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Separation axioms in interval-valued soft topological spaces

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Abstract. Our purpose of the research is to investigate two aspects: First, we introduce the concept of separation axioms in an interval-valued soft topological space, and obtain some of its properties and give some examples. Second, we propose new separation axioms in an interval-valued soft topological space by using interval-valued α-open sets basic topological structures based on interval-valued soft set, and study some of their basic properties. Furthermore, we deal with hereditary problems of each separation axiom.

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Keywords: Interval-valued soft topological space, IVST_i(j)-space (i=0,1,2,3,4; j=i,ii), Interval-valued α-open [resp. closed] set, IVSαT_i(j)-space (i=0,1,2,3,4; j=i,ii).

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1. Introduction

In 1999, Molodtsov [1] introduced the concept of soft sets which has rich potential for practical applications in several domains as a tool for dealing with uncertainties. After then, Maji et al. [2] proposed some basic operations on soft sets and studied some of their properties (See [3, 4, 5] for the further researches). Moreover, many researchers have applied the notion of soft sets to various fields, for example abstract algebra (See [6, 7, 8, 9]), logical algebra (See [10, 11, 12]), decision making problems (See [13, 14, 15]), topology (See [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]), topological group (See [27, 28, 29, 30, 31, 32]), etc. In particular, Bayramov et al. [33] introduced the concepts of interval-valued fuzzy soft topological spaces, interval-valued fuzzy soft neighborhoods, interval-valued fuzzy soft continuities, etc. and studied some of their properties.

Topology is an important area of mathematics with many applications in the domains of computer and physical science. In particular, separation axioms in topology not only provide a systematic way to classify and study topological spaces but also ensure that the spaces behave in predictable and desirable ways for further mathematical exploration and application. Moreover, understanding the separation properties of a space is often the first step in analyzing its structure and behavior in both theoretical and applied settings. Then their necessity and importance cannot be overstated. In 2021, Lee et al. [34] investigated interval-valued soft topological structures as a generalization of soft topologies.

We intend to study in the following two aspects: First, we define separation axioms in an interval-valued soft topological space by modifying separation axioms introduced by Shabir and Naz [16]. Second, we introduce the concept of separation axioms in an interval-valued soft topological space by modifying separation axioms proposed by Akdag and Ozkan [35]. In order to realize our aim, this paper is composed of six sections. In Section 2, we recall some basic concepts required in each section. In Section 3, we introduce the notions of separation axioms in an interval-valued soft topological space in the sense of Shabir and Naz [16], and obtain its basic properties and give some examples. In Section 4, we deal with hereditary properties of separation axioms discussed in Section 3. In Section 5, we propose the concept of interval-valued soft α-open set by modifying soft α-open set introduced by Akdag and Ozkan [36] and study some of its properties. In Section 6, we define separation axioms in an interval-valued soft topological space in the viewpoint of Akdag and Ozkan [35] and discuss some of their properties.

2. Preliminaries

In this section, we recall basic concepts needed in next sections. Throughout this paper, let $X,\ Y,\ Z, \cdots$ denote non-empty universe sets, let $E,\ E',\ E'', \cdots$ denote non-empty sets of parameters and let 2^X denote the power set of X.

Definition 2.1 ([37, 38])**.** The form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an *interval-valued set* (briefly, IVS) or *interval set* in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the *interval-valued empty* [resp. *whole*] *set* in X and will be denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for a classical subset A of X. Then we consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_{_A} = [\chi_{_{A^-}}, \chi_{_{A^+}}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [39], where $\chi_{_A}$ denotes the characteristic function of A. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 ([37, 38])**.** Let $A,\ B \in IVS(X)$. Then
(i) we say that A *contained in* B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,
(ii) we say that A *equals to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,
(iii) the *complement* of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the *union* of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the *intersection* of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

Definition 2.3 ([37])**.** Let $a \in X$ and $A \in IVS(X)$. Then the notation $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an *interval-valued* [resp. *vanishing*] *point* in X and denoted by a_1 [resp. a_0]. We denote the set of all interval-valued points in X as $IV_P(X)$.
(i) We say that a_1 *belongs to* A, denoted by $a_1 \in A$, if $a \in A^-$.
(ii) We say that a_0 *belongs to* A, denoted by $a_0 \in A$, if $a \in A^+$.

Definition 2.4 ([37])**.** Let τ be a non-empty family of IVSs on X. Then τ is called an *interval-valued topology* (briefly, IVT) on X, if it satisfies the following axioms:
(IVO_1) $\widetilde{\varnothing},\ \widetilde{X} \in \tau$,
(IVO_2) $A \cap B \in \tau$ for any $A,\ B \in \tau$,
(IVO_3) $\bigcup_{j \in J} A_j \in \tau$ for any family $(A_j)_{j \in J}$ of members of τ.
In this case, the pair (X, τ) is called an *interval-valued topological space* (briefly, IVTS) and each member of τ is called an *interval-valued open set* (briefly, IVOS) in X. An IVS A is called an *interval-valued closed set* (briefly, IVCS) in X, if $A^c \in \tau$.

It is obvious that $\{\widetilde{\varnothing}, \widetilde{X}\}$ is an IVT on X, and is called the interval-valued indiscrete topology on X and denoted by $\tau_{IV,0}$. Also $IVS(X)$ is an IVT on X, and is called the *interval-valued discrete topology* on X and is denoted by $\tau_{IV,1}$. The pair $(X, \tau_{IV,0})$ [resp. $(X, \tau_{IV,1})$] is called the *interval-valued indiscrete* [resp. *discrete*] *space*.

We denote the set of all IVTs on X as $IVT(X)$. For an IVTS X, we denote the set of all IVOSs [resp. IVCSs] in X as $IVO(X)$ [resp. $IVC(X)$].

Definition 2.5 ([1, 17])**.** An F_A is called a *soft set* over X, if $F_A : A \to 2^X$ is a mapping such that $F_A(e) = \varnothing$ for each $e \notin A$, where $A \in 2^E$.
In other words, a soft set over X is a parametrized family of subsets of X. For each $e \in A$, $F_A(e)$ may be considered as the set of e-approximate elements of the soft set F_A. It is clear that a soft set is not a set. We will denote the set of all soft sets over X as $SS(X)$ while $SS(X)_E$ will denote the set of all soft sets over with respect to a fixed set E of parameters.

It has been well-known [1] that every Zadeh's fuzzy set A may be considered as the soft set $F_{[0,1]}$.

Definition 2.6 ([2, 17])**.** Let $F_A,\ F_B \in SS(X)$ and $A,\ B \in 2^E$. Then we say that:
(i) F_A is a *soft subset* of F_B, denoted by $F_A \widetilde{\subset} F_B$, if $A \subset B$ and $F_A(e) \subset F_B(e)$ for each $e \in A$,
(ii) F_A is a *soft super set* of F_B, denoted by $F_A \widetilde{\supset} F_B$, if $F_B \widetilde{\subset} F_A$,
(iii) F_A and F_B are *soft equal*, if $F_A \widetilde{\subset} F_B$ and $F_A \widetilde{\supset} F_B$.

Definition 2.7 ([2, 3])**.** Let $F_A \in SS(X)$. Then F_A is called:
(i) a *null soft set* or a *relative null soft set* (with respect to A), denoted by $\varnothing_A$, if $F_A(e) = \varnothing$ for each $e \in A$,
(ii) an *absolute soft set* or a *relative whole soft set* (with respect to A), denoted by X_A, if $F_A(e) = X$ for each $e \in A$.
We will denote the null [resp. absolute] soft set in $SS_E(X)$ by X_E [resp. $\varnothing_E$].

Definition 2.8 ([16])**.** Let τ be a collection of members of $SS_E(X)$. Then τ is called a *soft topology* on X, if it satisfies the following conditions:
(i) $\varnothing_E,\ X_E \in \tau$,
(ii) $A \cap B \in \tau$ for any $A, B \in \tau$,
(iii) $\bigcup_{j\in J} A_j \in \tau$ for each $(A_j)_{j\in J} \subset \tau$, where J denotes an index set.
Each member of τ is called a *soft open set in* X and the complement of each member of τ is called a *closed soft set in* X. The triple (X, τ, E) is called a *soft topological space* over X.

Definition 2.9 ([34])**.** Let $A \in 2^E$. Then an $\mathbf{F}_A = [F_A^-, F_A^+]$ is called an *interval-valued soft set* (briefly, IVSS) over X, if $\mathbf{F}_A : A \to IVS(X)$ is a mapping such that $\mathbf{F}_A(e) = \widetilde{\varnothing}$ for each $e \notin A$, i.e., $F_A^-,\ F_A^+ \in SS(X)$ such that $F_A^-(e) \subset F_A^+(e)$ for each $e \in A$.

In other words, an IVSS over X is a parametrized family of IVSs of X. For each $e \in A$, $\mathbf{F}_A(e) = [F_A^-(e), F_A^+(e)]$ may be considered as an interval-valued set of e-approximate elements of the IVSS $\mathbf{F}_A$. We denote the set of all IVSSs over X as $IVSS(X)$.

It is obvious that if $F_A \in SS(X)$, then $[F_A, F_A] \in IVSS(X)$. Now we can see that an IVSS is the generalization of a soft set. Moreover, if $\mathbf{F}_A \in IVSS(X)$, then clearly, $\chi_{\mathbf{F}_A}$ is an interval-valued fuzzy soft set (briefly, IVFSS) over X introduced by Yang et al. [40]. Thus an IVSS is the special case of an IVFSS.

Definition 2.10 ([34])**.** Let $A \in 2^E$ and $\mathbf{F}_A \in IVSS(X)$. $\mathbf{F}_A$ is called:
(i) a *relative null interval-valued soft set* (with respect to A), denoted by $\widetilde{\varnothing}_A$, if $\mathbf{F}_A(e) = \widetilde{\varnothing}$ for each $e \in A$,
(ii) a *relative whole interval-valued soft set* (with respect to A), denoted by $\widetilde{X}_A$, if $\mathbf{F}_A(e) = \widetilde{X}$ for each $e \in A$.
We denote the set of all IVSSs over X with respect to the fixed parameter set A as $IVSS_A(X)$.

Now we will denote the members of $IVSS_E(X)$ by $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \cdots$. In fact, $\mathbf{A},\ \mathbf{B},\ \mathbf{C} : E \to IVS(X)$. In particular, the interval-valued soft empty [resp. whole] set over X with respect to E, denoted by $\widetilde{\varnothing}_E$ [resp. $\widetilde{X}_E$], is the IVS in X defined by $\widetilde{\varnothing}_E(e) = \widetilde{\varnothing}$ [resp. $\widetilde{X}_E(e) = \widetilde{X}$] for each $e \in E$.

Definition 2.11 ([34])**.** Let $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. We say that

(i) $\mathbf{A}$ is an *interval-valued soft subset* of $\mathbf{B}$, denoted by $\mathbf{A} \subset \mathbf{B}$, if $\mathbf{A}(e) \subset \mathbf{B}(e)$ for each $e \in E$,

(ii) $\mathbf{A}$ and $\mathbf{B}$ are *interval-valued soft equal*, denoted by $\mathbf{A} = \mathbf{B}$, if $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$,

(iii) the *interval-valued soft complement* of $\mathbf{A}$, denoted by $\mathbf{A}^c$, is the mapping $\mathbf{A}^c : E \to IVS(X)$ defined as: for each $e \in E$,

$$\mathbf{A}^c(e) = (\mathbf{A}(e))^c.$$

3. Interval-valued soft separation axioms

In this section, we introduce the concepts of separation axioms in interval-valued soft topological spaces, for example, IVST$_j$(i)-space for $j = 0,\ 1,\ 2,\ 3,\ 4$ and $i = 1,\ 2$. And we study some of their properties and some relationships among them. Also, we give some examples.

Definition 3.1. Let $\mathbf{A} \in IVSS_E(X)$ and $x \in X$.

(i) $x_{_1}$ said to *belong* or *totally belong* to $\mathbf{A}$, denoted by $x_{_1} \in \mathbf{A}$, if $x \in A^-(e)$ for each $e \in E$.

(ii) $x_{_0}$ said to *belong* or *totally belong* to $\mathbf{A}$, denoted by $x_{_0} \in \mathbf{A}$, if $x \in A^+(e)$ for each $e \in E$.

Note that for any $x \in X$, $x_{_1} \notin \mathbf{A}$ [resp. $x_{_0} \notin \mathbf{A}$], if $x \notin A^-(e)$ [resp. $x \notin A^+(e)$] for some $e \in E$. It is obvious that if $x_{_1} \in \mathbf{A}$, then $x_{_0} \in \mathbf{A}$. But the converse is not true in general (See Example 3.2).

Example 3.2. Let $X = \{a, b, c, x, y, z\}$ be a universe set and $E = \{e, f, g\}$ a set of parameters. Consider the IVSS $\mathbf{A}$ defined by:

$$\mathbf{A}(e) = [\{a, b\}, \{a, b, c\}],\ \mathbf{A}(f) = [\{a\}, \{a, c, z\}],\ \mathbf{A}(g) = [\{a, c, x\}, \{a, c, x\}].$$

Then it can be easily checked that $a_{_1},\ a_{_0} \in \mathbf{A}$, $c_{_0} \in \mathbf{A}$ but $b_{_1},\ c_{_1},\ x_{_1},\ y_{_1},\ z_{_1} \notin \mathbf{A}$ and $b_0,\ x_{_0}\ y_0,\ z_0 \notin \mathbf{A}$.

Definition 3.3 ([34])**.** Let τ be a family of IVSSs over X with respect to E. Then τ is called an *interval-valued soft topology* (briefly, IVST) *on* X *with respect to* E, if it satisfies the following axioms:

[IVSO$_1$] $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau$,

[IVSO$_2$] $\mathbf{A} \cap \mathbf{B} \in \tau$ for any $\mathbf{A},\ \mathbf{B} \in \tau$,

[IVSO$_3$] $\bigcup_{j \in J} \mathbf{A}_j \in \tau$ for each $(\mathbf{A}_j)_{j \in J} \subset \tau$.

The triple (X, τ, E) is called an *interval-valued soft topological space* (briefly, IVSTS). Every member of τ is called an *interval-valued soft open set* (briefly, IVSOS) and the complement of an IVSOS is called an *interval-valued soft closed set* (briefly, IVSCS) in X. The set of all IVSOSs [resp. IVSCSs] in X is denoted by $IVSO(X)$ [resp. $IVSC(X)$]. It is obvious that $\{\widetilde{\varnothing}_E, \widetilde{X}_E\}$, $IVSS_E(X) \in IVST_E(X)$, where $IVST_E(X)$ denotes the set of all IVSTs on X with respect to E. In this case, $\{\widetilde{\varnothing}_E, \widetilde{X}_E\}$ [resp. $IVSS_E(X)$] is called an *interval-valued soft indiscrete* [resp. *discrete*] *topology* on X and is denoted by $\widetilde{\tau}_{_0}$ [resp. $\widetilde{\tau}_{_1}$]. We will denote the set of all IVSTSs over X with respect to E as $IVSTS_E(X)$ and denote the set of all IVSCSs in an IVSTS (X, τ, E) by τ^c. In fact,

$$\tau^c = \{\mathbf{A} \in IVSS_E(X) : \mathbf{A}^c \in \tau\}.$$

It is obvious that if $\tau \in IVST_E(X)$, then $\chi_\tau = \{\chi_{\mathbf{U}} : \mathbf{U} \in \tau\}$ is an interval-valued fuzzy soft topology (briefly, IVFST) on X defined by Ali et al. [41]. Thus an IVFST is the generalization of an IVST.

From Remark 4.3 in [34], we can consider (X, τ^-, τ^+) as a soft bi-topological space in the sense of Kelly [42] for each $\tau \in IVST_E(X)$, where

$$\tau^- = \{U^- \in IVS(X) : \mathbf{U} \in \tau\},\ \tau^+ = \{U^+ \in IVS(X) : \mathbf{U} \in \tau\}.$$

Result 3.4 (Proposition 4.5, [34])**.** *Let* (X, τ, E) *be an IVSTS and for each* $e \in E$, *let*

$$\tau_e = \{\mathbf{U}(e) \in IVS(X) : \mathbf{U} \in \tau\}.$$

Then τ_e *is an interval-valued topology* (briefly, IVT) *on* X *introduced by Kim et al.* [37]. In this case, τ_e will be called a *parametric interval-valued topology* and (X, τ_e) will be called a *parametric interval-valued topological space*.

Furthermore, we obtain two classical topologies on X for each IVSTS (X, τ, E) and each $e \in E$ given as follows (See Remark 4.6 (1), [34]):

$$\tau_e^- = \{A(e)^- \in 2^X : \mathbf{A}(e) \in \tau_e\} \text{ and } \tau_e^+ = \{A(e)^+ \in 2^X : \mathbf{A}(e) \in \tau_e\}.$$

In this case, τ_e^- and τ_e^+ will be called *parametric topologies* on X.

Definition 3.5. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft* T_0(i)*-space* (briefly, IVST_0(i)-space), if for any $x,\ y \in X$ with $x \neq y$, there is $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_{_1} \in \mathbf{U}$, $y_{_1} \notin \mathbf{U}$ or $y_{_1} \in \mathbf{V}$, $x_{_1} \notin \mathbf{V}$,

(ii) *interval-valued soft* T_0(ii)*-space* (briefly, IVST_0(ii)-space), if for any $x,\ y \in X$ with $x \neq y$, there is $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_{_0} \in \mathbf{U}$, $y_{_0} \notin \mathbf{U}$ or $y_{_0} \in \mathbf{V}$, $x_{_0} \notin \mathbf{V}$,

(iii) *interval-valued soft* T_1(i)*-space* (briefly, IVST_1(i)-space), if for any $x,\ y \in X$ with $x \neq y$, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \notin \mathbf{U}$ and $y_{_1} \in \mathbf{V}$, $x_{_1} \notin \mathbf{V}$,

(iv) *interval-valued soft* T_1(ii)*-space* (briefly, IVST_1(ii)-space), if for any $x,\ y \in X$ with $x \neq y$, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \notin \mathbf{U}$ and $y_{_0} \in \mathbf{V}$, $x_{_0} \notin \mathbf{V}$.

Remark 3.6. (1) Every IVST_1(i) [resp. IVST_1(ii)]-space is an IVST_0(i) [resp. IVST_0(ii)]-space. The converse is not true in general (See Example 3.7)

(2) Every IVST_0(i) [resp. IVST_1(i)]-space is an IVST_0(ii) [resp. IVST_1(ii)]-space. The converse is not true in general (See Example 3.7)

Example 3.7. Let $X = \{a, b, c\}$ be a universe set and $E = \{e, f\}$ a set of parameters.

(1) Consider IVST τ_1 on X given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = \mathbf{A}(f) = [\varnothing, \{c\}]$, $\mathbf{B}(e) = \mathbf{B}(f) = \{\varnothing, \{b, c\}]$,
$\mathbf{C}(e) = \mathbf{C}(f) = \{\{a\}, \{a, c\}]$, $\mathbf{D}(e) = \mathbf{D}(f) = [\{c\}, \{b, c\}]$,
$\mathbf{E}(e) = \mathbf{E}(f) = [\{a, b\}, X]$, $\mathbf{F}(e) = \mathbf{F}(f) = [\{a, c\}, X]$,
$\mathbf{G}(e) = \mathbf{G}(f) = [\{a\}, X]$.

Then clearly, (X, τ_1, E) is an IVST_0(i)-space. On the other hand, for $a \neq b \in X$, we cannot find $\mathbf{U} \in \tau_1$ such that $a_{_1} \in \mathbf{U}$, $b_{_1} \notin \mathbf{U}$. Thus X is not an IVST_1(i)-space.

(2) Consider IVST τ_2 on X given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{I}, \mathbf{J}, \widetilde{X}_E\},$$

where $\mathbf{A}(e)=\mathbf{A}(f)=[\varnothing,\{a\},\ \mathbf{B}(e)=\mathbf{B}(f)=\{\varnothing,\{b\}],\ \mathbf{C}(e)=\mathbf{C}(f)=\{\varnothing,\{c\}],$
$\mathbf{D}(e)=\mathbf{D}(f)=[\{a\}],\{a,c\}],\ \mathbf{E}(e)=\mathbf{E}(f)=[\{b\},\{a,b\}],$
$\mathbf{F}(e)=\mathbf{F}(f)=[\{c\},\{b,c\}],\ \mathbf{G}(e)=\mathbf{G}(f)=[\{a,b\},X],$
$\mathbf{H}(e)=\mathbf{H}(f)=[\{a,c\},X],\ \mathbf{I}(e)=\mathbf{I}(f)=[\{b,c\},X],$
$\mathbf{J}=(e)=\mathbf{J}(f)=[\varnothing,\{b,c\}].$

Then we can easily check that (X,τ_2,E) is an IVST_1(i)-space and also an IVST_1(ii)-space

(3) Consider IVST τ_3 on X given by:

$$\tau_3=\{\widetilde{\varnothing}_E,\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G},\mathbf{H},\widetilde{X}_E\},$$

where $\mathbf{A}(e)=\mathbf{A}(f)=[\varnothing,\{a\}],\ \mathbf{B}(e)=\mathbf{B}(f)=\{\{a\},\{a\}],$
$\mathbf{C}(e)=\mathbf{C}(f)=\{\{b\},\{b\}],\ \mathbf{D}(e)=\mathbf{D}(f)=[\{c\},\{a,c\}],$
$\mathbf{E}(e)=\mathbf{E}(f)=[\{a,b\},\{a,b\}],\ \mathbf{F}(e)=\mathbf{F}(f)=[\{a,c\},\{a,c\}],$
$\mathbf{G}(e)=\mathbf{G}(f)=[\{b,c\},X],\ \mathbf{H}(e)=\mathbf{H}(f)=[\{b\},\{a,b\}].$

Observe that (X,τ_3,E) is an IVST_0(ii)-space but there are not $\mathbf{U}$, $\mathbf{U}\in\tau_3$ such that $a_0\in\mathbf{U}$, $b_0\notin\mathbf{U}$ and $b_0\in\mathbf{V}$, $a_0\notin\mathbf{V}$ for $a\neq b\in X$. Then X is not an IVST_1(ii)-space.

(4) Consider IVST τ_4 on X given by:

$$\tau_4=\{\widetilde{\varnothing}_E,\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E},\mathbf{F},\mathbf{G},\widetilde{X}_E\},$$

where $\mathbf{A}(e)=\mathbf{A}(f)=[\varnothing,\{a\}],\ \mathbf{B}(e)=\mathbf{B}(f)=\{\varnothing,\{b\}],$
$\mathbf{C}(e)=\mathbf{C}(f)=\{\varnothing,\{c\}],\ \mathbf{D}(e)=\mathbf{D}(f)=[\varnothing,\{a,b\}],$
$\mathbf{E}(e)=\mathbf{E}(f)=[\varnothing,\{a,c\}],\ \mathbf{F}(e)=\mathbf{F}(f)=[\varnothing,\{b,c\}],$
$\mathbf{G}(e)=\mathbf{G}(f)[\varnothing,X].$

Observe that (X,τ,E) is an IVST_1(ii)-space but there are not $\mathbf{U}$, $\mathbf{U}\in\tau_4$ such that $a_1\in\mathbf{U}$, $b_1\notin\mathbf{U}$ and $b_1\in\mathbf{V}$, $a_1\notin\mathbf{V}$ for $a\neq b\in X$. Then X is not an IVST_1(i)-space.

Remark 3.8. (1) If (X,τ,E) is an IVST_0(i) [resp. IVST_1(i)]-space, then (X,τ^-,E) and (X,τ^+,E) are soft T_0 [resp. T_1]-spaces in the sense of Shabir and Naz [16].

(2) If (X,τ,E) is an IVST_0(ii) [resp. IVST_1(ii)]-space, then (X,τ^+,E) is a soft T_0 [resp. T_1]-space in the sense of Shabir and Naz [16].

Definition 3.9. Let X be a set, E be a set of parameters and $x\in X$. An IVSS $\mathbf{x_1}$ [resp. $\mathbf{x_0}$] is defined by $\mathbf{x_1}(e)=x_1$ [resp. $\mathbf{x_0}(e)=x_0$] for each $e\in E$.

Proposition 3.10. *Let (X,τ,E) be an IVSTS.*

(1) *If $\mathbf{x_1}$ is an IVSCS in X for each $x\in X$, then X is an IVST_1*(i)*-space.*

(2) *If $\mathbf{x_0}$ is an IVSCS in X for each $x\in X$, then X is an IVST_1*(ii)*-space.*

Proof. (1) Suppose $\mathbf{x_1}$ is an IVSCS in X for each $x\in X$ and let $x\neq y\in X$. Then clearly, ${\mathbf{x_1}}^c=[X\setminus\{x\},X\setminus\{x\}]\in\tau$ and ${\mathbf{y_1}}^c=[X\setminus\{y\},X\setminus\{y\}]\in\tau$. Moreover, $x_1\notin{\mathbf{x_1}}^c$, $y_1\in{\mathbf{x_1}}^c$ and $y_1\notin{\mathbf{y_1}}^c$, $x_1\in{\mathbf{y_1}}^c$. Thus X is IVST_1(i)-space.

(2) The poof is similar to (1). □

Remark 3.11. The converses of Proposition 3.10 is not true in general (See Example 3.12).

Example 3.12. Let $X=\{x,y\}$ and $E=\{e,f\}$.

(1) Consider the collection τ_1 of IVSSs over X given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = \widetilde{X}$, $\mathbf{A}(f) = [\{y\}, \{y\}]$, $\mathbf{B}(e) = [\{x\}, \{x\}]$, $\mathbf{B}(f) = \widetilde{X}$,
$\mathbf{C}(e) = [\{x\}, \{x\}]$, $\mathbf{C}(f) = [\{y\}, \{y\}]$.
Then clearly, (X, τ_1, E) is an IVSTS on X. Moreover, $\mathbf{A}$, $\mathbf{B} \in \tau$ such that $y_{_1} \in \mathbf{A}$, $x_{_1} \notin \mathbf{A}$ and $x_{_1} \in \mathbf{B}$, $y_{_1} \notin \mathbf{B}$. Thus (X, τ_1, E) is an IVST_1(i)-space.

On the other hand, since

$$\mathbf{x_1}(e) = \mathbf{x_1}(f) = x_{_1} = [\{x\}, \{x\}]$$

and

$$\mathbf{y_1}(e) = \mathbf{y_1}(f) = y_{_1} = [\{y\}, \{y\}],$$

we have

$$\mathbf{x_1}^c(e) = \mathbf{x_1}^c(f) = y_{_1} = [\{y\}, \{y\}]$$

and

$$\mathbf{y_1}^c(e) = \mathbf{y_1}^c(f) = x_{_1} = [\{x\}, \{x\}].$$

So $\mathbf{x_1}^c$, $\mathbf{y_1}^c \notin \tau_1$. Hence the converse of Proposition 3.10 (1) does not hold.

(2) Consider the collection τ_2 of IVSSs over X given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = [\varnothing, X]$, $\mathbf{A}(f) = [\varnothing, \{y\}]$, $\mathbf{B}(e) = [\varnothing, \{x\}]$, $\mathbf{B}(f) = [\varnothing, X]$,
$\mathbf{C}(e) = [\varnothing, \{x\}]$, $\mathbf{C}(f) = [\varnothing, \{y\}]$, $\mathbf{D}(e) = \mathbf{D}(f) = [\varnothing, X]$.
Then clearly, (X, τ_2, E) is an IVSTS on X. Moreover, $\mathbf{A}$, $\mathbf{B} \in \tau$ such that $y_{_0} \in \mathbf{A}$, $x_{_0} \notin \mathbf{A}$ and $x_{_0} \in \mathbf{B}$, $y_{_0} \notin \mathbf{B}$. Thus (X, τ_2, E) is an IVST_1(ii)-space.

On the other hand, since

$$\mathbf{x_0}(e) = \mathbf{x_0}(f) = x_{_0} = [\varnothing, \{x\}]$$

and

$$\mathbf{y_0}(e) = \mathbf{y_0}(f) = y_{_0} = [\varnothing, \{y\}],$$

we have

$$\mathbf{x_1}^c(e) = \mathbf{x_1}^c(f) = [\{y\}, X]$$

and

$$\mathbf{y_1}^c(e) = \mathbf{y_1}^c(f) = [\{x\}, X].$$

It follows that $\mathbf{x_0}^c$, $\mathbf{y_0}^c \notin \tau_2$. This means that the converse of Proposition 3.10 (2) does not hold.

Proposition 3.13. *Let (X, τ, E) be an IVSTS and $x \neq y \in X$.*

(1) *If there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{U}^c$ or $y_{_1} \in \mathbf{V}$, $x_{_1} \in \mathbf{V}^c$, then (X, τ, E) is an IVST_0(i)-space and for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_0(i)-space in the sense of Lee et al. [43].*

(2) *If there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \in \mathbf{U}^c$ or $y_{_0} \in \mathbf{V}$, $x_{_0} \in \mathbf{V}^c$, then (X, τ, E) is an IVST_0(ii)-space and for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_0(ii)-space in the sense of Lee et al. [43].*

Proof. (1) Suppose there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{U}^c$ or $y_{_1} \in \mathbf{V}$, $x_{_1} \in \mathbf{V}^c$. If $y_{_1} \in \mathbf{U}^c$, then $y \in (U^-(e))^c$. Thus $y \notin U^-(e)$ for each $e \in E$. So $y_{_1} \notin \mathbf{U}$. Similarly, we can see that if $x_{_1} \in \mathbf{V}^c$, then $x_{_1} \notin \mathbf{V}$. Hence (X, τ, E) is an IVST_0(i)-space.

Now let $e \in E$. Then clearly, by Result 3.4, $\tau_{_e}$ is an IVT on X. Since $x_{_1} \in \mathbf{U}$ and $y_{_1} \in \mathbf{U}^c$ or $y_{_1} \in \mathbf{V}$ and $x_{_1} \in \mathbf{V}^c$, $x_{_1} \in \mathbf{U}(e)$ and $y_{_1} \notin \mathbf{U}(e)$ or $y_{_1} \in \mathbf{V}(e)$ and $x_{_1} \notin \mathbf{V}(e)$. Thus $(X, \tau_{_e})$ is an interval-valued T_0(ii)-space.

(2) The proof is analogous to (1). □

The following is an immediate consequence of Proposition 3.13 and Proposition 13 in [16].

Corollary 3.14. *Let (X, τ, E) be an IVSTS and $x \neq y \in X$.*

(1) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{U}^c$ or $y_{_1} \in \mathbf{V}$, $x_{_1} \in \mathbf{V}^c$, then $(X, \tau^-_{_e})$ and $(X, \tau^+_{_e})$ are T_0-spaces for each $e \in E$.*

(2) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \in \mathbf{U}^c$ or $y_{_0} \in \mathbf{V}$, $x_{_0} \in \mathbf{V}^c$, then $(X, \tau^+_{_e})$ is a T_0-space for each $e \in E$.*

Proposition 3.15. *Let (X, τ, E) be an IVSTS and $x \neq y \in X$.*

(1) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{U}^c$ and $y_{_1} \in \mathbf{V}$, $x_{_1} \in \mathbf{V}^c$, then (X, τ, E) is an IVST_1(i)-space and for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_1(i)-space in the sense of Lee et al.* [43].

(2) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \in \mathbf{U}^c$ and $y_{_0} \in \mathbf{V}$, $x_{_0} \in \mathbf{V}^c$, then (X, τ, E) is an IVST_0(ii)-space and for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_1(ii)-space in the sense of Lee et al.* [43].

Proof. The proofs are similar to Proposition 3.13. □

The following is an immediate consequence of Proposition 3.15 and Proposition 14 in [16].

Corollary 3.16. *Let (X, τ, E) be an IVSTS and $x \neq y \in X$.*

(1) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{U}^c$ and $y_{_1} \in \mathbf{V}$, $x_{_1} \in \mathbf{V}^c$, then $(X, \tau^-_{_e})$ and $(X, \tau^+_{_e})$ are T_1-spaces for each $e \in E$.*

(2) *If there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \in \mathbf{U}^c$ and $y_{_0} \in \mathbf{V}$, $x_{_0} \in \mathbf{V}^c$, then $(X, \tau^+_{_e})$ is an T_1-space for each $e \in E$.*

Remark 3.17. (X, τ, E) is an IVST_1(i) [resp. IVST_1(ii)]-space but $(X, \tau_{_e})$ may not be an interval-valued T_1(i) [resp. T_1(ii)]-space (See Example 3.18).

Example 3.18. Let (X, τ_1, E) [resp. (X, τ_2, E)] be the IVST_1(i) [resp. IVST_1(ii)]-space given in Example 3.12. Then we have

$$\tau_{1_e} = \{\widetilde{\varnothing}, [\{x\}, \{x\}], \widetilde{X}\},\ \tau_{1_f} = \{\widetilde{\varnothing}, [\{y\}, \{y\}], \widetilde{X}\}$$

$$[\text{resp. } \tau_{2_e} = \{\widetilde{\varnothing}, [\varnothing, X], [\varnothing, \{x\}], \widetilde{X}\},\ \tau_{2_f} = \{\widetilde{\varnothing}, [\varnothing, X], [\varnothing, \{y\}], \widetilde{X}\}].$$

Thus we can easily check that neither τ_{1_e} nor τ_{1_f} [resp. neither τ_{2_e} nor τ_{2_f}] is an interval-valued T_1(i) [resp. T_1(ii)].

Definition 3.19. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft T_2(i)-space* (briefly, IVST_2(i)-space), if for any $x,\ y \in X$ with $x \neq y$, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$,

(ii) *interval-valued soft* T_2(ii)*-space* (briefly, IVST_2(ii)-space), if for any $x,\ y \in X$ with $x \neq y$, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U},\ y_{_0} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.

Remark 3.20. (1) Every IVST_2(i) [resp. IVST_2(ii)]-space is an IVST_1(i) [resp. IVST_1(ii)]-space. The converse is not true in general (See Example 3.21).

(2) Every IVST_2(i)-space is an IVST_2(ii)-space. The converse is not true in general (See Example 3.21).

Example 3.21. (1) Let (X, τ_1, E) [resp. (X, τ_2, E)] be the IVST_1(i) [resp. IVST_1(ii)]-space given as Example 3.12 (1) [resp. (2)]. Then we cannot have $\mathbf{U},\ \mathbf{V} \in \tau_1$ such that $x_{_1} \in \mathbf{U},\ y_{_1} \notin \mathbf{U}$ and $y_{_1} \in \mathbf{V},\ x_{_1} \notin \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$ [resp. $\mathbf{U},\ \mathbf{V} \in \tau_2$ such that $x_{_0} \in \mathbf{U},\ y_{_0} \notin \mathbf{U}$ and $y_{_0} \in \mathbf{V},\ x_{_0} \notin \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$]. Thus (X, τ_1, E) [resp. (X, τ_2, E)] is not an IVST_2(i) [resp. IVST_2(ii)]-space.

(2) Let $X = \{x, y\}$, $E = \{e, f\}$ and consider the IVST τ given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = \mathbf{A}(f) = [\varnothing, \{x\}]$, $\mathbf{B}(e) = \mathbf{B}(f) = [\varnothing, \{y\}]$, $\mathbf{C}(e) = \mathbf{C}(f) = [\varnothing, X]$. Then observe that (X, τ, E) is an IVST_2(ii)-space. On the other hand, there are not $\mathbf{U},\ \mathbf{V} \in \tau_1$ such that $x_{_1} \in \mathbf{U},\ y_{_1} \notin \mathbf{U}$ and $y_{_1} \in \mathbf{V},\ x_{_1} \notin \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus X is not an IVST_2(i)-space.

Remark 3.22. (1) If (X, τ, E) is an IVST_2(i)-space, then (X, τ^-, E) and (X, τ^+, E) are soft T_2-spaces in the sense of Shabir and Naz [16].

(2) If (X, τ, E) is an IVST_2(ii)-space, then (X, τ^+, E) is a soft T_2-space in the sense of Shabir and Naz [16].

Proposition 3.23. *Let (X, τ, E) be an IVSTS.*

(1) *If (X, τ, E) is an IVST_2*(i)*-space, then for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_2*(i)*-space in the sense of Lee et al.* [43].

(2) *(X, τ, E) is an IVST_2*(ii)*-space, then for each $e \in E$, $(X, \tau_{_e})$ is an interval-valued T_2*(ii)*-space in the sense of Lee et al.* [43].

Proof. (1) Suppose (X, τ, E) is an IVST_2(i)-space, let $x \neq y \in X$ and $e \in E$. Then there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U},\ y_{_1} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus $x_{_1} \in \mathbf{U}(e)$, $y_{_1} \in \mathbf{V}(e)$ and $\mathbf{U}(e) \cap \mathbf{V}(e) = \widetilde{\varnothing}$. By Result 3.4, $\mathbf{U}(e),\ \mathbf{V}(e) \in \tau_{_e}$. So $(X, \tau_{_e})$ is an interval-valued T_2(i)-space.

(2) The proof is analogous to (1). □

The following is an immediate consequence of Proposition 3.23 and Proposition 17 in [16].

Proposition 3.24. *Let (X, τ, E) be an IVSTS.*

(1) *If (X, τ, E) is an IVST_2*(i)*-space, then $(X, \tau^-_{_e})$ and $(X, \tau^+_{_e})$ are T_2-space for each $e \in E$.*

(2) *(X, τ, E) is an IVST_2*(ii)*-space, then $(X, \tau^+_{_e})$ is a T_2-space for each $e \in E$.*

Proof. (1) Suppose (X, τ, E) is an IVS$T_2(i)$-space, let $x \neq y \in X$ and let $e \in E$. Then there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U},\ y_{_1} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus Definition 3.1 and Result 3.4, $x \in U^-(e),\ y \in V^-(e)$, $U^-(e) \cap V^-(e) = \varnothing$ and $U^-(e),\ V^-(e) \in \tau^-_{_e}$. So $(X, \tau^-_{_e})$ is a T_2-space. Note that $\tau^-_{_e} \subset \tau^+_{_e}$. Hence $(X, \tau^+_{_e})$ is a T_2-space.

(2) The proof is similar to (1). □

Definition 3.25. Let (X, τ, E) be an IVSTS and $\mathbf{A}$ an IVSCS in X. Then (X, τ, E) is called:

(i) an *interval-valued soft regular* (i)*-space* (briefly, IVSR(i)-space), if for each $x \in X$ and each $\mathbf{A} \in \tau^c$ with $x_{_1} \notin \mathbf{A}$, there are $\mathbf{U}, \mathbf{V} \in \tau$ such that $x_{_1} \in \mathbf{U}$, $\mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$,

(ii) an *interval-valued soft regular* (ii)*-space* (briefly, IVSR(ii)-space), if for each $x \in X$ and each $\mathbf{A} \in \tau^c$ with $x_{_0} \notin \mathbf{A}$, there are $\mathbf{U}, \mathbf{V} \in \tau$ such that $x_{_0} \in \mathbf{U}$, $\mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.

Definition 3.26. An IVSTS (X, τ, E) is called:

(i) an *interval-valued soft* T_3(i)*-space* (briefly, IVST_3(i)-space), if it is an IVSR(i) and IVST_1(i)-space,

(ii) an *interval-valued soft* T_3(ii)*-space* (briefly, IVST_3(ii)-space), if it is an IVSR(ii) and IVST_1(ii)-space.

Remark 3.27. (1) Every IVST_3(i)-space is an IVST_3(ii)-space but the converse is not true in general (See Example 3.28).

(2) An IVST_3(i) [resp. IVST_3(ii)]-space may not be an IVST_2(i) [resp. IVST_2(ii)]-space (See Example 3.28).

(3) If (X, τ, E) is an IVST_3(i) [resp. IVST_3(ii)]-space, then $(X, \tau_{_e})$ may not be an interval-valued T_3(i) [resp. T_3(ii)]-space for each $e \in E$ in the sense of Lee et al. [43] Furthermore, $(X, \tau^-_{_e})$ and $(X, \tau^+_{_e})$ may not be T_3-spaces (See Example 3.28).

Example 3.28. (1) Let $X = \{a, b, c, d\}$, let $E = \{e, f\}$ and consider the IVST τ_1 on X given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{15}, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a, b\}, \{a, b, c\}]$, $\mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{c, d\}, \{a, c, d\}]$,
$\mathbf{A}_3(e) = \mathbf{A}_3(f) = [\{c, d\}, \{a, b, d\}]$, $\mathbf{A}_4(e) = \mathbf{A}_4(f) = [\{b, c\}, \{b, c, d\}]$,
$\mathbf{A}_5(e) = \mathbf{A}_5(f) = [\varnothing, \{a, c\}]$, $\mathbf{A}_6(e) = \mathbf{A}_6(f) = [\{a\}, \{a, b\}]$,
$\mathbf{A}_7(e) = \mathbf{A}_7(f) = [\{b\}, \{b, c\}]$, $\mathbf{A}_8(e) = \mathbf{A}_8(f) = [\varnothing, \{a, d\}]$,
$\mathbf{A}_9(e) = \mathbf{A}_9(f) = [\{c\}, \{c, d\}]$, $\mathbf{A}_{10}(e) = \mathbf{A}_{10}(f) = [\varnothing, \{b, d\}]$,
$\mathbf{A}_{11}(e) = \mathbf{A}_{11}(f) = [\{a, b, d\}, X]$, $\mathbf{A}_{12}(e) = \mathbf{A}_{12}(f) = [\{a, b, c\}, X]$,
$\mathbf{A}_{13}(e) = \mathbf{A}_{13}(f) = [\{a, c, d\}, X]$, $\mathbf{A}_{14}(e) = \mathbf{A}_{14}(f) = [\{b, c, d\}, X]$,
$\mathbf{A}_{15}(e) = \mathbf{A}_{15}(f) = [\{a\}, \{a, b\}]$.

Observe that (X, τ_1, E) is an IVST_3(ii)-space but not an IVST_3(i)-space.

Now consider the IVST τ_2 on X given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{17}, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a\}, \{a, d\}]$, $\mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{b\}, \{a, b\}]$,
$\mathbf{A}_3(e) = \mathbf{A}_3(f) = [\{c\}, \{b, c\}]$, $\mathbf{A}_4(e) = \mathbf{A}_4(f) = [\{d\}, \{c, d\}]$,
$\mathbf{A}_5(e) = \mathbf{A}_5(f) = [\varnothing, \{a\}]$, $\mathbf{A}_6(e) = \mathbf{A}_6(f) = [\varnothing, \{b\}]$,
$\mathbf{A}_7(e) = \mathbf{A}_7(f) = [\varnothing, \{c\}]$, $\mathbf{A}_8(e) = \mathbf{A}_8(f) = [\varnothing, \{d\}]$,
$\mathbf{A}_9(e) = \mathbf{A}_9(f) = [\{a, b\}, \{a, b, d\}]$, $\mathbf{A}_{10}(e) = \mathbf{A}_{10}(f) = [\{a, d\}, \{a, c, d\}]$,
$\mathbf{A}_{11}(e) = \mathbf{A}_{11}(f) = [\{c, d\}, \{b, c, d\}]$, $\mathbf{A}_{12}(e) = \mathbf{A}_{12}(f) = [\{b, c\}, \{a, b, c\}]$,
$\mathbf{A}_{13}(e) = \mathbf{A}_{13}(f) = [\{a, c\},]$, $\mathbf{A}_{14}(e) = \mathbf{A}_{14}(f) = [\{b, d\}, X]$,
$\mathbf{A}_{15}(e) = \mathbf{A}_{15}(f) = [\varnothing, \{a, b\}]$, $\mathbf{A}_{16}(e) = \mathbf{A}_{16}(f) = [\varnothing, \{a, c\}]$,

$\mathbf{A}_{17}(e) = \mathbf{A}_{17}(f) = [\varnothing, \{a, d\}]$.
Then observe that (X, τ_2, E) is an IVST_3(i)-space.

(2) Let $X = \{a, b, c\}$, $E = \{e, f\}$ and consider the IVST τ on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{32}, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \widetilde{X},\ \mathbf{A}_1(f) = \widetilde{\varnothing},\ \mathbf{A}_2(e) = [\{a\}, \{a\}],\ \mathbf{A}_2(f) = \widetilde{\varnothing},$
$\mathbf{A}_3(e) = [\{b\}, \{b\}],\ \mathbf{A}_2(f) = \widetilde{\varnothing},\ \mathbf{A}_4(e) = [\{c\}, \{c\}],\ \mathbf{A}_4(f) = \widetilde{\varnothing},$
$\mathbf{A}_5(e) = [\{a, b\}, \{a, b\}],\ \mathbf{A}_5(f) = \widetilde{\varnothing},\ \mathbf{A}_6(e) = [\{a, c\}, \{a, c\}],\ \mathbf{A}_6(f) = \widetilde{\varnothing},$
$\mathbf{A}_7(e) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_7(f) = \widetilde{\varnothing},\ \mathbf{A}_8(e) = \widetilde{X},\ \mathbf{A}_8(f) = [\{a\}, \{a\}],$
$\mathbf{A}_9(e) = \mathbf{A}_9(f) = [\{a\}, \{a\}],\ \mathbf{A}_{10}(e) = [\{b\}, \{b\}],\ \mathbf{A}_{10}(f) = [\{a\}, \{a\}],$
$\mathbf{A}_{11}(e) = [\{c\}, \{c\}],\ \mathbf{A}_{11}(f) = [\{a\}, \{a\}],\ \mathbf{A}_{12}(e) = [\{a, b\}, \{a, b\}],$
$\mathbf{A}_{12}(f) = [\{a\}, \{a\}],\ \mathbf{A}_{13}(e) = [\{a, c\}, \{a, c\}],\ \mathbf{A}_{13}(f) = [\{a\}, \{a\}],$
$\mathbf{A}_{14}(e) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{14}(f) = [\{a\}, \{a\}],\ \mathbf{A}_{15}(e) = \widetilde{\varnothing},$
$\mathbf{A}_{15}(f) = [\{a\}, \{a\}],\ \mathbf{A}_{16}(e) = \widetilde{X},\ \mathbf{A}_{16}(f) = [\{b, c\}, \{b, c\}],$
$\mathbf{A}_{17}(e) = [\{a\}, \{a\}],\ \mathbf{A}_{17}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{18}(e) = [\{b\}, \{b\}],$
$\mathbf{A}_{18}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{19}(e) = [\{c\}, \{c\}],\ \mathbf{A}_{19}(f) = [\{b, c\}, \{b, c\}],$
$\mathbf{A}_{20}(e) = [\{a, b\}, \{a, b\}],\ \mathbf{A}_{20}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{21}(e) = [\{a, c\}, \{a, c\}],$
$\mathbf{A}_{21}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{22}(e) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{22}(f) = [\{b, c\}, \{b, c\}],$
$\mathbf{A}_{23}(e) = \widetilde{\varnothing},\ \mathbf{A}_{23}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{24}(e) = [\{a\}, \{a\}],\ \mathbf{A}_{24}(f) = \widetilde{X},$
$\mathbf{A}_{25}(e) = [\{b\}, \{b\}],\ \mathbf{A}_{25}(f) = \widetilde{X},\ \mathbf{A}_{26}(e) = [\{c\}, \{c\}],\ \mathbf{A}_{26}(f) = \widetilde{X},$
$\mathbf{A}_{27}(e) = [\{a, b\}, X],\ \mathbf{A}_{27}(f) = \widetilde{X},\ \mathbf{A}_{28}(e) = [\{a, c\}, X],\ \mathbf{A}_{28}(f) = \widetilde{X},$
$\mathbf{A}_{29}(e) = [\{b, c\}, X],\ \mathbf{A}_{29}(f) = \widetilde{X},\ \mathbf{A}_{30}(e) = \widetilde{\varnothing},\ \mathbf{A}_{30}(f) = \widetilde{X}.$
$\mathbf{A}_{31}(e) = \widetilde{X},\ \mathbf{A}_{31}(f) = \widetilde{\varnothing},\ \mathbf{A}_{32}(e) = \widetilde{X},\ \mathbf{A}_{32}(f) = [\{a\}, \{a\}].$

Then observe that (X, τ, E) is an IVST_3(i)-space but not an IVST_2(i)-space.

On the other hand, by Result 3.4, we have

$$\tau_e = \{\widetilde{\varnothing}, [\{a\}, \{a\}], [\{b\}, \{b\}], [\{c\}, \{c\}], [\{a, b\}, \{a, b\}], [\{a, c\}, \{a, c\}], [\{b, c\}, \{b, c\}], \widetilde{X}\}$$

and

$$\tau_f = \{\widetilde{\varnothing}, [\{a\}, \{a\}], [\{b, c\}, \{b, c\}], \widetilde{X}\}.$$

Obviously, (X, τ_f) is not an interval-valued T_3(i)-space. Moreover, we have

$$\tau_e^- = \tau_e^+ = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$$

and

$$\tau_f^- = \tau_f^+ = \{\varnothing, \{a\}, \{b, c\}, X\}.$$

Thus we can see that (X, τ_f^-) is not a T_3-space.

Definition 3.29. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft normal space* (briefly, IVSNS), If for any IVSCSs $\mathbf{F}_1$, $\mathbf{F}_2$ in X with $\mathbf{F}_1 \cap \mathbf{F}_2 = \widetilde{\varnothing}_E$, there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $\mathbf{F}_1 \subset \mathbf{U}$, $\mathbf{F}_2 \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$,

(ii) *interval-valued soft T_4(i)-space* (briefly, IVST_4(i)-space), if it is an T_1(i)-space and an IVSNS,

(iii) *interval-valued soft T_4(ii)-space* (briefly, IVST_4(ii)-space), if it is an T_1(ii)-space and an IVSNS.

Remark 3.30. (1) An IVST_4(i) [resp. IVST_4(ii)]-space need not an IVST_3(i) [resp. IVST_3(ii)]-space (See Example 3.31).

(2) If (X, τ, E) is an IVST_4(i) [resp. IVST_4(ii)]-space, then (X, τ_e) may not be an interval-valued T_4-space for each $e \in E$ in the sense of Lee et al. [43] Furthermore, (X, τ_e^-) and (X, τ_e^+) may not be T_4-spaces (See Example 3.31).

Example 3.31. Let $X = \{a, b, c, d\}$, let $E = \{e, f\}$ and consider the IVST τ on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7, \mathbf{A}_8, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [\{a, b, d\}, \{a, b, d\}],\ \mathbf{A}_1(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_2(e) = [\{a, c, d\}, \{a, c, d\}],\ \mathbf{A}_2(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_3(e) = [\{a, d\}, \{a, d\}],\ \mathbf{A}_3(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_4(e) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_4(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_5(e) = [\{b\}, \{b\}],\ \mathbf{A}_5(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_6(e) = [\{c\}, \{c\}],\ \mathbf{A}_6(f)) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_7(e) = \widetilde{\varnothing},\ \mathbf{A}_7(f) = [\{a, b, c\}, \{a, b, c\}],$
$\mathbf{A}_8(e) = \widetilde{X},\ \mathbf{A}_8(f) = [\{a, b, c\}, \{a, b, c\}].$

Then we can check that (X, τ, E) is an IVST_4(i)-space but not an IVST_3(i)-space.

On the other hand, by Result 3.4, we have

$$\tau_e = \{\widetilde{\varnothing}, [\{b\}, \{b\}], [\{c\}, \{c\}], [\{a, d\}, \{a, d\}], [\{b, c\}, \{b, c\}], [\{a, c, d\}, \{a, c, d\}], [\{a, b, d\}, \{a, b, d\}], \widetilde{X}\}$$

and

$$\tau_f = \{\widetilde{\varnothing}, [\{a, b, c\}, \{a, b, c\}], \widetilde{X}\}.$$

Then we can easily see that (X, τ_e) and (X, τ_f) are not interval-valued T_4-spaces. Furthermore, we get

$$\tau_e^- = \tau_e^+ = \{\varnothing, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, \{a, b, d\}, X\}$$

and

$$\tau_f^- = \tau_f^+ = \{\varnothing, \{a, b, c\}, X\}.$$

Thus we can check that (X, τ_e^-) and (X, τ_f^-) are not T_4-spaces.

4. Interval-valued soft subspaces

In this section, we deal with some hereditary problems in an IVSTS. For this reason, we introduce the notion of interval-valued soft subspaces of an IVSTS.

Definition 4.1. Let Y be a nonempty subset of X and $\mathbf{A} \in IVSS_E(X)$. Then

(i) the *interval valued soft set* (Y, E) *over* X, denoted by $\widetilde{Y}_E$, is defined as follows:

$$\widetilde{Y}_E(e) = [Y, Y] \text{ for each } e \in E,$$

(ii) the *interval-valued soft subset of* $\mathbf{A}$ *over* Y, denoted by $\mathbf{A}_Y$, is defined as follows:

$$\mathbf{A}_Y = \widetilde{Y}_E \cap \mathbf{A}, \text{ i.e., } \mathbf{A}_Y(e) = [Y \cap A^-(e), Y \cap A^+(e)] \text{ for each } e \in E.$$

Example 4.2. Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $E = \{e, f\}$ and $\mathbf{A}$ be the IVSS over X given by:

$$\mathbf{A}(e) = [\{a, b\}, \{a, b, d\}],\ \mathbf{A}(f) = [\{a, d\}, \{a, c, d\}].$$

Then clearly, $\mathbf{A}_Y(e)=[\{a,b\},\{a,b\}],\ \mathbf{A}_Y(f)=[\{a\},\{a,c\}]$.

Proposition 4.3. *Let (X,τ,E) be an IVSTS and Y a nonempty subset of X. Then $\tau_Y=\{\mathbf{A}_Y:\mathbf{A}\in\tau\}$ is an IVST on Y.*

In this case, τ_Y is called the *interval-valued soft relative topology* on Y and (Y,τ_Y,E) is called an *interval-valued soft subspace* (briefly, IVS-subspace) of (X,τ,E). Each member of τ_Y is called an *IVSOS in* Y and an IVSS $\mathbf{A}$ over Y is called an *IVSCS in* Y, if $[Y,Y]\setminus\mathbf{A}=[Y\setminus A^+,Y\setminus A^-]\in\tau_Y$.

Proof. The proof is straightforward. □

We can see that (Y,τ_Y,E) is a special interval-valued soft subspace of an IVSTS (X,τ,E) (See Proposition 4.15, [34]).

Remark 4.4. Every IVS-subspace of an interval-valued soft discrete [resp. indiscrete] space is an interval-valued soft discrete [resp. indiscrete] space.

Lemma 4.5. *Let (X,τ) be an IVTS and Y a nonempty subset of X. Then $\tau_{[Y,Y]}=\{[Y,Y]\cap A:A\in\tau\}=\{[Y\cap A^-,[Y\cap A^+]:A\in\tau\}$ is an IVT on Y.*

In this case, $\tau_{[Y,Y]}$ is called the *interval-valued relative topology* on Y and $([Y,Y],\tau_{[Y,Y]})$ is called an *interval-valued subspace* (briefly, IV-subspace) of (X,τ).

Proof. The proof is straightforward. □

Proposition 4.6. *Let (X,τ,E) be an IVSTS and Y a nonempty subset of X. Then $(Y,(\tau_Y)_e)=([Y,Y],\tau_{e,[Y,Y]})$, i.e., $(Y,(\tau_Y)_e)$ is an interval-valued subspace of IVTS (X,τ_e) for each $e\in E$, where $\tau_{e,[Y,Y]}=\{[Y,Y]\cap\mathbf{A}(e):\mathbf{A}(e)\in\tau_e\}$.*

Proof. The proof follows from Proposition 4.3 and Lemma 4.5. □

Corollary 4.7 (See Proposition 11, [16])**.** *Let (X,τ,E) be an IVSTS and Y a nonempty subset of X. Then $(Y,(\tau_Y)^-_e)=(Y,(\tau^-_e)_Y)$ [resp. $(Y,(\tau_Y)^+_e)=(Y,(\tau^+_e)_Y)$], i.e., $(Y,(\tau_Y)^-_e)$ [resp. $(Y,(\tau_Y)^+_e)$] is a subspace of (X,τ^-_e) [resp. (X,τ^+_e)] for each $e\in E$.*

Proposition 4.8. *Let (Y,τ_Y,E) be an IVS-subspace of an IVSTS (X,τ,E) and $\mathbf{A}$ an IVSOS in Y. If $\widetilde{Y}_E\in\tau$, then $\mathbf{A}\in\tau$.*

Proof. The proof is obvious. □

Theorem 4.9. *Let (Y,τ_Y,E) be an IVS-subspace of an IVSTS (X,τ,E) and $\mathbf{A}\in IVSS_E(X)$.*

(1) *$\mathbf{A}\in\tau_Y$ if and only if there is $\mathbf{B}\in\tau$ such $\mathbf{A}=\widetilde{Y}_E\cap\mathbf{B}$.*

(2) *$\mathbf{A}\in\tau^c_Y$ if and only if there is $\mathbf{B}\in\tau^c$ such $\mathbf{A}=\widetilde{Y}_E\cap\mathbf{B}$.*

Proof. (1) The proof follows from Proposition 4.3.

(2) Suppose $\mathbf{A}\in\tau^c_Y$. Then clearly, $[Y,Y]-\mathbf{A}=[Y-A^+,Y-A^-]\in\tau_Y$. Thus by (1), there is $\mathbf{B}\in\tau$ such that $[Y-A^+,Y-A^-]=\widetilde{Y}_E\cap\mathbf{B}$. Now let $e\in E$. Then

$$[Y-A^+,Y-A^-](e)=\widetilde{Y}_E\cap\mathbf{B}(e),\ \text{i.e.,}$$

$$[Y - A^+(e), Y - A^-(e)] = [Y \cap B^-(e), Y \cap B^-(e)], \text{i.e.,}$$

$$Y - A^+(e) = Y \cap B^-(e),\ Y - A^-(e) = Y \cap B^-(e).$$

Thus we have

$$A^+(e) = Y - (Y - A^+(e)) = Y - (Y \cap B^-(e)) = Y \cap (Y - B^-(e)) = Y \cap (X - B^-(e)),$$

$$A^-(e) = Y - (Y - A^-(e)) = Y - (Y \cap B^+(e)) = Y \cap (Y - B^+(e)) = Y \cap (X - B^+(e)).$$

So $\mathbf{A}(e) = [A^-(e), A^+(e) = [Y \cap (X - B^+(e)), Y \cap (X - B^-(e))] = (\widetilde{Y}_E \cap \mathbf{B}^c)(e)$. Hence $\mathbf{A} = \widetilde{Y}_E \cap \mathbf{B}^c$. Since $\mathbf{B} \in \tau$, $\mathbf{B}^c \in \tau^c$. Therefore the necessary condition holds.

Conversely, suppose the necessary condition holds, i.e., there is $\mathbf{B} \in \tau^c$ such that $\mathbf{A} = \widetilde{Y}_E \cap \mathbf{B}$. Then $\mathbf{B}^c = \widetilde{X}_E - \mathbf{B} \in \tau$. We will show that $[Y, Y] - \mathbf{A} = \widetilde{Y}_E \cap \mathbf{B}^c$. Let $e \in E$. Then we get

$$\begin{aligned}([Y, Y] - \mathbf{A})(e) &= [Y - A^+(e), Y - A^-(e)]\\ &= [Y - Y \cap B^-(e), Y - Y \cap B^+(e)]\\ &= [Y \cap (Y - B^+(e)), Y \cap (Y - B^-(e))]\\ &= [Y \cap (X - B^+(e)), Y \cap (X - B^-(e)).\end{aligned}$$

Thus $[Y, Y] - \mathbf{A} = \widetilde{Y}_E \cap (\widetilde{X}_E - \mathbf{B}) = \widetilde{Y}_E \cap \mathbf{B}^c$. Since $\mathbf{B}^c \in \tau$, we have $[Y, Y] - \mathbf{A} \in \tau_{_Y}$. So $\mathbf{A} \in \tau^c_{_Y}$. This completes the proof. □

Proposition 4.10. *Let* (X, τ, E) *be an IVSTS and* Y *a nonempty subset of* X.

(1) *If* X *is an* $IVST_0$(i) [resp. $IVST_0$(ii)]*-space, then* $(Y, \tau_{_Y}, E)$ *is an* $IVST_0$(i) [resp. $IVST_0$(ii)]*-space.*

(2) *If* X *is an* $IVST_1$(i) [resp. $IVST_1$(ii)]*-space, then* $(Y, \tau_{_Y}, E)$ *is an* $IVST_1$(i) [resp. $IVST_1$(ii)]*-space.*

Proof. (1) Suppose X is an IVST_0(i)-space and let $x \neq y \in Y$. Then there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_{_1} \in \mathbf{U}$, $y_{_1} \notin \mathbf{U}$ or $y_{_1} \in \mathbf{U}$, $x_{_1} \notin \mathbf{U}$, say $x_{_1} \in \mathbf{U}$, $y_{_1} \notin \mathbf{U}$. Since $x \in Y$, $x_{_1} \in \widetilde{Y}_E$. Since $x_{_1} \in \mathbf{U}$, we have $x_{_1} \in \widetilde{Y}_E \cap \mathbf{U} = \mathbf{U}_Y$. Since $y_{_1} \notin \mathbf{U}$, there is $e \in E$ such that $y \notin \mathbf{U}(e)$, i.e., $y \notin U^-(e)$. Thus $y \notin Y \cap U^-(e) = (Y \cap U^-)(e)$. So $y_{_1} \notin \widetilde{Y}_E \cap \mathbf{U} = \mathbf{U}_Y$. Since $\mathbf{U} \in \tau$, $\mathbf{U}_Y \in \tau_{_Y}$. Hence $(Y, \tau_{_Y}, E)$ is an IVST_1(i)-space. The proof of the second part is similar.

(2) The proof is similar to (1). □

Proposition 4.11. *Let* (X, τ, E) *be an IVSTS and* Y *a nonempty subset of* X. *If* X *is an* $IVST_2$(i) [resp. $IVST_2$(ii)]*-space, then* $(Y, \tau_{_Y}, E)$ *is an* $IVST_2$(i) [resp. $IVST_2$(ii)]*-space.*

Proof. Suppose X is an IVST_2(i)-space and let $x \neq y \in Y$. Then there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{V}$, $\mathbf{U} \cap \mathbf{V} = \widetilde{X}_E$. Then clearly, we have

$$x \in U^-(e),\ y \in V^-(e),\ U^+(e) \cap V^+(e) = \varnothing.$$

Thus $x \in Y \cap U^-(e),\ y \in Y \cap V^-(e),\ (Y \cap U^+(e)) \cap (Y \cap V^+(e)) = \varnothing$. So we get

$$x \in U^-_{_Y}(e),\ y \in V^-_{_Y}(e),\ U^+_{_Y}(e) \cap V^+_{_Y}(e) = \varnothing.$$

Hence $x_{_1} \in \mathbf{U}_{_Y},\ y_{_1} \in \mathbf{V}_{_Y},\ \mathbf{U}_{_Y} \cap \mathbf{V}_{_Y} = \widetilde{\varnothing}_E$. Therefore $(Y, \tau_{_Y}, E)$ is an IVST_2(i)-space. The proof of the second part is similar. □

Proposition 4.12. *Let (X, τ, E) be an IVSTS and Y a nonempty subset of X. If X is an $IVST_3$*(i) [resp. *$IVST_3$*(ii)]*-space, then $(Y, \tau_{_Y}, E)$ is an $IVST_3$*(i) [resp. *$IVST_3$*(ii)]*-space.*

Proof. Suppose X is an IVST_3(i)-space. Then by Proposition 4.10 (2), $(X, \tau_{_Y}, E)$ is an IVST_1(i)-space. Let $y \in Y$ and $\mathbf{A} \in \tau^c_{_Y}$ such that $y_{_1} \notin \mathbf{A}$. Then by Theorem 4.9 (2), there is $\mathbf{B} \in \tau^c$ such that $\mathbf{A} = \widetilde{Y}_E \cap \mathbf{B}$. Since $y_{_1} \notin \mathbf{A}$, $y_{_1} \notin \widetilde{Y}_E \cap \mathbf{B}$. But $y_{_1} \in \widetilde{Y}_E$. Thus $y_{_1} \notin \mathbf{B}$. By the hypothesis, there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $y_{_1} \in \mathbf{U}$, $\mathbf{B} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. So by Proposition 4.10 (1), $\widetilde{Y}_E \cap \mathbf{U}$, $\widetilde{Y}_E \cap \mathbf{V} \in \tau_{_Y}$. Furthermore, $y_{_1} \notin \widetilde{Y}_E \cap \mathbf{U}$, $\mathbf{A} \subset \widetilde{Y}_E \cap \mathbf{V}$ and $(\widetilde{Y}_E \cap \mathbf{U}) \cap (\widetilde{Y}_E \cap \mathbf{V} = \widetilde{\varnothing}_E$. Hence $(Y, \tau_{_Y}, E)$ is an IVST_3(i)-space. The proof of the second part is similar. □

Remark 4.13. Let (X, τ, E) be an IVSTS and let Y be a nonempty subset of X. When (X, τ, E) is an IVST_4(i) [resp. IVST_4(ii)]-space and Y is a nonempty subset of X, $(Y, \tau_{_Y}, E)$ need not be an IVST_3(i) [resp. IVST_3(ii)]-space (See Example 4.14).

Example 4.14. Consider the IVST_4(i)-space (X, τ, E) given in Example 3.31 and let $Y = \{a, b, c\}$. Then clearly, we have

$$\tau_{_Y} = \{\widetilde{\varnothing}_E, \mathbf{A}_{1_Y}, \mathbf{A}_{2_Y}, \mathbf{A}_{3_Y}, \mathbf{A}_{4_Y}, \mathbf{A}_{5_Y}, \mathbf{A}_{6_Y}, \mathbf{A}_{7_Y}, \mathbf{A}_{7_Y}, \widetilde{Y}_E\},$$

where $\mathbf{A}_{1_Y}(e) = [\{a, b\}, \{a, b\}]$, $\mathbf{A}_{1_Y}(f) = [Y, Y]$,
$\mathbf{A}_{2_Y}(e) = [\{a, c\}, \{a, c\}]$, $\mathbf{A}_{2_Y}(f) = [Y, Y]$,
$\mathbf{A}_{3_Y}(e) = [\{a\}, \{a\}]$, $\mathbf{A}_{3_Y}(f) = [Y, Y]$,
$\mathbf{A}_{4_Y}(e) = [\{b, c\}, \{b, c\}]$, $\mathbf{A}_{4_Y}(f) = [Y, Y]$,
$\mathbf{A}_{5_Y}(e) = [\{b\}, \{b\}]$, $\mathbf{A}_{5_Y}(f) = [Y, Y]$,
$\mathbf{A}_{6_Y}(e) = [\{c\}, \{c\}]$, $\mathbf{A}_{6_Y}(f) = [Y, Y]$,
$\mathbf{A}_{7_Y}(e) = \widetilde{\varnothing}$, $\mathbf{A}_{7_Y}(f) = [Y, Y]$,
$\mathbf{A}_{8_Y}(e) = \widetilde{X}$, $\mathbf{A}_{7_Y}(f) = [Y, Y]$.
Thus we can check that $(Y, \tau_{_Y}, E)$ is not an IVST_3(i)-space.

5. Interval-valued soft α-closures [resp. interiors]

In this section, we define an interval-valued α-closure [resp. interior] of an interval-valued soft set and discuss some of their properties, and give some examples.

Definition 5.1. Let (X, τ, E) be a soft topological space over X and A, $B \in SS_E(X)$.

(i) The *soft closure* of A [16], denoted by $cl_\tau(A)$ or $cl(A)$ or $\overline{A}$, is a soft set over X defined as follows:

$$cl_\tau(A) = \bigcap\{F \in \tau^c : A \subset F\}.$$

(ii) The *soft interior* of A [20], denoted by $int_\tau(A)$ or $int(A)$ or A°, is a soft set over X defined as follows:

$$int_\tau(A) = \bigcup\{U \in \tau : U \subset A\}.$$

It is obvious that $cl(A)$ is the smallest soft closed set in X which contains A and $int(A)$ is the largest soft open set in X which is contained in A.

Definition 5.2 ([34])**.** Let (X, τ, E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$.

(i) The *interval-valued soft closure* of $\mathbf{A}$ w.r.t. τ, denoted by $IVScl(\mathbf{A})$, is an IVSS over X defined as:

$$IVScl(\mathbf{A}) = \bigcap\{\mathbf{K} \in \tau^c : \mathbf{A} \subset \mathbf{K}\}.$$

(ii) The *interval-valued soft interior* of $\mathbf{A}$ w.r.t. τ, denoted by $IVSint(\mathbf{A})$, is an IVSS over X defined as:

$$IVint(\mathbf{A}) = \bigcup\{\mathbf{U} : \mathbf{U} \in \tau \text{ and } \mathbf{U} \subset \mathbf{A}\}.$$

We can see that $IVScl(\mathbf{A})$ is the smallest IVSCS in X containing $\mathbf{A}$ and $IVSint(\mathbf{A})$ is the largest IVSOS in X contained $\mathbf{A}$.

Remark 5.3. $IVScl(\mathbf{A}) = [cl_{\tau^-}(A^-), cl_{\tau^+}(A^+)], IVSint(\mathbf{A}) = [int_{\tau^-}(A^-), int_{\tau^+}(A^+)]$.

Example 5.4. Let $X = \{a, b, c\}$, $E = \{e, f\}$ and consider τ the IVST on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{U}, \widetilde{X}_E\},$$

where $\mathbf{U}(e) = [\{a\}, \{a, b\}],\ \mathbf{U}(f) = [\{b\}, \{b, c\}]$. Then we have

$$\tau^- = \{\varnothing_E, U^-, X_E\},\ \tau^+ = \{\varnothing_E, U^+, X_E\},$$

where $U^-(e) = \{a\},\ U^-(f) = \{b\},\ U^+(e) = \{a, b\},\ U^+(f) = \{b, c\}$. Moreover,

$$\tau^c = \{\widetilde{\varnothing}_E, \mathbf{U}^c, \widetilde{X}_E\},$$

where $\mathbf{U}^c(e) = [\{c\}, \{b, c\}],\ \mathbf{U}^c(f) = [\{a\}, \{a, c\}]$.

Consider the IVSS $\mathbf{A}$ over X defined by $\mathbf{A}(e) = [\{c\}, \{c\}],\ \mathbf{A}(f) = [\{a\}, \{a\}]$. Then clearly, $\mathbf{A} \subset \mathbf{U}^c$ and $\mathbf{A} \subset \widetilde{X}_E$. Thus $cl(\mathbf{A}) = \mathbf{U}^c \cap \widetilde{X}_E = \mathbf{U}^c$. On the other hand, we get

$$cl_{\tau^-}(A^-) = U^{c,-},\ cl_{\tau^+}(A^+) = U^{c,+},$$

where $U^{c,-}(e) = \{c\},\ U^{c,-}(f) = \{a\},\ U^{c,+}(e) = \{b, c\},\ U^{c,+}(f) = \{a, c\}$. So $cl(\mathbf{A}) = [cl_{\tau^-}(A^-), cl_{\tau^+}(A^+)]$.

Definition 5.5 ([36])**.** Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. A is called a *soft α-open set* in X, if $A \subset int(cl(int(A)))$. The complement of a soft α-open set is called a *soft α-closed set* in X.

The set of all soft α-open [resp. closed] sets in a soft topological space (X, τ, E) is denoted by SαOS(X) [resp. SαCS(X)].

Definition 5.6. (i) Let (X, τ, E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then $\mathbf{A}$ is called an *interval-valued soft α-open set* (briefly, IVSαOS) in X, if it satisfies the following conditions:

$$\mathbf{A} \subset IVSint(IVScl(IVSint(\mathbf{A}))).$$

The complement of an IVSαOS is called an *interval-valued soft α-closed set* (briefly, IVSαCS) in X.

(ii) Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then A is called an *interval-valued α-open set* (briefly, IVαOS) in X, if $A \subset IVint(IVcl(IVint(A)))$, where $IVint(A)$ and $IVcl(A)$ denote the interval-valued interior and the interval-valued closure of A (See [37]). The complement of an IVαOS is called an *interval-valued α-closed set* (briefly, IVαCS) in X.

We will denote the set of all IVSαOSs [resp. IVSαCS] by IVSαOS(X) [resp. IVSαCS(X)]. Also, We will denote the set of all IVαOSs [resp. IVαCS] by IVαOS(X) [resp. IVαCS(X)].

From Remark 5.3, it is obvious that $\mathbf{A} \in IVS\alpha OS(X)$, then $A^- \in S\alpha OS(X, \tau^-, E)$ and $A^+ \in S\alpha OS(X, \tau^+, E)$.

Example 5.7. Let (X, τ, E) be the IVSTS given in Example 5.4. Observe that $\mathbf{U} \in IVS\alpha OS(X)$. Furthermore we can confirm that $U^- \in S\alpha OS(X, \tau^-, E)$ and $U^+ \in S\alpha OS(X, \tau^+, E)$.

Proposition 5.8. *Let (X, τ, E) be an IVSTS. Then*
(1) $\bigcup_{j \in J} \mathbf{A}_j \in IVS\alpha OS(X)$ *for each* $(\mathbf{A}_j)_{j \in J} \subset IVS\alpha OS(X)$,
(2) $\bigcap_{j \in J} \mathbf{A}_j \in IVS\alpha CS(X)$ *for each* $(\mathbf{A}_j)_{j \in J} \subset IVS\alpha CS(X)$.

Proof. The proofs are straightforward. □

Remark 5.9. In an IVSTS (X, τ, E), $\tau \subset IVS\alpha OS(X)$ and $\tau^c \subset IVS\alpha CS(X)$. But the converse need not be true (See Example 5.10)

Example 5.10. Let $X = \{a, b, c, d\}$, $E = \{e, f, g\}$ and consider the IVST τ given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{17}, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [\{a\}, \{a\}],\ \mathbf{A}_1(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_1(g) = [\{a, d\}, \{a, d\}]$,
$\mathbf{A}_2(e) = [\{b, d\}, \{b, d\}],\ \mathbf{A}_2(f) = [\{a, c, d\}, \{a, c, d\}],\ \mathbf{A}_2(g) = [\{a, b, d\}, \{a, b, d\}]$,
$\mathbf{A}_3(e) = [\varnothing, \varnothing],\ \mathbf{A}_3(f) = [\{c\}, \{c\}],\ \mathbf{A}_3(g) = [\{a\}, \{a\}]$,
$\mathbf{A}_4(e) = [\{a, b, d\}, \{a, b, d\}],\ \mathbf{A}_4(f) = [X, X],\ \mathbf{A}_4(g) = [X, X]$,
$\mathbf{A}_5(e) = [\{a, c\}, \{a, c\}],\ \mathbf{A}_5(f) = [\{b, d\}, \{b, d\}],\ \mathbf{A}_5(g) = [\{b\}, \{b\}]$,
$\mathbf{A}_6(e) = [\{a\}, \{a\}],\ \mathbf{A}_6(f) = [\{b\}, \{b\}],\ \mathbf{A}_6(g) = [\varnothing, \varnothing]$,
$\mathbf{A}_7(e) = [\{a, c\}, \{a, c\}],\ \mathbf{A}_7(f) = [\{b, d, d\}, \{b, c, d\}],\ \mathbf{A}_7(g) = [\{a, b, d\}, \{a, b, d\}]$,
$\mathbf{A}_8(e) = [\varnothing, \varnothing],\ \mathbf{A}_8(f) = [\{d\}, \{d\}],\ \mathbf{A}_8(g) = [\{b\}, \{b\}]$,
$\mathbf{A}_9(e) = [X, X],\ \mathbf{A}_9(f) = [X, X],\ \mathbf{A}_9(g) = [\{a, b, c\}, \{a, b, c\}]$,
$\mathbf{A}_{10}(e) = [\{a, c\}, \{a, c\}],\ \mathbf{A}_{10}(f) = [\{b, c, d\}, \{b, c, d\}],\ \mathbf{A}_{10}(g) = [\{a, b\}, \{a, b\}]$,
$\mathbf{A}_{11}(e) = [\{b, c, d\}, \{b, c, d\}],\ \mathbf{A}_{11}(f) = [X, X],\ \mathbf{A}_{11}(g) = [\{a, b, c\}, \{a, b, c\}]$,
$\mathbf{A}_{12}(e) = [\{a\}, \{a\}],\ \mathbf{A}_{12}(f) = [\{b, c, d\}, \{b, c, d\}],\ \mathbf{A}_{12}(g) = [\{a, b, d\}, \{a, b, d\}]$,
$\mathbf{A}_{13}(e) = [\{a\}, \{a\}],\ \mathbf{A}_{13}(f) = [\{b, d\}, \{b, d\}],\ \mathbf{A}_{13}(g) = [\{b\}, \{b\}]$,
$\mathbf{A}_{14}(e) = [\{c, d\}, \{c, d\}],\ \mathbf{A}_{13}(f) = [\{a, b\}, \{a, b\}],\ \mathbf{A}_{14}(g) = [\varnothing, \varnothing]$,
$\mathbf{A}_{15}(e) = [\{a\}, \{a\}],\ \mathbf{A}_{15}(f) = [\{b, c\}, \{b, c\}],\ \mathbf{A}_{15}(g) = [\{a\}, \{a\}]$
$\mathbf{A}_{16}(e) = [\varnothing, \varnothing],\ \mathbf{A}_{16}(f) = [\{c\}, \{c\}],\ \mathbf{A}_{16}(g) = [\{a, d\}, \{a, d\}]$,
$\mathbf{A}_{17}(e) = [\{a, b, d\}, \{a, b, d\}],\ \mathbf{A}_{15}(f) = [X, X],\ \mathbf{A}_{15}(g) = [\{a, b, d\}, \{a, b, d\}]$.
Then we have

$$\tau^c = \{\widetilde{\varnothing}_E, \mathbf{A}_1^c, \mathbf{A}_2^c, \cdots, \mathbf{A}_{15}^c, \widetilde{X}_E\}.$$

Let $\mathbf{A}$ be the IVSS over X defined by:

$$\mathbf{A}(e) = [\{a, b, c\}, \{a, b, c\}],\ \mathbf{A}(f) = [\{b, c, d\}, \{b, c, d\}]\ \mathbf{A}(g) = [\{a, b\}, \{a, b\}].$$

Then clearly, $IVSint(\mathbf{A}) = \mathbf{A}_{10}$. Thus $IVScl(IVSint(\mathbf{A})) = \widetilde{X}_E$. So we have

$$IVSint(IVScl(IVSint(\mathbf{A}))) = \widetilde{X}_E, \text{ i.e., } \mathbf{A} \subset IVSint(IVScl(IVSint(\mathbf{A}))).$$

Hence $\mathbf{A} \in IVS\alpha OS(X)$ but $\mathbf{A} \notin \tau$. Therefore $IVS\alpha OS(X) \not\subset \tau$.

Definition 5.11 ([36]). Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then

(i) the *soft α-closure* of A, denoted by $cl_{S\alpha}(A)$ or $cl_{S\alpha,\tau}(A)$, is a soft set over X defined as follows:

$$cl_{S\alpha}(A) = \bigcap\{F \in S\alpha CS(X) : A \subset F\},$$

(ii) the *soft α-interior* of A, denoted by $int_{S\alpha}(A)$ or $int_{S\alpha,\tau}(A)$, is a soft set over X defined as follows:

$$cl_{S\alpha}(A) = \bigcup\{U \in S\alpha OS(X) : U \subset A\}.$$

It is clear that $cl_{S\alpha}(A)$ is the smallest soft α-closed set over X which contains A and $int_{S\alpha}(A)$ is the largest soft α-open set over X which is contained in A.

Definition 5.12. Let (X, τ, E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then

(i) the *interval-valued soft α-closure* of $\mathbf{A}$, denoted by $cl_{IVS\alpha}(\mathbf{A})$, is an IVSS over X defined as follows:

$$cl_{IVS\alpha}(\mathbf{A}) = \bigcap\{\mathbf{F} \in IVS\alpha CS(X) : \mathbf{A} \subset \mathbf{F}\},$$

(ii) the *interval-valued soft α-interior* of $\mathbf{A}$, denoted by $int_{IVS\alpha}(\mathbf{A})$, is an IVSS over X defined as follows:

$$int_{IVS\alpha}(\mathbf{A}) = \bigcup\{\mathbf{U} \in IVS\alpha OS(X) : \mathbf{U} \subset \mathbf{A}\}.$$

It is obvious that $cl_{IVS\alpha}(\mathbf{A})$ is the smallest interval-valued soft α-closed set over X containing $\mathbf{A}$ and $int_{IVS\alpha}(\mathbf{A})$ is the largest interval-valued soft α-open set over X contained in $\mathbf{A}$.

Remark 5.13. Let (X, τ, E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then

(1) $cl_{IVS\alpha}(\mathbf{A}) = [cl_{S\alpha,\tau^-}(A^-), cl_{S\alpha,\tau^+}(A^+)]$,

(2) $int_{IVS\alpha}(\mathbf{A}) = [int_{S\alpha,\tau^-}(A^-), int_{S\alpha,\tau^+}(A^+)]$.

Example 5.14. Let (X, τ, E) be the IVSTS given in Example 5.4. To find IVSαOSs in X, consider the following IVSSs containing $\mathbf{U}$:

$$\mathbf{U},\ \mathbf{A}_1,\ \mathbf{A}_2,\ \cdots,\ \mathbf{A}_{20},$$

where $\mathbf{A}_1(e) = [\{a\}, \{a,b\}],\ \mathbf{A}_1(f) = [\{b\}, X],\ \mathbf{A}_2(e) = [\{a\}, \{a,b\}],$
$\mathbf{A}_2(f) = [\{b,c\}, \{b,c\}],\ \mathbf{A}_3(e) = [\{a\}, \{a,b\}],\ \mathbf{A}_3(f) = [\{b,c\}, X],$
$\mathbf{A}_4(e) = [\{a\}, \{a,b\}],\ \mathbf{A}_4(f) = [\{b,c\}, X], \mathbf{A}_5(e) = [\{a\}, \{a,b\}],$
$\mathbf{A}_5(f) = [X, X],\ \mathbf{A}_6(e) = [\{a,b\}, \{a,b\}],\ \mathbf{A}_6(f) = [\{b\}, \{b,c\}],$
$\mathbf{A}_6(e) = [\{a,b\}, \{a,b\}],\ \mathbf{A}_6(f) = [\{b\}, \{b,c\}],\ \mathbf{A}_7(e) = [\{a,b\}, \{a,b\}],$
$\mathbf{A}_6(f) = [\{b\}, X],\ \mathbf{A}_8(e) = [\{a,b\}, \{a,b\}],\ \mathbf{A}_8(f) = [\{b,c\}, \{b,c\}],$
$\mathbf{A}_9(e) = [\{a,b\}, \{a,b\}],\ \mathbf{A}_9(f) = [\{b,c\}, X],\ \mathbf{A}_{10}(e) = [\{a,b\}, \{a,b\}],$
$\mathbf{A}_{10}(f) = [X, X],\ \mathbf{A}_{11}(e) = [\{a,b\}, X],\ \mathbf{A}_{11}(f) = [\{b\}, \{b,c\}],$
$\mathbf{A}_{12}(e) = [\{a,b\}, X],\ \mathbf{A}_{12}(f) = [\{b\}, X],\ \mathbf{A}_{13}(e) = [\{a,b\}, X],$
$\mathbf{A}_{13}(f) = [\{b,c\}, \{b,c\}],\ \mathbf{A}_{14}(e) = [\{a,b\}, X],\ \mathbf{A}_{14}(f) = [\{b,c\}, X],$
$\mathbf{A}_{15}(e) = [\{a,b\}, X],\ \mathbf{A}_{15}(f) = [X, X],\ \mathbf{A}_{16}(e) = [X, X],$
$\mathbf{A}_{16}(f) = [\{b\}, \{b,c\}],\ \mathbf{A}_{17}(e) = [X, X],\ \mathbf{A}_{17}(f) = [\{b\}, X],$
$\mathbf{A}_{18}(e) = [X, X],\ \mathbf{A}_{18}(f) = [\{b,c\}, \{b,c\}],\ \mathbf{A}_{19}(e) = [X, X],$

$\mathbf{A}_{19}(f) = [\{b,c\},X],\ \mathbf{A}_{20}(e) = [X,X],\ \mathbf{A}_{20}(f) = [X,X]$.
Then clearly, $IVSint(\mathbf{A}_i) = \mathbf{U}$ for each $i \in \{1,2,\cdots,19\}$. Since $\widetilde{X}_E$ is the only IVSCS in X containing $\mathbf{U}$, $IVScl(IVSint(\mathbf{A}_i)) = \widetilde{X}_E$ for each $i \in \{1,2,\cdots,20\}$. Thus by Proposition 5.26 in [34], $IVSint(IVScl(IVSint(\mathbf{A}_i))) = \widetilde{X}_E$ for each $i \in \{1,2,\cdots,20\}$. Since $\mathbf{A}_i \subset \widetilde{X}_E$, $\mathbf{A}_i \in IVS\alpha OS(X)$ for each $i \in \{1,2,\cdots,20\}$. So $cl_{_{IVS\alpha}}(\mathbf{A}_i) = \widetilde{X}_E$ for each $i \in \{1,2,\cdots,20\}$.

Theorem 5.15. *Let (X,τ,E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then*
(1) $\mathbf{A} \in IVS\alpha CS(X)$ *if and only if* $A = cl_{_{IVS\alpha}}(\mathbf{A})$,
(2) $\mathbf{A} \in IVS\alpha OS(X)$ *if and only if* $A = int_{_{IVS\alpha}}(\mathbf{A})$.

Proof. The proofs are straightforward. □

Proposition 5.16. *Let (X,τ,E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then*
(1) $[cl_{_{IVS\alpha}}(\mathbf{A})]^c = int_{_{IVS\alpha}}(\mathbf{A}^c)$,
(2) $[int_{_{IVS\alpha}}(\mathbf{A})]^c = cl_{_{IVS\alpha}}(\mathbf{A}^c)$.

Proof. The proofs follow from Definition 5.12. □

Proposition 5.17. *Let (X,τ,E) be an IVSTS and $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. Then*
(1) $cl_{_{IVS\alpha}}(\widetilde{\varnothing}_E) = \widetilde{\varnothing}_E,\ cl_{_{IVS\alpha}}(\widetilde{X}_E) = \widetilde{X}_E$,
(2) $cl_{_{IVS\alpha}}(\mathbf{A}) \in IVS\alpha CS(X)$,
(3) *if* $\mathbf{A} \subset \mathbf{B}$, *then* $cl_{_{IVS\alpha}}(\mathbf{A}) \subset cl_{_{IVS\alpha}}(\mathbf{B})$,
(4) $cl_{_{IVS\alpha}}(cl_{_{IVS\alpha}}(\mathbf{A})) = cl_{_{IVS\alpha}}(\mathbf{A})$,
(5) $cl_{_{IVS\alpha}}(\mathbf{A} \cup \mathbf{B}) = cl_{_{IVS\alpha}}(\mathbf{A}) \cup cl_{_{IVS\alpha}}(\mathbf{B})$,
(6) $cl_{_{IVS\alpha}}(\mathbf{A} \cap \mathbf{B}) \subset cl_{_{IVS\alpha}}(\mathbf{A}) \cap cl_{_{IVS\alpha}}(\mathbf{B})$.

Proof. The proofs are straightforward. □

Proposition 5.18. *Let (X,τ,E) be an IVSTS and $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. Then*
(1) $int_{_{IVS\alpha}}(\widetilde{\varnothing}_E) = \widetilde{\varnothing}_E,\ int_{_{IVS\alpha}}(\widetilde{X}_E) = \widetilde{X}_E$,
(2) $int_{_{IVS\alpha}}(\mathbf{A}) \in IVS\alpha OS(X)$,
(3) *if* $\mathbf{A} \subset \mathbf{B}$, *then* $int_{_{IVS\alpha}}(\mathbf{A}) \subset int_{_{IVS\alpha}}(\mathbf{B})$,
(4) $int_{_{IVS\alpha}}(int_{_{IVS\alpha}}(\mathbf{A})) = int_{_{IVS\alpha}}(\mathbf{A})$,
(5) $int_{_{IVS\alpha}}(\mathbf{A} \cup \mathbf{B}) \subset int_{_{IVS\alpha}}(\mathbf{A}) \cup int_{_{IVS\alpha}}(\mathbf{B})$,
(6) $int_{_{IVS\alpha}}(\mathbf{A} \cap \mathbf{B}) = int_{_{IVS\alpha}}(\mathbf{A}) \cap int_{_{IVS\alpha}}(\mathbf{B})$.

Proof. The proofs are straightforward. □

6. Interval-valued soft α-separation axioms

In this section, we propose some of new separation axioms such as the IVSαT$_0$(j), IVSαT$_1$(j), IVSαT$_2$(j), IVSαT$_3$(j) and IVSαT$_4$(j) axioms for j=i, ii as a generalization of separation axioms discussed in Section 3. Furthermore, we study some of their properties and the relations between them in the general framework of IVSTSs.

Definition 6.1 ([35])**.** A soft topological space (X,τ,E) is called a *soft αT_0-space*, if for any $x \neq y \in X$, there are $U,\ V \in S\alpha OS(X)$ such that either $x \in U,\ y \notin U$ or $y \in V,\ x \notin V$.

Definition 6.2. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft* αT_0(i)-*space* (briefly, IVSαT_0(i)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that either $x_{_1} \in \mathbf{U},\ y_{_1} \notin \mathbf{U}$ or $y_{_1} \in \mathbf{V},\ x_{_1} \notin \mathbf{V}$,

(ii) *interval-valued soft* αT_0(ii)-*space* (briefly, IVSαT_0(ii)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that either $x_{_0} \in \mathbf{U},\ y_{_0} \notin \mathbf{U}$ or $y_{_0} \in \mathbf{V},\ x_{_0} \notin \mathbf{V}$.

Remark 6.3. (1) Every IVST_0(i) [resp. IVST_0(ii)]-space is an IVSαT_0(i) [resp. IVSαT_0(ii)]-space.

(2) Every IVSαT_0(i)-space is an IVSαT_0(ii)-space. The converse is not true in general (See Example 6.4).

(3) If an IVSTS (X, τ, E) is an IVSαT_0(i) [resp. IVSαT_0(ii)]-space, then (X, τ^-, E) and (X, τ^+, E) are soft αT_0-spaces [resp. (X, τ^+, E) is a soft αT_0-space].

Example 6.4. Let $X = \{a, b\}$, $E = \{e, f\}$ and consider the IVST τ_1 given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = \mathbf{A}(f) = [\{a\}, \{a\}]$. Then we can easily check that (X, τ_1, E) is an IVSαT_0(i)-space. Moreover, we can confirm that $\tau_1^- = \tau_1^+$ and (X, τ_1^-, E) is a soft αT_0-space.

Now consider the IVST τ_2 given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = \mathbf{A}(f) = \{\varnothing, \{a\}]$. Then clearly, (X, τ_2, E) is an IVSαT_0(ii)-space and (X, τ_2^+, E) is a soft αT_0-space. But (X, τ_2, E) is not an IVSαT_0(i)-space.

The following is a similar consequence of Proposition 3.13.

Proposition 6.5. *Let* (X, τ, E) *be an IVSTS and let* $x \neq y \in X$.

(1) *If there are* $\mathbf{U},\ \mathbf{U} \in IVS\alpha OS(X)$ *such that either* $x_{_1} \in \mathbf{U},\ y_{_1} \in \mathbf{U}^c$ *or* $y_{_1} \in \mathbf{V},\ x_{_1} \in \mathbf{V}^c$, *then* X *is an IVS*αT_0(i)-*space.*

(2) *If there are* $\mathbf{U},\ \mathbf{U} \in IVS\alpha OS(X)$ *such that either* $x_{_0} \in \mathbf{U},\ y_{_0} \in \mathbf{U}^c$ *or* $y_{_0} \in \mathbf{V},\ x_{_0} \in \mathbf{V}^c$, *then* X *is an IVS*αT_0(ii)-*space.*

Proof. The proofs are similar to Proposition 3.13. □

Also, we obtain a similar consequence of Proposition 4.10 (1).

Proposition 6.6. *Let* (X, τ, E) *be an IVSTS and let* Y *be a nonempty subset of* X. *If* X *is an IVS*αT_0(i) [resp. IVSαT_0(ii)]-*space, then* $(Y, \tau_{_Y}, E)$ *is an IVS*αT_0(i) [resp. IVSαT_0(ii)]-*space.*

Proof. The proofs are similar to Proposition 4.10 (1). □

Definition 6.7 ([35])**.** A soft topological space (X, τ, E) is called a *soft* αT_1-*space*, if for any $x \neq y \in X$, there are $U,\ V \in S\alpha OS(X)$ such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

Definition 6.8. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft* αT_1(i)-*space* (briefly, IVSαT_1(i)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U},\ y_{_1} \notin \mathbf{U}$ and $y_{_1} \in \mathbf{V},\ x_{_1} \notin \mathbf{V}$,

(ii) *interval-valued soft* αT_1(ii)*-space* (briefly, IVSαT_1(ii)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_0} \in \mathbf{U},\ y_{_0} \notin \mathbf{U}$ and $y_{_0} \in \mathbf{V},\ x_{_0} \notin \mathbf{V}$.

Remark 6.9. (1) Every IVSαT_1(i)-space is an IVSαT_1(ii)-space.

(2) If an IVSTS (X, τ, E) is an IVSαT_1(i) [resp. IVSαT_1(ii)]-space, then (X, τ^-, E) and (X, τ^+, E) are soft αT_1-spaces [resp. (X, τ^+, E) is a soft αT_1-space].

Example 6.10. Let $X = \{a, b\}$, $E = \{e, f\}$ and consider the IVST τ_1 given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [X, X],\ \mathbf{A}_1(f) = [\{b\}, \{b\}]$,
$\mathbf{A}_2(e) = [\{a\}, \{a\}],\ \mathbf{A}_2(f) = [X, X]$,
$\mathbf{A}_3(e) = [\{a\}, \{a\}],\ \mathbf{A}_3(f) = [\{b\}, \{b\}]$.
Then we can easily see that (X, τ_1, E) is an IVSαT_1(i)-space. Moreover, (X, τ_1^-, E) and (X, τ_1^+, E) are soft αT_1-spaces.

Now consider the IVST τ_2 given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [\varnothing, X],\ \mathbf{A}_1(f) = [\varnothing, \{b\}]$,
$\mathbf{A}_2(e) = [\varnothing, \{a\}],\ \mathbf{A}_2(f) = [\varnothing, X]$,
$\mathbf{A}_3(e) = [\varnothing, \{a\}],\ \mathbf{A}_3(f) = [\varnothing, \{b\}]$.
Then clearly (X, τ_2, E) is an IVSαT_1(ii)-space but not an IVSαT_1(i)-space. Furthermore, (X, τ_2^+, E) is a soft αT_1-spaces.

The following is an immediate consequence of Definitions 6.5 and 6.8.

Proposition 6.11. *Every IVST*$_1$(i) [resp. IVST$_1$(ii)]*-space is an IVS*αT_1(i) [resp. IVSαT_1(ii)]*-space. But the converse is not true in general* (See Example 6.12).

Proof. The proof is obvious. □

Example 6.12. Let $X = \{a, b, c, d\}$, $E = \{e, f\}$ and consider the IVST τ_1 given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a, b, c\}, \{a, b, c\}]$,
$\mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{b, c\}, \{b, c\}]$,
$\mathbf{A}_3(e) = \mathbf{A}_2(f) = [\{b\}, \{b\}]$.
Then we can see that (X, τ_1, E) is an IVSαT_0(i)-space but not an IVSαT_1(i)-space.

Now consider IVST τ_2 given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e)\mathbf{A}_1(f) = [\varnothing, \{a, b, c\}]$,
$\mathbf{A}_2(e) = \mathbf{A}_2(f) = [\varnothing, \{b, c\}]$,
$\mathbf{A}_3(e) = \mathbf{A}_2(f) = [\varnothing, \{b\}]$.
Then we can easily check that (X, τ_2, E) is an IVSαT_0(i)-space but not an IVSαT_1(ii)-space.

We have a similar consequence of Proposition 3.10.

Proposition 6.13. *Let* (X, τ, E) *be an IVSTS.*

(1) *If* $\mathbf{x_1}$ *is an IVS*α*CS in* X *for each* $x \in X$, *then* X *is an IVS*αT_1(i)*-space.*

(2) *If* $\mathbf{x_0}$ *is an IVS*α*CS in* X *for each* $x \in X$, *then* X *is an IVS*αT_1(ii)*-space.*

Proof. The proofs are similar to Proposition 3.10. □

Theorem 6.14. *Let (X, τ, E) be an IVSTS.*
(1) *X is an IVSαT_1(i)-space if and only if $x_{_1} \in IVS\alpha CS(X)$ for each $x \in X$.*
(2) *X is an IVSαT_1(ii)-space if and only if $x_{_0} \in IVS\alpha CS(X)$ for each $x \in X$.*

Proof. (1) Suppose X is an IVSαT_1(i)-space and let $x \neq y \in X$. Then there is $\mathbf{U}_{x_{_1}} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U}_{x_{_1}}$, $y_{_1} \notin \mathbf{U}_{x_{_1}}$. Thus by Proposition 5.8 (1), $x_{_1} \in \bigcup_{x \in X} \mathbf{U}_{x_{_1}} \in IVS\alpha OS(X)$. So $x_{_1} = [\bigcup_{x \in X} \mathbf{U}_{x_{_1}}]^c \in IVS\alpha CS(X)$.

Conversely, suppose the necessary condition holds and let $x \neq y \in X$. Then clearly, $y_{_1} \in [X - \{x\}, X - \{x\}] \in IVS\alpha OS(X)$ and $x_{_1} \notin [X - \{x\}, X - \{x\}]$. Similarly, $x_{_1} \in [X - \{y\}, X - \{y\}] \in IVS\alpha OS(X)$ and $y_{_1} \notin [X - \{y\}, X - \{y\}]$. Thus X is an IVSαT_1(i)-space.

(2) The proof is similar to (1). □

The following is a similar consequence of Proposition 4.10 (2).

Proposition 6.15. *Let (X, τ, E) be an IVSTS and let Y be a nonempty subset of X. If X is an IVSαT_1(i) [resp. IVSαT_1(ii)]-space, then $(Y, \tau_{_Y}, E)$ is an IVSαT_1(i) [resp. IVSαT_1(ii)]-space*

Proof. The proof is similar to Proposition 4.10 (2). □

Definition 6.16 ([35]). A soft topological space (X, τ, E) is called a *soft αT_2-space*, if for any $x \neq y \in X$, there are $U,\ V \in S\alpha OS(X)$ such that $x \in U$, $y \in V$ and $U \cap C = \varnothing_E$.

Definition 6.17. An IVSTS (X, τ, E) is called an:
(i) *interval-valued soft αT_2(i)-space* (briefly, IVSαT_2(i)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$,
(ii)*interval-valued soft αT_2(ii)-space* (briefly, IVSαT_2(ii)-space), if for any $x \neq y \in X$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_0} \in \mathbf{U}$, $y_{_0} \in \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.

Remark 6.18. (1) Every IVSαT_2(i)-sapce is an IVSαT_2(ii)-space.
(2) If an IVSTS (X, τ, E) is an IVSαT_2(i) [resp. IVSαT_2(ii)]-space, then (X, τ^-, E) and (X, τ^+, E) are soft αT_2-spaces [resp. (X, τ^+, E) is a soft αT_2-space].

Example 6.19. Let $X = \{a, b\}$, $E = \{e, f\}$ and consider the IVST τ_1 given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a\}, \{a\}],\ \mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{b\}, \{b\}]$.
Then clearly, (X, τ_1, E) is an IVSαT_2(i)-space, and, (X, τ_1^-, E) and (X, τ_1^+, E) are soft αT_2-spaces.

Now consider the IVST τ_2 given by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\varnothing, \{a\}],\ \mathbf{A}_2(e) = \mathbf{A}_2(f) = [\varnothing, \{b\}]$.
Then it is conspicuous that (X, τ_2, E) is an IVSαT_2(ii)-space but not an IVSαT_2(i)-space. Moreover, (X, τ_2^+, E) is a soft αT_2-spaces.

Proposition 6.20. *Every IVSαT_2(i) [resp. IVSαT_2(ii)]-space is an IVSαT_1(i) [resp. IVSαT_1(ii)]-space. But the converse is not true in general* (See Example 6.21).

Proof. The proof is clear. □

Example 6.21. Consider the IVSαT_1(i)-space (X, τ_1, E) [resp. IVSαT_1(ii)-space (X, τ_2, E) given in Example 6.10. Then we can see that (X, τ_1, E) is not an IVSαT_2(i) [resp. IVSαT_2(ii)]-space.

We have a similar consequence of Proposition 4.11.

Proposition 6.22. *Let (X, τ, E) be an IVSTS and let Y be a nonempty subset of X. If X is an IVSαT_2*(i) [resp. *IVSαT_2*(ii)]*-space, then $(X, \tau_{_Y}, E)$ is an IVSαT_2*(i) [resp. *IVSαT_2*(ii)]*-space.*

Proof. The proof is similar to Proposition 4.11. □

Definition 6.23 ([35])**.** A soft topological space (X, τ, E) is called a *soft α-regular space*, if for each $A \in S\alpha CS(X)$ and each $x \in X$ with $x \notin A$, there are $U,\ V \in S\alpha OS(X)$ such that $x \in U,\ A \subset V$ and $U \cap V = \varnothing_E$.

Definition 6.24. An IVSTS (X, τ, E) is called an:
(i) *interval-valued soft α-regular*(i)*-space* (briefly, IVSαR(i)-space), if for each $\mathbf{A} \in IVS\alpha CS(X)$ and each $x \in X$ with $x_{_1} \notin \mathbf{A}$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U},\ \mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$,
(ii)*interval-valued soft α-regular*(ii)*-space* (briefly, IVSαR(ii)-space), if for each $\mathbf{A} \in IVS\alpha CS(X)$ and each $x \in X$ with $x_{_0} \notin \mathbf{A}$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_0} \in \mathbf{U},\ \mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.

It is obvious that if an IVSTS (X, τ, E) is an IVSR(i) [resp. IVSR(ii)]-space, then it is an IVSαR(i) [resp. IVSαR(ii)]-space.

Example 6.25. Let $X = \{a, b, c\}$, $E = \{e, f\}$ and consider the IVST τ_1 on X defined by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a\}, \{a\}],\ \mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{b, c\}, \{b, c\}]$.
Then we can easily check that (X, τ_1, E) is an IVSαR(i)-space but not an IVSαT_1(i)-space. Now consider the IVST τ_2 on X defined by:

$$\tau_2 = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\varnothing, \{a\}],\ \mathbf{A}_2(e) = \mathbf{A}_2(f) = [\varnothing, \{b, c\}]$.
Then clearly, (X, τ_1, E) is an IVSαR(ii)-space but not an IVSαT_1(ii)-space.

Definition 6.26 ([35])**.** A soft topological space (X, τ, E) is called a *soft αT_3-space*, if it is a soft αT_1-space and a soft α-regular space.

Definition 6.27. An IVSTS (X, τ, E) is called an:
(i) *interval-valued soft αT_3*(i)*-space* (briefly, IVSαT_3(i)-space), if it is an IVSαT_1(i)-space and an IVSαR(i)-space,
(ii) *interval-valued soft αT_3*(ii)*-space* (briefly, IVSαT_3(ii)-space), if it is an IVSαT_1(ii)-space and an IVSαR(ii)-space.

Remark 6.28. Every IVSαT_3(i)-space is an IVSαT_3(ii)-space.

Proposition 6.29. *Every IVSαT_3*(i) [resp. IVSαT_3(ii)]*-space is an IVSαT_2*(i) [resp. IVSαT_2(ii)]*-space. But the converse is not true in general* (See Example 6.30)*.*

Proof. Let (X, τ, E) be an IVSαT_3(i)-space and let $x \neq y \in X$. Since X is an IVSαT_1(i)-space, by Theorem 6.14, $y_{_1} \in IVS\alpha CS(X)$ and $x_{_1} \notin y_{_1}$. Since X is an IVSαR(i)-space, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U}$, $y_{_1} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Then (X, τ, E) is an IVSαT_2(i)-space. The proof of the second part can be done by the same token. □

We obtain a similar consequence of Proposition 4.12.

Proposition 6.30. *Let (X, τ, E) be an IVSTS and let Y be a nonempty subset of X. If X is an IVSαT_3(i) [resp. IVSαT_3(ii)]-space, then $(X, \tau_{_Y}, E)$ is an IVSαT_3(i) [resp. IVSαT_3(ii)]-space.*

Proof. The proof is similar to Proposition 4.12. □

Definition 6.31 ([35])**.** A soft topological space (X, τ, E) is called a *soft α-normal space*, if for each $A,\ B \in S\alpha CS(X)$ with $A \cap B = \varnothing_E$, there are $U,\ V \in S\alpha OS(X)$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing_E$.

Definition 6.32. An IVSTS (X, τ, E) is called an *interval-valued soft α-normal-space* (briefly, IVSαN-space), if for each $\mathbf{A},\ \mathbf{B} \in IVS\alpha CS(X)$ with $\mathbf{A} \cap \mathbf{B} = \widetilde{\varnothing}_E$, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U}$, $\mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.

Remark 6.33. Every IVSαN-space may be neither an IVSαR(i) [resp. IVSαR(ii)]-space nor an IVSαT_1(i) [resp. IVSαT_1(ii)]-space (See Example 6.34).

Example 6.34. Let $X = \{a, b, c\}$, $E = \{e, f\}$ and consider the IVST τ on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = [\{a\}, \{a\}],\ \mathbf{A}_2(e) = \mathbf{A}_2(f) = [\{b\}, \{b\}]$,
$\mathbf{A}_3(e) = \mathbf{A}_3(f) = [\{a, b\}, \{a, b\}]$.
Then we can see that (X, τ, E) is an IVSαN-space but neither an IVSαR(i) [resp. IVSαR(ii)]-space nor an IVSαT_1(i) [resp. IVSαT_1(ii)]-space.

Definition 6.35 ([35])**.** A soft topological space (X, τ, E) is called a *soft αT_4-space*, if it is a soft αT_1-space and a soft α-normal space.

Definition 6.36. An IVSTS (X, τ, E) is called an:

(i) *interval-valued soft αT_4(i)-space* (briefly, IVSαT_4(i)-space), if It is an IVSαT_1(i)-space and an IVSαN-space,

(ii) *interval-valued soft αT_4(ii)-space* (briefly, IVSαT_4(ii)-space), if It is an IVSαT_1(ii)-space and an IVSαN-space.

Example 6.37. In Example 6.19, we can easily check that an IVSTS (X, τ_1, E) [resp. (X, τ_2, E)[is an IVSαT_4(i) [resp. IVSαT_4(ii)]-space.

Proposition 6.38. *Every IVSαT_4(i) [resp. IVSαT_4(ii)]-space is an IVSαT_3(i) [resp. IVSαT_3(ii)]-space.*

Proof. Let X be an IVSαT_4(i)-space. Since X is an IVSαT_1(i)-space, it is enough to prove that X is an IVSαR(i)-space. Let $\mathbf{A} \in IVS\alpha CS(X)$ with $x_{_1} \notin \mathbf{A}$. Since X is an IVSαT_1(i)-space, by Theorem 6.14 (1), $x_{_1} \in IVS\alpha CS(X)$. Note that $\mathbf{A} \cap x_{_1} = \widetilde{\varnothing}_E$. Since X is an IVSαN-space, there are $\mathbf{U},\ \mathbf{V} \in IVS\alpha OS(X)$ such that $x_{_1} \in \mathbf{U}$,

$\mathbf{A} \subset \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus X is an IVSαT_3(i)-space. The proof of the second part is analogous and therefore omitted. □

Remark 6.39. From Propositions 6.11, 6.20, 6.29 and 6.38, we have relationships among IVSαT_0(i) [resp. IVSαT_0(ii)], IVSαT_1(i) [resp. IVSαT_1(ii)], IVSαT_2(i) [resp. IVSαT_2(ii)], IVSαT_3(i) [resp. IVSαT_3(ii)] and IVSαT_4(i) [resp. IVSαT_4(ii)]:

$$\begin{aligned} \text{IVS}\alpha T_4\text{(i) [resp. IVS}\alpha T_4\text{(ii)]} &\Longrightarrow \text{IVS}\alpha T_3\text{(i) [resp. IVS}\alpha T_3\text{(ii)]} \\ &\Longrightarrow \text{IVS}\alpha T_2\text{(i) [resp. IVS}\alpha T_2\text{(ii)]} \\ &\Longrightarrow \text{IVS}\alpha T_1\text{(i) [resp. IVS}\alpha T_1\text{(ii)]} \\ &\Longrightarrow \text{IVS}\alpha T_0\text{(i) [resp. IVS}\alpha T_0\text{(ii)].} \end{aligned}$$

7. Conclusions

First, we defined separation axioms, i.e., IVST_0(i) [resp. IVST_0(ii)], IVST_1(i) [resp. IVST_1(ii)], IVST_2(i) [resp. IVST_2(ii)], IVST_3(i) [resp. IVST_3(ii)] and IVST_4(i) [resp. IVST_4(ii)], and studied some of their relationships. Second, by using interval-valued soft α-open sets, we introduced some of new separation axioms, i.e., IVSαT_0(i) [resp. IVSαT_0(ii)], IVSαT_1(i) [resp. IVSαT_1(ii)], IVSαT_2(i) [resp. IVSαT_2(ii)], IVSαT_3(i) [resp. IVSαT_3(ii)] and IVSαT_4(i) [resp. IVSαT_4(ii)], and discussed relationships among them. Moreover, we dealt with hereditary properties of each separation axiom.

In the future, we would like to study new separation axioms as well as decision making problems that were not covered in our study.

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