Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (x 2023), pp. 1–x
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Γ-BCH-algebras and their application to topology

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Received 21 January 2024; Revised 5 March 2024; Accepted 26 March 2024

Abstract. In this paper, we define Γ-BCH-algebras as a subclass of Γ-BCK-algebras and study their various properties. Next, we propose the notion of Γ-BCH-ideals and discuss some of its properties. Finally, we deal with some topological structures on Γ-BCH-algebras.

2020 AMS Classification: 06F35, 54B10, 54B05

Keywords: Γ-BCH-algebra, Medial Γ-BCH-algebra, Γ-BCH-ideal, Quotient Γ-BCH-algebra, Left mapping, Semitopological Γ-BCH-algebra.

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1. Introduction

The exploration and development of algebraic structures have been a cornerstone in the advancement of both theoretical and applied mathematics. The introduction of BCK-algebras by Iséki and Tanaka [1] in 1978 marked a significant milestone in this journey, providing a framework that has since been extended and refined through concepts like BCI-algebras (Iséki, [2]), BCC-algebras (Dudek, [3] and Thomys, [4]), QS-algebras (Ahn and Kim, [5]), Q-algebras (Neggers et al. [6]), BCH-algebras (Hu and Li [7]) and BE-algebras (Kim and Kim [8]). These developments not only enriched the algebraic theory but also paved the way for applying algebraic structures to diverse fields such as topology and group theory. In particular, Jansi and Thiruveni [9, 10] applied BCH-algebras to topology and topological group (See [11, 12, 13, 14, 15, 16, 17, 18] for further researches).

Classical algebraic structures with Γ concept is another interest for most of the researchers in algebra one of them is Γ-Semirings introduced by Rao [19] and further studied by Kaushik and Moin [20]. Similar motivation comes from classical to logical algebras to study different structures using Γ concept. For example, Saeid et al. [21] introduced the concept of Γ-BCK-algebras as a generalization of BCK-algebras and investigated some of its properties. Shi et al. [22] redefined a Γ-BCK-algebra proposed by Saeid et al. [21] and studied its various properties. After then, Ibedou et al. [23] studied topological structures on Γ-BCK-algebras. Shi et al. [24] proposed the notion of Γ-BCI-algebras as a generalization of BCI-algebras, and discussed some of its basic properties and some topological structures on Γ-BCI-algebras.

In this vein, our research aims to contribute to this evolving landscape by introducing the concept of Γ-BCH-algebras, a novel subclass within the realm of Γ-BCI-algebras. Our focus is not only on defining and elucidating the properties of Γ-BCH-algebras but also on exploring their application to topological structures. This dual emphasis on theoretical foundation and practical application reflects our broader objectives: to enrich the algebraic theory with new insights and to demonstrate the utility of these insights in understanding and solving complex problems in topology and beyond.

By defining Γ-BCH-ideals, the Γ-center, and the Γ-branch of a Γ-BCH-algebra, we aim to provide a comprehensive framework that extends the applicability of algebraic structures to topological concepts. Our investigation into the properties of quotient Γ-BCH-algebras and their topological properties is motivated by a desire to bridge the gap between abstract algebra and practical applications, fostering a deeper understanding of the underlying principles that govern both fields.

In summary, our research is driven by a commitment to advancing the frontiers of algebraic studies through the introduction of Γ-BCH-algebras and applying these structures within the domain of topology. Our goal is to provide a rich, theoretically sound foundation that not only adds to the algebraic discourse but also equips other researchers with new tools for exploring the interplay between algebra and topology, thus contributing to the broader scientific community's understanding of these fundamental concepts.

2. Preliminaries

We recall some definitions needed in next sections.

Definition 2.1 ([1, 2])**.** Let X be a nonempty set with a constant 0 and a binary operation $*$. Consider the following axioms: for any $x,\ y,\ z \in X$,

(A_1) $[(x*y)*(x*z)]*(z*y)=0$,
(A_2) $[x*(x*y)]*y=0$,
(A_3) $x*x=0$,
(A_4) $x*y=0$ and $y*x=0$ imply $x=y$,
(A_5) $0*x=0$,
(A_6) $(x*y)*z=(x*z)*y$.

Then X is called a:

(i) *BCI-algebra* [2], if it satisfies axioms (A_1)–(A_4),
(ii) *BCK-algebra* [1], if it satisfies axioms (A_1)–(A_5),
(iii) *BCH-algebra* [7], if it satisfies axioms (A_3), (A_4), (A_6).

It is well-known that the followings hold (See [7]):

The class of BCK-alebras ⊂ The class of BCI-alebras ⊂ The class of BCH-alebras.

In $BCK/BCI/BCH$-algebra X, we define a binary operation $\leq$ on X as follows: for any $x,\ y \in X$,

$$x \leq y \text{ if and only if } x*y=0.$$

Definition 2.2 ([7]). A BCH-algebra X is said to be *proper*, if it is not a BCI-algebra.

Definition 2.3 (See [7, 25]). A BCI/BCH-algebra X is said to be *associative*, if it satisfies the following condition:

$$(x * y) * z = x * (y * z) \ for \ any \ x, \ y, \ z \in X. \tag{2.1}$$

Definition 2.4 ([25]). A BCI/BCH-algebra X is said to be *medial*, if it satisfies the following condition:

$$(x * y) * (z * u) = (x * z) * (y * u) \ for \ any \ x, \ y, \ u, \ z \in X. \tag{2.2}$$

Definition 2.5 ([19]). Let X and Γ be two nonempty sets. Then X is called a Γ-*semigroup*, if there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$, such that it satisfies the following condition: for any $x, \ y, \ z \in X$ and any $\alpha, \ \beta \in \Gamma$,

$$x\alpha(y\beta z) = (x\alpha y)\beta z. \tag{2.3}$$

3. Basic properties of Γ-BCH-algebras

In this section, we introduce the notions of Γ-BCH-algebras and medial Γ-BCH-algebras, and study some of their properties.

Definition 3.1. Let X be a set with a constant 0 and let Γ be a nonempty set. Then X is called a Γ–*algebra*, if it is Γ-groupoid, i.e., there is a mapping $f : X \times \Gamma \times X \to X$, denoted by $f(x, \alpha, y) = x\alpha y$ for each $(x, \alpha, y) \in X \times \Gamma \times X$.

Definition 3.2. Let Γ-algebra X satisfy the following axioms: for any $x, \ y, \ z \in X$ and $\alpha, \ \beta \in \Gamma$,

(ΓA_1) $[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) = 0$,
(ΓA_2) $[x\alpha(x\beta y)]\alpha y = 0$,
(ΓA_3) if $x\alpha y = 0 = y\alpha x$, then $x = y$,
(ΓA_4) $x\alpha x = 0$,
(ΓA_5) $0\alpha x = 0$,
(ΓA_6) $(x\alpha y)\beta z = (x\alpha z)\beta y$.

Then X is called a:

(i) Γ-*BCK-algebra* [22], if it satisfies the axioms (ΓA_1)–(ΓA_5),
(ii) Γ-*BCI-algebra* [24], if it satisfies the axioms (ΓA_1)–(ΓA_4),
(iii) Γ-*BCH-algebra*, if it satisfies the axioms (ΓA_3), (ΓA_4), (ΓA_6).

For a Γ-$BCK/BCI/BCH$-algebra X and a fixed $\alpha \in \Gamma$, we define the operation $* : X \times X \to X$ as follows: for any $x, \ y \in X$,

$$x * y = x\alpha y.$$

Then it is clear $(X, *, 0)$ is a $BCK/BCI/BCH$-algebra and is denoted by X_α.

Example 3.3. (1) Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $X = \{0, 1, 2\}$ be a set with the ternary operation defined as the following table:
Then clearly, X is a Γ-BCH-algebra.

(2) Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3\}$ be a set with the ternary operation defined as the following table:
Then we can easily check that X is a Γ-BCH-algebra.

α	0	1	2
0	0	0	2
1	1	0	2
2	2	1	0

β	0	1	2
0	0	1	1
1	1	0	1
2	2	1	0

γ	0	1	2
0	0	2	2
1	1	0	2
2	2	2	0

Table 3.1

α	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

β	0	1	2	3
0	0	0	0	0
1	1	0	0	3
2	2	3	0	3
3	3	0	0	0

Table 3.2

(3) Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3\}$ be a set with the ternary operation defined as the following table:

α	0	1	2	3
0	0	0	0	3
1	1	0	1	3
2	2	2	0	3
3	3	3	3	0

β	0	1	2	3
0	0	0	0	3
1	1	0	3	2
2	2	3	0	3
3	3	3	3	0

Table 3.3

Then clearly, X is a Γ-BCH-algebra.

The followings are immediate consequences of Definition 3.2 (iii).

Lemma 3.4. *Let X be a Γ-BCH-algebra. Then the axiom* (ΓA_2) *holds.*

The following is an immediate consequence of Definition 3.2 and Lemma 3.4.

Corollary 3.5. *Every Γ-BCH-algebra satisfying the axiom* (ΓA_1) *is a Γ-BCI-algebra*

Lemma 3.6. *Let X be a Γ-BCH-algebra. Then the following condition hold:*

(3.1) $$\text{for each } x \in X \text{ and each } \alpha \in \Gamma,\ x\alpha 0 = 0 \text{ implies } x = 0.$$

Thus the following condition hold:

(3.2) $$x\alpha 0 = x \text{ for each } x \in X \text{ and each } \alpha \in \Gamma.$$

Proof. The proof is straightforward from Definition 3.2. □

The following is an immediate consequence of Definition 3.2 (ii) and (iii).

Proposition 3.7. *Every Γ-BCI-algebra is a Γ-BCH-algebra. But the converse is not true* (See Example 3.8).

Example 3.8. Let X be the Γ-BCH-algebra given in Example 3.3 (2). Then

$$[(2\alpha 3)\beta(2\alpha 1)]\beta(1\alpha 3) = (2\beta 0)\beta 3 = 2\beta 3 = 3 \neq 0.$$

Thus the axiom (ΓA_1) does not hold. So X is not a Γ-BCI-algebra

Proposition 3.9. *Let X be a Γ-BCH-algebra. Then the following identity:*

$$(xαy)βx = 0βy \text{ for any } x,\ y \in X \text{ and any } α,\ β \in \Gamma. \tag{3.3}$$

Proof. The proof follows from the axioms (ΓA_6) and (ΓA_4). □

The followings are immediate consequences of Proposition 3.9.

Corollary 3.10. *Let X be a Γ-BCH-algebra. Then the following identities:*

$$(0αx)β0 = 0βx,\ (xα0)βx = 0 \text{ for each } x \in X \text{ and any } α,\ β \in \Gamma. \tag{3.4}$$

Proposition 3.11. *Let X be a Γ-BCH-algebra. Then the following identity:*

$$0α(xβy) = (0αx)β(0αy) \text{ for any } x,\ y \in X \text{ and any } α,\ β \in \Gamma. \tag{3.5}$$

Proof. Let $x,\ y \in X$ and let $α,\ β \in \Gamma$. Then we have

$$\begin{aligned} 0α(xβy) &= [(0αy)β(xβy)]α(0αy) \text{ [By the axiom } (\Gamma A_6)] \\ &= [((xβy)αx)β(xβy)]β[(xβy)αx] \\ &= [(((xβy)α(xβy))αx]β[(xβx)αy] \text{ [By the condition (3.3)]} \\ &= (0αx)β(0αy). \text{ [By the axiom } (\Gamma A_4)] \end{aligned}$$

Thus the condition (3.4) holds. □

Proposition 3.12. *Let X be a Γ-BCH-algebra. Then the following identity:*

$$0α(0β(0αx)) = 0αx \text{ for each } x \in X \text{ and any } α,\ β \in \Gamma. \tag{3.6}$$

Proof. From the axiom (ΓA_6), we get

$$[0α(0β(0αx))]β(0αx) = 0. \tag{3.7}$$

On the other hand, we get

$$\begin{aligned} (0αx)β[0α(0β(0αx))] &= [0α(0β(0αx))]β[0α(0β(0αx))] \text{ [By (3.7)]} \\ &= 0. \text{ [By the axiom } (\Gamma A_4)] \end{aligned}$$

Thus by the axiom (ΓA_3), the identity (3.6) holds. □

We have a characterization of Γ-BCH-algebras.

Theorem 3.13. *Let X be a Γ-algebra. Then X is a Γ-BCH-algebra if and only if it satisfies the axioms (ΓA_3), (ΓA_4) and the following condition:*

$$[(xαy)βz]α[(xαz)βy] = 0 \text{ for any } x,\ y,\ z \in X \text{ and any } α,\ β \in \Gamma. \tag{3.8}$$

Proof. Suppose X is a Γ-BCH-algebra. Since the axioms (ΓA_3) and (ΓA_4) hold, it is sufficient to prove that the condition (3.8) holds. Let $x,\ y,\ z \in X$ and let $α,\ β \in γ$. Then by the axioms (ΓA_6) and (ΓA_4), we get

$$[(xαy)βz]α[(xαz)βy] = [(xαy)βz]α[(xαy)βz] = 0.$$

Thus the condition (3.8) holds.

Conversely, suppose the axioms (ΓA_3), (ΓA_4) and the condition (3.8) hold. □

In Γ-$BCK/BCI/BCH$-algebra X, we define a binary operation $\leq$ on X as follows: for any $x,\ y \in X$ and each $α \in \Gamma$,

$$x \leq y \text{ if and only if } xαy = 0.$$

Proposition 3.14. *Let X be a Γ-BCH-algebra. Then the followings hold: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) $x \leq y,\ y \leq x$ *imply* $x = y$,
(2) $x \leq x$.

Proof. The proofs are straightforward from Definition 3.2 (iii). □

Definition 3.15. A Γ-BCH-algebra X is said to be *proper*, if it is not a Γ-BCI-algebra.

Remark 3.16. From Example 3.8, we can easily see that there is a proper Γ-BCH-algebra.

The following provides criteria for determining whether Γ-BCH-algebra is proper or not.

Theorem 3.17. *A Γ-BCH-algebra X is proper if and only if the axioms* (ΓA_1) *does not hold.*

Proof. The proof is straightforward. □

Definition 3.18. A Γ-BCI/BCH-algebra X is said to be *associative*, if it is Γ-semigroup, i.e., the following condition holds:

(3.9) $$(x\alpha y)\beta z = x\alpha(y\beta z) \text{ for any } x,\ y,\ z \in X \text{ and any } \alpha,\ \beta \in \Gamma.$$

It is obvious that if X is an associative Γ-BCI/BCH-algebra, then for each $\alpha \in \Gamma$, X_α is an associative BCI/BCH-algebra.

Proposition 3.19. *Every associative Γ-BCH-algebra is an associative Γ-BCI-algebra.*

Proof. Let X be an associative Γ-BCH-algebra. From Lemma 3.4, it is sufficient to show that the axiom (ΓA_1) holds. Let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we get
$$\begin{aligned}[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) &= [(x\alpha y)\beta x]\alpha[(z\beta z)\alpha y] \text{ [By (3.5)]}\\ &= [(x\alpha x)\beta y]\alpha[(z\alpha z)\beta y] \text{ [By the axiom } (\Gamma A_6)]\\ &= (0\beta y)\alpha(0\beta y) \text{ [By the axiom } (\Gamma A_3)]\\ &= 0.\end{aligned}$$
Thus the axiom (ΓA_1) holds. So X is an associative Γ-BCI-algebra. □

The following is an immediate consequence of Definition 3.18 and Proposition 3.19.

Corollary 3.20. *If X is an associative Γ-BCH-algebra, then for each $\alpha \in \Gamma$, X_α is an associative BCH-algebra and thus an associative BCI-algebra.*

Definition 3.21. A Γ-BCI/BCH-algebra X is said to be *medial*, if it satisfies the following condition holds:

(3.10) $$(x\alpha y)\beta(z\alpha u) = (x\alpha z)\beta(y\alpha u) \text{ for any } x,\ y,\ u,\ z \in X \text{ and any } \alpha,\ \beta \in \Gamma.$$

It is clear that if X is a medial Γ-BCI/BCH-algebra, then for each $\alpha \in \Gamma$, X_α is a medial BCI/BCH-algebra.

Proposition 3.22. *Every medial Γ-BCH-algebra is a medial Γ-BCI-algebra.*

Proof. Let X be a medial Γ-BCH-algebra. It is sufficient to prove that the axiom (ΓA_1) holds. Let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we get

$$\begin{aligned}[(x\alpha y)\beta(x\alpha z)]\beta(z\alpha y) &= [(x\alpha y)\beta(z\alpha y)]\beta(x\alpha z) \text{ [By the axiom } (\Gamma A_6)] \\ &= [(x\alpha z)\beta(y\alpha y)]\beta(x\alpha z) \text{ [By (3.10)]} \\ &= [(x\alpha z)\beta 0]\beta(x\alpha z) \text{ [By the axiom } (\Gamma A_4)] \\ &= (x\alpha z)\beta(x\alpha z) \text{ [By Lemma 3.6]} \\ &= 0. \text{ [By the axiom } (\Gamma A_4)]\end{aligned}$$

Thus the axiom (ΓA_1) holds. So X is a medial Γ-BCI-algebra. □

The following is an immediate consequence of Definition 3.21 and Proposition 3.22.

Corollary 3.23. *If X is a medial Γ-BCH-algebra, then for each $\alpha \in \Gamma$, X_α is a medial BCH-algebra and thus a medial BCI-algebra.*

We give a characterization of medial Γ-BCH-algebras.

Theorem 3.24. *A Γ-BCH-algebra is medial if and only if it satisfies one of the following conditions: for any $x,\ y,\ z \in X$ and $\alpha,\ \beta \in \Gamma$,*

$$x\alpha y = 0\beta(y\alpha x), \tag{3.11}$$

$$x\alpha(y\beta z) = z\alpha(y\beta x) \tag{3.12}$$

$$x\alpha(x\beta y) = y, \tag{3.13}$$

$$0\alpha(0\beta y) = y. \tag{3.14}$$

Proof. Suppose X is a medial Γ-BCH-algebra, and let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by Lemma 3.6, the axiom (ΓA_4) and (3.10), we have

$$x\alpha y = (x\alpha y)\beta 0 = (x\alpha y)\beta(x\alpha x) = (x\alpha x)\beta(y\alpha x) = 0\beta(y\alpha x).$$

Thus the condition (3.11) holds.

Now suppose the condition (3.11) holds, let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \gamma$. Then by (3.11) and the axiom (ΓA_6), we get

$$(x\alpha y)\beta z = 0\beta[(y\alpha z)\beta x] = 0\beta[(y\alpha x)\beta z] = z\alpha(y\beta x).$$

Thus the condition (3.12) holds. It is clear that $[x\alpha(x\beta y)]\beta y = 0$. On the other hand, $y\beta[x\alpha(x\beta y)] = 0\beta[(x\alpha(x\beta y))\beta y] = 0$. By the axiom ($\Gamma A_3$), $x\alpha(x\beta y) = y$. So the condition (3.13) holds. the condition (3.12) follows from the condition (3.13).

Finally suppose the condition (3.14) holds, let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}x\alpha y &= 0\alpha[0\alpha(x\beta y)] \text{ [By the hypothesis]} \\ &= 0\alpha[(0\alpha x)\beta(0\alpha y)] \text{ [By (3.5)]} \\ &= 0\alpha[(0\alpha(0\alpha y))\beta x] \text{ [By the axiom } (\Gamma A_6)] \\ &= 0\alpha(y\beta x). \text{ [By the hypothesis]}\end{aligned}$$

Thus the condition (3.11) holds.

Conversely, suppose the necessary condition (3.12) holds, let $x,\ y,\ u,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$u\alpha[z\beta(x\alpha y)] = u\alpha[y\varepsilon(x\alpha z)] = (x\alpha z)\beta(y\alpha u).$$

Since $u\alpha[z\beta(x\alpha y)] = (x\alpha y)\beta(z\alpha u)$, $(x\alpha y)\beta(z\alpha u) = (x\alpha z)\beta(y\alpha u)$. Thus the condition (3.10) holds. So X is medial. □

4. Γ-BCH-ideals of Γ-BCH-algebras

In this section, we define a Γ-BCH-ideal, the Γ-center and the Γ-branch of a Γ-BCH-algebra, and deal with some of their properties

Definition 4.1 (See [24]). Let X be a Γ-BCI/BCH-algebra and let S be a nonempty subset of X. Then S is called a *Γ-subalgebra* of X, if S itself is a Γ-BCI/BCH-algebra.

It is obvious that X and $\{0\}$ are Γ-subalebras of X. In this case, X and $\{0\}$ will be called the *trivial Γ-subalgebras* of X. A nonempty subset S is called a *proper Γ-subalgebra* of X, if S is a Γ-subalgebra of X and $S \subsetneqq X$. It is clear that $\{0\}$ is a proper Γ-subalgebra of X.

From Definition 4.1, we obtain easily the following.

Theorem 4.2 (See Theorem 3.25, [24]). *Let X be a Γ-BCI/BCH-algebra and let S be a nonempty subset of X. Then S is a Γ-subalgebra of X if and only if $x\alpha y \in X$ for any $x,\ y \in S$ and each $\alpha \in \Gamma$.*

Definition 4.3 (See [24]). Let X be a Γ-BCI/BCH-algebra and let I be a nonempty subset of X. Then I is called a *Γ-BCH-ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$ and $\alpha \in \Gamma$,
(ΓI_1) $0 \in I$,
(ΓI_2) if $x\alpha y \in I$ and $y \in I$, then $x \in I$.
We will denote the set of all Γ-BCH-ideals of X by $\Gamma\mathcal{I}(X)$.

Example 4.4. (1) Let X be the Γ-BCH-algebra given in Example 3.3 (2). Then clearly, $\{0, 1\}$ is a Γ-subalgebra of X but $\{0, 1\} \notin \Gamma\mathcal{I}(X)$ since $2\alpha 1 \in \{0, 1\}$ and $1 \in \{0, 1\}$ but $2 \notin \{0, 1\}$.
(2) Let X be the Γ-BCI-algebra given in Example 3.3 (3). Then we can see that

$$\{0, 1\},\ \{0, 2\},\ \{0, 3\} \in \Gamma\mathcal{I}(X).$$

However, $\{0, 1, 2\} \notin \Gamma\mathcal{I}(X)$ because $3\beta 2 = 2 \in \{0, 1, 2\}$ and $2 \in \{0, 1, 2\}$ but $3 \notin \{0, 1, 2\}$.

Definition 4.5 (See [24]). Let X be a Γ-BCI/BCH-algebra X and let $I \in \Gamma\mathcal{I}(X)$. Then I is called a *closed Γ-BCH-ideal* of X, if $x \in I$ implies $0\alpha x \in I$ for each $\alpha \in \Gamma$.
We will denote the set of all closed Γ-ideals of X by $\Gamma\mathcal{I}_c(X)$.

Example 4.6. Let X be the Γ-BCH-algebra given in Example 3.3 (2). The we can check that $\{0, 1\} \in \Gamma\mathcal{I}_c(X)$ but $\{0, 2\} \notin \Gamma\mathcal{I}_c(X)$ because $2 \in \{0, 2\}$ and $\{0, 2\} \in \Gamma\mathcal{I}(X)$ but $0\beta 2 = 1 \notin \{0, 2\}$.

Proposition 4.7 (See Proposition 4.10, [24]). *Every closed Γ-BCH-ideal of a Γ-BCH-algebra X is a Γ-subalgebra of X. But the converse is not true.*

Proof. Let I be a closed Γ-BCH-ideal of X. Since $0 \in I$, $I \neq \varnothing$. Let $x,\ y \in I$ and let $\alpha \in \Gamma$. Then $(x\alpha y)\beta x = (x\alpha x)\beta y = 0\beta y \in I$. Since $I \in \Gamma\mathcal{I}(X)$, $x\alpha y \in I$. Thus I is a Γ-subalgebra of X.

Consider the Γ-BCH-algebra X given in Example 3.3 (2). Then $\{0, 3\}$ is a Γ-subalgebra of X but $\{0, 3\} \notin \Gamma\mathcal{I}_c(X)$. □

The following is a characterization of closed Γ-ideals.

Theorem 4.8 (See Theorem 4.11, [24])**.** *Let X be a Γ-BCH-algebra and let I be a subset of X. Then $I \in \Gamma\mathcal{I}_c(X)$ if and only if it satisfies the following conditions: for any $x,\ y,\ z \in X$ and each $\alpha \in \Gamma$,*
(1) $0 \in I$,
(2) $x\alpha z,\ y\alpha z,\ z \in I$ imply $x\alpha y \in I$.

Proof. The proof is similar to one of Theorem 4.11 in [24]. □

Definition 4.9. Let X be a Γ-BCH-algebra. Then the subset of X defined by:

$$\{x \in X : 0\alpha x = 0\ for\ each\ \alpha \in \Gamma\}$$

is called a *Γ-BCA-part* of X and denoted by $\Gamma\mathrm{BCA}(X)$.

If X is a Γ-BCK-algebra, then the subset of X is called a *Γ-BCK-part* of X and denoted by $\Gamma\mathrm{BCK}(X)$.

It is obvious that $\Gamma\mathrm{BCA}(X) \neq \varnothing$ and if X is a Γ-BCI-algebra, then $\Gamma\mathrm{BCA}(X) = \Gamma\mathrm{BCK}(X)$.

Remark 4.10. $\Gamma\mathrm{BCA}(X)$ is not necessarily a Γ-BCK-algebra (See Example 4.11).

Example 4.11. Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3, 4\}$ be a set with the ternary operation defined as the following table:

α	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

β	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	2	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Table 4.1

Then clearly, X is a Γ-BCH-algebra and $\Gamma\mathrm{BCA}(X) = \{0, 1, 2, 3\}$. On the other hand,

$$[(1\alpha 3)\beta(1\alpha 2)]\beta(2\alpha 3) = 1 \neq 0.$$

Thus the axiom ($\Gamma\mathrm{A}_1$) does not hold. So $\Gamma\mathrm{BCA}(X)$ is not a Γ-BCK-algebra.

Proposition 4.12. *Let X be a Γ-BCH-algebra X. Then $\Gamma BCA(X) \in \Gamma\mathcal{I}_c(X)$ and thus $\Gamma BCA(X)$ is a Γ-subalgebra of X. Furthermore, if $x \in \Gamma BCA(X)$ and $y \in \Gamma BCA(X)^c$, then $x\alpha y,\ y\alpha x \in \Gamma BCA(X)^c$ for each $\alpha \in \Gamma$.*

Proof. By the definition of $\Gamma\text{BCA}(X)$, $0 \in \Gamma\text{BCA}(X)$. Then the condition ($\Gamma\text{I}_1$) holds. Suppose $x\alpha y,\ y \in \Gamma\text{BCA}(X)$ for each $\alpha \in \Gamma$. By the definition of $\Gamma\text{BCA}(X)$, we have

$$0\beta(x\alpha y) = 0,\ 0\beta y = 0 \text{ for each } \beta \in \Gamma.$$

Then $(x\alpha y)\beta x = (x\alpha x)\beta y = 0\beta y = 0$. Thus, we get

$$0 = 0\beta(x\alpha y) = [(x\alpha y)\beta x]\beta(x\alpha y) = [(x\alpha y)\beta(x\alpha y)]\beta x = 0\beta x.$$

So $x \in \Gamma\text{BCA}(X)$, i.e., the condition ($\Gamma\text{I}_2$) holds. Hence $\Gamma\text{BCA}(X) \in \Gamma\mathcal{I}(X)$. Finally, let $x \in \Gamma\text{BCA}(X)$. Then clearly, $0\alpha x = 0$. Thus $0\beta(0\alpha x) = 0\beta 0 = 0$. So $0\alpha x \in \Gamma\text{BCA}(X)$. Hence $\Gamma\text{BCA}(X) \in \Gamma\mathcal{I}_c(X)$.

Now suppose $x \in \Gamma\text{BCA}(X)$ and $y \in \Gamma\text{BCA}(X)^c$. Assume that $x\alpha y \in \Gamma\text{BCA}(X)$ for each $\alpha \in \Gamma$. Since $\Gamma\text{BCA}(X) \in \Gamma\mathcal{I}_c(X)$, $(x\alpha y)\beta x = 0\beta y \in \Gamma\text{BCA}(X)$ for each $\beta \in \Gamma$. Thus $0\alpha(0\beta y) = 0$, i.e., $0 = [0\alpha(0\beta y)]\alpha y = 0\alpha y$. So $y \in \Gamma\text{BCA}(X)$. This is a contradiction. Hence $x\alpha y \notin \Gamma\text{BCA}(X)$ for each $\alpha \in \Gamma$. Similarly, $y\alpha x \notin \Gamma\text{BCA}(X)$ for each $\alpha \in \Gamma$. □

For a Γ-BCH-algebra X, the subset $\Gamma\text{Med}(X)$ of X defined by:

$$\Gamma Med(X) = \{x \in X : 0\alpha(0\beta x) = x \ for\ any\ \alpha,\ \beta \in \Gamma\}$$

is called the Γ-*medial part* of X. Each member of $\Gamma\text{Med}(X)$ is called a Γ-*medial element* of X. It is obvious that 0 is a Γ-medial element of X and then $\Gamma\text{Med}(X) \neq \varnothing$.

Definition 4.13. Let X be a Γ-BCH-algebra. Then $\Gamma\text{Med}(X)$ is called the Γ-center of X, if it is a medial Γ-subalgebra of X. In this case, we will denote $\Gamma\text{Med}(X)$ by ΓI_X.

It is obvious that ΓI_X is a Γ-subalgebra of X.

Example 4.14. Let X be the Γ-BCH-algebra given in Example 4.11. Then clearly, $\Gamma I_X = \{0, 4\}$. Moreover, we can confirm that ΓI_X is a Γ-subalebra of X.

Proposition 4.15. *Let X be a Γ-BCH-algebra. Then for each $x \in X$ and each $\alpha \in \Gamma$, there is a unique $x_0 \in \Gamma I_X$ such that $x_0\alpha x = 0$, i.e., $x_0 \leq x$.*

Proof. Let $x \in X$ and let $\alpha,\ \beta \in \Gamma$. It is clear that $[0\alpha(0\beta x)]\alpha x = 0$. Let $x_0 = 0\alpha(0\beta x)$. Then we have

$$0\beta[0\alpha(0\beta x)] = [(0\alpha(0\beta x))\alpha x]\beta[0\alpha(0\beta x)] = 0\beta x.$$

Thus $0\alpha[0\beta(0\alpha(0\beta x))] = 0\alpha(0\beta x) = x_0$. So $x_0 \in \Gamma I_X$ and $x_0 \leq x$.

Now suppose $y_0 \in \Gamma I_X$ such that $y_0 \leq x$, i.e., $y_0\alpha x = 0$ for each $\alpha \in \Gamma$. Then $0\beta y_0 = (y_0\alpha x)\beta y_0 = (y_0\alpha y_0)\beta x = 0\beta x$. Thus by the hypothesis, $0\alpha(0\beta x) = 0\alpha(0\beta y_0) = y_0$. So $y_0 = x_0$. Hence x_0 is unique. □

For each $x \in X$ and any $\alpha,\ \beta \in \Gamma$, the point $0\alpha(0\beta x) = x_0 \in \Gamma I_X$ is called the *medial* Γ-*point* or *central* Γ-*point* of x and will be denoted by $\Gamma med(x)$.

The following is an immediate consequence of Proposition 4.15.

Corollary 4.16. *Let X be a Γ-BCH-algebra. If $x,\ y \in X$ such that $x \leq y$ and $x_0 = \Gamma med(x),\ y_0 = \Gamma med(y)$, then $x_0 = y_0$.*

Remark 4.17. If $x_0 \in \Gamma I_X$ and $y \leq x_0$, then clearly, $y = x_0$ by Corollary 4.15. Thus each central Γ-point of a Γ-BCH-algebra X is also a minimal point. Moreover, we have the following identity:

$$z\alpha(z\beta x_0) = x_0 \text{ for each } z \in X \text{ and any } \alpha,\ \beta \in \Gamma. \tag{4.1}$$

Proposition 4.18. *Let X be a Γ-BCH-algebra. If for any $x,\ y \in X$ and each $\alpha \in \Gamma$, $x_0 = \Gamma med(x),\ y_0 = \Gamma med(y)$, then $\Gamma med(x\alpha y) = x_0\alpha y_0$, i.e., for any $\alpha,\ \beta \in \Gamma$,*

$$(x\alpha y)_0 = 0\alpha[0\beta(x\alpha y)] = [0\alpha(0\beta x)]\alpha[0\alpha(0\beta y)] = x_0\alpha y_0. \tag{4.2}$$

Proof. Suppose $x_0 = \Gamma med(x),\ y_0 = \Gamma med(y)$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then clearly, $x_0,\ y_0 \in \Gamma I_X$. Since ΓI_X is a subalgebra of X, $x_0\alpha y_0 \in \Gamma I_X$. Thus there is $z \in X$ such that $x_0\alpha y_0 = \Gamma med(z)$. It is sufficient to prove that $z = x\alpha y$, i.e., $x_0\alpha y_0 \leq x\alpha y$. Let $\beta \in \Gamma$. Then we get

$$\begin{aligned}
(x_0\alpha y_0)\beta(x\alpha y) &= [(0\alpha(0\beta x))\alpha(0\alpha(0\beta y))]\beta(x\alpha y) \text{ [Since } x_0,\ y_0 \in \Gamma I_X] \\
&= [(0\alpha(0\alpha(0\beta y)))\alpha(0\beta x)]\beta(x\alpha y) \text{ [By the axiom } (\Gamma A_6)] \\
&= [(0\beta y)\alpha(0\beta x)]\beta(x\alpha y) \text{ [By Proposition 3.12]} \\
&= [(0\beta(0\beta x))\alpha y]\beta(x\alpha y) \text{ [By the axiom } (\Gamma A_6)] \\
&= (x_0\alpha y)\beta(x\alpha y) \text{ [Since } x_0,\ y_0 \in \Gamma I_X] \\
&= [(x\alpha(x\beta x_0))\alpha y]\beta(x\alpha y) \text{ [By (4.1)]} \\
&= [(x\alpha y)\alpha(x\beta x_0)]\beta(x\alpha y) \\
&= [(x\alpha y)\alpha(x\alpha y)]\beta(x\beta x_0) \\
&= 0\beta(x\beta x_0) \\
&= (x\alpha x)\beta(x\beta x_0) \\
&= [x\alpha(x\beta x_0)]\beta x \\
&= x_0\beta x \\
&= 0.
\end{aligned}$$

Thus (4.2) holds. □

Definition 4.19. Let X be a Γ-BCH-algebra and let $x_0 \in \Gamma I_X$. Then the subset of X, denoted by $\Gamma B(x_0)$, defined by:

$$\Gamma B(x_0) = \{x \in X : x_0 \leq x\} = \{x \in X : x_0\alpha x = 0 \text{ for each } \alpha \in \Gamma\}$$

is called the Γ-*branch* of X determined by x_0.

Remark 4.20. From Remark 4.17, if $y \leq x_0$, then $y = x_0$. Thus we can consider x_0 as the starting point of $\Gamma B(x_0)$. Moreover, $x \in \Gamma B(x_0)$ if and only if $\Gamma med(x) = x_0$. So $x_0 \in \Gamma B(x_0)$ and hence $\Gamma B(x_0) \neq \varnothing$. Furthermore, we can see that $\Gamma B(0) = \Gamma BCA(X)$.

Example 4.21. Let X be the Γ-BCH-algebra given in Example 4.11. Then we can easily check that $\Gamma B(0) = \{0, 1, 2, 3, 4\} = \Gamma BCA(X)$. Moreover, $\Gamma B(4) = \{4\}$.

Proposition 4.22 (See Theorem 6, [25])**.** *Let X be a Γ-BCH-algebra and let $x,\ y \in X$. Then*

(1) $X = \bigcup_{x_0 \in \Gamma I_X} \Gamma B(x_0)$,

(2) $\Gamma B(x_0) \cap \Gamma B(y_0) = \varnothing$ *for any* $x_0,\ y_0 \in \Gamma I_X$,

(3) $x\alpha y,\ y\alpha x \in \Gamma BCA(X)$ *for each* $\alpha \in \Gamma$ *if and only if there is* $x_0 \in \Gamma I_X$ *such that* $x,\ y \in \Gamma B(x_0)$,

(4) *$x\alpha y,\ y\alpha x \in \Gamma BCA(X)^c$ for each $\alpha \in \Gamma$ if and only if $x \in \Gamma B(x_0),\ y \in \Gamma B(y_0)$ and $x_0 \neq y_0$,*

(5) *$x \in \Gamma B(x_0),\ y \in \Gamma B(y_0)$ imply $x\alpha y \in \Gamma B(x_0\alpha y_0)$ for each $\alpha \in \Gamma$.*

Proof. (1) It is clear that $\Gamma B(x_0) \subset X$ for each $x_0 \in \Gamma I_X$. Then $\bigcup_{x_0\in\Gamma I_X} \Gamma B(x_0) \subset X$. Now let $y \in X$. Then by Proposition 4.15, there is unique $y_0 = 0\alpha(0\beta y) \in \Gamma I_X$ such that $y_0 \leq y$ for any $\alpha,\ \beta \in \Gamma$. Thus $y \in \Gamma B(y_0) \subset \bigcup_{x_0\in\Gamma I_X} \Gamma B(x_0)$. So $X \subset \bigcup_{x_0\in\Gamma I_X} \Gamma B(x_0)$. Hence $X = \bigcup_{x_0\in\Gamma I_X} \Gamma B(x_0)$.

(2) Assume that $\Gamma B(x_0) \cap \Gamma B(y_0) \neq \varnothing$ for some $x_0,\ y_0 \in \Gamma I_X$, say $z \in \Gamma B(x_0) \cap \Gamma B(y_0)$. Then $z \in \Gamma B(x_0$ and $z \in \Gamma B(y_0$ such that $z \leq x_0$ and $z \leq y_0$. Thus $\Gamma med(z) = \{x_0, y_0\}$. This is a contradiction to Proposition 4.15. So $\Gamma B(x_0) \cap \Gamma B(y_0) = \varnothing$.

(3) ($\Rightarrow$): Suppose $x\alpha y,\ y\alpha x \in \Gamma BCA(X)$ for each $\alpha \in \Gamma$ and let $x \in \Gamma B(x_0)$, $y \in \Gamma B(y_0)$. Then $x_0 = \Gamma med(x),\ y_0 = \Gamma med(y)$. Thus by Proposition 4.18, we have

$$\Gamma med(x\alpha y) = \Gamma med(x)\alpha\Gamma med(y) = x_0\alpha y_0$$

and

$$\Gamma med(y\alpha x) = \Gamma med(y)\alpha\Gamma med(x) = y_0\alpha x_0.$$

Since $\Gamma B(0) = \Gamma BCA(X)$,, by the hypothesis, $x\alpha y,\ y\alpha x \in \Gamma B(0)$. So $\Gamma med(x\alpha y) = 0 = \Gamma med(y\alpha x)$. Since a medial Γ-point is unique, $x_0\alpha y_0 = y_0\alpha x_0$. Hence $x_0 = y_0$. Therefore $x,\ y \in \Gamma B(x_0)$ for some $x_0 \in \Gamma I_X$.

($\Leftarrow$): Conversely, suppose there is $x_0 \in \Gamma I_X$ such that $x,\ y \in \Gamma B(x_0)$. Then clearly, $x_0 \leq x,\ x_0 \leq y$, i.e., $x_0\alpha x = 0,\ x_0\alpha y = 0$ for each $\alpha \in \Gamma$. Thus we get: for each $\beta \in \Gamma$,

$$0\beta(x\alpha x_0) = (x\alpha x)\beta(x\alpha x_0) = [x\alpha(x\alpha x_0)]\beta x = x_0\beta x = 0.$$

Thus $x\alpha x_0 \in \Gamma BCA(X)$. Similarly, $y\alpha x_0 \in \Gamma BCA(X)$. On the other hand, we have

$$(x\alpha y)\beta(x\alpha x_0) = [x\alpha(x\alpha x_0)]\beta y = x_0\beta y = 0 \in \Gamma BCA(X).$$

Note that $\Gamma BCA(X)$ is a Γ-ideal of X by Proposition 4.12. Since $x\alpha x_0 \in \Gamma BCA(X)$, $x\alpha y \in \Gamma BCA(X)$. Similarly, $y\alpha x \in \Gamma BCA(X)$. So the sufficient condition holds.

(4) ($\Rightarrow$): Suppose $x\alpha y,\ y\alpha x \in \Gamma BCA(X)^c$ for each $\alpha \in \Gamma$. Assume that $x,\ y \in \Gamma B(x_0)$. Then by (3), $x\alpha y,\ y\alpha x \in \Gamma BCA(X)$. This is a contradiction. Thus the necessary conditions hold.

($\Leftarrow$): Suppose $x \in \Gamma B(x_0),\ y \in \Gamma B(y_0)$ and $x_0 \neq y_0$. Assume that $x\alpha y \in \Gamma BCA(X) = \Gamma B(0)$ for some $\alpha \in \Gamma$. Then by Proposition 4.18, $\Gamma med(x\alpha y) = x_0\alpha y_0$. Thus $x\alpha y \in \Gamma B(x_0\alpha y_0)$. Since $x\alpha y \in \Gamma B(0)$, $x_0\alpha y_0 = 0$. So $(x_0\alpha y_0)\beta x_0 = 0\beta x_0$, i.e., $0\beta y_0 = 0\beta x_0$ for some $\beta \in \Gamma$. I follows that $0\alpha(x_0\alpha y_0) = 0\alpha(0\beta x_0)$. Hence $x_0 = y_0$. This is a contradiction. Therefore $x\alpha y \in \Gamma BCA(X)^c$. Similarly, $y\alpha x \in \Gamma BCA(X)^c$.

(5) The proof is straightforward. □

From Theorem 4.22 (1) and (2), we can see that each Γ-BCH-algebra is a disjoint union of its Γ-branches determined by its medial Γ-points.

Theorem 4.23 (See Theorem 7, [25])**.** *Let X be a Γ-BCH-algebra and let $D \subset \Gamma I_X$. Then the followings are equivalent:*

(1) *$J = \bigcup_{d_0\in D} \Gamma B(d_0) \in \Gamma\mathcal{I}_c(X)$,*

(2) *D is a closed Γ-ideal in ΓI_X.*

Proof. (1) $\Rightarrow$ (2): Suppose $J = \bigcup_{d_0 \in D} \Gamma B(d_0) \in \Gamma\mathcal{I}(X)$. Since $J \neq \varnothing$, $\varnothing \neq D \subset \Gamma I_X$. Let $x_0 \in D$. Then $x_0 \in \Gamma B(x_0) \subset \bigcup_{d_0 \in D} \Gamma B(d_0) = J$. Since $J \in \Gamma\mathcal{I}_c(X)$, $0\alpha x_0 \in J$ for each $\alpha \in \Gamma$. Thus there is $d_{0,1} \in D$ such that $0\alpha x_0 \in \Gamma B(d_{0,1})$. So $\Gamma med(0\alpha x_0) = d_{0,1}$. Since $0\alpha x_0 \in \Gamma I_X$, $\Gamma med(0\alpha x_0) = 0\alpha x_0$. Hence $0\alpha x_0 = d_{0,1} \in D$.

Now suppose $y_0\alpha x_0,\ x_0 \in D$ for each $\alpha \in \Gamma$. Then $y_0\alpha x_0 \in \Gamma B(y_0\alpha x_0) \subset J$, $x_0 \in \Gamma B(x_0) \subset J$. Since $J \in \Gamma\mathcal{I}_c(X)$, $y_0 \in J$. Thus there is $d_{0,2} \in D \subset \Gamma I_X$ such that $y_0 \in \Gamma B(d_{0,2})$. So $d_{0,2} = \Gamma med(y_0) = y_0 \in D$. Hence D is a closed Γ-ideal in ΓI_X.

(2) $\Rightarrow$ (1): Conversely, suppose D is a closed Γ-ideal in ΓI_X. Then clearly, $D \neq \varnothing$. Thus $J \neq \varnothing$. Let $x \in J$. Then there is a unique $d_{0,3} \in D$ such that $x \in \Gamma B(d_{0,3})$. Thus by the hypothesis, $0\alpha d_{0,3} \in D$. Since $0 \in \Gamma B(0)$ and $x \in \Gamma B(d_{0,3})$, by Proposition 4.22 (5), $0\alpha x \in \Gamma B(0\alpha d_{0,3})$. So $0\alpha x \in \Gamma B(0\alpha d_{0,3}) \subset J$. Hence $0\alpha x \in J$.

Now suppose $y\alpha x,\ x \in J$ for each $\alpha \in \Gamma$. Then there are unique $d_{0,3},\ d_{0,4} \in D$ such that $y\alpha x \in \Gamma B(d_{0,3})$ and $x \in \Gamma B(d_{0,4})$. Let $\Gamma med(y) = y_0$. Then we have

$$d_{0,4} = \Gamma med(x) = x_0,\ d_{0,3} = \Gamma med(y\alpha x) = y_0\alpha x_0 = y_0\alpha d_{0,4}.$$

Thus $(y_0\alpha d_{0,4})\beta y_0 = d_{0,3}\beta y_0$ for each $\beta \in \Gamma$. Note that $0\beta d_{0,4} = d_{0,3}\beta y_0$. So $d_{0,3}\alpha(0\beta d_{0,4}) = 0\beta d_{0,3}\alpha(d_{0,3}\beta y_0) = y_0$. Since D is a Γ-ideal of X, D is a Γ-subalgebra of X. Since $0,\ d_{0,3},\ d_{0,4} \in D$, $y_0 = d_{0,3}\alpha(0\beta d_{0,4}) \in D$. Hence we have

$$y \in \Gamma B(y_0) = \Gamma B(d_{0,3}\alpha(0\beta d_{0,4})) \subset \bigcup_{d_0 \in D} \Gamma B(d_0) = J.$$

Therefore D is a closed Γ-ideal in X. □

5. Quotient Γ-BCH-algebras

Definition 5.1 (See [21])**.** Let $X,\ Y$ be two Γ-BCH-algebras. Then a mapping $f : X \to Y$ is called a Γ-*homomorphism*, if it satisfies the following condition:

$$f(x\alpha y) = f(x)\alpha f(y) \text{ for any } x,\ y \in X \text{ and each } \alpha \in \Gamma. \tag{5.1}$$

In particular, a Γ-homomorphism $f : X \to X$ is called a Γ-*endomorphism* on X. We will denote the set of all Γ-endomorphisms on a Γ-BCH-algebra X as $\Gamma End(X)$.

The subset of X [resp. Y], denoted by $\Gamma ker(f)$ [resp. $\Gamma Im(f)$], defined by:

$$\Gamma ker(f) = \{x \in X : f(x) = 0\} \text{ [resp. } \Gamma Im(f) = \{f(x) : x \in X\}]$$

is called the Γ-*kernel* [resp. Γ-*image*] of f.

Lemma 5.2. *Let X be a Γ-BCH-algebra and let $\varphi : X \to X$ be the mapping defined by: for each $x \in X$ and each $\alpha \in \Gamma$,*

$$\varphi(x) = 0\alpha x. \tag{5.2}$$

Then $\varphi \in \Gamma End(X)$.

Proof. Let $x,\ y \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by Proposition 3.11, we have

$$\varphi(x\alpha y) = 0\beta(x\alpha y) = (0\beta x)\alpha(0\beta x) = \varphi(x)\beta\varphi(y).$$

Thus φ is a Γ-enomorphism on X. So $\varphi \in \Gamma End(X)$. □

Proposition 5.3. *Let X be a Γ-BCH-algebra and let $f \in \Gamma End(X)$. Then*
(1) $f(0) = 0$,
(2) $f(0\alpha x) = 0\alpha f(x)$ *for each* $x \in X$ *and each* $\alpha \in \Gamma$,
(3) *if* $x\alpha y = 0$, *then* $f(x)\alpha f(y) = 0$ *for any* $x,\ y \in X$ *and each* $\alpha \in \Gamma$,
(4) *if* A *is a* Γ*-subalgebra of* X, *then so is* $f(A)$,
(5) *if* $I \in \Gamma\mathcal{I}(X)$, *then* $f(I) \in \Gamma\mathcal{I}(X)$,
(6) $\Gamma ker(f) \in \Gamma\mathcal{I}_c(X)$.

Proof. The proofs of (1)–(3) are straightforward from Definition 5.1.

(4) Suppose A is a Γ-subalgebra of X and let $x,\ y \in f(A)$, $\alpha \in \Gamma$. Then there are $a,\ b \in A$ such that $x = f(a)$ and $y = f(b)$. Thus $x\alpha y = f(a)\alpha f(b) = f(a\alpha b)$. Since A is a Γ-subalgebra of X, $a\alpha b \in A$, i.e., $f(a\alpha b) \in f(A)$. So $x\alpha y \in f(A)$. Hence $f(A)$ is Γ-subalgebra of X.

(5) Suppose $I \in \Gamma\mathcal{I}(X)$. Then clearly, $0 \in f(I)$. Now suppose $x\alpha y,\ y \in f(I)$ for each $\alpha \in \Gamma$. Then there are $a,\ b \in I$ such that $x = f(a)$ and $y = f(b)$. Thus $x\alpha y = f(a\alpha b)$. Since $a\alpha b \in I$, $b \in I$ and $I \in \Gamma\mathcal{I}(X)$, $a \in I$. So $x = f(a) \in f(I)$. Hence $f(I) \in \Gamma\mathcal{I}(X)$.

(6) From (1), it is clear that $0 \in \Gamma ker(f)$. Suppose $x\alpha y,\ y \in \Gamma ker(f)$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then $f(x\alpha y) = f(x)\alpha f(y) = 0$ and $f(y) = 0$. Thus by Lemma 3.6, $f(x) = 0$, i.e., $x \in \Gamma ker(f)$. So $\Gamma ker(f) \in \Gamma\mathcal{I}(X)$. Now let $x \in \Gamma ker(f)$. Then $f(x) = 0$. On the other hand, by (2), $f(0\alpha x) = 0\alpha f(x)$ for each $\alpha \in \Gamma$. Thus $f(0\alpha x) = 0$. So $0\alpha x \in \Gamma ker(f)$. Hence $\Gamma ker(f) \in \Gamma\mathcal{I}_c(X)$. □

From Lemma 5.2 and Proposition 5.3 (3), we have the following.

Corollary 5.4. *Let φ be the Γ-endomorphism on a Γ-BCH-algebra X given in Lemma 5.2. Then $\Gamma ker(\varphi) \in \Gamma\mathcal{I}_c(X)$.*

Lemma 5.5. *Let X be a Γ-BCH-algebra and let $\sim$ be the binary relation on X defined as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,*

$$x \sim y \text{ if and only if } x\alpha y,\ y\alpha x \in \Gamma ker(\varphi), \text{ i.e., } \varphi(x\alpha y) = \varphi(y\alpha x) = 0. \tag{5.3}$$

Then $\sim$ is a congruence relation on X. In this case, $\sim$ is called a Γ*-congruence relation on* X *determined by* $\Gamma ker(\varphi)$.

Proof. The proof is straightforward. □

For a congruence relation $\sim$ on a Γ-BCH-algebra X and each $x \in X$, a subset C_x of X defined by

$$C_x = \{y \in X : x \sim y\} = \{y \in X : \varphi(x) = \varphi(y)\}$$

is called the *congruence class* in X determined by x with respect to $\sim$. The set of all congruence classes in X is denoted by $X/\Gamma ker(\varphi)$ or $X/\sim$.

Proposition 5.6. *Let X be a Γ-BCH-algebra and let $\sim$ be a Γ-congruence relation on X determined by by $\Gamma ker(\varphi)$. We define a mapping $f : X/\sim \times\Gamma\times X/\sim \to X/\sim$ as follows: for each $(C_x, \alpha, C_y) \in X/\sim \times\Gamma\times X/\sim$,*

$$f(C_x, \alpha, C_y) = C_x\alpha C_y = C_{x\alpha y} = \{z \in X : \varphi(z) = \varphi(x\alpha y)\} = \{z \in X : 0\beta z = 0\beta(x\alpha y)\}.$$

Then $X/\sim$ is a Γ-BCH-algebra. In this case, $X/\sim$ is called the *quotient* Γ*-*BCH*-algebra of* X *by* $\Gamma ker(\varphi)$.

Proof. By the definition of φ and Corollary 5.4, it is obvious that f is well-defined and $C_0 = \Gamma ker(\varphi)$. Let $x \in X$ and let $\alpha \in \Gamma$. Then $C_x \alpha C_x = C_{x\alpha x} = C_0$. Thus the axiom ($\Gamma A_4$) holds.

Let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma A_6$), we have

$$(C_x \alpha C_y)\beta C_z = C_{(x\alpha y)\beta z} = C_{x\alpha z)\beta y)} = (C_x \alpha C_z)\beta C_y.$$

Thus the (ΓA_6) holds.

Finally, suppose $C_x \alpha C_y = C_0 = C_y \alpha C_x$ for any $x,\ y \in X$ and each $\alpha \in \Gamma$. Then $\varphi(x)\alpha\varphi(y) = \varphi(x\alpha y) = 0 = \varphi(y\alpha x) = \varphi(y)\alpha\varphi(x)$. Thus by the axiom ($\Gamma A_3$), $\varphi(x) = \varphi(y)$, i.e., $C_x = C_y$. So the axiom (ΓA_3) holds. Hence $X/\sim$ is a Γ-BCH-algebra. □

We define a partial ordering $\leq$ on $X/\sim$ as follows: for any $x,\ y \in X$ and each $\alpha \in \Gamma$,

$$C_x \leq C_y \text{ if and only if } C_x \alpha C_y = C_0 = \Gamma ker(\varphi).$$

Then we have similar consequences of Proposition 3.14.

Proposition 5.7. *Let X be a Γ-BCH-algebra and let $X/\sim$ be the quotient Γ-BCH-algebra of X by $\Gamma ker(\varphi)$. Then the followings hold: for any $x,\ y \in X$ and any $\alpha,\ \beta \in \Gamma$,*
(1) *$C_x \leq C_y,\ C_y \leq C_x$ imply $C_x = C_y$,*
(2) *$C_x \leq C_y$.*

Proposition 5.8. *If X is an associative Γ-BCH-algebra, then so is $X/\sim$.*

Proof. Suppose X is an associative Γ-BCH-algebra and let $x,\ y,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}(C_x \alpha C_y)\beta C_z &= C_{(x\alpha y)\beta z}\\ &= C_{x\alpha(y\beta z)} \text{ [By the hypothesis]}\\ &= C_x \alpha (C_y \beta C_z).\end{aligned}$$

Thus $X/\sim$ is an associative Γ-BCH-algebra. □

The following is an immediate consequence of Propositions 3.19 and 5.8.

Corollary 5.9. *If X is an associative Γ-BCH-algebra, then $X/\sim$ is an associative Γ-BCI-algebra.*

Proposition 5.10. *If X is a medial Γ-BCH-algebra, then so is $X/\sim$.*

Proof. Suppose X is a medial Γ-BCH-algebra and let $x,\ y,\ u,\ z \in X$ and let $\alpha,\ \beta \in \Gamma$. Then we have

$$\begin{aligned}(C_x \alpha C_y)\beta(C_z \alpha C_u) &= C_{(x\alpha y)\beta(z\alpha u)}\\ &= C_{(x\alpha z)\beta(y\alpha u)} \text{ [By the hypothesis]}\\ &= (C_x \alpha C_z)\beta(C_y \alpha C_u).\end{aligned}$$

Thus $X/\sim$ is a medial Γ-BCH-algebra. □

The following is an immediate consequence of Propositions 3.22 and 5.10.

Corollary 5.11. *If X is a medial Γ-BCH-algebra, then $X/\sim$ is a medial Γ-BCI-algebra.*

Definition 5.12. Let X, Y be two Γ-BCH-algebras. Then a mapping $f : X \to Y$ is called an *isomorphism*, if it is bijective and a homomorphism.

Two Γ-BCH-algebras are said to be *isomorphic*, denoted by $X \cong Y$, if there is an isomorphism $f : X \to Y$.

Let $\Gamma\mathcal{A}$ denote the class of all Γ-BCH-algebras.

Proposition 5.13. (1) $\cong$ *is an equivalence relation on* $\Gamma\mathcal{A}$*, i.e.,*

(*a*) $X \cong X$ *for each* Γ-BCH*-algebra* X*,*

(*b*) $X \cong Y$ *implies* $Y \cong X$ *for any* Γ-BCH*-algebras* X *and* Y*,*

(*c*) $X \cong Y$ *and* $Y \cong Z$ *imply* $X \cong Z$ *for any* Γ-BCH*-algebras* X*,* Y *and* Z*.*

(2) *Let* X*,* Y *be two* Γ-BCH*-algebras. If* X *is proper and* $X \cong Y$*, then* Y *is proper.*

Proof. The proof are straightforward. □

Let $\mathbf{\Gamma_{BCHA}}$ be the family of the class of all Γ-BCH-algebras and isomorphisms between them.

Remark 5.14. From Proposition 5.13 (1) and (2), we can easily see that the followings hold:

(1) $\mathbf{\Gamma_{BCHA}}$ forms a concrete category,

(2) $\mathbf{P\Gamma_{BCHA}}$ is a full subcategory of $\mathbf{\Gamma_{BCHA}}$, where $\mathbf{P\Gamma_{BCHA}}$ denotes the family of the class of all proper Γ-BCH-algebras and isomorphisms between them.

6. Topological structures on Γ-BCH-algebras

Definition 6.1 ([9])**.** Let $(X, *, 0)$ be a BCH-algebra and let τ be a topology on X. Then $(X, *, \tau)$ is called a *topological* BCH*-algebra* (briefly, $TBCH$-algebra), if $* : (X \times X, \tau \times \tau) \to (X, \tau)$ is continuous, i.e., for any $x,\ y \in X$ and each $W \in \tau$ with $x * y \in W$, there are $U, V \in \tau$ such that $x \in U,\ y \in V$ and $U * V \subset W$, where $U * V = \{x * y \in X : x \in U,\ y \in V\}$ (See [26]).

Definition 6.2. Let X be a Γ-BCH-algebra and let τ be a topology on X. Then (X, τ) is called a *semitopological* Γ-BCH*-algebra* (briefly, STΓ-BCH-algebra), if the mapping $f : (X, \tau) \times \Gamma \times (X, \tau) \to (X, \tau)$ is continuous at each $(x, \alpha, y) \in X \times \Gamma \times X$, i.e., for each $\alpha \in \Gamma$, any $x,\ y \in X$ and each $W \in \tau$ with $x\alpha y \in W$, there are $U,\ V \in \tau$ such that $x \in U,\ y \in V$ and $U\alpha V \subset W$, where $U\alpha V = \{x\alpha y : x \in U,\ y \in V\}$.

It is obvious that if X is a STΓ-BCH-algebra, then X_α is a $TBCH$-algebra for each $\alpha \in \Gamma$ in the sense of Definition 6.1.

Example 6.3. (1) Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3\}$ be the Γ-BCH-algebra having the the ternary operation defined as the following table:
Consider the topology τ on X defined by:

$$\tau = \{\varnothing, \{2\}, \{3\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, X\}.$$

Then we can easily confirm that (X, τ) is a STΓ-BCH-algebra. Moreover, X_α and X_β are $TBCH$-algebras.

(2) Let $X = \{0, 1, 2, 3\}$ be the Γ-BCH-algebra given in Example 3.3 (2). Let us consider the topology τ on X defined by:

$$\tau = \{\varnothing, \{2\}, \{2, 3\}, X\}.$$

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	0	2	1	1	0	0	2
2	2	2	0	0	2	2	0	0	0
3	3	3	3	0	3	3	3	3	0

Table 6.1

Let $W = \{2\} \in \tau$. Then clearly, $2\alpha 0 = 2 \in \{2\} = W$. Now let $U = \{2\},\ V = X \in \tau$. Then clearly, $2 \in U,\ 0 \in X$. But $U\alpha X = \{0, 2\} \not\subset W$. Thus (X, τ) is not a STΓ-BCH-algebra.

Definition 6.4 (See [27]). Let X be a STΓ-BCH-algebra and let $a \in X$. Then a mapping $l_a : X \to X$ defined as follows:

$$l_a(x) = a\alpha x \text{ for each } x \in X \text{ each } \alpha \in \Gamma$$

is called a *left mapping* on X. We will denote the set of all left mappings on X by $l(X)$.

Proposition 6.5. *Every left mapping on a STΓ-BCH-algebra X is continuous.*

Proof. Let $a,\ x \in X$, let $l_a : (X, \tau) \to (X, \tau)$ be a left mapping on X and let $W \in \tau$ such that $l_a(x) = a * x \in W$. Since X is STΓ-BCH-algebra, there are $U,\ V \in \tau$ such that $a \in U$, $x \in V$ and $U\alpha V \subset W$ for each $\alpha \in \Gamma$. Then clearly, $l_a(V) = a\alpha V \subset U\alpha V \subset W$. Thus l_a is continuous. □

Definition 6.6 (See [9]). Let X be a Γ-BCH-algebra and let $\alpha \in \Gamma$. Then the ternary operation $\overline{\alpha}$ on $l(X)$ as follows: for any $l_a,\ l_b \in l(X)$ and each $x \in X$,

$$(l_a \overline{\alpha} l_b)(x) = l_a(x)\alpha l_b(x), \text{ i.e., } (l_a \overline{\alpha} l_b)(x) = (a\alpha x)\alpha(b\alpha x).$$

Example 6.7. Let $\Gamma = \{\alpha, \beta\}$ and let $X = \{0, 1, 2, 3\}$ be the Γ-BCH-algebra having the the ternary operation defined as the following table:

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	1	0	0	0
2	2	2	0	0	2	2	2	0	0
3	3	3	3	0	3	3	3	3	0

Table 6.2

Then clearly, $l(X) = \{l_0, l_1, l_2, l_3\}$. Moreover, we can easily check that $l_1 \overline{\alpha} l_2 = l_0$ for $l_1,\ l_2 \in l(X)$.

Proposition 6.8. *If X is a Γ-BCH-algebra, then $l(X)$ is a Γ-BCH-algebra with thee zero element l_0.*

Proof. Let $l_a,\ l_b \in l(X)$, let $x \in X$ and let $\alpha \in \Gamma$. Then we have

$$\begin{aligned}(l_a \overline{\alpha} l_b)(x) &= (a\alpha x)\alpha(b\alpha x) \text{ [By Definition 6.6]} \\ &= (a\alpha b)\alpha z \text{ [Since } X \text{ is positive implicative]} \\ &= l_{a\alpha b}(x).\end{aligned}$$

Thus $l_a \overline{\alpha} l_b = l_{a\alpha b}$.

Now let $a,\ b,\ c \in X$ and let $\alpha,\ \beta \in \Gamma$. Then by the axiom ($\Gamma$A$_4$), we get

$$l_a \overline{\alpha} l_a = l_{a\alpha a} = l_0.$$

Thus $l(X)$ satisfies the axiom (ΓA$_4$). On the other hand, by the axiom (ΓA$_6$), we have

$$(l_a \overline{\alpha} l_b)\overline{\beta} l_c = l_{(a\alpha b)\beta c} = l_{(a\alpha c)\beta b} = (l_a \overline{\alpha} l_c)\overline{\beta} l_b.$$

So $l(X)$ satisfies the axiom (ΓA$_6$).

Suppose $l_a \overline{\alpha} l_b = l_0$ and $l_b \overline{\alpha} l_a = l_0$. Then $l_{a\alpha b} = l_0$ and $l_{b\alpha a} = l_0$. Thus $a\alpha b = 0 = b\alpha a$. By the axiom ($\Gamma$A$_3$), $a = b$. So $l_a = l_b$. Hence $l(X)$ satisfies the axiom (ΓA$_3$). Therefore $l(X)$ is a Γ-BCH-algebra. □

Definition 6.9 (See [9]). Let X be a Γ-BCH-algebra. Then X is called a *positive implicative* Γ-*BCH-algebra*, if it satisfies the following condition:

(6.1) $\quad (x\alpha z)\beta(y\alpha z) = (x\alpha y)\beta z$ for any $x,\ y,\ z \in X$ and any $\alpha,\ \beta \in \Gamma$.

Example 6.10. Let X be the Γ-BCH-algebra given in Example 6.7. Then we can easily see that X is a positive implicative Γ-BCH-algebra.

Definition 6.11. A Γ-algebra X is called a *positive implicative semitopological* Γ-*BCH-algebra* (briefly, positive implicative STΓ-BCH-algebra), if it is a positive implicative Γ-BCH-algebra and a STΓ-BCH-algebra.

Definition 6.12. Let X be a Γ-BCH-algebra. Then the mapping $\Phi : X \to l(X)$ defined by: for each $x \in X$,

$$\Phi(x) = l_x$$

is called the *natural mapping on* X.

Example 6.13. Let X be the Γ-BCH-algebra given in Example 6.10 and consider the topology τ on X defined by:

$$\tau = \{\varnothing, \{0,1\}, \{0,1,2\}, \{0,1,3\}, X\}.$$

Then we can easily check that X is a positive implicative STΓ-BCH-algebra.

For a Γ-BCH-algebra X and any subset A of X, the subset l_A of $l(X)$ is defined as follows:

$$l_A = \{l_a \in l(X) : a \in A\}.$$

It is clear that $\Phi(A) = l_A$.

The following is an immediate consequence of Definition 6.12.

Lemma 6.14. *Let X be a Γ-BCH-algebra and let $A,\ B \subset X$. Then the followings hold:*

(1) $\Phi(A) \subset \Phi(B)$, *i.e.,* $l_A \subset l_B$,
(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$, *i.e.,* $l_{A\cup B} = l_A \cup l_B$,
(3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, *i.e.,* $l_{A\cap B} = l_A \cap l_B$,
(4) $\Phi(A\alpha B) = \Phi(A)\overline{\alpha}\Phi(B)$, *i.e.,* $l_{A\alpha B} = l_A \overline{\alpha} l_B$ *for each* $\alpha \in \Gamma$.

Proposition 6.15. *If X is a positive implicative Γ-BCH-algebra, then the natural mapping Φ is a Γ-isomorphism of Γ-BCH-algebras.*

Proof. Suppose X is a positive implicative Γ-BCH-algebra. Then clearly, by Proposition 6.9, $l(X)$ is a Γ-BCH-algebras with the zero element l_0. Let $a,\ b,\ xinX$ and let $\alpha \in \Gamma$. Then we have

$$\begin{aligned}\Phi(a\alpha b)(x) &= l_{(a\alpha b)}(x)\\ &= (a\alpha b)\alpha x \text{ [By the definition of the left mapping]}\\ &= (a\alpha x)\alpha(b\alpha x) \text{ [By Definition 6.8]}\\ &= l_a\overline{\alpha}l_b\\ &= \Phi(a)\overline{\alpha}\Phi(b).\end{aligned}$$

Thus Φ is a Γ-homomorphism of Γ-BCH-algebras. It is obvious that Φ is bijective. So Φ is a Γ-isomorphism. □

Proposition 6.16. *If (X,τ) is a positive implicative $ST\Gamma$-BCH-algebra, then*

$$\tau_{\Phi} = \{\Phi(U) \subset l(X) : U \in \tau\} = \{l_U \subset l(X) : U \in \tau\}$$

is a topology on $l(X)$.

In this case, τ_{Φ} is called a *left Γ-topology* on $l(X)$.

Proof. It is clear that $l(X),\ \varnothing \in \tau_{\Phi}$. Suppose $A,\ B \in \tau_{\Phi}$. Then there are $U,\ V \in \tau$ such that $A = \Phi(U),\ B = \Phi(V)$. Thus by Lemma 6.14 (3), $A \cap B = \Phi(U \cap V)$ and $U \cap V \in \tau$. So $A \cap B \in \tau_{\Phi}$. Now let $(A_j)_{j\in J}$ be any family of members of τ_{Φ}, where J is an index set. Then for each $j \in J$, there is $U_j \in \tau$ such that $A_j = \Phi(U_j)$. Thus $\bigcup_{j\in J} U_j \in \tau$ and $\bigcup_{j\in J} A_j = \bigcup_{j\in J}\Phi(U_j)$ by Lemma 6.14 (2). So $\bigcup_{j\in J} A_j \in \tau_{\Phi}$. Hence τ_{Φ} is a topology on $l(X)$. □

Example 6.17. Let $\Gamma = \{\alpha,\beta\}$ and let $X = \{0,1,2,3\}$ be the positive implicative Γ-BCH-algebra having the the ternary operation defined as the following table:

α	0	1	2	3	β	0	1	2	3
0	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	1	0	0	1
2	2	2	0	2	2	2	2	0	0
3	3	3	3	0	3	3	3	3	0

Table 6.3

Consider the topology $\tau = \{\varnothing, \{2\}, \{3\}, \{0,1\}, \{2,3\}, \{0,1,2\}, \{0,1,3\}, X\}$ on X. Note that $l(X) = \{l_0, l_1, l_2, l_3\}$. Then the left Γ-topology τ_{Φ} on $l(X)$ as follows:

$$\tau_{\Phi} = \{\varnothing, \{l_2\}, \{l_3\}, \{l_0,l_1\}, \{l_2,l_3\}, \{l_0,l_1,l_2\}, \{l_0,l_1,l_3\}, l(X)\}.$$

Proposition 6.18. *If (X,τ) is a positive implicative $ST\Gamma$-BCH-algebra, then $(l(X),\tau_{\Phi})$ is a $ST\Gamma$-BCH-algebra.*

In this case, $(l(X),\tau_{\Phi})$ is called a *left semitopological Γ-BCH-algebra* (briefly, LSTΓ-BCH-algebra.

Proof. Suppose (X,τ) is a positive implicative STΓ-BCH-algebra and let $a,\ b \in X$. Then clearly, $\Phi(a) = l_a,\ \Phi(b) = l_b \in l(X)$. Let $\alpha \in \Gamma$ and let $W' \in \tau_{\Phi}$ such that $l_a\overline{\alpha}l_b \in W'$. Then there is $W \in \tau$ such that $W' = \Phi(W)$. Since X is a STΓ-BCH-algebra, there are there are $U,\ V \in \tau$ such that $a \in U,\ b \in V$ and $U\alpha V \subset W$. Thus $\Phi(U\alpha V) = \Phi(U)\overline{\alpha}\Phi(V) subset \Phi(W) = W'$. Moreover, $\Phi(U),\ \Phi(V) \in \tau_{\Phi}$

and $l_a \in \Phi(U)$, $l_b \in \Phi(V)$. So $\overline{\alpha}$ is continuous. Hence $(l(X), \tau_{_\Phi})$ is a STΓ-BCH-algebra. □

Proposition 6.19. *Let (X, τ) be a positive implicative STΓ-BCH-algebra. If $\{0\} \in \tau$, then $(l(X), \tau_{_\Phi})$ is a discrete space.*

Proof. Suppose $\{0\} \in \tau$, let $l_x \in l(X)$ and let $\alpha \in \Gamma$. Then clearly, $l_x \overline{\alpha} l_x = l_0$. Since $x\alpha x = 0$, $\{0\} \in \tau$ and X is a positive implicative STΓ-BCH-algebra, there are $U,\ V \in \tau$ such that $x \in U \cap V$ and $U\alpha V \subset \{0\}$. Thus $\Phi(U),\ \Phi(V) \in \tau_{_\Phi}$, $l_x \in \Phi(U) \cap \Phi(V)$ and $\Phi(U \cap V) = \Phi(U) \cap \Phi(V) \subset \Phi(\{0\}) = \Phi(\{l_0\})$. Now let $W = U \cap V$. Then clearly, $\Phi(W\alpha W) = \Phi(\{l_0\})$. Thus $\Phi(W) = \Phi(\{l_x\})$. Since $W \in \tau$, $\Phi(W) \in \tau_{_\Phi}$. So $(l(X), \tau_{_\Phi})$ is a discrete space. □

Theorem 6.20. *Let (X, τ) be a positive implicative STΓ-BCH-algebra. Then $\{0\}$ is closed in X if and only if X is Hausdorff.*

Proof. Suppose $\{0\}$ is closed in X, let $a \neq b \in X$ and let $\alpha \in \Gamma$. Then clearly, either $a\alpha b \neq 0$ or $b\alpha b \neq 0$, say $a\alpha b \neq 0$. Since X is a STΓ-BCH-algebra and $\{0\}^c \in \tau$, $a\alpha b \in \{0\}^c$, there are U , $V \in \tau$ such that $a \in U$, $b \in V$ and $U\alpha V \subset \{0\}^c$. Thus $U \cap V = \varnothing$. So X is Hausdorff.

Conversely, suppose X is Hausdorff and let $a \in \{0\}^c$. Then clearly, $a \neq 0$. Thus by the hypothesis, there are $U,\ V \in \tau$ such that $a \in U_a$, $0 \in V_a$ and $U_a \cap V_a = \varnothing$, i.e., $0 \notin U_a$. So $U_a \subset \{0\}^c$, i.e., $\{0\}^c = \bigcup_{a \in \{0\}^c} U_a \tau$. Hence $\{0\}$ is closed in X. □

The following is an immediate consequence of Lemma 6.14 (3).

Theorem 6.21. *Let (X, τ) be a positive implicative STΓ-BCH-algebra. Then X is Hausdorff if and only if $(l(X), \tau_{_\Phi})$ is Hausdorff.*

We obtain the following from Theorems 6.20 and 6.21.

Corollary 6.22. *Let (X, τ) be a positive implicative STΓ-BCH-algebra. Then $\{0\}$ is closed in X if and only if $(l(X), \tau_{_\Phi})$ is Hausdorff.*

7. Conclusions

We have established foundational properties of Γ-BCH-algebras, including the introduction and examination of concepts such as Γ-ideals, the Γ-center, and the Γ-branch, alongside exploring several attributes of quotient Γ-BCH-algebras. Furthermore, we have ventured into analyzing some topological characteristics inherent to Γ-BCH-algebras.

Moving forward, our objective will be to advance our study of Γ-BCH-algebras within the framework of topological groups, aiming to enrich and refine our understanding of their structural and topological nuances.

Acknowledgements. We wish to thank the reviewers for kind and excellent suggestions that have been incorporated into the paper.

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