore, to solve the IT2FLP problem with vagueness in the TCs, the problem (4.13) for various α values in $[0,1]$ should be solved. Finally, two values are obtained for each common α.

It is worth noting that the constraints in the problems (4.12) and (4.13) include αx_j product, thus, these problems are nonlinear. Therefore, the fuzzy decisive set method (algorithm α) is used to solve these problems. The idea of the fuzzy decisive set is based on the conception that the problems (4.12) and (4.13), turns into a linear problem with a fixed value of α, respectively.

The alpha algorithm is as follows:

1. Put $\alpha = 1$ and consider if the problem is feasible using the simplex method or not. If so, insert $\alpha = 1$; otherwise, suppose $\alpha^L = 0$, $\alpha^R = 1$, and go to the next step.
2. Suppose that $\alpha = \frac{\alpha^L + \alpha^R}{2}$ and update the values α^L and α^R . If the problem is feasible for a new α, you can put $\alpha^L = \alpha$. If not, insert $\alpha^R = \alpha$.

As a result, for each α , examine whether the above problem is feasible or not and the highest value of α^* is true in the above-mentioned constraints.

Mode 2: Suppose $\tilde{\tilde{a}}_{ij} \in \left[\frac{\bar{a}_{ij}^{\vee} + \underline{a}_{ij}^{\vee}}{2}, \frac{\underline{a}_{ij}^{\wedge} + \bar{a}_{ij}^{\wedge}}{2}\right]$ and $\Delta_{ij} \in \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right]$ intervals for the upper and lower MFs. Then, this problem is displayed as an interval non-LP problem:

$$
\begin{aligned}
&\max \quad \alpha \\
&s.t. \quad \alpha(z_b - z_w) - \sum_{j=1}^{n} c_j x_j + z_w \le 0, \\
&\sum_{j=1}^{n} \left(\left[\frac{\bar{a}_{ij}^{\vee} + \underline{a}_{ij}^{\vee}}{2}, \frac{\underline{a}_{ij}^{\wedge} + \bar{a}_{ij}^{\wedge}}{2}\right] + \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right] \alpha\right) x_j \le b_i, \quad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1]\,,\ x_j \ge 0, \qquad\qquad j = 1, \cdots, n,
\end{aligned} \tag{4.14}
$$

which, given interval programming, the best problem of the problem (4.14) is expressed as follows:

$$\begin{aligned}
&\max \quad \alpha \\
&s.t. \quad \alpha(z_b - z_w) - \sum_{j=1}^{n} c_j x_j + z_w \le 0, \\
&\qquad \sum_{j=1}^{n} \left(\tfrac{\bar{a}^{\vee}_{ij}+\underline{a}^{\vee}_{ij}}{2} + \underline{\Delta}_{ij}\alpha\right) x_j \le b_i, \quad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1],\ x_j \ge 0, \qquad\qquad j = 1, \cdots, n.
\end{aligned} \tag{4.15}$$

The problem (4.15) is also a nonlinear. As a result, solving the IT2FLP problem with vagueness in the TCs is sufficient to solve the problem (4.15) for $\alpha \in [0, 1]$.

4.2. **The second new method.** Consider the IT2FLP problem with vagueness in the TCs:

$$\begin{aligned}
&\max \ \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \le b_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \ge 0, \qquad j = 1, \cdots, n,
\end{aligned} \tag{4.16}$$

for $j = 1, 2, ..., n$, $\tilde{\tilde{a}}_{ij}$ are interval type-2 fuzzy TCs. As the MF of the constraints (4.12), the problem (4.16) is considered as follows:

$$\begin{aligned}
&\max \ \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \mu_{\tilde{\tilde{G}}_i}(x) \ge \alpha, \qquad i = 1, \cdots, m, \\
&\qquad\quad \alpha \in [0,1],\ x_j \ge 0, \quad j = 1, \cdots, n.
\end{aligned} \tag{4.17}$$

which problem (4.17) is equivalent to:

$$\begin{aligned}
&\max \ \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} (a_{ij} + \Delta_{ij}\alpha) x_j - b_i \le 0, \qquad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1],\ x_j \ge 0, \qquad\qquad j = 1, \cdots, n,
\end{aligned} \tag{4.18}$$

for the upper and lower MFs, suppose that $\tilde{\tilde{a}}_{ij} \in \left[\frac{\bar{a}^{\vee}_{ij}+\underline{a}^{\vee}_{ij}}{2}, \frac{\underline{a}^{\wedge}_{ij}+\bar{a}^{\wedge}_{ij}}{2}\right]$ and $\Delta_{ij} \in \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right]$. Then the problem (4.18) is displayed as an interval non-LP problem:

$$\begin{aligned}
&\max \ \sum_{j=1}^{n} c_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \left(\left[\tfrac{\bar{a}^{\vee}_{ij}+\underline{a}^{\vee}_{ij}}{2}, \tfrac{\underline{a}^{\wedge}_{ij}+\bar{a}^{\wedge}_{ij}}{2}\right] + \left[\underline{\Delta}_{ij}, \bar{\Delta}_{ij}\right] \alpha\right) x_j \le b_i, \quad i = 1, \cdots, m, \\
&\qquad \alpha \in [0,1],\ x_j \ge 0, \quad j = 1, \cdots, n,
\end{aligned} \tag{4.19}$$

According to the interval programming, the optimal problem of (4.19) is expressed as follows:

$$\begin{aligned} & \max \ \sum_{j=1}^{n} c_j x_j \\ (4.20) \qquad & s.t. \ \sum_{j=1}^{n} \left(\frac{\bar{a}_{ij}^{\vee}+\underline{a}_{ij}^{\vee}}{2} + \underline{\Delta}_{ij}\alpha\right)x_j \le b_i, \quad i = 1, \cdots, m, \\ & \alpha \in [0,1]\,,\ x_j \ge 0, \qquad j = 1, \cdots, n, \end{aligned}$$

problem (4.20) is also a nonlinear. As a result, it is solved for $\alpha \in [0, 1]$ to solve the IT2FLP problem with vagueness in the TCs.

Example 4.3. We have the following data for IT2FLP problem with vagueness in the TCs:

$$\bar{a}_{ij}^{\vee} = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 1 & 6 \\ 1 & 3 & 1 \\ 0 & 4 & 4 \\ 2 & 3 & 8 \end{bmatrix}, \bar{a}_{ij}^{\wedge} = \begin{bmatrix} 9 & 7 & 11 \\ 14 & 8 & 13 \\ 8 & 10 & 7 \\ 6 & 11 & 11 \\ 9 & 10 & 15 \end{bmatrix}, \underline{a}_{ij}^{\vee} = \begin{bmatrix} 3 & 2 & 5 \\ 8 & 2 & 7 \\ 2 & 4 & 2 \\ 1 & 5 & 5 \\ 3 & 4 & 9 \end{bmatrix}, \underline{a}_{ij}^{\wedge} = \begin{bmatrix} 7 & 5 & 9 \\ 12 & 6 & 11 \\ 6 & 8 & 5 \\ 4 & 9 & 9 \\ 7 & 8 & 13 \end{bmatrix},$$

$$c = \begin{bmatrix} 12 \\ 7 \\ 9 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 66 \\ 92 \\ 60 \\ 85 \\ 68.5 \end{bmatrix}.$$

Solving the Example 4.3 using the first new method: As mentioned in the solution method, first, for each $\tilde{\tilde{a}}_{ij}$, the optimal objective function value $(\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge})$ are obtained, and then we have

$$\begin{aligned} z_b &= \max\{\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge}\} = \max\{239.9211, 94.8000, 179.8462, 79.0526\} = 239.9211, \\ z_w &= \min\{\bar{z}_{ij}^{\vee}, \bar{z}_{ij}^{\wedge}, \underline{z}_{ij}^{\vee}, \underline{z}_{ij}^{\wedge}\} = \min\{239.9211, 94.8000, 179.8462, 79.0526\} = 79.0526, \end{aligned}$$

by replacing the obtained values with other values in the problem (4.15), the following nonlinear problem is obtained:

$$\begin{aligned} & \max \ \alpha \\ & s.t. \ -12x_1 - 7x_2 - 9x_3 + 160.8685\alpha + 79.0526 \le 0, \\ & (2.5+\alpha)x_1 + (1.5+\alpha)x_2 + (4.5+\alpha)x_3 - 66 \le 0, \\ & (7.5+\alpha)x_1 + (1.5+\alpha)x_2 + (6.5+\alpha)x_3 - 92 \le 0, \\ (4.21) \qquad & (1.5+\alpha)x_1 + (3.5+\alpha)x_2 + (1.5+\alpha)x_3 - 60 \le 0, \\ & (0.5+\alpha)x_1 + (4.5+\alpha)x_2 + (4.5+\alpha)x_3 - 85 \le 0, \\ & (2.5+\alpha)x_1 + (3.5+\alpha)x_2 + (5.5+\alpha)x_3 - 68.5 \le 0, \\ & \alpha \in [0,1]\,,\ x_j \ge 0,\ j = 1, 2, 3, \end{aligned}$$

using algorithm α: first, we put $\alpha = 1$ and solve the problem (4.21) using the simplex method. This problem has an infeasible solution, thus we put $\alpha^L = 0$ and $\alpha^R = 1$, and $\alpha = \frac{\alpha^L+\alpha^R}{2} = 0.5$ is obtained according to the second step. The value $\alpha = 0.5$ is inserted and solved in the problem (4.21), and it is a feasible. Thus, $\alpha^L = 0.5$ and $\alpha^R = 1$, and hence $\alpha = \frac{\alpha^L+\alpha^R}{2} = 0.75$ is obtained. By putting the value $\alpha = 0.75$ in the problem (4.21), it become infeasible.

$\alpha^L = 0.5$ and $\alpha^R = 0.75$, and hence $\alpha = \frac{\alpha^L+\alpha^R}{2} = 0.625$ is obtained. By putting the value $\alpha = 0.625$ in the problem (4.21), it become infeasible. Therefore, in the 15th iteration, $\alpha^* = 0.5999$ is the maximum value in which the constraints satisfaction occurs. In addition, the optimal solutions are obtained as $(x_1^*, x_2^*, x_3^*) = (8.7403, 10.0975, 0)$, and from the problem (4.1) the optimal value is obtained as 175.5661.

In this method, only one optimal value for α^* is obtained, which is actually the best value to satisfy the constraints. We put the values of the optimal solutions from this method into the row of the main problem's OFV. Then, we obtain the optimal value of the OFV.

Solving the Example 4.3 using the second new method: Using the existing data in the Example 4.3, the following problem is obtained.

$$(4.22)\quad \begin{array}{ll} \max & 12x_1 + 7x_2 + 9x_3 \\ s.t. & (2.5+\alpha)x_1 + (1.5+\alpha)x_2 + (4.5+\alpha)x_3 - 66 \le 0, \\ & (7.5+\alpha)x_1 + (1.5+\alpha)x_2 + (6.5+\alpha)x_3 - 92 \le 0, \\ & (1.5+\alpha)x_1 + (3.5+\alpha)x_2 + (1.5+\alpha)x_3 - 60 \le 0, \\ & (0.5+\alpha)x_1 + (4.5+\alpha)x_2 + (4.5+\alpha)x_3 - 85 \le 0, \\ & (2.5+\alpha)x_1 + (3.5+\alpha)x_2 + (5.5+\alpha)x_3 - 68.5 \le 0, \\ & \alpha \in [0,1]\,,\ x_j \ge 0,\ j = 1,2,3. \end{array}$$

Table 3. Solution of the problem (4.22)

$\alpha \in [0,1]$	$\mathbf{x}_1^*$	$\mathbf{x}_2^*$	$\mathbf{x}_3^*$	z^*
0	9.7444	12.6111	0	205.2111
0.1	9.5517	12.1293	0	199.5259
0.2	9.3703	11.6757	0	194.1736
0.3	9.1992	11.2480	0	189.1260
0.4	9.0375	10.8439	0	184.3577
0.5	0	9.4687	0	179.8462
0.6	8.7379	10.0993	0	175.5712
0.7	8.6022	9.7555	0	171.5146
0.8	8.4737	9.4211	0	167.6313
0.9	8.3590	9.0769	0	163.8462
1	8.4407	8.1017	0	158

For different values of $\alpha \in [0,1]$, we can see in Table 3, as smaller α are chosen, the values on the left side of the inequality become smaller; hence, a better optimal objective is obtained. One of the advantages of this method is that for different values of $\alpha \in [0,1]$, the corresponding values for the optimal objective function and optimal solutions are obtained. Therefore, the DM can choose one of the optimal solutions according to the real conditions of the problem and what he/she is considering. This feature is one of the advantages of the second proposed method.

5. The IT2FLP problem with vagueness in the OFV and RsV

In this section, the IT2FLP problem with vagueness in the OFV and RsV is introduced. The general form of this type of problem is as follows:

$$\begin{array}{lll} \max & \sum_{j=1}^{n} \tilde{\tilde{c}}_j x_j & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j \leq \tilde{\tilde{b}}_i, & i = 1, \cdots, m, \\ & x_j \geq 0, & j = 1, \cdots, n, \end{array} \tag{5.1}$$

where $\tilde{\tilde{c}}_j$, $(j = 1, 2, \cdots, n)$ and $\tilde{\tilde{b}}_i$, $(i = 1, 2, \cdots, m)$ are IT2FSs. The details of the MFs of OFV and RsV are presented in the Section 3 and [15], respectively. In the following, we propose three new methods for solving the IT2FLP problem with vagueness in the OFV and RsV. The first two methods are proposed on the idea of the Chandra and Aggarwal in solving the FLP problem [19].

5.1. **The first new method.** This method is based on Chandra and Aggarwal's idea for solving the IT2FLP problem with vagueness in the OFV and RsV and has two phases [19]. In the first phase, the MF corresponding to IT2F constraints is placed under the max-min operator and β is the maximum value of the satisfaction degree of IT2F constraints, which is calculated as follows:

$$\begin{array}{lll} \max & \beta & \\ s.t. & \beta \leq \mu((Ax)_i, u), & i = 1, \cdots, m, \\ & x_j \geq 0, \beta \in [0,1], & j = 1, \cdots, n, \end{array} \tag{5.2}$$

the problem (5.2) corresponds with the following problem:

$$\begin{array}{lll} \max & \beta & \\ s.t. & (Ax)_i \leq b_i + (1-\alpha)\Delta_i, & i = 1, \cdots, m, \\ & x_j \geq 0, \beta \in [0,1], & j = 1, \cdots, n, \end{array} \tag{5.3}$$

by substituting the intervals $\left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right]$ and $\left[\underline{\Delta}_i, \bar{\Delta}_i\right]$ with b_i and Δ_i, problem (5.3) turns into an ILP problem:

$$\begin{array}{lll} \max & \beta & \\ s.t. & (Ax)_i \leq \left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right] + (1-\beta)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], & i = 1, \cdots, m, \\ & x_j \geq 0, \beta \in [0,1], & j = 1, \cdots, n, \end{array} \tag{5.4}$$

the best solution for the problem (5.4) is acquired by solving the following LP problem:

$$\begin{array}{lll} \max & \beta & \\ s.t. & (Ax)_i + \underline{\Delta}_i \beta \leq \bar{b}_i^{\wedge}, & i = 1, \cdots, m, \\ & x_j \geq 0, \beta \in [0,1], & j = 1, \cdots, n, \end{array} \tag{5.5}$$

suppose that $(\bar{x}, \bar{\beta})$ is the optimal solution of the problem (5.5). Therefore, $\bar{x}$ is a $\bar{\beta}$ -feasible solution for the IT2FLP problem. $S(\bar{\beta})$ is the set of all $\bar{\beta}$ -feasible solutions of the main LP problem. Now, the problem (5.6) in the second phase is defined to obtain the maximum MF of the IT2F objective function on $S(\bar{\beta})$ feasible set as follows:

$$
\begin{array}{ll}
\max & \alpha \\
s.t. & cx \geq [\underline{z}_0^{\vee}, \bar{z}_0^{\wedge}] - (1-\alpha)\left[\underline{\Delta}_0, \bar{\Delta}_0\right] \\
& (Ax)_i \leq \left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right] + (1-\bar{\beta})\left[\underline{\Delta}_i, \bar{\Delta}_i\right], \quad i = 1, \cdots, m, \\
& x_j \geq 0, \ \alpha \in [0,1], \qquad\qquad j = 1, \cdots, n,
\end{array} \tag{5.6}
$$

the best solution of the problem (5.6) is obtained by solving the following problem:

$$
\begin{array}{ll}
\max & \alpha \\
s.t. & cx \geq \underline{z}_0^{\vee} - \bar{\Delta}_0 + \alpha\underline{\Delta}_0, \\
& (Ax)_i \leq \bar{b}_i^{\wedge} + \bar{\Delta}_i - \bar{\beta}\underline{\Delta}_i, \quad i = 1, \cdots, m, \\
& x_j \geq 0, \ \alpha \in [0,1], \qquad j = 1, \cdots, n,
\end{array} \tag{5.7}
$$

that $(\hat{x}, \hat{\alpha})$ is the optimal solution to the problem (5.7).

5.2. **The second new method.** In this method, a weight is assigned to each degree of membership of the objective function and constraints. Then, by using it, we write the new objective function and consider the constraints of this problem to be similar to the constraints of problem (5.7). Therefore, we obtained the following problem:

$$
\begin{array}{ll}
\max & \lambda\alpha + (1-\lambda)\beta \\
s.t. & cx \geq [\underline{z}_0^{\vee}, \bar{z}_0^{\wedge}] - (1-\alpha)\left[\underline{\Delta}_0, \bar{\Delta}_0\right] \\
& (Ax)_i \leq \left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right] + (1-\beta)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], \quad i = 1, \cdots, m, \\
& x_j \geq 0, \ \alpha, \beta, \lambda \in [0,1], \qquad\qquad j = 1, \cdots, n,
\end{array} \tag{5.8}
$$

where the best solution of the problem (5.8) is obtained by solving the following problem:

$$
\begin{array}{ll}
\max & \lambda\alpha + (1-\lambda)\beta \\
s.t. & cx \geq \underline{z}_0^{\vee} - \bar{\Delta}_0 + \alpha\underline{\Delta}_0, \\
& (Ax)_i \leq \bar{b}_i^{\wedge} + \bar{\Delta}_i - \beta\underline{\Delta}_i, \quad i = 1, \cdots, m, \\
& x_j \geq 0, \ \alpha, \beta, \lambda \in [0,1], \quad j = 1, \cdots, n.
\end{array} \tag{5.9}
$$

5.3. **The third new method.** The Section 3 related to MFs of OFV and Ref. [15] of the RsV are placed in the problem, and finally, the following problem with vagueness in OFV and RsV is obtained:

$$
\begin{array}{ll}
\max & \sum\limits_{j=1}^{n} (\underline{c}_j^{\vee} + \alpha(\underline{c}_j^{\wedge} - \underline{c}_j^{\vee}))x_j \\
s.t. & \sum\limits_{j=1}^{n} a_{ij}x_j \leq \bar{b}_i^{\wedge} - \alpha\underline{\Delta}_i, \quad i = 1, \cdots, m, \\
& x_j \geq 0, \ \alpha \in [0,1], \qquad j = 1, \cdots, n.
\end{array} \tag{5.10}
$$

The problem (5.10) is a nonlinear. The results are obtained by allocating various values in $\alpha \in [0,1]$.

Example 5.1. Consider the TCs and OFV data from Example 3.1, and the following RsV data for the IT2FLP problem with vagueness in the OFV and RsV:

$$\underline{b}^{\vee}=\begin{bmatrix}50\\70\\40\\60\\40\end{bmatrix},\ \underline{b}^{\wedge}=\begin{bmatrix}72\\104\\65\\95\\80\end{bmatrix},\ \bar{b}^{\wedge}=\begin{bmatrix}95\\110\\77\\102\\98\end{bmatrix},\ \bar{b}^{\vee}=\begin{bmatrix}60\\80\\55\\75\\57\end{bmatrix}.$$

Solving the Example 5.1 using the first new method: For the first phase, the following LP problem is obtained.

$$\begin{aligned}
&\max\ \ \beta\\
&s.t.\ \ \ 5x_1+3x_2+7x_3+10\beta\leq 95,\\
&\qquad\ \ 10x_1+4x_2+9x_3+10\beta\leq 110,\\
&\qquad\ \ 4x_1+6x_2+3x_3+15\beta\leq 77,\\
&\qquad\ \ 2x_1+7x_2+7x_3+15\beta\leq 102,\\
&\qquad\ \ 5x_1+6x_2+11x_3+17\beta\leq 98,\\
&\qquad\ \ \alpha\in[0,1]\,,\ x_j\geq 0,\ \ j=1,2,3,
\end{aligned}\tag{5.11}$$

by solving the problem (5.11) the value of $\bar{\beta}=1$ obtained. Now, we go to the second phase. The value $\bar{\beta}=1$ is placed in the problem (5.7), and we have

$$\begin{aligned}
&\max\ \ \alpha\\
&s.t.\ \ -12x_1-7x_2-9x_3+60.0749\alpha+119.7713\leq 0,\\
&\qquad\ 5x_1+3x_2+7x_3\leq 95,\\
&\qquad\ 10x_1+4x_2+9x_3\leq 110,\\
&\qquad\ 4x_1+6x_2+3x_3\leq 77,\\
&\qquad\ 2x_1+7x_2+7x_3\leq 102,\\
&\qquad\ 5x_1+6x_2+11x_3\leq 98,\\
&\qquad\ \alpha\in[0,1]\,,\ x_j\geq 0,\ \ j=1,2,3,
\end{aligned}\tag{5.12}$$

then $\hat{\alpha}=0.4782$ and $\hat{x}_j=(\hat{x}_1,\hat{x}_2,\hat{x}_3)=(8,7.5,0)$ are obtained as the best solutions of the problem (5.7). In the first phase, we calculate the maximum degree of satisfaction of the constraints once, and in the second step, we replace this value in the constraints. The problem in the second phase, which has the highest degree of satisfaction with the constraints, is solved by calculating the maximum degree of the constraints' satisfaction of the problem and constraint related to the objective function.

Solving the Example 5.1 using the second new method: Consider the data mentioned in the Example 5.1 at the problem (5.9), then we have

$$\begin{aligned}
&\max\ \ \lambda\alpha+(1-\lambda)\beta\\
&s.t.\ \ -12x_1-7x_2-9x_3+60.0749\alpha+119.7713\leq 0,\\
&\qquad\ 5x_1+3x_2+7x_3+10\beta\leq 105,\\
&\qquad\ 10x_1+4x_2+9x_3+10\beta\leq 120,\\
&\qquad\ 4x_1+6x_2+3x_3+15\beta\leq 92,\\
&\qquad\ 2x_1+7x_2+7x_3+15\beta\leq 117,\\
&\qquad\ 5x_1+6x_2+11x_3+17\beta\leq 115,\\
&\qquad\ \alpha,\beta\in[0,1]\,,\ x_j\geq 0,\ \ j=1,2,3.
\end{aligned}\tag{5.13}$$

The following results were obtained by assigning various weighting values to $\lambda\in[0,1]$ in the problem (5.13), (see the Table 4). It can be seen that for $\lambda=0.1$ until $\lambda=0.7$,

TABLE 4. Solution of the problem (5.13)

$\lambda \in [0,1]$	$\mathbf{x}_1^*$	$\mathbf{x}_2^*$	$\mathbf{x}_3^*$	α^*	β^*	z^*
0	0	0	0	0	1	1
0.1	8	7.5	0	0.4782	1	0.9478
0.2	8	7.5	0	0.4782	1	0.8956
0.3	8	7.5	0	0.4782	1	0.8435
0.4	8	7.5	0	0.4782	1	0.7913
0.5	8	7.5	0	0.4782	1	0.7391
0.6	8	7.5	0	0.4782	1	0.6869
0.7	8	7.5	0	0.4782	1	0.6348
0.8	8	10	0	0.7695	0	0.6156
0.9	8	10	0	0.7695	0	0.6926
1	8	10	0	0.7695	0	0.7695

we obtained $x_j^* = (x_1^*, x_2^*, x_3^*) = (8, 7.5, 0)$, $\alpha^* = 0.4782$, and $\beta^* = 1$. These values are the same as the values obtained in the example of the first new method.

Solving the Example 5.1 using the third new method:

Consider the data mentioned in the Example 5.1 for IT2FLP problem with vagueness in the OFV and RsV as follows:

$$\begin{array}{ll} \max & (8+7\alpha)x_1 + (13+9\alpha)x_2 + (5+8\alpha)x_3 \\ s.t. & 5x_1 + 3x_2 + 7x_3 + 10\alpha \le 95, \\ & 10x_1 + 4x_2 + 9x_3 + 10\alpha \le 110, \\ & 4x_1 + 6x_2 + 3x_3 + 15\alpha \le 77, \\ & 2x_1 + 7x_2 + 7x_3 + 15\alpha \le 102, \\ & 5x_1 + 6x_2 + 11x_3 + 17\alpha \le 98, \\ & \alpha \in [0,1],\ x_j \ge 0,\ \ j = 1,2,3, \end{array} \tag{5.14}$$

for $\alpha \in [0, 1]$, we solved problem (5.14) and the obtained results are shown in Table 5: The maximum objective function was obtained in $\alpha = 1$, as expected, given

TABLE 5. Solution of the problem (5.14)

$\alpha \in [0,1]$	$\mathbf{x}_1^*$	$\mathbf{x}_2^*$	$\mathbf{x}_3^*$	z^*
0	0	12.8333	0	166.8333
0.1	0	12.5833	0	174.9083
0.2	0	12.3333	0	182.5333
0.3	0	12.0833	0	189.7083
0.4	0	11.8333	0	196.4333
0.5	0	10.3333	2.5000	203.3333
0.6	0	10.0958	2.4750	210.0183
0.7	0	9.8583	2.4500	216.2358
0.8	0	9.6208	2.4250	221.9858
0.9	6.4826	5.4668	1.5897	227.4439
1	6.4774	5.0032	1.6903	229.2065

the objective function maximization and the constraints being less or equal. Also, the DM can choose one of the α values and the corresponding optimal value and solutions according to the existing conditions of the real problem.

6. The IT2FLP problem with vagueness in RsV and TCs

This subsection deals with the IT2FLP problem with the following vagueness constraints.

$$\begin{aligned} &\max \ \sum_{j=1}^{n} c_j x_j \\ &s.t. \ \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \leq \tilde{\tilde{b}}_i, \quad i = 1, \cdots, m, \\ &\qquad x_j \geq 0, \qquad\qquad j = 1, \cdots, n, \end{aligned} \tag{6.1}$$

where (for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$), $\tilde{\tilde{a}}_{ij}$ and $\tilde{\tilde{b}}_i$ are IT2FSs. By applying α-cut on IT2FSs and obtaining the left and right bounds, as expressed in ILP problems, the best objective function on the largest feasible area and the worst objective function value on the smallest feasible area are defined. To solve the problem (6.1), four ILP problems or equivalently eight LP problems are required.

Theorem 6.1 ([30])**.** *For each $i = 1, ..., m$, the interval inequality $\sum_{j=1}^{n} \left[\alpha_{\bar{\tilde{a}}^{\vee}_{ij}}, \alpha_{\underline{\tilde{\tilde{a}}}^{\vee}_{ij}}\right] x_j \leq \left[\alpha_{\bar{\tilde{b}}^{\vee}_{i}}, \alpha_{\bar{\tilde{b}}^{\wedge}_{i}}\right]$ and the interval inequality $\sum_{j=1}^{n} \left[\alpha_{\underline{\tilde{\tilde{a}}}^{\wedge}_{ij}}, \alpha_{\bar{\tilde{a}}^{\wedge}_{ij}}\right] x_j \leq \left[\alpha_{\underline{\tilde{\tilde{b}}}^{\vee}_{i}}, \alpha_{\underline{\tilde{\tilde{b}}}^{\wedge}_{i}}\right]$ are the biggest and smallest feasible areas, respectively.*

Theorem 6.2 ([30])**.** *For each $i = 1, \cdots, m$, the biggest and smallest feasible areas are equal to $\sum_{j=1}^{n} \alpha_{\underline{\tilde{\tilde{a}}}^{\wedge}_{ij}} x_j \leq \alpha_{\bar{\tilde{b}}^{\wedge}_{i}}$ and $\sum_{j=1}^{n} \alpha_{\bar{\tilde{a}}^{\wedge}_{ij}} x_j \leq \alpha_{\underline{\tilde{\tilde{b}}}^{\wedge}_{i}}$, respectively.*

Theorem 6.3 ([30])**.** *For each α-cut, the UMF of the objective function is obtained from solving the LP problem of the best objective function value z_b^{α} and the worst objective function value z_w^{α}:*

$$\begin{aligned} z_b^{\alpha} : &\max \ \sum_{j=1}^{n} c_j x_j \\ &s.t. \ \sum_{j=1}^{n} \alpha_{\bar{\tilde{a}}^{\vee}_{ij}} x_j \leq \alpha_{\bar{\tilde{b}}^{\wedge}_{i}}, \quad i = 1, \cdots, m, \\ &\qquad x_j \geq 0, \qquad\qquad j = 1, \cdots, n, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} z_w^{\alpha} : &\max \ \sum_{j=1}^{n} c_j x_j \\ &s.t. \ \sum_{j=1}^{n} \alpha_{\bar{\tilde{a}}^{\wedge}_{ij}} x_j \leq \alpha_{\underline{\tilde{\tilde{b}}}^{\vee}_{i}}, \quad i = 1, \cdots, m, \\ &\qquad x_j \geq 0, \qquad\qquad j = 1, \cdots, n. \end{aligned} \tag{6.3}$$

In the next subsection, a new solution method is suggested for the IT2FLP problem with vagueness in the TCs and RsV using the Farhadinia's method idea [20].

6.1. **The new solving method.** In this subsection, the asymmetrical problem (6.1) changes to the symmetrical problem using the max-min operator. In this method, using the idea of Farhadinia's method for solving FLP problems with vagueness in the RsV and TCs [20], we proposed a new method for solving IT2FLP problems with vagueness in the RsV and TCs. Farhadinia presented a formula for fuzzy constraints MF given the note in the fuzzy concept.

Note: An appropriate fuzzy MF is zero if the constraints are strongly rejected in the finite mode; otherwise, it is one. In addition, it should uniformly increase from zero to one.

According to Farhadinia's idea, the following modes hold for the constraints of the problem:

$$\begin{aligned}
&\sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j \leqslant b_i, && i=1,\cdots,m,\\
&\sum_{j=1}^{n}a_{ij}x_j-\Delta_i \leqslant b_i, && i=1,\cdots,m,\\
&\sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j-\Delta_i \leqslant b_i, && i=1,\cdots,m,\\
&\sum_{j=1}^{n}a_{ij}x_j \leqslant b_i, && i=1,\cdots,m.
\end{aligned}$$

For any $x=(x_j)_{1\times n}$, define the followings:

$$\begin{aligned}
b_i^{\max}(x) = \max\Bigg\{&\sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j, \sum_{j=1}^{n}a_{ij}x_j-\Delta_i\\
&, \sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j \ -\Delta_i, \sum_{j=1}^{n}a_{ij}x_j\Bigg\},\\
b_i^{\min}(x) = \min\Bigg\{&\sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j, \sum_{j=1}^{n}a_{ij}x_j-\Delta_i,\\
&, \sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j-\Delta_i, \sum_{j=1}^{n}a_{ij}x_j\Bigg\}.
\end{aligned}$$

Then considering the above note, the FS of the ith constraint is shown as below:

$$\mu_{\tilde{\tilde{G}}_i}(x)=\left\{\begin{array}{cc}
1, & b_i > b_i^{\max},\\
\in[0,1], & b_i^{\min}\leqslant b_i\leqslant b_i^{\max},\\
0, & b_i\leqslant b_i^{\min},
\end{array}\right.$$

for each $x\geqslant 0$

$$b_i^{\max}(x)=\sum_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j,$$

$$b_i^{\min}(x)=\sum_{j=1}^{n}a_{ij}x_j-\Delta_i.$$

Thus the MF of the problem's constraints can be expressed as below:

$$\mu_{\tilde{\tilde{G}}_i}(x)=\begin{cases} 1, & b_i \geq \sum\limits_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j, \\ \frac{b_i-\sum\limits_{j=1}^{n}a_{ij}x_j+\Delta_i}{\sum\limits_{j=1}^{n}\Delta_{ij}x_j+\Delta_i}, & \sum\limits_{j=1}^{n}a_{ij}x_j-\Delta_i \leq b_i \leq \sum\limits_{j=1}^{n}(a_{ij}+\Delta_{ij})x_j, \\ 0, & b_i \leq \sum\limits_{j=1}^{n}a_{ij}x_j-\Delta_i. \end{cases} \tag{6.4}$$

However, to symmetrical it, the problem is solved for $b_i^{\max}$ and $b_i^{\min}$, and two values are obtained for the objective function, which are displayed as z_b^{α} and z_w^{α}, respectively. Using max-min operator, we have

$$\begin{aligned} &\max \quad \alpha \\ &s.t. \quad \alpha(z_b^{\alpha}-z_w^{\alpha})-\sum_{j=1}^{n}c_jx_j+z_w^{\alpha}\leq 0, \\ &\qquad \sum_{j=1}^{n}(a_{ij}+\Delta_{ij}\alpha)x_j+\Delta_i\alpha-\Delta_i-b_i\leq 0, \quad i=1,\cdots,m, \\ &\qquad \alpha\in[0,1],\ x_j\geq 0, \qquad\qquad j=1,\cdots,n, \end{aligned} \tag{6.5}$$

since the TCs and RsV have upper and lower MFs, as mentioned in the Section 4 and Ref. [15], the mean upper and lower MFs for the TCs, $\tilde{\tilde{a}}_{ij}\in\left[\frac{\bar{a}_{ij}^{\vee}+\underline{a}_{ij}^{\vee}}{2},\frac{\underline{a}_{ij}^{\wedge}+\bar{a}_{ij}^{\wedge}}{2}\right]$ and $\Delta_{ij}\in\left[\underline{\Delta}_{ij},\bar{\Delta}_{ij}\right]$, for the RsV $\tilde{\tilde{b}}_i\in\left[\underline{b}_i^{\vee},\bar{b}_i^{\vee}\right]$ and $\Delta_i\in\left[\underline{\Delta}_i,\bar{\Delta}_i\right]$ are considered, respectively. Thus, the problem (6.6) changes to an interval nonlinear problem as follows:

$$\begin{aligned} &\max \quad \alpha \\ &s.t. \quad \alpha(z_b^{\alpha}-z_w^{\alpha})-\sum_{j=1}^{n}c_jx_j+z_w^{\alpha}\leq 0, \\ &\qquad \sum_{j=1}^{n}(\left[\frac{\bar{a}_{ij}^{\vee}+\underline{a}_{ij}^{\vee}}{2},\frac{\underline{a}_{ij}^{\wedge}+\bar{a}_{ij}^{\wedge}}{2}\right]+\left[\underline{\Delta}_{ij},\bar{\Delta}_{ij}\right]\alpha)x_j+\left[\underline{\Delta}_i,\bar{\Delta}_i\right]\alpha-\left[\underline{\Delta}_i,\bar{\Delta}_i\right] \\ &\qquad\qquad -\left[\underline{b}_i^{\vee},\bar{b}_i^{\vee}\right]\leq 0, \quad i=1,\cdots,m, \\ &\qquad \alpha\in[0,1],\ x_j\geq 0, \qquad\qquad j=1,\cdots,n, \end{aligned} \tag{6.6}$$

based on interval programming problem, the best problem of (6.6) given the maximization of objective function and equal or less constraints is as follows:

$$\begin{aligned} &\max \quad \alpha \\ &s.t. \quad \alpha(z_b^{\alpha}-z_w^{\alpha})-\sum_{j=1}^{n}c_jx_j+z_w^{\alpha}\leq 0, \\ &\qquad \sum_{j=1}^{n}(\frac{\bar{a}_{ij}^{\vee}+\underline{a}_{ij}^{\vee}}{2}+\underline{\Delta}_{ij}\alpha)x_j+\underline{\Delta}_i\alpha-\bar{\Delta}_i-\bar{b}_i^{\wedge}\leq 0, \quad i=1,\cdots,m, \\ &\qquad \alpha\in[0,1],\ x_j\geq 0, \qquad\qquad j=1,\cdots,n, \end{aligned} \tag{6.7}$$

the problem (6.7) is nonlinear and we can solve it using the α algorithm.

Example 6.4. Consider the TCs and the OFV data from Example 4.3, and RsV data from Example 5.1. At first, by using the problems (6.2) and (6.3), we have

$$\begin{aligned} z_b^\alpha : \max \quad & 12x_1 + 7x_2 + 9x_3 \\ s.t. \quad & 3x_1 + 2x_2 + 5x_3 \leqslant 50, \\ & 8x_1 + 2x_2 + 7x_3 \leqslant 70, \\ & 2x_1 + 4x_2 + 2x_3 \leqslant 40, \\ & x_1 + 5x_2 + 5x_3 \leqslant 60, \\ & 3x_1 + 4x_2 + 9x_3 \leqslant 40, \\ & x_j \geqslant 0, \;\; j = 1, 2, 3, \end{aligned}$$

and

$$\begin{aligned} z_w^\alpha : \max \quad & 12x_1 + 7x_2 + 9x_3 \\ s.t. \quad & 2x_1 + x_2 + 4x_3 \leqslant 95, \\ & 7x_1 + x_2 + 6x_3 \leqslant 110, \\ & x_1 + 3x_2 + x_3 \leqslant 77, \\ & 4x_2 + 4x_3 \leqslant 102, \\ & 2x_1 + 3x_2 + 8x_3 \leqslant 98, \\ & x_j \geqslant 0, \;\; j = 1, 2, 3, \end{aligned}$$

we calculate the values of $z_b^\alpha = 301.9500$ and $z_w^\alpha = 68.5714$, respectively. Then by putting data in problem (6.7), we obtain a nonlinear problem as follows:

$$(6.8) \quad \begin{aligned} \max \quad & \alpha \\ s.t. \quad & -12x_1 - 7x_2 - 9x_3 + 233.3786\alpha + 68.57140 \leq 0, \\ & (2.5+\alpha)x_1 + (1.5+\alpha)x_2 + (4.5+\alpha)x_3 \leq 95 + 10(1-\alpha), \\ & (7.5+\alpha)x_1 + (1.5+\alpha)x_2 + (6.5+\alpha)x_3 \leq 110 + 10(1-\alpha), \\ & (1.5+\alpha)x_1 + (3.5+\alpha)x_2 + (1.5+\alpha)x_3 \leq 77 + 15(1-\alpha), \\ & (0.5+\alpha)x_1 + (4.5+\alpha)x_2 + (4.5+\alpha)x_3 \leq 102 + 15(1-\alpha), \\ & (2.5+\alpha)x_1 + (3.5+\alpha)x_2 + (8.5+\alpha)x_3 \leq 98 + 17(1-\alpha), \\ & \alpha \in [0,1]\,, \; x_j \geq 0, \quad j = 1, 2, 3, \end{aligned}$$

since the problem (6.8) is a a nonlinear, we use an algorithm α to solve it. First, we put $\alpha = 1$ and solve problem using simplex. This is an infeasible problem, thus $\alpha^L = 0$, $\alpha^R = 1$, and based on the second step, $\alpha = \frac{\alpha^L + \alpha^R}{2} = 0.5$ is obtained. We put and solve $\alpha = 0.5$ value in the last problem. The above problem is feasible per $\alpha = 0.5$. Thus, $\alpha^L = 0.5$ and $\alpha^R = 1$, and thereby $\alpha = \frac{\alpha^L + \alpha^R}{2} = 0.75$. By inserting the value $\alpha = 0.75$ in the problem, it becomes infeasible. Then, $\alpha^L = 0.5$, $\alpha^R = 0.75$, and as a result $\alpha = \frac{\alpha^L + \alpha^R}{2} = 0.625$. By placing the value $\alpha = 0.625$ in the problem, it becomes feasible. Consequently, in the 13th iteration, the maximum constraints satisfaction value has occurred in $\alpha^* = 0.6589$. In addition, the optimal solutions are obtained as $(x_1^*, x_2^*, x_3^*) = (10.0578, 14.5216, 0)$, and from the problem (6.1) the optimal value is obtained as 222.3448.

7. The IT2FLP problem with vagueness in OFV, RsV and TCs

The IT2FLP problem with vagueness in OFV, RsV and TCs is as follows:

$$
\begin{aligned}
&\max \ \sum_{j=1}^{n} \tilde{\tilde{c}}_j x_j \\
&s.t. \quad \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \leq \tilde{\tilde{b}}_i, \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \qquad\quad j = 1, \cdots, n,
\end{aligned} \tag{7.1}
$$

where $\tilde{\tilde{c}}_j$, $\tilde{\tilde{a}}_{ij}$ and $\tilde{\tilde{b}}_i$ are IT2FSs. They represent the interval type-2 fuzzy of the OFV, TCs and RsV with an impression of vagueness type, respectively. Then the MF of such problems is introduced and a new solving method is suggested.

By applying α-cut on IT2FSs, the left and right bounds are obtained from the ILP problems. To solve problem (7.1), we need eight ILP equivalents for sixteen LP problems. Using the Theorem 7.1, we show that the MF of the problem (7.1) can be obtained only by solving two-LP problems related to each α-cut. Given the Theorem 6.1 and ILP problems, obtaining the objective function in the largest and smallest feasible area is required to calculate the best and worst objective function values. Therefore, instead of eight ILP problems related to each α-cut, the MF of the objective function can be obtained only by solving two ILP problems:

$$
\begin{aligned}
&\max \ \sum_{j=1}^{n} \left[\alpha_{\bar{\tilde{\tilde{c}}}_j^{\vee}}, \alpha_{\bar{\tilde{\tilde{c}}}_j^{\wedge}}\right] x_j \\
&s.t. \quad \sum_{j=1}^{n} \left[\alpha_{\underline{\tilde{\tilde{a}}}_{ij}^{\wedge}}, \alpha_{\underline{\tilde{\tilde{a}}}_{ij}^{\vee}}\right] x_j \leq \left[\alpha_{\bar{\tilde{\tilde{b}}}_i^{\vee}}, \alpha_{\bar{\tilde{\tilde{b}}}_i^{\wedge}}\right], \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \ \ j = 1, \cdots, n.
\end{aligned}
$$

and

$$
\begin{aligned}
&\max \ \sum_{j=1}^{n} \left[\alpha_{\underline{\tilde{\tilde{c}}}_j^{\wedge}}, \alpha_{\underline{\tilde{\tilde{c}}}_j^{\vee}}\right] x_j \\
&s.t. \quad \sum_{j=1}^{n} \left[\alpha_{\bar{\tilde{\tilde{a}}}_{ij}^{\vee}}, \alpha_{\bar{\tilde{\tilde{a}}}_{ij}^{\wedge}}\right] x_j \leq \left[\alpha_{\underline{\tilde{\tilde{b}}}_i^{\wedge}}, \alpha_{\underline{\tilde{\tilde{b}}}_i^{\vee}}\right], \quad i = 1, \cdots, m, \\
&\qquad\quad x_j \geq 0, \ \ j = 1, \cdots, n.
\end{aligned}
$$

Theorem 7.1 ([30])**.** *For each α-cut, the UMF of the objective function is obtained by solving two LP problems:*

$$
\begin{aligned}
&z_b^{\alpha} : \max \ \sum_{j=1}^{n} \alpha_{\bar{\tilde{\tilde{c}}}_j^{\wedge}} x_j \\
&\qquad s.t. \quad \sum_{j=1}^{n} \alpha_{\underline{\tilde{\tilde{a}}}_{ij}^{\wedge}} x_j \leq \alpha_{\bar{\tilde{\tilde{b}}}_i^{\wedge}}, \quad i = 1, \cdots, m, \\
&\qquad\qquad x_j \geq 0, \ \ j = 1, \cdots, n.
\end{aligned} \tag{7.2}
$$

and

$$
\begin{aligned}
&z_w^{\alpha} : \max \ \sum_{j=1}^{n} \alpha_{\underline{\tilde{\tilde{c}}}_j^{\wedge}} x_j \\
&\qquad s.t. \quad \sum_{j=1}^{n} \alpha_{\bar{\tilde{\tilde{a}}}_{ij}^{\wedge}} x_j \leq \alpha_{\underline{\tilde{\tilde{b}}}_i^{\wedge}}, \quad i = 1, \cdots, m, \\
&\qquad\qquad x_j \geq 0, \ \ j = 1, \cdots, n.
\end{aligned} \tag{7.3}
$$

which the problems (7.2) and (7.3) are the best and worst LP problems, respectively.

7.1. **The new solving method.** In this subsection, a new method for solving such problems is proposed by integrating the methods applied in the preceding sections. Based on the Section 6.1 for constraint MFs (the Eq. (6.4)), the materials related to Section 3 and Section 7, as well as all the rules used in interval programming, the following problem can be used to solve the IT2FLP problem with vagueness in OFV, TCs, and RsV:

$$
\begin{array}{lll}
\max & \sum_{j=1}^{n} (\underline{c}_i^{\vee} + \alpha(\underline{c}_i^{\wedge} - \underline{c}_i^{\vee}))x_j & \\
s.t. & \sum_{j=1}^{n} (\frac{\bar{a}_{ij}^{\vee}+\underline{a}_{ij}^{\vee}}{2} + \underline{\Delta}_{ij}\alpha)x_j + \underline{\Delta}_i\alpha - \bar{\Delta}_i - \bar{b}_i^{\wedge} \leq 0, & i = 1, \cdots, m, \\
& \alpha \in [0,1]\,,\; x_j \geq 0, & j = 1, \cdots, n,
\end{array}
\tag{7.4}
$$

the problem (7.4) is a nonlinear, and an arbitrary value to $\alpha \in [0,1]$ is assigned to solve it.

Example 7.2. For the IT2FLP problem with vagueness in OFV, TCs, and RsV, we consider the OFV data from Example 3.1, the TCs data from Example 4.3 and the RsV data from Example 5.1. By placing the above data in the non-LP problem (7.4), we have

$$
\begin{array}{ll}
\max & (8+7\alpha)x_1 + (13+9\alpha)x_2 + (5+8\alpha)x_3 \\
s.t. & (2.5+\alpha)x_1 + (1.5+\alpha)x_2 + (4.5+\alpha)x_3 \leqslant 95 + 10(1-\alpha), \\
& (7.5+\alpha)x_1 + (1.5+\alpha)x_2 + (6.5+\alpha)x_3 \leqslant 110 + 10(1-\alpha), \\
& (1.5+\alpha)x_1 + (3.5+\alpha)x_2 + (1.5+\alpha)x_3 \leqslant 77 + 15(1-\alpha), \\
& (0.5+\alpha)x_1 + (4.5+\alpha)x_2 + (4.5+\alpha)x_3 \leqslant 102 + 15(1-\alpha), \\
& (2.5+\alpha)x_1 + (3.5+\alpha)x_2 + (8.5+\alpha)x_3 \leqslant 98 + 15(1-\alpha), \\
& \alpha \in [0,1]\,,\; x_j \geqslant 0, \qquad j = 1,2,3,
\end{array}
\tag{7.5}
$$

by solving the problem (7.5), the results are showed in Table 6. Obviously, we

TABLE 6. Solution of the problem (7.5)

$\alpha \in [0,1]$	$\mathbf{x}_1^*$	$\mathbf{x}_2^*$	$\mathbf{x}_3^*$	z^*
0	11.7500	21.2500	0	370.2500
0.1	11.43556	16.1048	0	378.2734
0.2	11.1445	18.9336	0	384.9758
0.3	10.8750	17.8750	0	390.4750
0.4	10.6250	16.8750	0	394.8750
0.5	10.3929	15.9286	0	398.2679
0.6	10.1771	15.0313	0	400.7354
0.7	9.9764	14.1791	0	402.3507
0.8	9.7895	13.3684	0	403.1789
0.9	9.6154	12.5962	0	403.2788
1	9.4531	11.8594	0	402.7031

find that by increasing the α value, the obtained objective function values increase.

The advantages of this method include its simplicity and lack of computational complexity. Also, corresponding to each alpha value, the optimal value of the objective function and optimal solutions are obtained. In addition, the DM can choose one of the α values and the corresponding optimal solutions according to the existing conditions.

8. Conclusion

This study presents the basic concepts related to the IT2FLP problem with vagueness in OFV, TCs, RsV, or any possible combination. In these types of problems, the input data are modeled using fuzzy preference-based MFs. Some ideas have been used to solve FLP problems, and we have proposed new methods to solve IT2FLP problems. Among the possible modes created based on the position of vagueness in the problem, IT2FLP problem with vagueness in OFV, IT2FLP problem with vagueness in the TCs, IT2FLP problem with vagueness in OFV and RsV, IT2FLP problem with vagueness in OFV and TCs, and finally IT2FLP problem with vagueness in OFV, RsV, and TCs can be mentioned. For each problem mentioned above, we presented the new solution method(s), and some examples are provided for a better understanding. In summary, the benefits of this research can be described as follows:

- There are not enough studies on IT2FLP problems with vagueness in coefficients.
- Solving IT2FLP problems with uncertainty by using and expanding the ideas used to solve FLP problems.
- The proposed method(s) involves simple steps and does not require extensive computational complexity.
- We solved the problems for different α values ($\alpha \in [0, 1]$) in most of our proposed methods. One of the advantages of these methods is that they provide the DM with a set of optimal solutions and the optimal value of the objective function for different α values. Therefore, the DM can choose a solution according to the real conditions of the problem.
- Our proposed methods are flexible and interpretable because the Bellman-Zadeh operator is used to find a crisp solution to the IT2FL problem. Then they are appropriate for numerous similar problems.

In future work, other existing methods for solving FLP problems could be generalized to solve IT2FLP problems with the imprecision of vagueness type. Our proposed methods could be extended to various real-life decision-making problems such as [31], [32] and [33]. In addition, given that several types of cuts are available for alpha, they could be evaluated to solve these types of problems.

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