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On intuitionistic fuzzy congruence of a near-ring module

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Abstract. This study aims to examine intuitionistic fuzzy congruences and intuitionistic fuzzy submodules on an R-module (near-ring module). The relationship between intuitionistic fuzzy congruences and intuitionistic fuzzy submodules of an R-module is also obtained. Furthermore, the intuitionistic fuzzy quotient R-module of an R-module over an intuitionistic fuzzy submodule is defined. The correspondence between intuitionistic fuzzy congruences on an R-module and intuitionistic fuzzy congruences on the intuitionistic fuzzy quotient R-module of an R-module over an intuitionistic fuzzy submodule of an R-module is also obtained.

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1. Introduction

The concept of a fuzzy set was introduced by Zadeh [1] in 1965. Since then, there has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behaviour studies. The concept of fuzzy relations on a set was defined by Zadeh [1, 2]. Fuzzy relations on group has been studied by Bhattacharya and Mukharjee [3], and those in rings and groups by Malik and Modeson in [4]. The detailed applications of fuzzy relations are given by Baets and Kerre in [5]. The construction of a fuzzy congruence relation generated by a fuzzy relation on a vector space was given by Khosravi et al. in [6]. Dutta and Biswas [7] applied the concept of fuzzy congruence in the near-ring module. As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [8] in 1983. After that time, several researchers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] applied the notion of intuitionistic fuzzy sets to relations, algebra, topology, and topological structures. In particular, Bustince and

Burillo [20], and Deschrijver and Kerre [21] applied the concept of intuitionistic fuzzy sets to relations. Also, Hur et al. [14] investigated several properties of intuitionistic fuzzy congruences. Moreover, Hur and his colleagues [22] introduce the notion of intuitionistic fuzzy congruences on a lattice and a semigroup, and investigate some of their properties. Basnet in [11, 12] studied many properties of intuitionistic fuzzy relations with respect to level cut sets. The notion of intuitionistic fuzzy congruence on groups was introduced by Emam [23], and that of universal algebra was studied by Cuvalcioglu and Tarsuslu (Yilmaz) in [24, 25]. Rasuli in [26, 27] studied intuitionistic fuzzy congruence on groups and rings under the t-norm, respectively.

Since the correspondence theorem provides a bridge between algebraic structures, and their quotients, allowing us to study properties, factorizations, and relationships in a more manageable way. Its applications extend beyond pure algebra and have implications in various mathematical areas. The main objective of this paper is to establish a connection between intuitionistic fuzzy congruences and intuitionistic fuzzy submodules on an R-module (where R is a near-ring) and quotient R-module over an intuitionistic fuzzy submodule of an R-module. This is achieved by establishing a one-to-one correspondence between the set of intuitionistic fuzzy submodules and the set of intuitionistic fuzzy congruences of an R-module. Lastly, we study the intuitionistic fuzzy congruence of a quotient R-module over an intuitionistic fuzzy submodule of an R-module and obtain a correspondence theorem. .

2. Preliminaries

We recall some definitions and results that are used in this paper. For details, see the references quoted therein.

Definition 2.1 ([28, 29])**.** A *near-ring* R is a system with two binary operations, addition and multiplication, such that:
(i) $(R,+)$ is a group,
(ii) $(R,\cdot)$ is a semigroup,
(iii) $x(y+z) = xy + xz$ for all $x,\ y, z \in R$.

Definition 2.2 ([28])**.** An *R-module* (i.e. *near-ring module*) M is a system consisting of an additive group M, a near-ring R, and a mapping $(m, r) \rightarrowtail mr$ of $M \times R$ into M such that
(i) $m(x + y) = mx + my$ for all $m \in M$ and for all $x,\ y \in R$,
(ii) $m(xy) = (mx)y$ for all $m \in M$ and for all $x,\ y \in R$.

Definition 2.3 ([28])**.** An *R-homomorphism* f of an R-module M into an R-module $M^{'}$ is a mapping from M to $M^{'}$ such that for all $m; m_1,\ m_2 \in M$ and for all $r \in R$,
(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$,
(ii) $f(m)r = f(mr)$.

A non-empty subset N of an R-module M that forms with the restrictions of the operations on M (addition and scalar multiplication) to N itself an R-module is called a *submodule* of M. The *kernel* of an R-module homomorphism $f : M \to M^{'}$, denoted by $ker f$ is defined as:

$$ker f = \{x \in M : f(x) = 0^{'}\}.$$

It may be noted that $kerf$ is a submodule of M.

The submodules of an R-module M are defined to be the kernels of R-homomorphisms.

Proposition 2.4 ([28])**.** *An additive normal subgroup N of an R-module M is a submodule if and only if $(m+b)r - mr \in N$ for each $m \in M,\ b \in N$ and $r \in R$.*

Definition 2.5 ([14])**.** A relation ρ on an R-module M is called a *congruence* on M, if it is an equivalence relation on M such that $(a,b) \in \rho$ and $(c,d) \in \rho$ imply that $(a+c, b+d) \in \rho$ and $(ar, br) \in \rho$ for all $a,\ b,\ c,\ d$ in M and for all r in R.

Definition 2.6 ([8, 9, 10])**.** An *intuitionistic fuzzy set* (IFS) A in X can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

The *intutionistic fuzzy whole* [resp. *empty*] *set* in X, denoted by $\overline{1}$ [resp. $\overline{0}$], is an IFS in X defined as follows: for each $x \in X$,

$$\overline{1}(x) = (1,0) \text{ [resp. } \overline{0}(x) = (0,1)].$$

The set of all IFSs of X will be written as $IFS(X)$.

Remark 2.7 ([10, 12])**.** (1) When $\mu_A(x) + \nu_A(x) = 1\ \forall x \in X$, A is a fuzzy set.

(2) If $p, q \in [0,1]$ such that $p + q \le 1$, then $A \in IFS(X)$ defined by $\mu_A(x) = p$ and $\nu_A(x) = q$ for all $x \in X$, is called a *constant IFS* of X.

If $A,\ B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)\ \forall x \in X$. For any subset Y of X, the intuitionistic fuzzy characteristic function (IFCF) χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1,0)\ \forall x \in Y$ and $\chi_Y(x) = (0,1)\ \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$. Then the crisp set

$$A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \ge \alpha \text{ and } \nu_A(x) \le \beta\}$$

is called the (α, β)*-level subset* of A (See[12]).

Definition 2.8 ([13])**.** A nonempty IFS of an additive group G is called an *intuitionistic fuzzy normal subgroup* of G, if for all $x,\ y$ in G,

(i) $\mu_A(x+y) \ge \min\{\mu_A(x), \mu_A(y)\}$, $\nu_A(x+y) \le \max\{\nu_A(x), \nu_A(y)\}$,

(ii) $\mu_A(-x) = \mu_A(x)$, $\nu_A(-x) = \nu_A(x)$,

(iii) $\mu_A(y + x - y) = \mu_A(x)$, $\nu_A(y + x - y) = \nu_A(x)$.

Definition 2.9 ([13])**.** Let A be an intuitionistic fuzzy normal subgroup of an additive group G and $x \in G$. Then the IFS $x + A$ in G defined by

$$\mu_{(x+A)}(y) = \mu_A(y-x) \text{ and } \nu_{(x+A)}(y) = \nu_A(y-x) \text{ for all } y \text{ in } G$$

is called the *intuitionistic fuzzy coset of A with respect to x*.

3. Intuitionistic fuzzy submodule (IFSM)

Definition 3.1. Let A be a non-empty IFS of an R-module M. Then A is said to be an *intuitionistic fuzzy submodule* (IFSM) of M, if for all $x,\ y \in M,\ r \in R$,

(i) A is an IF normal subgroup of M,

(ii) $\mu_A((x+y)r - xr) \ge \mu_A(y)$ and $\nu_A((x+y)r - xr) \le \nu_A(y)$.

Example 3.2. Consider the additive group $(M = \{0, 1, 2, 3\}, +_4)$ and the near-ring $R = (M_{2\times 2}, +, .)$ of 2×2 matrices over the real numbers under the usual operation of addition and multiplication of matrices. Let $. : M \times R \to M$ be the mapping defined as $mX = m + m + m + ... + m(|X|$ times), where $|X|$ denotes the determinant of matrix $X \in R$. Then M is an R-module.
Define the IFS A on M as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0, 2 \\ 0.6, & \text{if } x = 1, 3 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0, 2 \\ 0.3, & \text{if } x = 1, 3. \end{cases}$$

Then it is a routine matter to check that the conditions (i) and (ii) of Definition 3.1 hold. Thus A is an IFSM of an R-module M.

Proposition 3.3. *Let K be a non-empty subset of an R-module M. Then the IF characteristic function χ_K is an IFSM of M if and only if K is a submodule of M.*

The proposition can be directly verified.

Proposition 3.4. *Let A be an IFSM of an R-module M. Then the (α, β)-level set $A_{(\alpha,\beta)} = \{x \in M : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is a submodule of M*

Proof. The proof is omitted. □

Definition 3.5. Let A be an IFSM of an R-module M. Then the submodule $A_{(\alpha,\beta)}$ is called the (α, β)*-level submodule* of M.

Proposition 3.6. *For a non-empty IFS A of an R-module M, the following assertion are equivalent:*
(1) *A is an IFSM of M,*
(2) *the (α, β)-level set $A_{(\alpha,\beta)}$ are submodules of M*

Proof. Since the proof is a simple matter of verification, we omit it. □

Proposition 3.7. *Let A is an IFNSG of an additive group G. Then $x + A = y + A$ if and only if $\mu_A(y - x) = \mu_A(0), \nu_A(y - x) = \nu_A(0)$ for all $x, y \in G$.*

Proof. The proof is straightforward. □

Theorem 3.8. *Let A be an IFSM of an R-module M. Then the set M/A of all IF cosets of A is an R-module w.r.t. the operation defined by*
$(x + A) + (y + A) = (x + y) + A$ and $(x + A)r = (xr + A)$ for all $x, y \in M, r \in R$.

If $f : M \to M/A$ is a surjective mapping defined by $f(x) = x + A$ for all $x \in M$, then f is an R-homomorphism with $Kerf = \{x \in M : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$.

Proof. We shall first show that the given operations are well-defined.
Let $x, y, u, v \in M$ be such that $x + A = u + A$ and $y + A = v + A$. Then $\mu_A(x - u) = \mu_A(0), \nu_A(x - u) = \nu_A(0)$ and $\mu_A(y - v) = \mu_A(0), \nu_A(y - v) = \nu_A(0)$.
On the other hand, we have

$$\begin{aligned} \mu_A(x + y - v - u) &= \mu_A(-u + x + y - v) \\ &\geq \min\{\mu_A(-u + x), \mu_A(y - v)\} \\ &= \mu_A(0) \end{aligned}$$

$$\geq \mu_A(x+y-v-u).$$

Thus $\mu_A(x+y-v-u) = \mu_A(0)$. Similarly, we can show that $\nu_A(x+y-v-u) = \nu_A(0)$. So $x+y+A = u+v+A$, i.e., $(x+A)+(y+A) = (u+A)+(v+A)$. Hence the first operation is well-defined.

Let x, y be two elements of M such that $x+A = y+A$. Let $r \in R$. Then $\mu_A(x-y) = \mu_A(0),\ \nu_A(x-y) = \nu_A(0)$. Thus we get

$$\mu_A(xr-yr) = \mu_A((y-y+x)r-yr) \geq \mu_A(y-x) = \mu_A(x-y) = \mu_A(0).$$

Again $\mu_A(0) = \mu_A((0+xr-yr)0_R - 00_R) \geq \mu_A(xr-yr)$. So $\mu_A(xr-yr) = \mu_A(0)$. Similarly, we can show that $\nu_A(xr-yr) = \nu_A(0)$. Hence $xr+A = yr+A$. Therefore the second operation is well-defined.

Now let f be the mapping from M to M/A defined by $f(x) = x+A$ for all x in M. Then we have

$$f(x+y) = x+y+A = (x+A)+(y+A) = f(x)+f(y).$$

Also, we get

$$f(xr) = xr+A = (x+A)r = f(x)r \text{ for } x, y \in M \text{ and } r \in R.$$

Obviously f is surjective. Thus f is an R-epimorphism.
Lastly, $x \in Kerf \Leftrightarrow f(x) = 0+A \Leftrightarrow x+A = 0+A \Leftrightarrow \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)$. So $Kerf = \{x \in M : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$. □

Definition 3.9. The R-module M/A is called the *quotient R-module* of M over its intuitionistic fuzzy submodule A.

4. Intuitionistic fuzzy congruence

Definition 4.1 ([11, 14])**.** Let M be an R-module. A nonempty intuitionistic fuzzy relation (IFR) ρ on M [i.e., a mapping $\rho : M \times M \to [0,1] \times [0,1]$] is called an *intuitionistic fuzzy equivalence relation* (IFER), if for all $x,\ y \in M$,

(i) $\mu_\rho(x,x) = 1, \nu_\rho(x,x) = 0$ (IF reflective),

(ii) $\mu_\rho(x,y) = \mu_\rho(y,x)$ and $\nu_\rho(x,x) = \nu_\rho(y,x)$ (IF symmetric),

(iii) $\mu_\rho(x,y) \geq \sup_{z\in M}\{\mu_\rho(x,z), \mu_\rho(z,y)\}$ and $\nu_\rho(x,y) \leq \inf_{z\in M}\{\nu_\rho(x,z), \nu_\rho(z,y)\}$ (IF transitive).

Let ρ be an IFER on a set M and $a \in M$ be any element. Then the IFS ρ_a on M defined by $\mu_{\rho_a}(x) = \mu_\rho(a,x)$ and $\nu_{\rho_a}(x) = \nu_\rho(a,x)\ \forall x \in M$ is called the *intuitionistic fuzzy equivalence class* of ρ containing a. The set $\{\rho_a : a \in M\}$ is called the *intuitionistic fuzzy quotient set* of M by ρ and is denoted by M/ρ.

Definition 4.2 ([14])**.** An IFER ρ on an R-module M is called an *intuitionistic fuzzy congruence* (IFC), if for all $a,\ b,\ c,\ d \in M$ and all $r \in R$.

(i) $\mu_\rho(a+c, b+d) \geq \min\{\mu_\rho(a,b), \mu_\rho(c,d)\}$, $\nu_\rho(a+c, b+d) \leq \max\{\nu_\rho(a,b), \nu_\rho(c,d)\}$,

(ii) $\mu_\rho(ar, br) \geq \mu_\rho(a,b)$, $\nu_\rho(ar, br) \leq \nu_\rho(a,b)$.

Example 4.3. Consider $M = \mathbf{Z_4}$, $R = (\{0,1\}, +_2, ._2)$. Then M is an R-module. Now we define the IFS ρ on $M \times M$ by

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.5, & \text{if } (x,y) \in \{(2,3),(3,2)\} \\ 0.3, & \text{otherwise,} \end{cases}$$

$$\nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.4, & \text{if } (x,y) \in \{(2,3),(3,2)\} \\ 0.6, & \text{otherwise.} \end{cases}$$

Then ρ is an IFER on M, but it is not an IFC, for $\mu_\rho(2{+}2, 3{+}2) \ngeq \min\{\mu_\rho(2,3), \mu_\rho(2,2)\}$.

Example 4.4. Consider the R-module M as in Example 3.2, we define the IFS ρ on $M \times M$ as follows

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.6, & \text{if } x \neq y, \end{cases} \quad \nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.3, & \text{if } x \neq y. \end{cases}$$

Then it is a routine matter to verify that the conditions of Definition 4.2 hold. Thus ρ is an IFC on M.

Theorem 4.5. *Let λ be a relation on an R-module M and χ_λ be its intuitionistic fuzzy characteristic function. Then λ is a congruence relation on M if and only if χ_λ is an IFC on M.*

Proof. The proof is omitted. □

Definition 4.6 ([11, 14])**.** Let ρ be an IFR on an R-module M. For each $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, the set

$$\rho_{(\alpha,\beta)} = \{(a,b) \in M \times M : \mu_\rho(a,b) \geq \alpha, \nu_\rho(a,b) \leq \beta\}$$

is called a (α,β)-*level set* of ρ.

It is shown in [11] that ρ is an IFER if and only if $\rho_{(\alpha,\beta)}$ is an equivalence relation on M for all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$."

Theorem 4.7. *Let ρ be an IFER on an R-module M. Then ρ is an IFC on M if and only if $\rho_{(\alpha,\beta)}$ is a congruence on M for each $(\alpha,\beta) \in img(\rho)$ (Image of ρ).*

Proposition 4.8. *Let ρ be an IFC on an R-module M and A_ρ be an IFS of M defined by*

$$\mu_{A_\rho}(a) = \mu_\rho(a,0) \text{ and } \nu_{A_\rho}(a) = \nu_\rho(a,0) \text{ for all } a \in M.$$

Then A_ρ is an IFSM of M.

Proof. Since $\mu_{A_\rho}(0) = \mu_\rho(0,0) = 1$ and $\nu_{A_\rho}(0) = \nu_\rho(0,0) = 0$.
Also, $\mu_{A_\rho}(a+b) = \mu_\rho(a+b,0) \geq \min\{\mu_\rho(a,0), \mu_\rho(b,0)\} = \min\{\mu_{A_\rho}(a), \mu_{A_\rho}(b)\}$
Similarly, $\nu_{A_\rho}(a+b) = \nu_\rho(a+b,0) \leq \max\{\nu_\rho(a,0), \nu_\rho(b,0)\} = \max\{\nu_{A_\rho}(a), \nu_{A_\rho}(b)\}$.
Then we have

$$\begin{aligned} \mu_{A_\rho}(-a) &= \mu_\rho(-a,0) = \mu_\rho(-a+0, -a+a) \\ &\geq \min\{\mu_\rho(-a,-a), \mu_\rho(0,a)\} \\ &= \mu_\rho(0,a) = \mu_\rho(a,0) \\ &= \mu_{A_\rho}(a). \end{aligned}$$

Similarly, we can show $\mu_{A_\rho}(a) \geq \mu_{A_\rho}(-a)$. Thus $\mu_{A_\rho}(-a) = \mu_{A_\rho}(a)$. In the same way, we can show that $\nu_{A_\rho}(-a) = \nu_{A_\rho}(a)$. Also, we have

$$\mu_{A_\rho}(a+b-a) = \mu_\rho(a+b-a,0) = \mu_\rho(a+b-a, a+0-a) \geq \mu_\rho(b,0) = \mu_{A_\rho}(b).$$

Similarly, we can show that $\nu_{A_\rho}(a+b-a) \leq \nu_{A_\rho}(b)$. So A_ρ is an IFNSG of an R-module M.
Now for $a,\ b \in M$ and $r \in R$, we have

$$\begin{aligned}
\mu_{A_\rho}((a+b)r - ar) &= \mu_\rho((a+b)r - ar, 0) = \mu_\rho((a+b)r - ar, ar - ar)\\
&\geq \min\{\mu_\rho((a+b)r, ar), \mu_\rho(-ar, -ar)\} = \mu_\rho((a+b)r, ar)\\
&\geq \mu_\rho((a+b), a)\\
&\geq \mu_\rho(b, 0)\\
&= \mu_{A_\rho}(b).
\end{aligned}$$

Similarly, we can show that $\nu_{A_\rho}((a+b)r - ar) \leq \nu_{A_\rho}(b)$.
Hence A_ρ is an IFSM of M. □

Example 4.9. Consider the R-module M as in Example 3.2, we define the IFS ρ on $M \times M$ as follows

$$\mu_\rho((x,y)) = \begin{cases} 1, & \text{if } x = y \\ 0.6, & \text{if } x \neq y \end{cases}; \quad \nu_\rho((x,y)) = \begin{cases} 0, & \text{if } x = y \\ 0.3, & \text{if } x \neq y. \end{cases}$$

Then ρ is an IFC on M (See Example 4.4). Let A_ρ be an IFS of M, defined by $\mu_{A_\rho}(a) = \mu_\rho(a,0)$ and $\nu_{A_\rho}(a) = \nu_\rho(a,0)$ for all $a \in M$. Then it is a easy to check that

$$\mu_{A_\rho}(a) = \begin{cases} 1, & \text{if } a = 0 \\ 0.6, & \text{if } a \neq 0, \end{cases} \quad \nu_{A_\rho}(a) = \begin{cases} 0, & \text{if } a = 0 \\ 0.3, & \text{if } a \neq 0. \end{cases}$$

Thus A_ρ is an IFSM of M.

Remark 4.10. From Examples 3.2 and 4.9, one can easily check that $(\rho)_{A_\rho} \neq A$.

Proposition 4.11. *Let A be an IFSM of an R-module M. Let ρ_A be an IFR on M defined by*

$$\mu_{\rho_A}(x,y) = \mu_A(x-y) \text{ and } \nu_{\rho_A}(x,y) = \nu_A(x-y) \text{ for all } x,\ y \in M.$$

Then ρ_A is an IFC on M.

Proof. Let $x \in M$. Then we have

$$\mu_{\rho_A}(x,x) = \mu_A(x-x) = \mu_A(0) = 1,\ \nu_{\rho_A}(x,x) = \nu_A(x-x) = \nu_A(0) = 0.$$

Thus ρ_A is IF reflexive. It is clear that ρ_A is IF symmetric. Now let $x,\ y \in M$. Then we get

$$\begin{aligned}
\mu_{\rho_A}(x,y) &= \mu_A(x-y) = \mu_A(x-z+z-y)\\
&\geq \min\{\mu_A(x-z), \mu_A(z-y)\}\\
&= \min\{\mu_{\rho_A}(x,z), \mu_{\rho_A}(z,y)\}\ \forall z \in M.
\end{aligned}$$

Thus $\mu_{\rho_A}(x,y) \geq \sup_{z\in M} \min\{\mu_{\rho_A}(x,z), \mu_{\rho_A}(z,y)\}$. Similarly, we can show that $\nu_{\rho_A}(x,y) \leq \inf_{z\in M} \max\{\nu_{\rho_A}(x,z), \nu_{\rho_A}(z,y)\}$. So ρ_A is an IF equivalence relation on M. On the other hand, we have

$$\begin{aligned}\mu_{\rho_A}(x+u, y+v) &= \mu_A(x+u-v-y) = \mu_A(-y+x+u-v) \\ &\geq \min\{\mu_A(-y+x), \mu_A(u-v)\} = \min\{\mu_A(x-y), \mu_A(u-v)\} \\ &= \min\{\mu_{\rho_A}(x,y), \mu_{\rho_A}(u,v)\}.\end{aligned}$$

Similarly, we can show that $\nu_{\rho_A}(x+u, y+v) \leq \max\{\nu_{\rho_A}(x,y), \nu_{\rho_A}(u,v)\}$. Again

$$\begin{aligned}\mu_{\rho_A}(xr, yr) &= \mu_A(xr-yr) = \mu_A((y-y+x)r-yr) \\ &\geq \mu_A(-y+x) = \mu_A(x-y) \\ &= \mu_{\rho_A}(x,y).\end{aligned}$$

Similarly, we can show that $\nu_{\rho_A}(xr-yr) \leq \nu_{\rho_A}(x,y)$. Hence ρ_A is an IFC on M. □

Remark 4.12. Note ρ_A is called the *IFC induced by* A and A_ρ is called the *IFSM induced by* ρ.

Example 4.13. Consider R, M and IFSM A as in Example 3.2. Define the IFS ρ_A on $M \times M$ as

$$\mu_{\rho_A}(x,y) = \mu_A(x-y) \text{ and } \nu_{\rho_A}(x,y) = \nu_A(x-y) \text{ for all } x,y \in M.$$

Then we have

$$\mu_{\rho_A}((x,y)) = \begin{cases} 1, & \text{if } x-y \in \{0,2\} \\ 0.6, & \text{if } x-y \in \{1,3\}, \end{cases} \quad \nu_{\rho_A}((x,y)) = \begin{cases} 0, & \text{if } x-y \in \{0,2\} \\ 0.3, & \text{if } x-y \in \{1,3\}. \end{cases}$$

It is easy to check that ρ_A is an IFC on M induced by A.

Remark 4.14. It can be easily seen that $(A_\rho)_A \neq \rho$.

Theorem 4.15. *Let M be an R-module. Then there exists an inclusion preserving bijection from the set $\mathcal{IFM}(\mathcal{M})$ of all IFSMs of M to the set of $\mathcal{IFC}(\mathcal{M})$ of all IFCs.*

Proof. We define mappings $\psi : \mathcal{IFM}(\mathcal{M}) \to \mathcal{IFC}(\mathcal{M})$ and $\phi : \mathcal{IFC}(\mathcal{M}) \to \mathcal{IFM}(\mathcal{M})$, respectively by $\psi(A) = \rho_A$ and $\phi(\rho) = A_\rho$ for $A \in \mathcal{IFM}(\mathcal{M})$ and $\rho \in \mathcal{IFC}(\mathcal{M})$. It is obvious that

$$(\phi \circ \psi)(A) = \phi(\psi(A)) = \phi(\rho_A) = A_{(\rho_A)}.$$

Let $a \in M$. Then we get

$$\mu_{A_{(\rho_A)}}(a) = \mu_{\rho_A}(a,0) = \mu_A(a-0) = \mu_A(a),\ \nu_{A_{(\rho_A)}}(a) = \nu_{\rho_A}(a,0) = \nu_A(a-0) = \nu_A(a).$$

Thus $A_{(\rho_A)} = A$. So $(\phi \circ \psi)(A) = A = \mathrm{Id}(\mathcal{IFC}(\mathcal{M}))(A)$. Hence ψ is injective.

Let $A_1,\ A_2 \in \mathcal{IFM}(\mathcal{M})$ be such that $A_1 \subseteq A_2$. Then for all $(x,y) \in M \times M$,

$$\mu_{\rho_{A_2}}(x,y) = \mu_{A_2}(x-y) \geq \mu_{A_1}(x-y) = \mu_{\rho_{A_1}}(x,y)$$

and

$$\nu_{\rho_{A_2}}(x,y) = \nu_{A_2}(x-y) \leq \nu_{A_1}(x-y) = \nu_{\rho_{A_1}}(x,y).$$

Thus $\rho_{A_1} \subseteq \rho_{A_2}$. So $\psi(A_1) \subseteq \psi(A_2)$. Hence ψ is inclusion preserving.

Now let $\rho \in \mathcal{IFC}(\mathcal{M})$. Then A_ρ is an IFSM of M. Note that ρ_{A_ρ} is an IFC of M. It is clear that

$$(\psi \circ \phi)(\rho) = \psi(\phi(\rho)) = \psi(A_\rho) = \rho_{A_\rho}.$$

Let $(x, y) \in M \times M$. Then we have

$$\mu_{\rho_{A_\rho}}(x, y) = \mu_{A_\rho}(x - y) = \mu_\rho(x - y, 0) = \mu_\rho(x, y)$$

and

$$\nu_{\rho_{A_\rho}}(x, y) = \nu_{A_\rho}(x - y) = \nu_\rho(x - y, 0) = \nu_\rho(x, y).$$

Thus $\rho_{A_\rho} = \rho$. So $(\psi \circ \phi)(\rho) = \psi(\phi(\rho)) = \rho = Id(\mathcal{IFC}(\mathcal{M}))(\rho)$ for all $\rho \in \mathcal{IFC}(\mathcal{M})$. Hence $\phi \circ \psi = \mathrm{Id}(\mathcal{IFC}(\mathcal{M}))$ which implies that ψ is surjective. Therefore ψ is an inclusion preserving bijection from $\mathcal{IFM}(\mathcal{M})$ to $\mathcal{IFC}(\mathcal{M})$. □

Proposition 4.16. *Let ρ be an IFC on an R-module M and A_ρ be an IFSM induced by ρ. Let $(\alpha, \beta) \in Img(\rho)$. Then*

$$A_{\rho_{(\alpha,\beta)}} = \{x \in M : x \equiv 0(\rho_{(\alpha,\beta)})\}$$

is a submodule induced by the congruence $\rho_{(\alpha,\beta)}$.

Proof. Let $a \in M$. Then $a \in A_{\rho_{(\alpha,\beta)}} \Leftrightarrow \mu_{A_\rho}(a) \geq \alpha, \nu_{A_\rho}(a) \leq \beta$
$\Leftrightarrow \mu_\rho(a, 0) \geq \alpha, \nu_\rho(a, 0) \leq \beta \Leftrightarrow (a, 0) \in \rho_{(\alpha,\beta)} \Leftrightarrow a \in \{x \in M : x \equiv 0(\rho_{(\alpha,\beta)})\}$.
Thus $A_{\rho_{(\alpha,\beta)}}$ is a submodule induced by the congruence $\rho_{(\alpha,\beta)}$. □

Proposition 4.17. *Let A be an IFSM of an R-module M and ρ_A be the IFC induced by A. Let $(\alpha, \beta) \in Img(A)$. Then $\rho_{A_{(\alpha,\beta)}}$ is the congruence on M induced by $A_{(\alpha,\beta)}$.*

Proof. Let λ be the congruence on M induced by $A_{(\alpha,\beta)}$. Then

$$(x, y) \in \lambda \Leftrightarrow x - y \in A_{(\alpha,\beta)}.$$

Let $(x, y) \in \rho_{A_{(\alpha,\beta)}}$. Then $\mu_{\rho_A}(x, y) \geq \alpha,\ \nu_{\rho_A}(x, y) \leq \beta$. Thus we have

$$\mu_A(x - y) \geq \alpha,\ \nu_A(x - y) \leq \beta \ \Rightarrow\ x - y \in A_{(\alpha,\beta)} \ \Rightarrow\ x - y \in \lambda.$$

So $\rho_{A_{(\alpha,\beta)}} \subseteq \lambda$. By reversing the above argument, we get $\lambda \subseteq \rho_{A_{(\alpha,\beta)}}$. Hence $\rho_{A_{(\alpha,\beta)}} = \lambda$. □

Definition 4.18. Let M be an R-module and ρ be an IFC on M. An IFC λ on M is said to be *ρ-invariant*, if for all $(x, y), (u, v) \in M \times M$,

$$\mu_\rho(x, y) = \mu_\rho(u, v) \text{ and } \nu_\rho(x, y) = \nu_\rho(u, v) \Rightarrow \mu_\lambda(x, y) = \mu_\lambda(u, v) \text{ and } \nu_\lambda(x, y) = \nu_\lambda(u, v).$$

Lemma 4.19. *Let M be an R-module and A be an IFSM of M. Let ρ_A be the IFC on M induced by A. Then the IFR ρ_A/ρ_A on M/A defined by*

$$(\rho_A/\rho_A)(x + A, y + A) = \rho_A(x, y)$$

is an IFC on M/A.

Proof. Assume that $x + A = u + A$ and $y + A = v + A$. Then we have

$$\mu_A(x - u) = \mu_A(0) = 1,\ \nu_A(x - u) = \nu_A(0) = 0$$

and

$$\mu_A(y - v) = \mu_A(0) = 1,\ \nu_A(y - v) = \nu_A(0) = 0.$$

Thus we get

$$\mu_{\rho_A}(x, y) \geq \min\{\mu_{\rho_A}(x, u), \mu_{\rho_A}(u, y)\}$$

$= \mu_{\rho_A}(u, y)$
$\geq \min\{\mu_{\rho_A}(u, v), \mu_{\rho_A}(v, y)\}$
$= \mu_{\rho_A}(u, v)$.
Similarly, we can show $\mu_{\rho_A}(u, v) \geq \mu_{\rho_A}(x, y)$. So $\mu_{\rho_A}(x, y) = \mu_{\rho_A}(u, v)$. Similarly, we get $\nu_{\rho_A}(x, y) = \nu_{\rho_A}(u, v)$. Hence ρ_A/ρ_A is meaningful.

Now the rest of the proof is a routine matter of verification. □

Theorem 4.20. *Let M be an R-module and A be an IFSM of M. Let ρ_A be the IFC on M induced by A. Then there exists a one-to-one correspondence between the set $\mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M})$ of ρ_A-invariant IFCs of M and the set $\mathcal{IFC}_{\rho_{\mathcal{A}}/\rho_{\mathcal{A}}}(\mathcal{M}/\mathcal{A})$ of ρ_A/ρ_A-invariant IFCs on M/A.*

Proof. Let λ be an ρ_A-invariant IFC on M. Define

$$(\lambda/\rho_A)(x + A, y + A) = \lambda(x, y) \text{ for all } x, y \in M.$$

Let $x+A = u+A$ and $y+A = v+A$. These imply that $\rho_A(x, y) = \rho_A(u, v)$. Since λ is ρ_A-invariant, we have $\lambda(x, y) = \lambda(u, v)$. Then the definition of λ/ρ_A is meaningful. It is easy to show that λ/ρ_A is ρ_A/ρ_A-invariant IFC on M/A. We define a map $\theta : \mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M}) \to \mathcal{IFC}_{\rho_{\mathcal{A}}/\rho_{\mathcal{A}}}(\mathcal{M}/\mathcal{A})$ by $\theta(\lambda) = \lambda/\rho_A$. Let $\lambda_1,\ \lambda_2 \in \mathcal{IFC}_{\rho_{\mathcal{A}}}(\mathcal{M})$ such that $\lambda_1(x, y) \neq \lambda_2(x, y)$. Then we have

$$(\lambda_1/\rho_A)(x + A, y + A) = \lambda_1(x, y) \neq \lambda_2(x, y) = (\lambda_2/\rho_A)(x + A, y + A).$$

Thus θ is injective.

Let $\lambda^{'}$ be a ρ_A/ρ_A-invariant IFC on M/A. We define an IFR λ on M as follow:

$$\lambda(x, y) = \lambda^{'}(x + A, y + A).$$

Then $\mu_\lambda(x, x) = \mu_{\lambda'}(x + A, x + A) = 1$ and $\nu_\lambda(x, x) = \nu_{\lambda'}(x + A, x + A) = 0$. Similarly, we can show that $\lambda(x, y) = \lambda(y, x)$. Also, we get

$$\mu_\lambda(x, y) \geq \sup_{z\in M} \min\{\mu_\lambda(x, z), \mu_\lambda(z, y)\}$$

and

$$\nu_\lambda(x, y) \leq \inf_{z\in M} \max\{\nu_\lambda(x, z), \nu_\lambda(z, y)\}.$$

Thus λ is an IFR on M. On the other hand, we get

$$\begin{aligned} \mu_\lambda(x + a, y + b) &= \mu_{\lambda'}(x + a + A, y + b + A) \\ &= \mu_{\lambda'}(x + A + a + A, y + A + b + A) \\ &\geq \min\{\mu_{\lambda'}(x + A, y + A), \mu_{\lambda'}(a + A, b + A)\} \\ &= \min\{\mu_\lambda(x, y), \mu_\lambda(a, b)\}. \end{aligned}$$

Similarly, we can show that $\nu_\lambda(x+a, y+b) \leq \max\{\nu_\lambda(x, y), \nu_\lambda(a, b)\}$. Also, we have

$$\begin{aligned} \mu_\lambda(xr, yr) &= \mu_{\lambda'}(xr + A, yr + A) \\ &= \mu_{\lambda'}((x + A)r, (y + A)r) \\ &\geq \mu_{\lambda'}(x + A, y + A). \end{aligned}$$

Similarly, we can show that $\nu_\lambda(xr, yr) \leq \nu_\lambda(x, y)$. So λ is an IFC on M. Since $\rho_A(x, y) = \rho_A(u, v),\ \rho_A/\rho_A(x + A, y + A) = \rho_A/\rho_A(u + A, v + A)$. This implies that $\lambda^{'}(x+A, y+A) = \lambda^{'}(u+A, v+A)$. Hence $\lambda(x,y) = \lambda(u,v)$. Therefore λ is ρ_A-invariant.

Now let $(x+A, y+A) \in M/A \times M/A$. Then $(\lambda/\rho)(x+A, y+A) = \lambda(x,y) = \lambda^{'}(x+A, y+A)$. Thus $\lambda^{'} = \lambda/\rho_A = \theta(\lambda)$. So θ is surjective. Hence θ is bijective. □

Conclusion

In this paper, we develop and study the notions of intuitionistic fuzzy submodules, intuitionistic fuzzy congruences of an R-module (where R is a near-ring) and quotient R-modules over an intuitionistic fuzzy submodule of an R-module. We exhibit a one-to-one correspondence between the set of intuitionistic fuzzy submodules and the set of intuitionistic fuzzy congruences of an R-module. Finally, we established an intuitionistic fuzzy congruence of the quotient R-module over an intuitionistic fuzzy submodule of the R-module and obtained a correspondence theorem.

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