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Semilattice implication algebras

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Abstract. We define a logical system, semilattice implication algebra, for a generalization of lattice implication algebras and Heyting semilattices and research some properties of this algebra, and the regular element in a semilattice implication algebra is defined and it is porved that the set of all regular elements is a distributive lattice. Also, we give some relationships of filters and multipliers, and give a filter constructed by a subsemigroup of monotone multipliers. And then it is showed that the filter generated by a subset is charaterized by simple multipliers.

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1. Introduction

Heyting algebra was introduced by A. Heyting to formalize the intuitunistic propositional calculus([1]). Heyting algebra has been known as several different names: Brouwerian lattic or Brouwerian logics (See [1, 2]). This algebra was generalized to Brouwerian semmilattice, which is the same algebra as implicative semilattice or Heyting semilattice (See [3, 4, 5]).

Lattice implication algebra was introduced in [6], and the notion of filters and prime filters of lattice implication algebras was defined and investigated in [7, 8, 9, 10]. The filter generated by a subset was costruced in [10] and the connection of filters and implicative filters was provided in [11].

Refer to [12, 13, 14] for additional related works on lattice implication algebrs.

All logical systems introduced above have propositional value in a lattice or a semilattice and the similar properties about implication operator.

The aim of this paper is to propose an algebraic system to generalize lattice implication algebras and Heyting semilattices and to research the common properties of these algebras. In section 3, we define the notion of semilattice implication algebra, which is a generalization of lattice implication algebras and Heyting semilattices, and research some basic properties that will be used in subsequent sections. In section 4, the regular element is defined and it is porved that the set of all regular elements is a distributive lattice. In section 5, we define a multiplier and a simple multiplier, and research some relationships of filters and multipliers. And then it is proved that the filter generated by a subset is charaterized by simple multipliers.

2. Preliminary

A *Heyting semilattice* (or *implicative semilattice* or *Brouwerian semilattice*) is an algebra $(L; \wedge, \rightarrow)$, where $(L; \wedge)$ is a semilattice and $\rightarrow$ is a binary operation on L satisfying: for any $x, y, z \in L$,

$$x \wedge y \leq z \text{ if and only if } x \leq y \rightarrow z.$$

Heyting algebra is an algebra $(L; \vee, \wedge, \rightarrow, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $(L; \wedge, \rightarrow)$ is a Heyting semilattice (See [3, 4, 5]). Every Heyting algebra is clearly a Heyting semilattice.

Lemma 2.1 ([3, 4, 5, 15])**.** *Let $(L, \wedge, \rightarrow, 1)$ be a Heyting semilattice. Then it satisfies the following properties: for every $x, y, z \in L$,*

(1) $x \rightarrow x = 1$ *and* $1 \rightarrow x = x$,
(2) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(3) $x \leq (x \rightarrow y) \rightarrow y$,
(4) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$,
(5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

A *lattice implication algebra* ([6]) is an algebra $(L; \vee, \wedge, \rightarrow, ', 0, 1)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $\rightarrow$ " a binary operation and " $'$ " an order-reversing involution satisfying the following axioms: for all $x, y, z \in L$,

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

Lemma 2.2 ([12, 14])**.** *Let L be a lattice implication algebra. Then it satisfies the following properties: for every $x, y, z \in L$,*

(1) $1 \rightarrow x = x$,
(2) $x \leq (xy)y$,
(3) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
(4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \vee (x \rightarrow z)$.

3. semilattice implication algebras

Definition 3.1. An *semilattice implication algebra* is a semilattice $(L, \wedge)$ with the greatest element 1 and a binary operation "$\cdot$" satisfying the following axioms: for every $x, y, z \in L$,

(SI1) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$,
(SI2) $1 \cdot x = x$,
(SI3) $x \wedge (xy)y = x$,
(SI4) $x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z)$.

We will denote $x \cdot y$ by xy shortly. Semilattice L has the binary operation $\wedge$ defined by $x \wedge y = \inf\{x, y\}$ for every $x, y \in L$, hence L is a partially ordered set with the partial order $\le$ defined by: $x \le y$ if and only if $x \wedge y = x$ for any $x, y \in L$.

Example 3.2. (1) Let $I_0 = (0, 1]$ be the interval in the set $\mathbb{R}$ of real numbers. Then it is a chain lattice with the usual order of the real numbers. If we define a binary operation $\cdot$ by

$$x \cdot y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$$

for any $x, y \in L$, then L is a semilattice implication algebra.

(2) Let $(\mathbb{R}, \mathcal{U})$ be the topological space with the usual topology $\mathcal{U}$ on $\mathbb{R}$. We define a binary operation $\cdot$ on $\mathcal{U}$ by

$$A \cdot B = int(A^c \cup B)$$

for any $A, B \in \mathcal{U}$, where $int(A)$ is interior of A. Then $\mathcal{U}$ is a semilattice implication algebra.

(3) For a poset $P = \{0, a, b\}$ with $0 \le a$, $0 \le b$, and a, b are non-comparable, and the interval $I_0 = (0, 1]$ in $\mathbb{R}$ in (1) of this example, let $L = P \oplus I_0$ be a semilattice with the partial order defined by $x < y$ for every $x \in P$ and $y \in I_0$. If we define a binary operation $\cdot$ on L by the following: for $0, a, b \in P$ and every $x, y \in I_0$,

$\cdot$	0	a	b	y
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
x	0	a	b	$x \cdot y$

and $x \cdot y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$,

then L be a semilattice implication algebra, and the least upper bound $a \vee b$ of a and b does not exist in L. Figure 1 is the Hasse diagram of L.

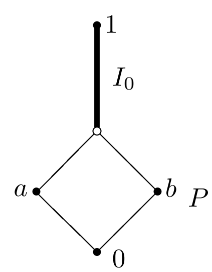

Figure 1. Hasse diagram of a lattice A

Lemma 3.3. *Let L be a semilattice implication algebra. Then it satisfies the following properties: for every $x, y \in L$,*

(1) $xx = 1$,
(2) $x \leq (xy)y$,

Proof. (1) Let $x \in L$. Then $1 = 1 \wedge (1x)x = 1 \wedge xx = xx$ by (SI3) and (SI2).
(2) It is clear from (SI3). □

Lemma 3.4. *Let L be a semilattice implication algebra. Then it satisfies the following property: for any $x, y \in L$,*

$$x \leq y \iff xy = 1.$$

Proof. Suppose $x \leq y$ in L for any $x,\ y \in L$. Then $x = x \wedge y$. Thus we have

$$1 = xx = x(x \wedge y) = (xx) \wedge (xy) = 1 \wedge (xy) = xy$$

by Lemma 3.3 (1) and (SI4).

Conversely, suppose $xy = 1$ for any $x,\ y \in L$. Then

$$x = x \wedge (xy)y = x \wedge 1y = x \wedge y.$$

by (SI3) and (SI2). This implies $x \leq y$. □

Lemma 3.5. *Let L be a semilattice implication algebra. Then it satisfies the following properties: for every $x, y, z \in L$,*

(1) $x1 = 1$,
(2) $x \leq yz$ *implies* $y \leq xz$,
(3) $y \leq xy$,
(4) $x \leq y$ *implies* $yz \leq xz$ *and* $zx \leq zy$,
(5) $xy = ((xy)y)y$,
(6) $xy \leq (yz)(xz)$,
(7) $xy \leq (zx)(zy)$.

Proof. (1) It is clear from Lemma 3.4, because 1 is the greatest element in L.

(2) Let $x \leq yz$. Then $1 = x(yz) = y(xz)$ by (SI1). Thus $y \leq xz$ by Lemma 3.4.

(3) Let $x, y \in L$. Then $x \leq 1 = yy$. This implies $y \leq xy$ by (2) of this lemma.

(4) Let $x \leq y$ in L. Then $x \leq y \leq (yz)z$ by Lemma 3.3 (2). Thus $yz \leq xz$ by (2) of this lemma. Also, since $x = x \wedge y$, $zx = z(x \wedge y) = (zx) \wedge (zy)$ by (SI4). So $zx \leq zy$.

(5) Let $x, y \in L$. Then $xy \leq ((xy)y)y$ by Lemma 3.3 (2). Also by Lemma 3.3 (2), $x \leq (xy)y \leq (((xy)y)y)y$. This implies $((xy)y)y \leq xy$ by (2) of this lemma. Thus $xy = ((xy)y)y$.

(6) Let $x, y, z \in L$. Then $y \leq (yz)z$ by Lemma 3.3 (2). Thus $xy \leq x((yz)z) = (yz)(xz)$ by (4) of this lemma and (SI1).

(7) Let $x, y, z \in L$. Then $zx \leq (xy)(zy)$ by (6) of this lemma. Thus $xy \leq (zx)(zy)$ by (2) of this lemma. □

Theorem 3.6. *Every lattice implication algebra is a semilattice implication algebra.*

Proof. It is clear from the definition of lattice implication algebra and Lemma 2.2. □

The semilattice implication algebra L of Example 3.2 (3) is not lattice implication algebra, because it is not lattice; $a \vee b$ does not exist. The converse direction of Theorem 3.6 is not true in general.

Theorem 3.7. *Every Heyting semilattice is a semilattice implication algebra.*

Proof. It is clear from Lemma 2.1. □

The converse of Theorem 3.7 is not true in general as the following example shows.

Example 3.8. Let $A = \{0, a, b, c, d, 1\}$ be a lattice with Hasse diagram of Figure 2. If we define a binary operation $\cdot$ on A as following:

·	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	a	c	d	c	1	1
1	0	a	b	c	d	1

then A is a lattice implication algebra, and so a semilattice implication algebra, but A is not Heyting semilattice, because $a \leq d = db$ but $a \wedge d = a \not\leq b$.

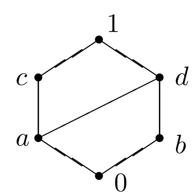

FIGURE 2. Hasse diagram of a lattice A

From Lemma 3.3 (2) and Lemma 3.5 (3), we know $(xy)y$ and $(yx)x$ are upper bounds of x and y, hence any two elements of a semilattice implication algebra has at lest one upper bound, and if $x \vee y$ exists, $x \vee y \leq (xy)y$ and $x \vee y \leq (yx)x$.

Lemma 3.9. *Let L be a semilattice implication algebra and $x, y, z \in L$. If $x \vee y$ exists, then*

$$(x \vee y)z = xz \wedge yz.$$

Proof. Let $x, y, z \in L$ and $x \vee y$ exists in L. Then $x \leq x \vee y$ and $y \leq x \vee y$. Thus by Lemma 3.5 (4), $(x \vee y)z \leq xz$ and $(x \vee y)z \leq yz$. So $(x \vee y)z$ is a lower bound of xz and yz.

Suppose that l is an lower bound of xz and yz. Since $l \leq xz$ and $l \leq yz$, $x \leq lz$ and $y \leq lz$ by Lemma 3.5 (2). Then $x \vee y \leq lz$ and $l \leq (x \vee y)z$. Thus $(x \vee y)z$ is the greatest lower bound of xz and yz, i.e., $(x \vee y)z = xz \wedge yz$. □

4. Regularity in semilattice implication algebras

For each element a in a semilattice implication algebra L, we can define a unary operation $*_a$ on L by $x^{*_a} = xa$ for every $x \in L$. Then $x^{*_a} \in [a, 1]$ for all $x \in L$ since $a \leq xa$. We denote $x^{*_a *_a}$ for $(x^{*_a})^{*_a}$.

Lemma 4.1. *Let L be a semilattice implication algebra and $a \in L$. Then the operation $*_a$ satisfies the following properties: for every $x, y \in L$,*

(1) $a^{*_a} = 1$ *and* $1^{*_a} = a$,
(2) $x \leq y$ *implies* $y^{*_a} \leq x^{*_a}$,
(3) $x \leq x^{*_a *_a}$,
(4) $xy^{*_a} = yx^{*_a}$,
(5) $x^{*_a *_a *_a} = x^{*_a}$,
(6) $xy \leq y^{*_a} x^{*_a}$.

Proof. It is clear from Lemma 3.3, (4)–(6) of Lemma 3.5 and (SI1). □

Let L be a semilattice implication algebra and $a \in L$. An element x in L is said to be *a-regular*, if $x^{*_a *_a} = x$ and $\mathcal{R}_a(L)$ will be denoted for the set of all a-regular elements in L.

Lemma 4.2. *Let L be a semilattice implication algebra and $a \in L$. Then*
(1) $1 \in \mathcal{R}_a(L)$ *and* $a \in \mathcal{R}_a(L)$,
(2) $xy = y^{*_a} x^{*_a}$ *for every* $x \in L$ *and every* $y \in \mathcal{R}_a(L)$,
(3) $x^{*_a} y = y^{*_a} x$ *for every* $x, y \in \mathcal{R}_a(L)$,
(4) $x^{*_a} \in \mathcal{R}_a(L)$ *for every* $x \in L$.

Proof. (1) It is clear from Lemma 4.1 (1).
(2) Let $x \in L$ and $y \in \mathcal{R}_a(L)$. Then $y^{*_a} x^{*_a} = x(y^{*_a})^{*_a} = xy$ by Lemma 4.1 (4) and the regularity of y.
(3) Let $x, y \in \mathcal{R}_a(L)$. Then $x^{*_a} y = y^{*_a} x^{*_a *_a} = y^{*_a} x$ by (2) of this lemma.
(4) It is clear from Lemma 4.1 (5). □

Theorem 4.3. *Let L be a semilattice implication algebra and $a \in L$. Then $\mathcal{R}_a(L)$ is closed under $\wedge$ of L.*

Proof. Let $x, y \in \mathcal{R}_a(L)$. Then $x \wedge y \leq (x \wedge y)^{*_a *_a}$ by Lemma 4.1 (3). Since $x \wedge y \leq x$ and $x \wedge y \leq y$, $(x \wedge y)^{*_a *_a} \leq x^{*_a *_a} = x$ and $(x \wedge y)^{*_a *_a} \leq y^{*_a *_a} = y$ by Lemma 4.1 (2). Thus $(x \wedge y)^{*_a *_a} \leq x \wedge y$. So $(x \wedge y)^{*_a *_a} = x \wedge y$, and $x \wedge y \in \mathcal{R}_a(L)$. □

Theorem 4.4. *Let L be a semilattice implication algebra and $a \in L$. Then $\mathcal{R}_a(L)$ is closed under the implication $\cdot$ of L.*

Proof. Let $x, y \in \mathcal{R}_a(L)$. Then $xy \leq (xy)^{*_a *_a}$ by Lemma 4.1 (3). Also, we have

$$\begin{aligned}
(xy)^{*_a *_a}(xy) &= x((xy)^{*_a *_a} y) \qquad \text{(by (SI1))} \\
&= x((y^{*_a}(xy)^{*_a *_a *_a}) \qquad \text{(by Lemma 4.2 (2))} \\
&= x(y^{*_a}(xy)^{*_a}) \qquad \text{(by Lemma 4.1 (5))} \\
&= x((xy)y^{*_a *_a}) \qquad \text{(by Lemma 4.1 (4))} \\
&= x((xy)y) \\
&= 1.
\end{aligned}$$

Thus $(xy)^{*_a *_a} \leq xy$. So $(xy)^{*_a *_a} = xy$ and $xy \in \mathcal{R}_a(L)$. □

From Theorem 4.3 and Theorem 4.4, we know that $\mathcal{R}_a(L)$ is a subalgebra of a semilattice implication algebra L.

Theorem 4.5. *Let L be a semilattice implication algebra and $a \in L$. Then $\mathcal{R}_a(L)$ is a join-semilattice with $x \vee_{\mathcal{R}_a(L)} y = (x^{*_a} \wedge y^{*_a})^{*_a}$ for every $x, y \in \mathcal{R}_a(L)$.*

Proof. Let $x, y \in \mathcal{R}_a(L)$. Since L is a semilattice, there exists $x^{*_a} \wedge y^{*_a}$. Then by Lemma 4.2 (4), there exists $(x^{*_a} \wedge y^{*_a})^{*_a}$. Also, since $x^{*_a} \wedge y^{*_a} \leq x^{*_a}$ and $x^{*_a} \wedge y^{*_a} \leq y^{*_a}$, $x = x^{*_a *_a} \leq (x^{*_a} \wedge y^{*_a})^{*_a}$ and $y = y^{*_a *_a} \leq (x^{*_a} \wedge y^{*_a})^{*_a}$.

If u is an upper bound of x and y in $\mathcal{R}_a(L)$, then $u^{*_a} \leq x^{*_a}$ and $u^{*_a} \leq y^{*_a}$ by Lemma 4.1 (2). Thus $u^{*_a} \leq x^{*_a} \wedge y^{*_a}$. So $(x^{*_a} \wedge y^{*_a})^{*_a} \leq u^{*_a *_a} = u$. Hence $(x^{*_a} \wedge y^{*_a})^{*_a}$ is the least upper bound of x and y in $\mathcal{R}_a(L)$. □

Let L be a semilattice implication algebra and $a \in L$. Then by Theorem 4.3 and Theorem 4.5, $\mathcal{R}_a(L)$ is a lattice with $x \wedge_{\mathcal{R}_a(L)} y = x \wedge y$ and $x \vee_{\mathcal{R}_a(L)} y = (x^{*_a} \wedge y^{*_a})^{*_a}$ for every $x, y \in \mathcal{R}_a(L)$.

Lemma 4.6. *Let L be a semilattice implication algebra and $a \in L$. Then for every $x, y, z \in \mathcal{R}_a(L)$,*

(1) $(x \vee_{\mathcal{R}_a(L)} y)^{*_a} = x^{*_a} \wedge y^{*_a}$,
(2) $(x \wedge y)^{*_a} = x^{*_a} \vee_{\mathcal{R}_a(L)} y^{*_a}$,
(3) $(x \vee_{\mathcal{R}_a(L)} y)z = (xz) \wedge (yz)$.

Proof. (1) Let $x, y \in \mathcal{R}_a(L)$. Then by Lemma 4.2 (4) and Theorem 4.3, we get

$$x^{*_a},\ y^{*_a} \in \mathcal{R}_a(L) \text{ and } x^{*_a} \wedge y^{*_a} \in \mathcal{R}_a(L).$$

Thus by Theorem 4.5, $(x \vee_{\mathcal{R}_a(L)} y)^{*_a} = (x^{*_a} \wedge y^{*_a})^{*_a *_a} = x^{*_a} \wedge y^{*_a}$.

(2) Let $x, y \in \mathcal{R}_a(L)$. Then $x^{*_a}, y^{*_a} \in \mathcal{R}_a(L)$. Thus by Theorem 4.5, we have

$$x^{*_a} \vee_{\mathcal{R}_a(L)} y^{*_a} = (x^{*_a *_a} \wedge y^{*_a *_a})^{*_a} = (x \wedge y)^{*_a}.$$

(3) Let $x, y, z \in \mathcal{R}_a(L)$. Then $x \vee_{\mathcal{R}_a(L)} y \in \mathcal{R}_a(L)$, and we have

$$\begin{aligned} (x \vee_{\mathcal{R}_a(L)} y)z &= z^{*_a}(x \vee_{\mathcal{R}_a(L)} y)^{*_a} && \text{(by Lemma 4.2 (2))} \\ &= z^{*_a}(x^{*_a} \wedge y^{*_a}) && \text{(by (1) of this lemma)} \\ &= (z^{*_a}x^{*_a}) \wedge (z^{*_a}y^{*_a}) && \text{(by (SI1))} \\ &= (xz) \wedge (yz) && \text{(by Lemma 4.2 (2))}. \end{aligned}$$

□

Theorem 4.7. *Let L be a semilattice implication algebra and $a \in L$. Then the following are equivalent:*

(1) *$x^{*_a}x = x$ for every $x \in \mathcal{R}_a(L)$,*
(2) *$x \vee_{\mathcal{R}_a(L)} y = x^{*_a}y$ and $x \wedge_{\mathcal{R}_a(L)} y = x \wedge y = (xy^{*_a})^{*_a}$ for every $x, y \in \mathcal{R}_a(L)$.*

Proof. (1)⇒(2): Suppose (1) holds and let $x, y \in \mathcal{R}_a(L)$. Then $x^{*_a}y \in \mathcal{R}_a(L)$ since $x^{*_a}, y \in \mathcal{R}_a(L)$. Also $a \leq y$ implies $x^{*_a} = xa \leq xy$ and $(xy)y \leq x^{*_a}y$. This implies $x^{*_a}y$ is an upper bound of x and y in $\mathcal{R}_a(L)$ since $(xy)y$ is an upper bound of x and y.

Suppose u is an upper bound of x and y in $\mathcal{R}_a(L)$. Then $u^{*_a} \leq x^{*_a}$ and $y \leq u$. This implies $x^{*_a}y \leq u^{*_a}y \leq u^{*_a}u$. Since $u^{*_a}u = u$, by the hypothesis, $x^{*_a}y \leq u$. Thus $x^{*_a}y$ is the least upper bound of x and y in $\mathcal{R}_a(L)$, i.e., $x \vee_{\mathcal{R}_a(L)} y = x^{*_a}y$. Moreover, by Lemma 4.6 (2), we have

$$x \wedge y = (x \wedge y)^{*_a *_a} = (x^{*_a} \vee_{\mathcal{R}_a(L)} y^{*_a})^{*_a} = (x^{*_a *_a}y^{*_a})^{*_a} = (xy^{*_a})^{*_a}.$$

(2)⇒(1): Suppose (2) holds and let $x \in \mathcal{R}_a(L)$. Then $x^{*_a}x = x \vee_{\mathcal{R}_a(L)} x = x$ by the hypothesis. □

In Example 3.8, $\mathcal{R}_0(A) = A$, but $a \vee d = d \neq 1 = dd = (a0)d = a^{*_0}d$, and $a^{*_0}a = (a0)a = da = c \neq a$.

In (2) of Example 3.2, $\mathcal{R}_\varnothing(\mathcal{U})$ is the family of all regular open sets in the topology $\mathcal{U}$ and it satisfies the property $U^{*_\varnothing}U = U$ for every $U \in \mathcal{R}_\varnothing(\mathcal{U})$, hence $U \vee_{\mathcal{R}_\varnothing(\mathcal{U})} V = U^{*_\varnothing}V$ for every $U, V \in \mathcal{R}_\varnothing(\mathcal{U})$.

Theorem 4.8. *Let L be a semilattice implication algebra and $a \in L$. If $x^{*_a}x = x$ for every $x \in \mathcal{R}_a(L)$, then $\mathcal{R}_a(L)$ is distributive.*

Proof. Let $x, y, z \in \mathcal{R}_a(L)$. Then we have

$$\begin{aligned} x \wedge (y \vee_{\mathcal{R}_a(L)} z) &= (x(y \vee_{\mathcal{R}_a(L)} z)^{*_a})^{*_a} && \text{(by Theorem 4.7)} \\ &= ((y \vee_{\mathcal{R}_a(L)} z)x^{*_a})^{*_a} && \text{(by Lemma 4.1 (4))} \\ &= ((yx^{*_a}) \wedge (zx^{*_a}))^{*_a} && \text{(by Lemma 4.6 (3))} \\ &= (yx^{*_a})^{*_a} \vee_{\mathcal{R}_a(L)} (zx^{*_a})^{*_a} && \text{(by Lemma 4.6 (2))} \\ &= (x \wedge y) \vee_{\mathcal{R}_a(L)} (x \wedge z) && \text{(by Theorem 4.7).} \end{aligned}$$

Hence $\mathcal{R}_a(L)$ is distributive. □

The converse direction of Theorem 4.8 is not true in general. For example, the semilattice implication algebra $A = \mathcal{R}_0(A)$ in Example 3.8 is distributive, but $a^{*_0}a = da = c \neq a$.

5. Multipliers and filters of semilattice implication algebras

Let L be a semilattice implication algebra. A map $\varphi : L \to L$ is called a *multiplier* of L, if it satisfies

$$\varphi(xy) = x\varphi(y)$$

for every $x, y \in L$.

Example 5.1. (1) Let L be the semilattice implication algebra and $a \in L$. If we define a map $\varphi_a : L \to L$ by

$$\varphi_a(x) = ax$$

for every $x \in L$. Then φ_a is a multiplier of L.

(2) Let $L = P \oplus I_0$ be the semilattice implication algebra of Example 3.2 (3). If we define a map $\varphi : L \to L$ by

$$\varphi(x) = \begin{cases} 1, & \text{if } x = 0, a, b \\ x, & \text{if } x \in I_0 \end{cases}$$

for every $x \in L$, then φ is a multiplier of L.

The multiplier φ_a in Example 5.1 (1) is called *simple multiplier* induced by a.

Lemma 5.2. *Let φ be a multipliers of a semilattice implication algebra L. Then it satisfies the following:*

(1) $\varphi(1) = 1$,
(2) $x \leq \varphi(x)$ *for every* $x \in L$.

Proof. (1) Let φ be a multiplier of L. Since 1 is the greatest element in L, $1 = \varphi(1)1$. Then $\varphi(1) = \varphi(\varphi(1)1) = \varphi(1)\varphi(1) = 1$.

(2) Let $x \in L$. Then $x\varphi(x) = \varphi(xx) = \varphi(1) = 1$ by (1) of this lemma. Thus $x \le \varphi(x)$. □

Lemma 5.3. *Let φ and ψ be multipliers of a semilattice implication algebra L. Then*

(1) *the composition $\psi\varphi := \psi \circ \varphi$ is a multiplier of L,*
(2) *if φ and ψ are monotone, then $\psi\varphi$ is monotone.*

Proof. It is clear from the definitions of multiplier and monotonity. □

Lemma 5.4. *Let L be a semilattice implication algebra. Then for each $a, b \in L$,*

(1) *φ_a is monotone,*
(2) *$\varphi_a\varphi_b = \varphi_b\varphi_a$.*

Proof. (1) Let $x \le y$ in L. Then $\varphi_a(x) = ax \le ay = \varphi(y)$ by Lemma 3.5 (4). Thus φ is monotone.

(2) Let $a, b \in L$. Then

$$(\varphi_a\varphi_b)(x) = \varphi_a(\varphi_b(x)) = a(bx) = b(ax) = \varphi_b(\varphi_a(x)) = (\varphi_b\varphi_a)(x)$$

for every $x \in L$. Thus$\varphi_a\varphi_b = \varphi_b\varphi_a$. □

The multiplier φ in Example 5.1 (2) is not simple, because it is not monotone. In fact, $a \le x$ for some $x \in I_0$ but $\varphi(a) = 1 \not\le x = \varphi(x)$. Then the multiplier φ is not simple by Lemma 5.4.

We will denote the family of all multipliers of L by $\mathbf{Mul}(L)$ and the family of all monotone multipliers of L by $\mathbf{mMul}(L)$. Then $\mathbf{Mul}(L)$ and $\mathbf{mMul}$ are semigroups under the composition $\circ$ of functions by Lemma 5.3, and has identity φ_1. In particular, $\mathbf{mMul}$ is a subsemigroup of $\mathbf{Mul}(L)$.

Let L be the semilattice implication algebra. A subset F of L is call a *filter* of L, if it satisfies the following: for any $x, y \in L$,

(i) $1 \in F$,
(ii) $x \in F$ and $xy \in F$ imply $y \in F$.

Lemma 5.5. *Let L be a semilattice implication algebra and F a filter of L. Then*

(1) *$\varphi(F) \subseteq F$ for ever multiplier φ of L,*
(2) *$\varphi_a^{-1}(F) = F$ for every $a \in F$.*

Proof. (1) Let $y \in \varphi(F)$. Then $y = \varphi(x)$ for some $x \in F$ and

$$xy = x\varphi(x) = \varphi(xx) = \varphi(1) = 1 \in F.$$

Since F is a filter and $x \in F$, $y \in F$. Thus $y = \varphi(x) \subseteq F$.

(2) Let $a \in F$. Then it is clear $F \subseteq \varphi_a^{-1}(F)$. If $x \in \varphi_a^{-1}(F)$, then $ax = \varphi_a(x) \in F$, and since $a \in F$ and F is a filter of L, $x \in F$. This implies $\varphi_a^{-1}(F) \subseteq F$. Thus $\varphi_a^{-1}(F) = F$ □

Theorem 5.6. *Let L be a semilattice implication algebra and $\mathcal{S}$ a subsemigroup of* $\mathbf{mMul}(L)$. *Then $F_{\mathcal{S}} := \{x \in L \mid \varphi(x) = 1$ for some $\varphi \in \mathcal{S}\}$ is a filter of L.*

Proof. For any $\varphi \in \mathcal{S}$, $\varphi(1) = 1$. Then $1 \in F_{\mathcal{S}}$.

Let $x \in F_{\mathcal{S}}$ and $xy \in F_{\mathcal{S}}$. Then $\varphi(x) = 1$ and $\psi(xy) = 1$ for some $\varphi, \psi \in \mathcal{S}$. Since $x\psi(y) = \psi(xy) = 1$ and φ is monotone, $x \leq \psi(y)$ and $1 = \varphi(x) \leq \varphi(\psi(y))$. This implies $(\varphi\psi)(y) = \varphi(\psi(y)) = 1$ and $\varphi\psi \in \mathcal{S}$ since $\mathcal{S}$ is a subsemigroup of $\mathbf{mMul}(L)$. Thus $y \in F_{\mathcal{S}}$. □

Let L be a semilatiice implication algebra and $S \subseteq L$. If $\mathcal{M}(S)$ is the subsemigroup of $\mathbf{Mul}(L)$ generated by the set

$$s(S) := \{\varphi_s \mid s \in S\}$$

of simple multipliers induced by $s \in S$, then

$$\mathcal{M}(S) = \{\varphi_{s_1}\varphi_{s_2}\cdots\varphi_{s_n} \mid \text{ for some } s_1, s_2, \cdots s_n \in S \text{ and for some } n \in \mathbb{N}\}$$

and $\mathcal{M}(S)$ is a subsemigroup of $\mathbf{mMul}(L)$ by Lemma 5.4 (1) and Lemma 5.3 (2).

Theorem 5.7. *Let L be a semilattice implication algebra and $S \subseteq L$. Then*

$$F_{\mathcal{M}(S)} = \{x \in L \mid \varphi(x) = 1 \textit{ for some } \varphi \in \mathcal{M}(S)\}$$

is the smallest filter of L containing S.

Proof. Since $\mathcal{M}(S)$ is a subgroup of $\mathbf{mMul}(L)$, $F_{\mathcal{M}(S)}$ is a filter by Theorem 5.6. Let $s \in S$. Then there is a multiplier $\varphi_s \in \mathcal{M}(S)$ such that $\varphi_s(s) = ss = 1$. Thus $s \in F_{\mathcal{M}(S)}$. So $S \subseteq F_{\mathcal{M}(S)}$.

Suppose that G be a filter of L and $S \subseteq G$. Let $x \in F_{\mathcal{M}(S)}$. Then there is a multiplier $\varphi = \varphi_{s_1}\cdots\varphi_{s_1} \in \mathcal{M}(S)$ such that $s_1, s_2, \cdots, s_n \in S$ and

$$(\varphi_{s_1}\cdots\varphi_{s_n})(x) = 1 \in G.$$

Since $s_i \in S \subseteq G$ for each $i = 1, 2, \cdots, n$, $x \in \varphi_{s_n}^{-1}(\cdots(\varphi_{s_1}^{-1}(G))\cdots) = G$ by Lemma 5.5 (2). Thus $F_{\mathcal{M}(S)} \subseteq G$. So $F_{\mathcal{M}(S)}$ is the smallest filter of L containing S. □

The set $F_{\mathcal{M}(S)}$ is the filter generated by a subset S of L from Theorem 5.7.

6. Conclusions

We defined the notion of semilattice implication algebras, which is a generalization of lattice implication algebras and Heyting semilattices, and researched the common properties of those algebras. In section 4, the regular element was defined and it was proved that the set of all regular elements is a distributive lattice. In section 5, we gave some relationships of filters and multipliers, and gave a filter construced by a subsemigroup in the semigroup of monotone multipliers. In particular we showed that the filter generated by a subset was charaterized by simple multipliers.

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