Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On primal ideal approximation spaces

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Received 24 January 2024; Revised 6 March 2024; Accepted 7 April 2024

Abstract. In this paper, we introduce "primal ideal" to generate new closure operators and, thus, new spaces named primal ideal approximation spaces. Some topological notions such as accumulation points, lower separation axioms, and connectedness in these primal ideal approximation spaces are defined and studied. Some examples are given to confirm the implications. Improving the accuracy measure and reducing the boundary region can be achieved easily by utilizing primal ideal in the construction of the approximations as it plays an important role in removing the vagueness of concept. The emergence of primal ideals leads to increase the lower approximations and decrease the upper approximations. Consequently, it minimizes the boundary and makes the accuracy higher than the previous based on ideals. Finally, a real life application induced from an information system is introduced to demonstrate the importance of this paper.

2020 AMS Classification: 54A05, 54B99, 54A10, 03E20

Keywords: Primal, Ideal, Primal ideal, Primal ideal separation axioms, Primal ideal connectedness..

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1. Introduction

Rough set [1, 2] is a nonstatistical technique to deal with the problems of incompleteness of knowledge and uncertainty in data. The rationale for this set is based on the fact that human knowledge is divided into three basic regions: inside, outside, and boundary. Therefore, the main idea of this set is to concentrate on the lower and upper approximations that are used to determine the boundary region and accuracy measure. Approximations in the standard rough set model are based on equivalence relations, however this requirement is not always true in many practical problems, and this restriction limits the set's vast uses.

An ideal is a nonempty collection of sets which is closed under hereditary property and the finite additivity [3]. It is a completely new technique to modelling vagueness and uncertainty by shrinking the boundary region and boosting the set's precision. Novel rough models called "ideal approximation spaces" have been introduced. In fact, these models enlarge the lower approximation and shrink the upper approximation of subsets, which means they increase their accuracy values. Some researchers followed this course of study and addressed some phenomena as presented in [4, 5, 6, 7]. Moreover, extensions of topology have been applied to provide new rough paradigms using certain topological structures and concepts like infra-topology, supra-topology, maximal and minimal neighbourhoods to deal with rough set notions and address some real-life problems [8, 9]. Moreover, many authors [10, 11] studied some topological notions in ideal approximation spaces such as closure spaces, separation axioms, continuity, connectedness, etc.

Recently, authors in [12] introduced a new structure called "primal". They obtain not only some fundamental properties related to primal but also some relationships between topological spaces and primal topological spaces. Moreover, [13, 14, 15] present types of operators have been defined by using the notion of primal with deep studies of their various properties.

In this paper, we combine ideal and primal structures to introduce a novel structure named "primal ideal". We introduce the interior and closure in primal ideal approximation spaces, generating primal ideal approximation topological spaces based on primal ideals. The relevant properties and results of these spaces are instituted. Moreover, accumulation points, separation axioms, and connectedness with respect to these primal ideal approximation spaces are reformulated and compared to the corresponding definitions given in [16] with examples to show their implications. Furthermore, we show that our primal ideal approximation space defined in Definition 3.12 produce better approximations and higher accuracy values than their counterparts ideal approximation spaces introduced in [1, 17, 18, 19, 20]. Finally, a real life application is provided to demonstrate the significance of adopting primal ideals in current techniques. In the proposed application, we illustrated that our technique in Definition 3.12 reduce boundary regions and improve the accuracy measure of the sets more than approach displayed in [20], which illustrate the importance of utilizing primal ideals in decision making problems.

2. Preliminaries

A *relation* R from a universe X to a universe X (a relation on X) is a subset of $X \times X$. The formula $(x, y) \in R$ is abbreviated as xRy and means that x is in relation R with y. Also, the *afterset* of $x \in X$ is $xR = \{y : xRy\}$ and the *forerset* of $x \in X$ is $Rx = \{y : yRx\}$.

Definition 2.1 ([18])**.** Let R be any binary relation on X. Then the set $< x > R$ is the intersection of all aftersets containing x, i.e.,

$$< x > R = \begin{cases} \cap_{x \in yR}(yR) & \text{if } \exists y : x \in yR, \\ \phi & \text{otherwise.} \end{cases}$$

Also, $R < x >$ is the intersection of all foresets containing x, i.e.,

$$R < x >= \begin{cases} \cap_{x \in yR}(Ry) & \text{if } \exists y : x \in Ry, \\ \phi & \text{otherwise.} \end{cases}$$

Definition 2.2 ([18]). Let R be binary relation on X. For any subset A of X, the *lower approximation* $L_R(A)$ and the *upper approximation* $U_R(A)$ of A are defined by:

$$L_R(A) = \{x \in A \ :< x > R \subseteq A\}, \tag{2.1}$$

$$U_R(A) = A \cup \{x \in X \ :< x > R \cap A \neq \phi\}. \tag{2.2}$$

Theorem 2.3 ([21]). *The upper approximation defined by Eq.* (2.2) *has the following properties:*

(1) $U_R(\phi) = \phi$,
(2) $L_R(A) \subseteq A \subseteq U_R(A)$, *for* $A \subseteq X$,
(3) $U_R(A \cup B) = U_R(A) \cup U_R(B)$, $\forall A, B \subseteq X$,
(4) $U_R(U_R(A)) = U_R(A)$, $\forall A \subseteq X$,
(5) $U_R(A) = (L_R(A^c))^c$, $\forall A \subseteq X$, *where* A^c *denotes the complement of* A.

Also, the operator U_R on $P(X)$ defined by Eq. (2.2) induced a topology on X denoted by τ_R and defined by $\tau_R = \{A \subseteq X : U_R(A^c) = A^c\}$.

Definition 2.4 ([3]). Let X be a non-empty set. Then $\mathcal{L} \subseteq P(X)$ is called an *ideal* on X, if it satisfies the following conditions:
(i) $\phi \in \mathcal{L}$,
(ii) if $A \in \mathcal{L}$ and $B \subseteq A$, then $B \in \mathcal{L}$,
(iii) If $A, B \in \mathcal{L}$, then $A \cup B \in \mathcal{L}$.

Definition 2.5 ([19]). Let R be a binary relation on X and $\mathcal{L}$ be an ideal defined on X and $A \subseteq X$. Then the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\overline{R}(A)$ of A by $\mathcal{L}$ are defined respectively by:

$$\underline{R}(A) = \{x \in A \ :< x > R \cap A^c \in \mathcal{L}\}, \tag{2.3}$$

$$\overline{R}(A) = A \cup \{x \in X \ :< x > R \cap A \notin \mathcal{L}\}. \tag{2.4}$$

Theorem 2.6 ([19]). *The upper approximation defined by Eq.* (2.4) *has the following properties: for* $A,\ B \subseteq X$,

(1) $\overline{R}(A) = (\underline{R}(A^c))^c$,
(2) $\overline{R}(\phi) = \phi$,
(3) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$,
(4) *if* $A \subseteq B$, *then* $\overline{R}(A) \subseteq \overline{R}(B)$,
(5) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,
(6) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,
(7) $\overline{R}(\overline{R}(A)) = \overline{R}(A)$.

Also, the operator $\overline{R}$ on $P(X)$ defined by (2.4) induced a topology on X denoted by τ_R^* and defined as $\tau_R^* = \{A \subseteq X : \overline{R}(A^c) = A^c\}$.

Definition 2.7 ([20]). Let R be a binary relation on X and $\mathcal{L}$ be an ideal on X and $A \subseteq X$. Then the *lower approximation* $\underline{\underline{R}}(A)$ and the *upper approximation* $\overline{\overline{R}}(A)$ of A by $\mathcal{L}$ are defined respectively by:

$$\underline{\underline{R}}(A) = \{x \in A : R < x > R \cap A^c \in \mathcal{L}\}, \tag{2.5}$$

$$\overline{\overline{R}}(A) = A \cup \{x \in X : R < x > R \cap A \notin \mathcal{L}\}, \tag{2.6}$$

where

$$R < x > R = R < x > \cap < x > R. \tag{2.7}$$

Theorem 2.8 ([20])**.** *The upper approximation defined by Eq.* (2.6) *has the following properties: for* $A, B \subseteq X$,

(1) $\overline{\overline{R}}(A) = (\underline{\underline{R}}(A^c))^c$,
(2) $\overline{\overline{R}}(\phi) = \phi$,
(3) $L_R(A) \subseteq \underline{R}(A) \subseteq \underline{\underline{R}}(A) \subseteq A \subseteq \overline{\overline{R}}(A) \subseteq \overline{R}(A) \subseteq U_R(A)$,
(4) *if* $A \subseteq B$, *then* $\overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B)$,
(5) $\overline{\overline{R}}(A \cap B) \subseteq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$,
(6) $\overline{\overline{R}}(A \cup B) = \overline{\overline{R}}(A) \cup \overline{\overline{R}}(B)$,
(7) $\overline{\overline{R}}(\overline{\overline{R}}(A)) = \overline{R}(A)$.

Also, the operator $\overline{\overline{R}}$ on $P(X)$ defined by (2.6) induced a topology on X denoted by τ_R^{**} and defined as $\tau_R^{**} = \{A \subseteq X : \overline{\overline{R}}(A^c) = A^c\}$. It is clear that $\tau_R \subseteq \tau_R^* \subseteq \tau_R^{**}$.

Definition 2.9 ([19])**.** Let R be a binary relation on X and $\mathcal{L}$ be an ideal defined on X and $A \subseteq X$. Then the *lower* and *upper approximations of A by $\mathcal{L}$*, denoted by $\underline{R}(A)$ and $\overline{R}(A)$, of A are defined as follows:

$$\underline{R}(A) = \{x \in A \ : \langle x \rangle\, R \cap A^c \in \mathcal{L}\}, \tag{2.8}$$

$$\overline{R}(A) = A \cup \{x \in X \ : \langle x \rangle\, R \cap A \notin \mathcal{L}\}. \tag{2.9}$$

The operator $\overline{R} : P(X) \longrightarrow P(X)$ defined by Eq. (2.9) induced a topology on X denoted by τ_R^* and defined as $\tau_R^* = \{A \subseteq X : \overline{R}(A^c) = A^c\}$.

Definition 2.10 ([20])**.** Let R be a binary relation on X and $\mathcal{L}$ be an ideal defined on X and $A \subseteq X$. Then the *lower* and *upper approximations*, $\underline{\underline{R}}(A)$ and $\overline{\overline{R}}(A)$, the *boundary region* and the accuracy measure of A are defined respectively by:

$$\underline{\underline{R}}(A) = \{x \in A \ : R < x > R \cap A^c \in \mathcal{L}\}, \tag{2.10}$$

$$\overline{\overline{R}}(A) = A \cup \{x \in X \ : R\,\langle x \rangle\, R \cap A \notin \mathcal{L}\}, \tag{2.11}$$

$$BND(A) = \overline{\overline{R}}(A) - \underline{\underline{R}}(A), \quad ACC(A) = \frac{|\underline{\underline{R}}(A)|}{|\overline{\overline{R}}(A)|}, \ |\overline{\overline{R}}(A)| \neq 0, \tag{2.12}$$

where $R\,\langle x \rangle\, R = R\,\langle x \rangle \cap \langle x \rangle\, R$ and $0 \leq ACC(A) \leq 1$.

The operator $\overline{\overline{R}} : P(X) \longrightarrow P(X)$ defined by Eq. (2.11) induced a topology on X denoted by τ_R^{**} and defined as $\tau_R^{**} = \{A \subseteq X : \overline{\overline{R}}(A^c) = A^c\}$.

Definition 2.11 ([18])**.** Let R be a binary relation on X. Then a point $x \in X$ is called an *accumulation point* of A, if $(\langle x \rangle\, R - \{x\}) \cap A \neq \phi$. The set of all accumulation points of A is denoted by $d(A)$, i.e.,

$$d(A) = \{x \in X : (< x > R - \{x\}) \cap A \neq \phi\}.$$

Definition 2.12 ([16])**.** Let $(X, R, \mathcal{L})$ be an ideal approximation space and $A \subseteq X$. Then a point $x \in X$ is said to be:

(i) $*$-*ideal accumulation point* of A, if $(\langle x\rangle R-\{x\})\cap A\notin\mathcal{L}$,
(ii) $**$-*ideal accumulation point* of A, if $(R\langle x\rangle R-\{x\})\cap A\notin\mathcal{L}$.
The set of all $*$-ideal accumulation points of A is denoted by $d^*(A)$, i.e.,

$$d^*(A)=\{x\in X:(\langle x\rangle R-\{x\})\cap A\notin\mathcal{L}\}.$$

The set of all $**$-ideal accumulation points of A is denoted by $d^{**}(A)$, i.e.,

$$d^{**}(A)=\{x\in X:(R\langle x\rangle R-\{x\})\cap A\notin\mathcal{L}\}.$$

Definition 2.13 ([16])**.** An ideal approximation space $(X,R,\mathcal{L})$ is called a:
(i) T_0^*-*space*, if $\forall x\neq y\in X$, there exists $A\subseteq X$ such that

$$(x\in\underline{R}(A),y\notin A)\ \textit{or}\ (y\in\underline{R}(A),x\notin A),$$

(ii) T_0^{**}-*space*, if $\forall x\neq y\in X$, there exists $A\subseteq X$ such that

$$(x\in\underline{\underline{R}}(A),y\notin A)\ \textit{or}\ (y\in\underline{\underline{R}}(A),x\notin A),$$

(iii) T_1^*-*space*, if $\forall x\neq y\in X$, there exist $A,\ B\subseteq X$ such that

$$(x\in\underline{R}(A),y\notin A)\ \textit{and}\ (y\in\underline{R}(B),x\notin B),$$

(iv) T_1^{**}-*space*, if $\forall x\neq y\in X$, there exist $A,\ B\subseteq X$ such that

$$(x\in\underline{\underline{R}}(A),y\notin A)\ \textit{and}\ (y\in\underline{\underline{R}}(B),x\notin B),$$

(v) An ideal approximation space $(X,R,\mathcal{L})$ is called a T_2^*-*space*, if $\forall x\neq y\in X$, there exist $A,\ B\subseteq X$ such that

$$x\in\underline{R}(A),\ y\in\underline{R}(B)\text{ and }A\cap B=\phi,$$

(vi) T_2^{**}-*space*, if $\forall x\neq y\in X$, there exist $A,\ B\subseteq X$ such that

$$x\in\underline{\underline{R}}(A),\ y\in\underline{\underline{R}}(B)\text{ and }A\cap B=\phi.$$

Definition 2.14 ([16])**.** Let $(X,R,\mathcal{L})$ be an ideal approximation space.
(i) $A,\ B\subseteq X$ are called $*$-*separated* (resp. $**$-*separated*) *sets*, if $\overline{R}(A)\cap B=A\cap\overline{R}(B)=\phi$ (resp. $\overline{\overline{R}}(A)\cap B=A\cap\overline{\overline{R}}(B)=\phi$).
(ii) $Y\subseteq X$ is called a $*$-*disconnected* (resp. $**$-*disconnected*) *set*, if there exists $*$-separated (resp. $**$-separated) sets $A,\ B\subseteq X$ such that $Y\subseteq A\cup B$.
(iii) $Y\subseteq X$ is said to be $*$-*connected* (resp. $**$-*connected*), if it is not $*$-disconnected (resp. $**$-disconnected).

Definition 2.15 ([22])**.** Let X be a nonempty set. A collection $\mathcal{F}\subseteq P(X)$ is called a *filter* on X, if it satisfies the following conditions:
(i) $\phi\notin\mathcal{F}$,
(ii) if $A\in\mathcal{F}$ and $A\subseteq B$, then $B\in\mathcal{F}$,
(iii) if $A,B\in\mathcal{F}$, then $A\cap B\in\mathcal{F}$.

Definition 2.16 ([23])**.** Let X be a nonempty set. A collection $\mathcal{G}\subseteq P(X)$ is called a *grill* on X, if it satisfies the following conditions:
(i) $\phi\notin\mathcal{G}$,
(ii) if $A\in\mathcal{G}$ and $A\subseteq B$, then $B\in\mathcal{G}$,
(iii) if $A\cup B\in\mathcal{G}$, then $A\in\mathcal{G}$ or $B\in\mathcal{G}$.

Definition 2.17 ([12])**.** Let X be a nonempty set. A collection $\mathcal{P} \subseteq P(X)$ is called a *primal* on X, if it satisfies the following conditions:
(i) $X \notin \mathcal{P}$,
(ii) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$,
(iii) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

Corollary 2.18 ([12])**.** *Let X be a nonempty set. A collection $\mathcal{P} \subseteq P(X)$ is a primal on X if and only if it satisfies the following conditions:*
(1) *$X \notin \mathcal{P}$,*
(2) *if $B \notin \mathcal{P}$ and $B \subseteq A$, then $A \notin \mathcal{P}$,*
(3) *if $A \notin \mathcal{P}$ and $B \notin \mathcal{P}$, then $A \cap B \notin \mathcal{P}$.*

3. Approximation spaces based on primal ideals

In this section of the manuscript, we define the notion of "primal ideal" and display the properties of the produced primal ideal approximation spaces.

Definition 3.1. Let $\mathcal{L} \neq P(X)$ be an ideal on X. An extension of $\mathcal{L}$ denoted by $\mathcal{L}^{\mathcal{P}} \subseteq P(X)$ is called a *primal ideal* on X, if it satisfies the following conditions:
(i) $X \notin \mathcal{L}^{\mathcal{P}}$,
(ii) if $A \in \mathcal{L}^{\mathcal{P}}$ and $B \subseteq A$, then $B \in \mathcal{L}^{\mathcal{P}}$,
(iii) if $A \cap B \in \mathcal{L}^{\mathcal{P}}$, then $A \in \mathcal{L}^{\mathcal{P}}$ or $B \in \mathcal{L}^{\mathcal{P}}$,
(iv) if $A, B \in \mathcal{L}^{\mathcal{P}}$, then $A \cup B \in \mathcal{L}^{\mathcal{P}}$.

Corollary 3.2. *Let X be a nonempty set and $\mathcal{L} \neq P(X)$ be an ideal on X. Then $\mathcal{L}^{\mathcal{P}}$ is a primal ideal on X if and only if it satisfies the following conditions:*
(1) *$X \notin \mathcal{L}^{\mathcal{P}}$,*
(2) *if $B \notin \mathcal{L}^{\mathcal{P}}$ and $B \subseteq A$, then $A \notin \mathcal{L}^{\mathcal{P}}$,*
(3) *if $A \notin \mathcal{L}^{\mathcal{P}}$ and $B \notin \mathcal{L}^{\mathcal{P}}$, then $A \cap B \notin \mathcal{L}^{\mathcal{P}}$,*
(4) *if $A \cup B \notin \mathcal{L}^{\mathcal{P}}$, then $A \notin \mathcal{L}^{\mathcal{P}}$ or $B \notin \mathcal{L}^{\mathcal{P}}$.*

Corollary 3.3. *Let X be a nonempty set and $\mathcal{L}^{\mathcal{P}}$ be a primal ideal on X. Then*
(1) *$\mathcal{L}^{\mathcal{P}}$ is a primal on X,*
(2) *$\mathcal{L}^{\mathcal{P}}$ is an ideal on X.*

Proof. Straightforward. □

Remark 3.4. Converse of Corollary 3.3 is not true in general as shown by the following example.

Example 3.5. Let $X = \{a, b, c\}$ with a primal $\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ and an ideal $\mathcal{L} = \{\phi, \{a\}\}$.
(1) $\mathcal{P}$ is not a primal ideal on X since $\{b\}, \{c\} \in \mathcal{P}$ but $\{b, c\} \notin \mathcal{P}$.
(2) $\mathcal{L}$ is not a primal ideal on X since $\{a\} = \{a, b\} \cap \{a, c\} \in \mathcal{L}$ but neither $\{a, b\} \in \mathcal{L}$ nor $\{a, c\} \in \mathcal{L}$.

Theorem 3.6. *Let $\mathcal{L}^{\mathcal{P}}$ be a primal ideal on X. Then, $\{A|A^c \in \mathcal{L}^{\mathcal{P}}\}$ is a filter on X.*

Proof. Let $\mathcal{L}^{\mathcal{P}}$ be a primal ideal of X and $\mathcal{F} = \{A|A^c \in \mathcal{L}^{\mathcal{P}}\}$. Then we are to show that $\mathcal{F}$ is a filter.
(i) Since $X \notin \mathcal{L}^{\mathcal{P}}$, $\phi \notin \mathcal{F}$.

(ii) Let $A \in \mathcal{F}$ and $A \subseteq B$. Then $B^c \subseteq A^c$. Since $A^c \in \mathcal{L}^{\mathcal{P}}$, $B^c \in \mathcal{L}^{\mathcal{P}}$. Thus $B \in \mathcal{F}$.

(iii) Let $A,\ B \in \mathcal{F}$. Then $A^c \cup B^c = (A \cap B)^c \in \mathcal{L}^{\mathcal{P}}$. Thus $A \cap B \in \mathcal{F}$. So by (i), (ii) and (iii), $\mathcal{F}$ is a filter on X. □

Theorem 3.7. *Let $\mathcal{L}^{\mathcal{P}}$ be a primal ideal on X. Then $\{A|A^c \in \mathcal{L}^{\mathcal{P}}\}$ is a grill on X.*

Proof. Let $\mathcal{L}^{\mathcal{P}}$ be a primal ideal of X and $\mathcal{G} = \{A|A^c \in \mathcal{L}^{\mathcal{P}}\}$. Then we are to show that $\mathcal{G}$ is a grill.

(i) Since $X \notin \mathcal{L}^{\mathcal{P}}$, $\phi \notin \mathcal{G}$.

(ii) Let $A \in \mathcal{G}$ and $A \subseteq B$. Then $B^c \subseteq A^c$. Since $A^c \in \mathcal{L}^{\mathcal{P}}$, $B^c \in \mathcal{L}^{\mathcal{P}}$. Thus $B \in \mathcal{G}$.

(iii) Let $A \cup B \in \mathcal{G}$. Then $A^c \cap B^c = (A \cup B)^c \in \mathcal{L}^{\mathcal{P}}$. Thus we get $A^c \in \mathcal{L}^{\mathcal{P}}$ or $B^c \in \mathcal{L}^{\mathcal{P}}$. So $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Hence by (i), (ii) and (iii), $\mathcal{G}$ is a grill on X. □

Definition 3.8. Let R be a binary relation on X and $\mathcal{L}^{\mathcal{P}}$ be a primal ideal defined on X and $A \subseteq X$. Then the *lower* and *upper approximations*, $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$ and $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$ of A are defined respectively by:

$$\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = \{x \in A\ :< x > R \cap A^c \in \mathcal{L}^{\mathcal{P}}\}, \tag{3.1}$$

$$\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A \cup \{x \in X\ :< x > R \cap A \notin \mathcal{L}^{\mathcal{P}}\}. \tag{3.2}$$

Lemma 3.9. *The lower approximation defined by (3.1) has the following properties: for $A,\ B \subseteq X$,*

(1) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = (\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A^c))^c$,
(2) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\phi) = \phi$ *and* $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(X) = X$,
(3) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq A$,
(4) *if* $A \subseteq B$, *then* $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq \underline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(5) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A \cap B) = \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \cap \underline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(6) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A \cup B) \supseteq \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \cup \underline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(7) $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)) = \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$.

Lemma 3.10. *The upper approximation defined by Eq. (3.2) has the following properties: for $A,\ B \subseteq X$,*

(1) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = (\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A^c))^c$,
(2) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\phi) = \phi$ *and* $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(X) = X$,
(3) $A \subseteq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$,
(4) *if* $A \subseteq B$, *then* $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(5) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A \cap B) \subseteq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \cap \overline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(6) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A \cup B) = \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \cup \overline{R_{\mathcal{L}^{\mathcal{P}}}}(B)$,
(7) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)) = \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$.

The operator $\overline{R_{\mathcal{L}^{\mathcal{P}}}} : P(X) \longrightarrow P(X)$ defined by Eq. (3.2) induced a topology on X denoted by $\tau_{\mathcal{L}^{\mathcal{P}}}$ and defined as $\tau_{\mathcal{L}^{\mathcal{P}}} = \{A \subseteq X : \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A^c) = A^c\}$. In addition, $(X, R, \mathcal{L}^{\mathcal{P}})$ is called a *primal ideal approximation space.*

Definition 3.11. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space. For $A \subseteq X$, the *boundary region* and the *accuracy measure*, $BND_{\mathcal{L}^{\mathcal{P}}}(A)$ and $ACC_{\mathcal{L}^{\mathcal{P}}}(A)$, are defined respectively, as follows:

$$BND_{\mathcal{L}^{\mathcal{P}}}(A) = \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) - \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A), \quad ACC_{\mathcal{L}^{\mathcal{P}}}(A) = \frac{|\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)|}{|\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)|}, \ |\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)| \neq 0,$$

where $0 \leq ACC_{\mathcal{L}^{\mathcal{P}}}(A) \leq 1$.

Definition 3.12. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then the *lower* and the *upper approximations*, $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$ and $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$ of A are defined respectively by:

$$\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = \{x \in A \ : R < x > R \cap A^c \in \mathcal{L}^{\mathcal{P}}\}, \tag{3.3}$$

$$\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = A \cup \{x \in X \ : R < x > R \cap A \notin \mathcal{L}^{\mathcal{P}}\}. \tag{3.4}$$

Lemma 3.13. *The lower approximation defined by Eq. (3.3) has the following properties: for $A,\ B \subseteq X$,*

(1) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = (\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A^c))^c$,
(2) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(\phi) = \phi$ *and* $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(X) = X$,
(3) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq A$,
(4) *if* $A \subseteq B$, *then* $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(5) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A \cap B) = \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \cap \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(6) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A \cup B) \supseteq \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \cup \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(7) $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)) = \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$.

Lemma 3.14. *The upper approximation defined by Eq. (3.4) has the following properties: for $A,\ B \subseteq X$,*

(1) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = (\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A^c))^c$,
(2) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(\phi) = \phi$ *and* $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(X) = X$,
(3) $A \subseteq \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$,
(4) *if* $A \subseteq B$, *then* $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(5) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A \cap B) \subseteq \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \cap \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(6) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A \cup B) = \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \cup \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(B)$,
(7) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)) = \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$.

The operator $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}} : P(X) \longrightarrow P(X)$ defined by Eq. (3.4) induced a topology on X denoted by $\tau_{\mathcal{L}^{\mathcal{P}}}$ and defined as $\tau^*_{\mathcal{L}^{\mathcal{P}}} = \{A \subseteq X : \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A^c) = A^c\}$.

Definition 3.15. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space. For $A \subseteq X$, the *boundary region* and the *accuracy measure*, $BND^*_{\mathcal{L}^{\mathcal{P}}}(A)$ and $ACC^*_{\mathcal{L}^{\mathcal{P}}}(A)$, are defined respectively, as follows:

$$BND^*_{\mathcal{L}^{\mathcal{P}}}(A) = \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) - \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A), \quad ACC^*_{\mathcal{L}^{\mathcal{P}}}(A) = \frac{|\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)|}{|\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)|}, \ |\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)| \neq 0,$$

where $0 \leq ACC^*_{\mathcal{L}^{\mathcal{P}}}(A) \leq 1$.

Theorem 3.16. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then*

$$L_R(A) \subseteq \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq A \subseteq \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq U_R(A).$$

Proof. Straightforward from Definitions 2.9, 3.8, and 3.12. □

Proposition 3.17. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space. Then $\tau_R \subseteq \tau_{\mathcal{L}^{\mathcal{P}}} \subseteq \tau^*_{\mathcal{L}^{\mathcal{P}}}$.*

Proof. Immediately by Theorem 3.16. □

Lemma 3.18. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then*
(1) $BND^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq BND_{\mathcal{L}^{\mathcal{P}}}(A)$,
(2) $ACC^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq ACC_{\mathcal{L}^{\mathcal{P}}}(A)$.

Proof. Straightforward from Definitions 3.11, and 3.15 using Theorem 3.16. □

Remark 3.19. (1) It is noted from Lemma 3.18 that Definition 3.12 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximation and decreasing the upper approximation with the comparison of the method in Definition 3.8.

(2) If we take grill in stead of primal ideal in Definitions 3.8 and 3.12, then the present accuracy measures produced by "primal ideal" are more accurate and higher than those produced by grill. Since, the boundary regions are decreased (or empty).

Example 3.20. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space with $X = \{a, b, c\}$. Let $\mathcal{L} = \{\phi, \{a\}\}$. be an ideal. Then $\mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ is a primal ideal on X with respect to $\mathcal{L}$. Consider $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$. Then $< a > R = \{a, b, c\}$, $< b > R = \{b, c\}$, $< c > R = \{c\}$. Also, we have $R < a >= \{a\}$, $R < b >= \{a, b\}$, $R < c >= \{a, b, c\}$. Thus we get

$$R < a > R = \{a\},\ R < b > R = \{b\},\ R < c > R = \{c\}.$$

(1) If $A = \{a, b\}$, then $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = \phi$, $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = \{a, b\}$, $BND_{\mathcal{L}^{\mathcal{P}}}(A) = \{a, b\}$ and $ACC_{\mathcal{L}^{\mathcal{P}}}(A) = 0$. But, $\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = \{a, b\}$, $BND^*_{\mathcal{L}^{\mathcal{P}}}(A) = \phi$ and $ACC^*_{\mathcal{L}^{\mathcal{P}}}(A) = 1$.

(2) Take $\mathcal{G} = \{A | A^c \in \mathcal{L}^{\mathcal{P}}\} = \{X, \{b, c\}, \{a, c\}, \{c\}\}$.

Table 1. Comparison between the boundary region and accuracy measure of Definition 3.15 produced by "primal ideal" and those produced by grill.

	Grill approximations				Primal ideal approximations			
2-56-9 $A \subseteq X$	$\underline{\underline{R_{\mathcal{G}}}}(A)$	$\overline{\overline{R_{\mathcal{G}}}}(A)$	$BND^*_{\mathcal{G}}(A)$	$ACC^*_{\mathcal{G}}(A)$	$\underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$	$\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$	$BND^*_{\mathcal{L}^{\mathcal{P}}}(A)$	$ACC^*_{\mathcal{L}^{\mathcal{P}}}(A)$
ϕ	ϕ	X	X	0	ϕ	ϕ	ϕ	1
$\{a\}$	ϕ	X	X	0	$\{a\}$	$\{a\}$	ϕ	1
$\{b\}$	ϕ	X	X	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	X	X	0	$\{c\}$	$\{c\}$	ϕ	1
$\{a, b\}$	ϕ	X	X	0	$\{a, b\}$	$\{a, b\}$	ϕ	1
$\{a, c\}$	ϕ	X	X	0	$\{a, c\}$	$\{a, c\}$	ϕ	1
$\{b, c\}$	ϕ	X	X	0	$\{b, c\}$	$\{b, c\}$	ϕ	1
X	ϕ	X	X	0	X	X	ϕ	1

The comparison between the introduced approach in Definition 3.12 utilizing "primal ideal" and the method in which "primal ideal" is replaced by grill is shown in Table 1. From Table 1, the approximation produced by "primal ideal" reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximation and decreasing the upper approximation with the comparison of the approximations produced by grill.

4. Modified topological concepts via primal ideal approximation spaces

In this section, we introduce accumulation points, separation axioms and connectedness via primal ideal as a generalization of accumulation points, separation axioms and connectedness via ideal given in [16]. We scrutinize its essential characterizations and infer some of its relationships associated with the primal ideal closure operators. Some illustrative examples are given. Furthermore, we compare between the current purposed technique in Definition 3.12 and technique in [20]. Then, we clarify that our approach in Definition 3.12 produces accuracy measures of subsets higher than their counterparts displayed in [20].

Theorem 4.1. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then*

(1) $\underline{R}(A) \subseteq \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$ *and* $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq \overline{R}(A)$,

(2) $\underline{\underline{R}}(A) \subseteq \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A)$ *and* $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) \subseteq \overline{\overline{R}}(A)$.

Proof. Straightforward. □

Definition 4.2. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then A point $x \in X$ is called a:

(i) *$*$-primal ideal accumulation point of A, if* $(< x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}$,

(ii) $**$-primal ideal accumulation point of A, if $(R < x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}$.

The set of all $*$-primal ideal accumulation points of A is denoted by $d^*(A)$,i.e.,

$$d^*_{\mathcal{L}^{\mathcal{P}}}(A) = \{x \in X : (< x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}\}.$$

The set of all $**$-primal ideal accumulation points of A is denoted by $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A)$,i.e.,

$$d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) = \{x \in X : (R < x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}\}.$$

Lemma 4.3. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space and $A \subseteq X$. Then we have*

(1) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A)$,

(2) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A$ *iff* $d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq A$,

(3) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = A \cup d^{**}_{\mathcal{L}^{\mathcal{P}}}(A)$,

(4) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(A) = A$ *iff* $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq A$.

Proof. (1) Let $x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$. Then $x \in (A \cup \{y \in X :< y > R \cap A \notin \mathcal{L}^{\mathcal{P}}\})$. Thus we have either $x \in A$, i.e,

$$x \in A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A) \tag{4.1}$$

or $x \notin A$. So $x \in \{y \in X :< y > R \cap A \notin \mathcal{L}^{\mathcal{P}}\}$. In the latter case, we have $(< x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}$. Hence $x \in d^*_{\mathcal{L}^{\mathcal{P}}}(A)$, i.e,

$$x \in A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A). \tag{4.2}$$

From Eq. (4.1) and Eq. (4.2), we have $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \subseteq A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A)$.

Conversely, let $x \in A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A)$. Then we have either $x \in A$, i.e,

$$x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \tag{4.3}$$

or $x \notin A$. Thus $x \in d^*_{\mathcal{L}^{\mathcal{P}}}(A)$. So $(< x > R - \{x\}) \cap A \notin \mathcal{L}^{\mathcal{P}}$. Hence $x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$,i.e,

$$x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A). \tag{4.4}$$

From Eq.(4.3) and Eq. (4.4), we have $A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$. Therefore $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A \cup d^*_{\mathcal{L}^{\mathcal{P}}}(A)$.

(2) For $x \notin A$, $x \notin \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$. Then $< x > R \cap A \in \mathcal{L}^{\mathcal{P}}$. Thus $(< x > R - \{x\}) \cap A \in \mathcal{L}^{\mathcal{P}}$ and $x \notin d^*_{\mathcal{L}^{\mathcal{P}}}(A)$.

Conversely, let $d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq A$. Then by (1), $d^*_{\mathcal{L}^{\mathcal{P}}}(A) \cup A = \overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A$.

The proofs of (3) and (4) are similar.

□

Corollary 4.4. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be any primal ideal approximation space and $A \subseteq X$. Then we have*

(1) $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d(A)$,
(2) $d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d^*(A)$ *and* $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d^{**}(A)$.

Proof. (1) Let $x \notin d(A)$. Then $(< x > R - \{x\}) \cap A = \phi$. Thus $(< x > R - \{x\}) \cap A \in \mathcal{L}^{\mathcal{P}}$. So $x \notin d^*_{\mathcal{L}^{\mathcal{P}}}(A)$ and $(R < x > R - \{x\}) \cap A \in \mathcal{L}^{\mathcal{P}}$, where $R < x > R \subseteq < x > R$. Hence $x \notin d^{**}_{\mathcal{L}^{\mathcal{P}}}(A)$. Therefore $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d^*_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq d(A)$.

(2) The proof is similar. □

Remark 4.5. The following example shows that the converse of Corollary 4.4 is not true in general.

Example 4.6. In Example 3.20, let $\mathcal{L} = \{\phi, \{a\}\}$. be an ideal. Then $\mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}, \{c\}, \{a, c\}\}$ is a primal ideal on X with respect to $\mathcal{L}$.

(1) Conciser $A = \{b, c\}$. Then we have

$$(< a > R - \{a\}) \cap A = \{b, c\} \neq \phi,$$
$$(< b > R - \{b\}) \cap A = \{c\} \neq \phi,$$
$$(< c > R - \{c\}) \cap A = \phi.$$

Thus $a \in d(A)$, $b \in d(A)$, $c \notin d(A)$. So $d(A) = \{a, b\}$.

On the other hand, we get

$$(< a > R - \{a\}) \cap A = \{b, c\} \notin \mathcal{L}^{\mathcal{P}},$$
$$(< b > R - \{b\}) \cap A = \{c\} \in \mathcal{L}^{\mathcal{P}},$$
$$(< c > R - \{c\}) \cap A = \phi \in \mathcal{L}^{\mathcal{P}}.$$

Then $a \in d^*_{\mathcal{L}^{\mathcal{P}}}(A)$, $b \notin d^*_{\mathcal{L}^{\mathcal{P}}}(A)$ $c \notin d^*_{\mathcal{L}^{\mathcal{P}}}(A)$. Thus $d^*_{\mathcal{L}^{\mathcal{P}}}(A) = \{a\}$. Also, we have

$$(R < a > R - \{a\}) \cap A = \phi \in \mathcal{L}^{\mathcal{P}},$$

$$(R < b > R - \{b\}) \cap A = \phi \in \mathcal{L}^{\mathcal{P}},$$
$$(R < c > R - \{c\}) \cap A = \phi \in \mathcal{L}^{\mathcal{P}}.$$

Then $a \notin d^{**}_{\mathcal{L}^{\mathcal{P}}}(A),\ b \notin d^{**}_{\mathcal{L}^{\mathcal{P}}}(A),\ c \notin d^{**}_{\mathcal{L}^{\mathcal{P}}}(A)$. Thus $d^{**}_{\mathcal{L}^{\mathcal{P}}}(A) = \phi$. So we get

$$d(A) \nsubseteq d^{*}_{\mathcal{L}^{\mathcal{P}}}(A) \nsubseteq d^{**}_{\mathcal{L}^{\mathcal{P}}}(A).$$

(2) From (1), $A = \{b, c\}$ implies that $d^{*}_{\mathcal{L}^{\mathcal{P}}}(A) = \{a\}$. But

$$(< a > R - \{a\}) \cap A = \{b, c\} \notin \mathcal{L},$$
$$(< b > R - \{b\}) \cap A = \{c\} \notin \mathcal{L},$$
$$(< c > R - \{c\}) \cap A = \phi \in \mathcal{L}.$$

Then $a \in d^*(A),\ b \in d^*(A),\ c \notin d^*(A)$. Thus $d^*(A) = \{a, b\}$. So $d^*(A) \nsubseteq d^{*}_{\mathcal{L}^{\mathcal{P}}}(A)$. In the same way, anyone can give an example to show that $d^{**}(A) \nsubseteq d^{**}_{\mathcal{L}^{\mathcal{P}}}(A)$.

Definition 4.7. A primal ideal approximation space $(X, R, \mathcal{L})$ is called a:

(i) *primal ideal-*T_0^**, if* $\forall x \neq y \in X$*, there exists* $A \subseteq X$ *such that*

$$(x \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A), y \notin A) \text{ or } (y \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A), x \notin A),$$

(ii) primal ideal-T_0^{**}*, if* $\forall x \neq y \in X$*, there exists* $A \subseteq X$ *such that*

$$(x \in \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A), y \notin A) \text{ or } (y \in \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A), x \notin A).$$

Proposition 4.8. *Let* $(X, R, \mathcal{L}^{\mathcal{P}})$ *be a primal ideal approximation space. For* $x \neq y \in X$*, then we have*

(1) $x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$ *iff* $< x > R \cap \{y\} \notin \mathcal{L}^{\mathcal{P}}$ *and* $x \notin \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$ *iff* $< x > R \cap \{y\} \in \mathcal{L}^{\mathcal{P}}$*.*

(2) $x \in \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{y\})$ *iff* $R < x > R \cap \{y\} \notin \mathcal{L}^{\mathcal{P}}$ *and* $x \notin \overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{y\})$ *iff* $R < x > R \cap \{y\} \in \mathcal{L}^{\mathcal{P}}$*.*

Proof. (1) Suppose $x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$. Then $x \in (\{y\} \cup \{z \in X :< z > R \cap \{y\} \notin \mathcal{L}^{\mathcal{P}}\})$. Thus $< x > R \cap \{y\} \notin \mathcal{L}^{\mathcal{P}}$.

Conversely, suppose $< x > R \cap \{y\} \notin \mathcal{L}^{\mathcal{P}}$. Then by Definition 3.8, $x \in \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$. The proof of the second part is similar.

(2) The proof is similar to (1). □

Proposition 4.9. *For a primal ideal approximation space* $(X, R, \mathcal{L}^{\mathcal{P}})$*, the following are equivalent:*

(1) X *is a primal ideal-*T_0^* *space,*

(2) $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}) \neq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$ *for each* $x \neq y \in X$.

Proof. (1) ⇒ (2): Suppose X is a primal ideal-T_0^* space and let $x \neq y \in X$. Then by the hypothesis, there exists $A \subseteq X$ such that $x \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A),\ y \notin A$. Then $< x > R \cap A^c \in \mathcal{L}^{\mathcal{P}},\ y \in A^c$. Thus $< x > R \cap \{y\} \in \mathcal{L}^{\mathcal{P}}$ and by Proposition 4.8 (1), $x \notin \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$. By the same way, we can prove that $y \notin \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\})$. So $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}) \neq \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\})$.

(2) ⇒ (1): Suppose the necessary condition (2) holds and let $x \neq y \in X$. Then by (2), we have $x \notin \overline{R}(\{y\})$ or $y \notin \overline{R}(\{x\})$. Thus by Proposition 4.8 (1), we get

$$< x > R \cap \{y\} \in \mathcal{L}^{\mathcal{P}} \text{ or } < y > R \cap \{x\} \in \mathcal{L}^{\mathcal{P}}, \text{ i.e.,}$$
$$(x \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\}^c), y \notin \{y\}^c) \text{ or } (y \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}^c), x \notin \{x\}^c).$$

So X is a primal ideal-T_0^* space. □

Corollary 4.10. *For a primal ideal approximation space* $(X, R, \mathcal{L}^{\mathcal{P}})$*, the following are equivalent:*

(1) *X is a primal ideal-T_0^{**} space,*
(2) $\overline{\overline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{x\}) \neq \overline{\overline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{y\})$ *for each* $x \neq y \in X$.

Corollary 4.11. *Let* $(X, R, \mathcal{L}^{\mathcal{P}})$ *be any primal ideal approximation space and* $A \subseteq X$*. Then we have*

(1) *primal ideal-T_0^* $\Rightarrow$ primal ideal-T_0^{**},*
(2) *T_0^* $\Rightarrow$ primal ideal-T_0^* and T_0^{**} $\Rightarrow$ primal ideal-T_0^{**}.*

Proof. The proof is straightforward from Theorems 3.16 and 4.1. □

Remark 4.12. The following example shows that the converse of Corollary 4.11 does not hold.

Example 4.13. Let $X = \{a, b, c\}, R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$. Consider $\mathcal{L} = \{\phi, \{c\}\}$ and $\mathcal{L}^{\mathcal{P}} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$ Then we have

$$< a > R = R < a > R = \{a, b\},$$

$$< b > R = R < b > R = \{a, b\},$$

$$< c > R = R < c > R = \{c\}.$$

Thus there exist $\{b\}, \{c\} \subseteq X$ such that

$$\underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \{b\} \text{ and } \underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \{c\}.$$

So we get

(i) For $a \neq b$, $b \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \{b\}$ and $a \notin \{b\}$,
(ii) For $b \neq c$, $b \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \{b\}$ and $c \notin \{b\}$,
(iii) For $a \neq c$, $c \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \{c\}$ and $\notin \{c\}$.

Hence X is primal ideal-T_0^* and primal ideal-T_0^{**}. But

$$\overline{R}(\{a\}) = \overline{\overline{R}}(\{a\}) = \overline{R}(\{b\}) = \overline{\overline{R}}(\{b\}) = \{a, b\}.$$

This means that, X is neither T_0^*-space nor T_0^{**}-space.

Definition 4.14. A primal ideal approximation space $(X, R, \mathcal{L})$ is called a:

(i) *primal ideal-T_1^**, if $\forall x \neq y \in X$, there exist A, $B \subseteq X$ such that

$$(x \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A), y \notin A) \text{ and } (y \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(B), x \notin B),$$

(ii) *primal ideal-T_1^{**}*, if $\forall x \neq y \in X$, there exist $A, B \subseteq X$ such that

$$(x \in \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(A), y \notin A) \text{ and } (y \in \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(B), x \notin B).$$

Proposition 4.15. *For a primal ideal approximation space* $(X, R, \mathcal{L}^{\mathcal{P}})$*, the following are equivalent:*

(1) *X is a primal ideal-T_1^*space,*
(2) $\overline{R}_{\mathcal{L}^{\mathcal{P}}}(\{x\}) = \{x\}$ *for each* $x \in X$,
(3) $d^*_{\mathcal{L}^{\mathcal{P}}}(\{x\}) = \phi$ *for each* $x \in X$.

Proof. (1) $\Rightarrow$ (2): Suppose $(X, R, \mathcal{L}^{\mathcal{P}})$ is T_1^*-space and let $x \in X$. Then for $y \in X - \{x\}$, $x \neq y$ and $\exists A \subseteq X$ such that $y \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A)$, $x \notin A$. Thus $\langle y \rangle R \cap A^c \in \mathcal{L}^{\mathcal{P}}$, $x \in A^c$. So $\langle y \rangle R \cap \{x\} \in \mathcal{L}^{\mathcal{P}}$, i.e., $y \notin \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\})$. Hence

$$\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}) = \{x\}.$$

(2) $\Rightarrow$ (3): Suppose (2) holds and let $x \in X$. Then $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}) = \{x\} \cup d^*_{\mathcal{L}^{\mathcal{P}}}(\{x\})$ but $x \notin d^*_{\mathcal{L}^{\mathcal{P}}}(\{x\})$. Thus $d^*_{\mathcal{L}^{\mathcal{P}}}(\{x\}) = \phi$.

(3) $\Rightarrow$ (1): Suppose (3) holds and let $x \neq y \in X$. Then by (3), $d^*_{\mathcal{L}^{\mathcal{P}}}(\{x\}) = d^*_{\mathcal{L}^{\mathcal{P}}}(\{y\}) = \phi$. Thus by Lemma 4.3 (1), we have $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}) = \{x\}$ and $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\}) = \{y\}$, i.e., $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}^c) = \{x\}^c$ and $\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\}^c) = \{y\}^c$. So there exist $\{x\}^c$ and $\{y\}^c \subseteq X$ such that $y \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{x\}^c)$, $x \notin \{x\}^c$ and $x \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{y\}^c)$, $y \notin \{y\}^c$. Hence X is a T_1^*-space. □

Corollary 4.16. *For a primal ideal approximation space $(X, R, \mathcal{L}^{\mathcal{P}})$, the following are equivalent:*

(1) *X is a primal ideal-T_1^{**} space,*
(2) $\overline{\overline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{x\}) = \{x\}$ *for each* $x \in X$,
(3) $d^{**}_{\mathcal{L}^{\mathcal{P}}}(\{x\}) = \phi$ *for each* $x \in X$.

Corollary 4.17. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be any primal ideal approximation space and $A \subseteq X$. Then we have*

(1) *primal ideal-T_1^* $\Rightarrow$ primal ideal-T_1^{**},*
(2) *T_1^* $\Rightarrow$ primal ideal-T_1^* and T_1^{**} $\Rightarrow$ primal ideal-T_1^{**}.*

Proof. The proof is straightforward from Theorems 3.16 and 4.1. □

Remark 4.18. The following example shows that the converse of Corollary 4.17 does not hold.

Example 4.19. In Example 4.13, consider $\mathcal{L} = \{\phi, \{a\}\}$ and $\mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}\{b\}, \{a, b\}\}$ Then there exist $\{a\}, \{b\}, \{c\} \subseteq X$ such that

$$\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{a\}) = \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{a\}) = \{a\},$$

,

$$\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{b\}) = \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{b\}) = \{b\},$$
$$\underline{R_{\mathcal{L}^{\mathcal{P}}}}(\{c\}) = \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(\{c\}) = \{c\}.$$

Thus X is primal ideal-T_1^* and primal ideal-T_1^{**}. But X is neither T_1^*-space nor T_1^{**}-space since we can not find a set $A \subseteq X$ such that $a \in \underline{R}(A)$, $b \notin A$.

Definition 4.20. A primal ideal approximation space $(X, R, \mathcal{L})$ is called a:

(i) *primal ideal-T_2^**, if $\forall x \neq y \in X$ there exist A, $B \subseteq X$ such that

$$x \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(A),\ y \in \underline{R_{\mathcal{L}^{\mathcal{P}}}}(B) \text{ and } A \cap B = \phi.$$

(ii) *primal ideal-T_2^{**}*, if $\forall x \neq y \in X$ there exist A, $B \subseteq X$ such that

$$x \in \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(A),\ y \in \underline{\underline{R_{\mathcal{L}^{\mathcal{P}}}}}(B) \text{ and } A \cap B = \phi.$$

Theorem 4.21. *For a primal ideal approximation space $(X, R, \mathcal{L}^{\mathcal{P}})$, the following are equivalent:*

(1) *X is a primal ideal-T_2^*space,*
(2) $\exists A \subseteq X : x \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A),\ y \in (\overline{R}_{\mathcal{L}^{\mathcal{P}}}(A))^c$ *for all* $x \neq y \in X$.

Proof. (1) $\Rightarrow$ (2): Suppose X is primal ideal-T_2^* space and let $x \neq y \in X$. Then there exist $A,\ B \subseteq X$ such that $x \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A),\ y \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(B)$ and $A \cap B = \phi$. Thus $\langle y \rangle R \cap B^c \in \mathcal{L}^{\mathcal{P}}$ and $A \subseteq B^c$. So $(\langle y \rangle R - \{x\}) \cap A \in \mathcal{L}^{\mathcal{P}}$, i.e., $y \notin d^*_{\mathcal{L}^{\mathcal{P}}}(A)$. Hence $\underline{R}_{\mathcal{L}^{\mathcal{P}}}(B) \cap d^*_{\mathcal{L}^{\mathcal{P}}}(A) = \phi$ and $\underline{R}_{\mathcal{L}^{\mathcal{P}}}(B) \cap A = \phi$, i.e., $\underline{R}_{\mathcal{L}^{\mathcal{P}}}(B) \cap \overline{R}_{\mathcal{L}^{\mathcal{P}}}(A) = \phi$. Therefore $x \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A), y \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(B) \subseteq (\overline{R}_{\mathcal{L}^{\mathcal{P}}}(A))^c$.

(2) $\Rightarrow$ (1): Suppose (2) holds and let $x \neq y \in X$. Then there exists $A \subseteq X$ such that $x \in \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A),\ y \in (\overline{R}_{\mathcal{L}^{\mathcal{P}}}(A))^c$. Let $B = (\overline{R}_{\mathcal{L}^{\mathcal{P}}}(A))^c$. Then $B = \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A^c)$ and thus $\underline{R}_{\mathcal{L}^{\mathcal{P}}}(B) = \underline{R}_{\mathcal{L}^{\mathcal{P}}}(\underline{R}_{\mathcal{L}^{\mathcal{P}}}(A^c)) = \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A^c) = B$. Also, we get

$$A \cap B = A \cap \underline{R}_{\mathcal{L}^{\mathcal{P}}}(A^c) \subseteq A \cap A^c = \phi.$$

So X is a primal ideal-T_2^*space. □

Corollary 4.22. *For a primal ideal approximation space $(X, R, \mathcal{L}^{\mathcal{P}})$, the following are equivalent:*

(1) *X is a primal ideal-T_2^{**}space,*
(2) $\exists A \subseteq X : x \in \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(A), y \in (\overline{\overline{R}}_{\mathcal{L}^{\mathcal{P}}}(A))^c$ *for all* $x \neq y \in X$.

Corollary 4.23. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be any primal ideal approximation space and $A \subseteq X$. Then we have*

(1) *primal ideal-T_2^* $\Rightarrow$ primal ideal-T_2^{**},*
(2) *T_2^* $\Rightarrow$ primal ideal-T_2^* and T_2^{**} $\Rightarrow$ primal ideal-T_2^{**}.*

Proof. The proof is straightforward from Theorems 3.16 and 4.1. □

Example 4.24. In Example 4.13, consider $\mathcal{L} = \{\phi, \{b\}\}$ and $\mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}\{b\}, \{a, b\}\}$. Then there exist $\{a\}, \{b\}, \{c\} \subseteq X$ such that

$$\underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{a\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{a\}) = \{a\},$$
$$\underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{b\}) = \{b\},$$
$$\underline{R}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \underline{\underline{R}}_{\mathcal{L}^{\mathcal{P}}}(\{c\}) = \{c\}.$$

Thus X is primal ideal-T_2^* and primal ideal-T_2^{**}. But X is neither T_2^*-space nor T_2^{**}-space since we can not find a set $A \subseteq X$ such that $b \in \underline{R}(A), a \notin A$.

Definition 4.25. Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be a primal ideal approximation space.

(i) $A,\ B \subseteq X$ are called $*$-*primal ideal separated* (resp. $**$-*primal ideal separated*) *sets*, if $\overline{R}_{\mathcal{L}^{\mathcal{P}}}(A) \cap B = A \cap \overline{R}_{\mathcal{L}^{\mathcal{P}}}(B) = \phi$ (resp. $\overline{\overline{R}}_{\mathcal{L}^{\mathcal{P}}}(A) \cap B = A \cap \overline{\overline{R}}_{\mathcal{L}^{\mathcal{P}}}(B) = \phi$).

(ii) $Y \subseteq X$ is called a $*$-*primal ideal disconnected* (resp. $**$-*primal ideal disconnected*) *set*, if there exist $*$-primal ideal separated (resp. $**$-primal ideal separated) sets $A,\ B \subseteq X$ such that $Y \subseteq A \cup B$.

(iii) $Y \subseteq X$ is said to be $*$-*primal ideal connected* (resp. $**$-*primal ideal connected*), if it is not $*$-primal ideal disconnected (resp. $**$-primal ideal disconnected).

(iv) $(X, R, \mathcal{L}^{\mathcal{P}})$ is called a $*$-*primal ideal disconnected* (resp. $**$-*primal ideal disconnected*) *space*, if there exist $*$-primal ideal separated (resp. $**$-primal ideal separated) sets $A,\ B \subseteq X$ such that $A \cup B = X$.

(v) $(X, R, \mathcal{L}^{\mathcal{P}})$ is called a *$*$-primal ideal connected* (resp. *$**$-primal ideal connected*) *space*, if it is not a $*$-primal ideal disconnected (resp. $**$-primal ideal disconnected) space.

Remark 4.26. We have the following implications:

$$*\text{-separated} \Longrightarrow *\text{-primal ideal separated},$$

$$**\text{-separated} \Longrightarrow **\text{-primal ideal separated},$$

$$*\text{-primal ideal connected} \Longrightarrow *\text{-connected},$$

$$**\text{-primal ideal connected} \Longrightarrow **\text{-connected}.$$

Next examples show that the Implication in the diagrams is not reversible.

Example 4.27. Let $X = \{a, b, c, d\}, R = \{(a, a), (a, b), (b, b), (b, c), (c, c), (d, d), (d, b)\}$ Then $< a > R = \{a, b\}, < b > R = \{b\}, < c > R = \{c\}, < d > R = \{b, d\}$. Consider an ideal $\mathcal{L} = \{\phi, \{d\}\}$. Then $\mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}\{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$ is a primal ideal with respect to $\mathcal{L}$.

If $A = \{b\}, B = \{a, d\}$, then $\overline{R}(A) = A \cup \{x \in X \ :< x > R \cap A \notin \mathcal{L}\} = \{a, b, d\}$, and $\overline{R}(B) = \{a, d\}$. Also, $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) = A \cup \{x \in X \ :< x > R \cap A \notin \mathcal{L}^{\mathcal{P}}\} = \{b\}$ and $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(B) = \{a, d\}$. Thus $\overline{R_{\mathcal{L}^{\mathcal{P}}}}(A) \cap B = A \cap \overline{R_{\mathcal{L}^{\mathcal{P}}}}(B) = \phi$ but $\overline{R}(A) \cap B = \{a\} \neq \phi$. So $A,\ B$ are $*-$primal ideal separated sets but are not $*$-separated sets. Similarly, any one can add an example to show that $**$-primal ideal separated $\not\Rightarrow$ $**$-separated.

Example 4.28. Let $X = \{a, b, c, d\},\ R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (d, d)\}$. Then $\langle a \rangle R = \{a, b, c\},\ \langle b \rangle R = \{b, c\},\ \langle c \rangle R = \{b, c\},\ \langle d \rangle R = \{d\}$. Consider $\mathcal{L} = \{\phi\}$ and $\mathcal{L}^{\mathcal{P}} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then we get

$$\overline{R}(\{b\}) = \overline{R}(\{c\}) = \overline{R}(\{b, c\}) = \overline{R}(\{a, b\}) = \overline{R}(\{a, c\}) = X,\ \overline{R}(\{a\}) = \{a\}.$$

Thus X is a $*$-connected space. But we have

$$X = \{a\} \cup \{b, c\},\ \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{a\}) \cap \{b, c\} = \{a\} \cap \overline{R_{\mathcal{L}^{\mathcal{P}}}}(\{b, c\}) = \phi.$$

So X is not a $*$-primal ideal connected space.

Theorem 4.29. *Let $(X, R, \mathcal{L}^{\mathcal{P}})$ be any primal ideal approximation space and $A \subseteq X$. Then*

(1) $BND_{\mathcal{L}^{\mathcal{P}}}(A) \subseteq BND(A)$,
(2) $ACC(A) \subseteq ACC_{\mathcal{L}^{\mathcal{P}}}(A)$.

Proof. Immediately by Theorem 4.1. □

Remark 4.30. Theorem 4.29 states that Definition 3.12 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximations and decreasing the upper approximations via primal ideal, compared to Definition 2.10 in [20] as shown in the next examples. Then our suggested method is more accurate than [20] in decision making.

Example 4.31. Let $X = \{a, b, c, d\},\ \mathcal{L} = \{\phi, \{a\}\},\ \mathcal{L}^{\mathcal{P}} = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ and $R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (b, d), (c, c), (d, a), (d, b), (d, d)\}$. Then $R < a > R = \{a, b, d\}, R < b > R = \{a, b, d\}, R < c > R = \{c\}, R < d > R = \{a, b, d\}$.

TABLE 2. Comparison between the boundary region and accuracy measure by using our method in Definition 3.12 and the previous one in Definition 2.10 in [20]

	Method in Definition 2.10 in [20].				Our method in Definition 3.12			
2-56-9 $A \subseteq X$	$\underline{\underline{R}}(A)$	$\overline{\overline{R}}(A)$	$BND(A)$	$ACC(A)$	$\underline{\underline{R_{\mathcal{LP}}}}(A)$	$\overline{\overline{R_{\mathcal{LP}}}}(A)$	$BND^*_{\mathcal{LP}}(A)$	$ACC^*_{\mathcal{LP}}(A)$
X	X	X	ϕ	1	X	X	ϕ	1
$\{a\}$	ϕ	$\{a\}$	$\{a\}$	0	$\{a\}$	$\{a\}$	ϕ	1
$\{b\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	$\{c\}$	$\{c\}$	ϕ	1	$\{c\}$	$\{c\}$	ϕ	1
$\{d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	$\{d\}$	$\{d\}$	ϕ	1
$\{a,b\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	$\{a,b\}$	$\{a,b\}$	ϕ	1
$\{a,c\}$	$\{c\}$	$\{a,c\}$	$\{a\}$	1/2	$\{a,c\}$	$\{a,c\}$	ϕ	1
$\{a,d\}$	ϕ	$\{a,b,d\}$	$\{a,b,d\}$	0	$\{a,d\}$	$\{a,d\}$	ϕ	1
$\{b,c\}$	$\{c\}$	X	$\{a,b,d\}$	1/4	$\{b,c\}$	$\{b,c\}$	ϕ	1
$\{b,d\}$	$\{b,d\}$	X	$\{a,c\}$	1/2	$\{b,d\}$	$\{b,d\}$	ϕ	1
$\{c,d\}$	$\{c\}$	X	$\{a,b,d\}$	1/4	$\{c,d\}$	$\{c,d\}$	ϕ	1
$\{a,b,c\}$	$\{c\}$	X	$\{a,b,d\}$	1/4	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1
$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	ϕ	1	$\{a,b,d\}$	$\{a,b,d\}$	ϕ	1
$\{a,c,d\}$	$\{c\}$	X	$\{a,b,d\}$	1/4	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1
$\{b,c,d\}$	$\{b,c,d\}$	X	$\{a\}$	3/4	$\{b,c,d\}$	$\{b,c,d\}$	ϕ	1

The comparison between the introduced method in Definition 3.12 and the previous method in Definition 2.10 in [20] is shown in Table 2. From Table 2, the approximation by Definition 3.12 reduces the boundary region and increases the accuracy measure of a set A by increasing the lower approximation and decreasing the upper approximation via primal ideal with the comparison of the approximation by Definition 2.10.

Example 4.32. Consider Example 8.2 in [24], where the data about six students is given as in Table 3. From Table 3, we have:

TABLE 3. Decision system of six students.

Student	Science	German	Mathematics	Decision
a_1	Bad	Good	Medium	Accept
a_2	Good	Bad	Medium	Accept
a_3	Good	Good	Good	Accept
a_4	Bad	Good	Bad	Reject
a_5	Good	Bad	Medium	Reject
a_6	Bad	Good	Good	Accept

- The set of universe:$X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$.
- The set of attributes: AT = {Science, German, Mathematics}.
- The sets of values: VA = {Bad, Good,Medium,Accept, Reject}.

The set of condition attributes, C = {Science, German, Mathematics}. Then the corresponding equivalence relation is

$$R = \{(a_1, a_1), (a_2, a_5), (a_3, a_3), (a_4, a_4), (a_5, a_2), (a_6, a_6)\}.$$

Thus $R < a_1 > R = \{a_1\}, R < a_2 > R = R < a_5 > R = \{a_2, a_5\}, R < a_3 > R = \{a_3\}, R < a_4 > R = \{a_4\}, R < a_6 > R = \{a_6\}$.
Consequently, anyone can present an ideal and a corresponding primal ideal to illustrate that the approximations in Definition 3.12 are superior to the previous

Definition 2.10 in [20] by extending a table similar to Table 2 by comparing the resultant accuracy.
For example, let $\mathcal{L} = \{\phi, \{a_1\}\}$ and $\mathcal{L}^{\mathcal{P}} = P(Y)$, where $Y = \{a_1, a_2, a_3, a_4, a_5\}$. From Table 3 $A = \{a_1, a_2, a_3, a_6\}$ (Decision: Accept). Thus we respectively computed the lower and upper approximations, the boundary and the accuracy measure of A to be as follows.

(1) The approach in Definition 2.10 yields $\{a_1, a_3, a_6\}$, $\{a_1, a_2, a_3, a_5, a_6\}$, $\{a_2, a_5\}$ and 3/5. This means that the student a_2 have decision (Reject), which contradicts the decision system in Table 3.

(2) The approach in Definition 3.12 yields $\{a_1, a_2, a_3, a_6\}$, $\{a_1, a_2, a_3, a_6\}$, ϕ and 1. This means that the students $\{a_1, a_2, a_3, a_5, a_6\}$, have decision (Accept) according to the proposed technique which is coincident with Table 3. Hence, a decision made according to the calculations of our current technique via primal ideals in Definition 3.12 is more accurate than the approach via ideals in Definition 2.10.

5. Conclusions

There is a close homogeneity between rough set theory and general topology. Ideal is a fundamental concept in topological spaces and played an important role in the study of a generalization of rough set. This investigation was dedicated to the discourse of "primal ideal". It is illustrated that using "primal ideal" to deal with the decision-making problems is more accurate than using ideal, grill, or primal.

In this paper, we define primal ideal as an extension of a given ideal. We introduced two new closure operators using primal ideal generating two primal ideal topological spaces. The properties of the proposed spaces were studied. Some topological notions such as accumulation points, lower separation axioms, and connectedness of such spaces were defined and compared to there corresponding notions defined by [16] in ideal approximation spaces. In the current results, primal ideals were very helpful for increasing the current lower approximations and decreasing the current upper approximations. Consequently, they reduced the boundary region and increased the accuracy measure. So, they removed the vagueness of a concept that is an essential goal for the rough set. Finally, a particle example was provided to clarify the technique of the present primal ideal approximation spaces and demonstrate their utility and efficiency.

Acknowledgements. The authors would like to thank the referees and editor for their insightful comments and suggestions, which aided in the improvement of this paper.

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