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Roughness based on filters in residuated multilattices.

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Abstract. This paper explores approximation spaces defined by residuated multilattices equipped with an equivalence relation induced by filters. It extends the work of J. Rachunek and D. Salounova on roughness in residuated lattices. Specifically, we use equivalence classes induced by filters, to define lower and upper approximations and, we provide several characterizations of these approximations.

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1. Introduction

It is well known that certain information processing, especially inferences based on certain information, is based on the classical logic (classical two-valued logic). Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched. In fact, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. Among these logical algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices, but in lattices there is a restriction namely, "the existence of least upper bounds and greatest lower bounds". Then Benado and Hassen [1, 2] introduced an algebraic structure called multilattice, where this restriction is relaxed by the "existence of minimal upper bounds and maximal lower bounds". Multilattices arise as well in other research areas, such as fuzzy extensions of logic programming: for instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs can be weakened only when the underlying set of truth-values is a multilattice. Also in formal concept analysis, Medina et al. [3] showed that only multilattices as the underlying set of truth degree can deal with certain kind of fuzzy data (such as classifications of hotels with same stars) to evaluate objects, attributes and their relationship(See [4]). Based on partially ordered commutative residuated integral monoid, Cabrera et al. [5] provided the notion of residuated multilattice and investigated their filters and congruences. A more interesting application of residuated multilattice in formal concept analysis can be found in which Koguep et al. [6] have built a residuated concepts multilattice.

The concept of rough set was originally proposed by Pawlak [7] as a formal tool for modelling and processing incomplete information. The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. It is a natural question to ask what happen if we substitute the universe by an algebraic structure. In this direction, B. Davvaz et al. explored roughness in a ring [8], roughness in MV-algebras [9], roughness in n-ary hypergroups [10] and also approximations based on fuzzy ideals in a ring [11]. Rachunek and Salounova [12] investigated applications of rough set theory to residuated lattice.

The aim of this paper is to extend the work of Rachunek and Salounova [12] by studying approximation spaces in residuated multilattices based on their filters. The contributions are the following items: Firstly, we apply the notion of filters of a residuated multilattice for definitions of the lower and upper approximations, then we provide relations between approximations of the multisupremum of two sets and the multisupremum of the approximations of these sets. Secondly, we establish some connections between approximation spaces, special filters, residuated full submultilatices and homomorphisms.

The organization of this paper is the following: in Section 2 we recall basic notions to make this paper self-contained. In Section 3 we define lower and upper approximations in a residuated multilattice with respect to filters, and we provide some properties of theses approximations.

2. Preliminaries

In this section, we recall several notions from multilattice theory and some basic notions of rough sets in order to make our paper self contained.

2.1. **Residuated Multilattice.**

A complete lattice is a poset such that the set of upper (respectively lower) bounds of every subset has a unique minimal (respectively maximal) element, that is, a minimum (respectively maximum). In a multilattice, this property is relaxed in the sense that minimal elements for the set of upper bounds should exist, but the uniqueness condition is dropped.

Let $(P, \leq)$ be a poset and $X \subseteq P$. We denote by $U(X)$ (resp. $L(X)$) the set of upper (resp. lower) bounds of X. A *multisupremum* (resp. *multiinfimum*) of X is a minimal (resp. maximal) element of $U(X)$ (resp. $L(X)$). The set of multisuprema (resp. multiinfima) of X is denoted by $\sqcup X$ (resp. $\sqcap X$). For $x,\ y \in P$, we simply write $U(x),\ L(x),\ x \sqcup y,\ x \sqcap y$ for $U(\{x\}),\ L(\{x\}), \sqcup\{x, y\}, \sqcap\{x, y\}$, respectively.

A *multilattice* is a poset $(M, \leq)$ satisfying the following conditions:, for any any $a,\ b \in M$,

(i) if $x \in M$ is an upper bound of $\{a, b\}$, there exits a multisupremum $m \in a \sqcup b$ such that $m \leq x$,

(ii) if $x \in M$ is a lower bound of $\{a, b\}$, there exits a multiinfinimum $m \in a \sqcap b$ such that $x \leq m$.

If the pair $\{a, b\}$ is changed by an arbitrary subset $X \subseteq M$, one obtains a *complete multilattice*. Figure 1 gives an example of multilattices.

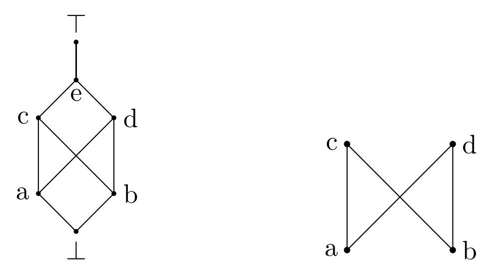

FIGURE 1. On the left: $(M_7, \leq)$, is a complete multilattice. and on the right: An example of multilattice which is not complete.

A multilattice is said to be *full*, if $a \sqcap b \neq \varnothing$ and $a \sqcup b \neq \varnothing$ for all $a, b \in M$.

Now, we are going to introduce the notion of *residuation*.

Definition 2.1 ([13])**.** A *partially ordered commutative residuated integral monoid* (briefly, pocrim) is a structure $(A, \leq, \odot, \rightarrow, \top)$ such that

(i) $(A, \odot, \top)$ is a commutative monoid with neutral element $\top$,

(ii) $(A, \leq)$ is a poset with a top element $\top$,

(iii) $a \odot b \leq c \iff a \leq b \rightarrow c$ for all $a,\ b,\ c \in A$. (adjointness condition)

The following properties hold in pocrims.

Proposition 2.2 ([5])**.** *Let $(A,\ \leq,\ \odot,\ \rightarrow,\ \top)$ be a pocrim and $a,\ b,\ c \in A$. We have the following:*

P1 $a \odot b \leq a$ *and* $a \odot b \leq b$,

P2 $a \odot (a \rightarrow b) \ \leq\ a\ \leq\ b \rightarrow (a \odot b)$ *and* $a \odot (a \rightarrow b)\ \leq\ b\ \leq\ a \rightarrow (a \odot b)$,

P3 *if* $a \leq b$, *then* $a \odot c\ \leq\ b \odot c$, $\quad c \rightarrow a\ \leq\ c \rightarrow b$, $\quad$ *and* $b \rightarrow c\ \leq\ a \rightarrow c$,

P4 $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = (a \odot b) \rightarrow c$,

P5 $\top \rightarrow a = a$ *and* $a \rightarrow \top = \top$,

P6 $a \leq b$ *if and only if* $a \rightarrow b = \top$.

Definition 2.3 ([13]). For given a pocrim $(A, \leq, \odot, \rightarrow, \top)$, a non-empty subset F of A is called a *deductive system*, if it satisfies the following conditions:
(i) $\top \in F$,
(ii) $a \rightarrow b \in F$ and $a \in F$ imply $b \in F$.

On any bounded, pocrim $(A, \leq, \odot, \rightarrow, \bot, \top)$ we can define an unary operator $*$ by $a^* := a \rightarrow \bot$ for any $a \in A$.

Definition 2.4 ([13]). A *residuated multilattice* is a pocrim, whose underlying poset is a multilattice. If in addition, there exists a bottom element, the residuated multilattice is said to be *bounded*.

A bounded residuated multilattice $(M, \leq, \rightarrow, \odot, \bot, \top)$ is say to be *regular*, if $x^{**} = x$ for all $x \in M$.

Definition 2.5 ([6]). Let $(M, \leq, \rightarrow, \odot, \bot, \top)$ be a bounded residuated multilattice and X a subset of M. We say that X is a *residuated full sub-multilattice*, if the following hold:
(**S1**) $\top \in X$,
(**S2**) For every $x,\ y \in X$, $x \odot y \in X$ and $x \rightarrow y \in X$,
(**S3**) X is a full sub-multilattice, i.e.,

$$\forall x, y \in X,\ x \sqcup y \subseteq X \text{ and } x \sqcap y \subseteq X.$$

From now, $\mathcal{M} := (M, \leq, \odot, \rightarrow, \bot, \top)$ will denote a complete residuated multilattice. The operations $\odot$, $\rightarrow, *$ and hyperoperations $\sqcap, \sqcup$ can be extended to $\mathcal{P}(M) - \{\varnothing\}$ as follows: for $A, B \in P(M) - \{\varnothing\}$,

$$A \odot B := \{a \odot b :\ a \in A \textit{ and } b \in B\}, \quad A \rightarrow B := \{a \rightarrow b :\ a \in A \textit{ and } b \in B\},$$
$$A \sqcup B := \bigcup_{a \in A, b \in B} a \sqcup b, \qquad A \sqcap B := \bigcup_{a \in A, b \in B} a \sqcap b,$$
$$A^* := \{a^* :\ a \in A\}.$$

Let R be a binary relation in $\mathcal{M}$, and $A, B \subseteq M$, then $A\widehat{R}B$ means that, for all $a \in A$, there exists $b \in B$ such that aRb and for all $b \in B$ there exists $a \in A$ such that aRb.

A *congruence* in $\mathcal{M}$ is any equivalence relation R in $\mathcal{M}$, that satisfied: if aRb, then $(a \odot c)R(b \odot c)$, $(a \rightarrow c)R(b \rightarrow c)$, $(c \rightarrow a)R(c \rightarrow b)$, $(a \sqcup c)\widehat{R}(b \sqcup c)$, $(a \sqcap c)\widehat{R}(b \sqcap c)$ $\forall a,\ b,\ c \in M$.

A map $f : M \rightarrow M'$ between two residuated multilattices is said to be a *homomorphism*, if for all $a, b \in M$,
(i) $f(a \sqcup b) = \big(f(a) \sqcup f(b)\big) \cap f(M)$ and $f(a \sqcap b) = \big(f(a) \sqcap f(b)\big) \cap f(M)$,
(ii) $f(a \odot b) = f(a) \odot f(b)$ and $f(a \rightarrow b) = f(a) \rightarrow f(b)$.

A non-empty subset $F \subseteq M$ is said to be a *filter*, if it is a deductive system and the following conditions hold:

$$a \rightarrow b \in F \text{ implies } (a \sqcup b) \rightarrow b \subseteq F \text{ and } a \rightarrow (a \sqcap b) \subseteq F.$$

Theorem 2.6 ([13]). *Let $h : M \rightarrow M'$ be a homomorphism between residuated multilattices.*
(1) *The Kernel relation, defined as:*

$$aR_{Kerh}b \textit{ if and only if } h(a) = h(b)$$

is a congruence on $\mathcal{M}$*.*

(2) *Let* F *be a filter of* $\mathcal{M}$*. The relation*

$$aR_Fb \text{ if and only if } a \to b, b \to a \in F$$

is a congruence relation of $\mathcal{M}$*.*

2.2. **Rough set theory.**

Rough set theory, propose a mathematical approach to imperfect knowledge, it can be defined by means of topological operations, interior and closure called *approximations*. Let U be a finite set of objects and $R \subseteq U \times U$ a binary equivalence relation. A pair (U, R) is called *approximation space* on U. For $X \subseteq U$,

(i) the set of all objects which can be certainty classified as members of X with respect to R is called the *lower approximation* of X and denoted by $\underline{R}(X)$, i.e.,

$$\underline{R}(X) = \{x \in X : \ [x] \subseteq X\},$$

(ii) the set of all objects which can be only classified as possible members of X with respect to R is called the *upper approximation* of a set X and denoted $\overline{R}(X)$, i.e.,

$$\overline{R}(X) = \{x \in X : \ [x] \cap X \neq \varnothing\},$$

(iii) the set of all objects which cannot be classified either to X or to $-X$ (the complement of X) is called *boundary region* of a set X and denoted $B(X)$, i.e.,

$$B(X) = \overline{R}(X) - \underline{R}(X)$$

(iv) a pair $(\overline{R}(X), \underline{R}(X))$ is called a *rough set* in (U, R).

(v) X is said to be *definable*, if $\underline{R}(X) = \overline{R}(X)$.

3. Approximations in a residuated multilattice

In this section, we denote by R_F a congruence relation induced by the filter F of $\mathcal{M}$. For any $A \subseteq M$, we denote $[A]_F = \bigcup\limits_{x \in A}[x]_F$, where $[x]_F$ is the equivalence class of x with respect to the congruence R_F.

We therefore define, for any subset A of M the upper and lower approximations as follow:

$$\overline{R_F}(A) = \{x \in M : \ [x]_F \cap A \neq \varnothing\},$$
$$\underline{R_F}(A) = \{x \in M : \ [x]_F \subseteq A\}.$$

Since R_F is an equivalence relation, from [7], we have the following proposition.

Proposition 3.1. *For every approximation space* (M, R_F) *and for every subsets* $A, \ B \subseteq M$*, we have the following:*

(1) $\underline{R_F}(A) \subseteq A \subseteq \overline{R_F}(A)$,

(2) $\underline{R_F}(\varnothing) = \overline{R_F}(\varnothing)$ *and* $\underline{R_F}(M) = \overline{R_F}(M)$,

(3) $\overline{R_F}(A \cup B) = \overline{R_F}(A) \cup \overline{R_F}(B)$ *and* $\underline{R_F}(A \cup B) \supseteq \underline{R_F}(A) \cup \underline{R_F}(B)$,

(4) $\underline{R_F}(A \cap B) = \underline{R_F}(A) \cap \underline{R_F}(B)$ *and* $\overline{R_F}(A \cap B) \subseteq \overline{R_F}(A) \cap \overline{R_F}(B)$,

(5) $A \subseteq B \Rightarrow \underline{R_F}(A) \subseteq \underline{R_F}(B)$ *and* $\overline{R_F}(A) \subseteq \overline{R_F}(B)$,

(6) $\underline{R_F}(\underline{R_F}(A)) = \overline{R_F}(\underline{R_F}(A)) = \underline{R_F}(A)$,

(7) $\overline{R_F}(\overline{R_F}(A)) = \underline{R_F}(\overline{R_F}(A)) = \overline{R_F}(A)$.

The following Corollary is a consequence of the fifth item of the previous proposition.

Corollary 3.2. *Let A and B be two non-empty subsets of M. Since for each $x \in A \cap B$, we have $x = x \sqcap x$ and $x = x \sqcup x$. Then*

$$A \cap B \subseteq A \sqcap B \text{ and } A \cap B \subseteq A \sqcup B,$$

we have the following:

(1) $\overline{R_F}(A \cap B) \subseteq \overline{R_F}(A \sqcap B)$ *and* $\overline{R_F}(A \cap B) \subseteq \overline{R_F}(A \sqcup B)$,
(2) $\underline{R_F}(A \cap B) \subseteq \underline{R_F}(A \sqcap B)$ *and* $\underline{R_F}(A \cap B) \subseteq \underline{R_F}(A \sqcup B)$.

In particular, if A and B are filters, we obtain

$$\overline{R_F}(A \cap B) = \overline{R_F}(A \sqcup B) \text{ and } \underline{R_F}(A \cap B) = \underline{R_F}(A \sqcup B).$$

Proposition 3.3. *Let A, B be two non-empty subsets of M and F be a filter of $\mathcal{M}$. Then*

(1) $\overline{R_F}(A) \cap \overline{R_F}(B) \subseteq \overline{R_F}(A \sqcup B)$,
(2) $\underline{R_F}(A) \cap \underline{R_F}(B) \subseteq \underline{R_F}(A \sqcup B)$.

In particular, if A and B are filters of $\mathcal{M}$, we obtain equalities.

Proof. (1) Let $x \in \overline{R_F}(A) \cap \overline{R_F}(B)$. Then we have $[x]_F \cap A \neq \varnothing$ and $[x]_F \cap B \neq \varnothing$. Thus there exist $a_1 \in [x]_F \cap A$ and $b_1 \in [x]_F \cap B$. Since M is complete, there exists $t \in a_1 \sqcup b_1$. So we obtain

$$t \in A \sqcup B \text{ and } t \in [x]_F \sqcup [x]_F = [x \sqcup x]_F = [x]_F.$$

Hence $[x]_F \cap (A \sqcup B) \neq \varnothing$. Therefore $\overline{R_F}(A) \cap \overline{R_F}(B) \subseteq \overline{R_F}(A \sqcup B)$.

(2) Let $x \in \underline{R_F}(A) \cap \underline{R_F}(B)$. Then we obtain $[x]_F \subseteq A$ and $[x]_F \subseteq B$. If $t \in [x]_F$, then $t \in A$ and $t \in B$. Thus $[x]_F \subseteq A \sqcup B$, since $t = t \sqcup t$. So $\underline{R_F}(A) \cap \underline{R_F}(B) \subseteq \underline{R_F}(A \sqcup B)$.

Now, suppose A and B are filters. Then $A \cap B = A \sqcup B$. Thus we have

$$\overline{R_F}(A \sqcup B) = \overline{R_F}(A \cap B) \subseteq \overline{R_F}(A) \cap \overline{R_F}(B)$$

and

$$\underline{R_F}(A \sqcup B) = \underline{R_F}(A \cap B) = \underline{R_F}(A) \cap \underline{R_F}(B).$$

So we obtain equality for the two items. □

We obtain the following corollary.

Corollary 3.4. *Let A, B be two non empty subsets of M and F be a filter of M. Then*

(1) $\overline{R_F}(A) \sqcup \overline{R_F}(B) \subseteq \overline{R_F}(A \sqcup B)$,
(2) $\underline{R_F}(A) \sqcup \underline{R_F}(B) \subseteq \underline{R_F}(A \sqcup B)$.

In particular, if A and B are filters of M, we obtain equalities.

Proof. (1) Let $x \in \overline{R_F}(A) \sqcup \overline{R_F}(B)$. We have $x \in a \sqcup b$ with $a \in \overline{R_F}(A)$ and $b \in \overline{R_F}(B)$. Then $[a]_F \cap A \neq \varnothing$ and $[b]_F \cap B \neq \varnothing$, i.e., there exist $a_1 \in [a]_F \cap A$ and $b_1 \in [b]_F \cap B$. It is clear that $a_1 R_F a$ and $b_1 R_F b$. Thus $a_1 \sqcup b \widehat{R_F} a \sqcup b$ and $a_1 \sqcup b_1 \widehat{R_F} a_1 \sqcup b$ So $a_1 \sqcup b_1 \widehat{R_F} a \sqcup b$. Since $x \in a \sqcup b$, there exists $t \in a_1 \sqcup b_1$ such that $t R_F x$. Hence $t \in [x]_F \cap (A \sqcup B)$. Therefore $\overline{R_F}(A) \sqcup \overline{R_F}(B) \subseteq \overline{R_F}(A \sqcup B)$.

(2) Let $x \in \underline{R_F}(A) \sqcup \underline{R_F}(B)$. We have $x \in a \sqcup b$ with $a \in \underline{R_F}(A)$ and $b \in \underline{R_F}(B)$. Then $x \in a \sqcup b$ implies $[x]_F \subseteq [a \sqcup b]_F = [a]_F \sqcup [b]_F$. Thus for $t \in [x]_F$, we obtain $t \in a_1 \sqcup b_1$ with $a_1 \in [a]_F \subseteq A$ and $b_1 \in [b]_F \subseteq B$, i.e., $[x]_F \subseteq A \sqcup B$. So $\underline{R_F}(A) \sqcup \underline{R_F}(B) \subseteq \underline{R_F}(A \sqcup B)$.

Now, suppose A and B are filters. Then we have

$$\overline{R_F}(A \sqcup B) = \overline{R_F}(A) \cap \overline{R_F}(B) \subseteq \overline{R_F}(A) \sqcup \overline{R_F}(B)$$

and

$$\underline{R_F}(A \sqcup B) = \underline{R_F}(A) \cap \underline{R_F}(B) \subseteq \underline{R_F}(A) \sqcup \underline{R_F}(B).$$

Thus the equalities hold. □

Proposition 3.5. *Let us consider F and G two filters of M such that $F \subseteq G$. Then for every non-empty subsets $A \subseteq M$, we have the following:*

(1) $\underline{R_G}(A) \subseteq \underline{R_F}(A)$,
(2) $\overline{R_F}(A) \subseteq \overline{R_G}(A)$.

Proof. (1) Let $x \in \underline{R_G}(A)$. Our goal is to show that $[x]_F \subseteq A$. For $y \in [x]_F$, we have $x \to y,\ y \to x \in F \subseteq G$. Then $y \in [x]_G$. Since $[x]_G \subseteq A$, we have $y \in A$. Thus we get the result.

(2) Let $x \in \overline{R_F}(A)$. Let us show that $[x]_G \cap A \neq \varnothing$. Since $[x]_F \cap A \neq \varnothing$, there exits $z \in [x]_F$ and $z \in A$. Then $x \to z, z \to x \in F \subseteq G$ and $z \in A$. Thus $[x]_G \cap A \neq \varnothing$. □

The following Corollary can be easily derived from the previously proposition.

Corollary 3.6. *Let F and G be two filters of M and a non-empty subset $A \subseteq M$. We have:*

(1) $\underline{R_F}(A) \cap \underline{R_G}(A) \subseteq \underline{R_{F\cap G}}(A)$,
(2) $\overline{R_{F\cap G}}(A) \subseteq \overline{R_F}(A) \cap \overline{R_G}(A)$.

Now we will show that, for a set A to be definable with respect to a filter F, it suffices $\underline{R_F}(A) = A$ or $\overline{R_F}(A) = A$.

Proposition 3.7. *Let A be a non-empty subset of M. Then*

$$\underline{R_F}(A) = A \Longleftrightarrow \overline{R_F}(A) = A.$$

Proof. Suppose $\underline{R_F}(A) = A$ and let $x \in \overline{R_F}(A)$. Then we have $[x]_F \cap A \neq \varnothing$. Thus there is $a \in [x]_F \cap A$. Since $\underline{R_F}(A) = A$, we obtain $[x]_F = [a]_F \subseteq A$. So $x \in A$. Hence $\overline{R_F}(A) = A$.

Conversely, suppose $\overline{R_F}(A) = A$ and let $x \in A$. Then we have $[x]_F \cap A \neq \varnothing$. Let $a \in [x]_F$. since $[a]_F = [x]_F$, we obtain $[a]_F \cap A \neq \varnothing$, i.e., $a \in \overline{R_F}(A) = A$. Then $A \subseteq \underline{R_F}(A)$. Thus $\underline{R_F}(A) = A$. □

Corollary 3.8. *A non-empty set A is definable, with respect to the filter F if and only if $\underline{R_F}(A) = A$ or $\overline{R_F}(A) = A$.*

Remark 3.9. Let F be a filter of M, and $x \in M$. We can observe that: for all $b \in F$;

$$[x]_F = [b \odot x]_F.$$

Proposition 3.10. *Let F be a filter of M and G be a residuated full sub-multilattice of M. Then $\overline{R_F}(G)$ is a residuated full sub-multilattice of M.*

Proof. (**S1**) $\top \in \overline{R_F}(G)$ by the definition.

(**S2**) For $x,\ y \in \overline{R_F}(G)$, we have $[x]_F \cap G \neq \varnothing$ and $[y]_F \cap G \neq \varnothing$. Then there is $x_1 \in [x]_F \cap G$ and $y_1 \in [y]_F \cap G$. Thus we have

$$x_1 \odot y_1 \in [x \odot y]_F \cap G \text{ and } x_1 \to y_1 \in [x \to y]_F \cap G.$$

So $x \odot y,\ x \to y \in \overline{R_F}(G)$.

(**S3**) For $x,\ y \in \overline{R_F}(G)$, according to Theorem 3.3 in [14], we have

$$x \sqcup y,\ x \sqcap y \subseteq \overline{R_F}(G).$$

□

As we can see in the following example, the lower approximation of a residuated full sub-multilattice of $\mathcal{M}$ is not always a residuated full sub-multilattice.

Example 3.11. Let $M = \{a_i, b_j, 0 \leq i \leq 3 \text{ and } 0 \leq j \leq 6\}$ be the multilattice described in the following figure and the following residuation operators:

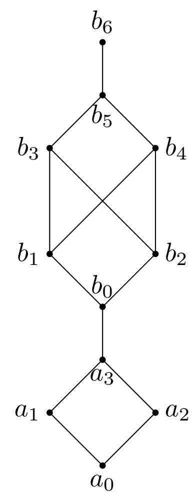

Figure 2. Multilattice.

Let us consider $A = \{a_i, 0 \leq i \leq 3\},\ B = \{b_j, 0 \leq j \leq 6\}$ and $C = \{b_j, 2 \leq j \leq 5\}$, the operators $\odot$ and $\to$ defined as follow:

$$x \odot y = \begin{cases} x \wedge y, & \text{if } x, y \in A \\ b_0, & \text{if } (x \in \{b_0, b_1\} \text{ and } y \in B - \{b_6\}) \text{ or } (y \in \{b_0, b_1\} \text{ and } x \in B - \{b_6\}) \\ x, & \text{if } y = b_6 \text{ or } (x \in A \text{ and } y \in B - \{b_6\}) \\ y, & \text{if } x = b_6 \text{ or } (y \in A \text{ and } x \in B - \{b_6\}) \\ b_2, & \text{otherwise} \end{cases}$$

and

$$x \to y = \begin{cases} b_6, & \text{if } x, y \in x \leq y \\ y, & \text{if } x = b_6 \text{ or } (x = a_3 \text{ and } y \in \{a_0, a_1, a_2\}) \text{ or } (x \in B - \{b_6\} \text{ and } y \in A) \\ a_1, & \text{if } x = a_2 \text{ and } y \in \{a_0, a_1\} \\ a_2, & \text{if } x = a_1 \text{ and } y \in \{a_0, a_2\} \\ b_1, & \text{if } x \in C \text{ and } y \in \{b_0, b_1\} \\ b_5, & \text{otherwise.} \end{cases}$$

It is straightforward to observe that we have a residuated multilattice.
The subset B is a filter and $D = A \cup \{b_6\}$ is a residuated full sub-multilattice, but $b_1 \in [b_6]_B$ and $b_1 \notin D$. Then $[b_6]_B \nsubseteq D$. Thus $\underline{R_B}(D)$ is not a residuated full sub-multilattice.

Proposition 3.12. *Let F and G be two filters of M. If $F \subseteq G$, then we have*

(1) *for all $x \in M$, $x \in G \Longleftrightarrow [x]_F \subseteq G$,*
(2) *moreover, $\underline{R_F}(G) = G = \overline{R_F}(G)$.*

Proof. (1) Suppose $x \in G$ and let $y \in [x]_F$. Then $y \to x,\ x \to y \in F \subseteq G$. Since $x \in G$, we have $y \in G$. Thus $[x]_F \subseteq G$. The reciprocity is straightforward.

(2) Let $x \in G$. Then by (1), we have $[x]_F \subseteq G$, i.e., $\underline{R_F}(G) = G$. On the other hand, for $x \in \overline{R_F}(G)$, we have $[x]_F \cap G \neq \varnothing$. Thus, there is $y \in [x]_F \cap G$. So $x \to y,\ y \to x \in F \subseteq G$. Since $y \in G$, we have $x \in G$. Hence $\overline{R_F}(G) = G$.

□

Proposition 3.13. *Let F and G be two filters of M such that $G \subseteq F$. Then $\overline{R_F}(G)$ is a filter of M.*

Proof. (i) $\top \in \overline{R_F}(G)$, since $\top \in [\top]_F \cap G$. Let $a,\ b \in M$ such that $a \to b \in \overline{R_F}(G)$ and $a \in \overline{R_F}(G)$. We show that $b \in \overline{R_F}(G)$, i.e., $[b]_F \cap G \neq \varnothing$. It is obvious that There exist $a_1 \in [a]_F$ and $b_1 \in [b]_F$ such that $a_1 \to b_1 \in [a \to b]_F \cap G$ and $a_2 \in [a]_F \cap G$. since $a_1, a_2 \in [a]_F$ we have $a_1 R_F a_2$. Then $a_1 \to a_2, a_2 \to a_1 \in F$. Thus $a_1 \in F$. Since $a_1 \to b_1 \in G \subseteq F$, we also have $b_1 \in F$. Since $b_1 \in [b]_F$, we have $b \in F$. So $\top \to b$ and $b \to \top$ are elements of F. Hence $\top \in [b]_F \cap G$.

(ii) It remains to show that for $a,\ b \in M$ such that $a \to b \in \overline{R_F}(G)$, $(a \sqcup b) \to b \subseteq \overline{R_F}(G)$ and $a \to (a \sqcap b) \subseteq \overline{R_F}(G)$. Let $t \in (a \sqcup b) \to b$ and let us show that $[t]_F \cap G \neq \varnothing$. Since $a \to b \in \overline{R_F}(G)$, there is $x \in [a \to b]_F \cap G$. Then we have $x \in G \subseteq F$. Since $x \leq t \to x$, and F is a filter, $t \to x \in F$. On the other hand, $x \in F$ and $x \in [a \to b]_F$ imply $a \to b \in F$. Thus $(a \sqcup b) \to b \subseteq F$. So $t \in F$. Hence $x \to t \in F$. We finally have $x \to t, t \to x \in F$ that is $x \in [t]_F$. Hence $x \in [t]_F \cap G$. By the same way, we show that $a \to (a \sqcap b) \subseteq \overline{R_F}(G)$.

□

The following example show that, under the same hypothesis, the lower approximation of a filter is not always a filter.

Example 3.14. Let us consider the residuated multilattice defined in Example 3.11. $B = \{b_i, 0 \leq i \leq 6\}$ and $M = \{a_i, b_j, 0 \leq i \leq 3, 0 \leq j \leq 6\}$ are two filters such that $B \subseteq M$. But $b_6 \notin \underline{R_M}(B)$, i.e., $\underline{R_M}(B)$ is not a filter.

Theorem 3.15. *If $\mathcal{M}$ is regular, and A is a non-empty subset of M, then the following hold:*

(1) $\big(\overline{R_F}(A)\big)^* = \overline{R_F}(A^*)$,
(2) $\big(\underline{R_F}(A)\big)^* = \underline{R_F}(A^*)$.

Proof. (1) Let $z \in \big(\overline{R_F}(A)\big)^*$. Then there is $a \in \overline{R_F}(A)$ such that $z = a^*$. Thus $z^* = a$, i.e., $z^* \in \overline{R_F}(A)$. So $[z^*]_F \cap A \neq \varnothing$, i.e., there is $z_1 \in [z^*]_F \cap A$ which means that $z_1 R_F z^*$ and $z_1 \in A$. Hence $z_1^* R_F z$ and $z_1^* \in A^*$, i.e., $[z]_F \cap A^* \neq \varnothing$. Therefore $z \in \big(\overline{R_F}(A^*)\big)$.

Conversely, let $z \in \overline{R_F}(A^*)$. Then $[z]_F \cap A^* \neq \varnothing$, i.e., there is $z_1 \in [z]_F \cap A^*$. Thus there is $a \in A$ such that $a^* R z$, i.e., $a R_F z^*$. So $a \in [z^*]_F$, i.e., $[z^*]_F \cap A \neq \varnothing$. Hence $z^* \in \overline{R}(A)$. Therefore $z \in \big(\overline{R}(A)\big)^*$.

(2) Let $z \in \underline{R_F}(A^*)$. Then $[z]_F \subseteq A^*$. Let $t \in [z^*]_F$. Then $t R_F z^*$ Thus $t^* R_F z$. So $t^* \in A^*$, i.e., $t \in A$. Hence $z^* \in \underline{R_F}(A)$. Therefore $z \in \big(\underline{R}(A)\big)^*$. The reciprocity is straightforward. □

Theorem 3.16. *Let M and M' be two residuated multilattices and $f : M \to M'$ a homomorphism. If A is a non-empty subset of M, then*

$$f\big(\overline{R_{Kerf}}(A)\big) = f(A)$$

Proof. $A \subseteq \overline{R_{Kerf}}(A)$ implies $f(A) \subseteq f\big(\overline{R_{Kerf}}(A)\big)$.

Reciprocally, let $y \in f\big(\overline{R_{Kerf}}(A)\big)$. Then there exists $x \in \overline{R_{Kerf}}(A)$ such that $y = f(x)$. Since $x \in \overline{R_{Kerf}}(A)$, we have $[x]_{Kerf} \cap A \neq \varnothing$. Thus there is $t \in [x]_{Kerf} \cap A$ such that $f(t) = f(x) = y$. So $y \in f(A)$. □

Proposition 3.17. *Let $f : M \to M'$ be a bijective homomorphism of residuated multilattice, A be a non-empty subset of M and F' a filter of $\mathcal{M}'$. The following hold:*

(1) $f\Big(\overline{R_{f^{-1}(F')}}(A)\Big) = \overline{R_{F'}}\big(f(A)\big)$,
(2) $f\Big(\underline{R_{f^{-1}(F')}}(A)\Big) = \underline{R_{F'}}\big(f(A)\big)$.

Proof. (1) Let $y \in f\Big(\overline{R_{f^{-1}(F')}}(A)\Big)$. Then there is $x \in \overline{R_{f^{-1}(F')}}(A)$ such that $y = f(x)$. On the other hand, we get

$$\begin{aligned} x \in \overline{R_{f^{-1}(F')}}(A) &\Longleftrightarrow [x]_{f^{-1}(F')} \cap A \neq \varnothing \\ &\Longleftrightarrow \text{ there exists } z \in [x]_{f^{-1}(F')} \cap A \\ &\Longleftrightarrow z \to x, x \to z \in f^{-1}(F') \text{ for some } z \in A \\ &\Longleftrightarrow f(z \to x), f(x \to z) \in F' \text{ for some } z \in A \\ &\Longleftrightarrow f(z) \to f(x), f(x) \to f(z) \in F' \text{ for some } z \in A \\ &\Longleftrightarrow f(z) \in [f(x)]_{F'}, z \in A \\ &\Longleftrightarrow [y]_{F'} \cap f(A) \neq \varnothing. \end{aligned}$$

(2) Let $y \in f\Big(\underline{R_{f^{-1}(F')}}(A)\Big)$. Then there exits $x \in \underline{R_{f^{-1}(F')}}(A)$ such that $y = f(x)$. Thus $[x]_{f^{-1}(F')} \subseteq A$. Let us show that

$$[y]_{F'} \subseteq f(A).$$

Let $t \in [y]_{F'}$. Then $t \to y,\ y \to t \in F'$. Since f is surjective, there exists a such that $t = f(a)$. Thus $a \to x,\ x \to a \in f^{-1}(F')$. So $a \in [x]_{f^{-1}(F')} \subseteq A$. Hence $t \in f(A)$.

Reciprocally, let $y \in \underline{R_{F'}}\big(f(A)\big)$. Then $[y]_{F'} \subseteq f(A)$. Thus there exists $x \in A$ such that $y = f(x)$. Let us show that

$$[x]_{f^{-1}(F')} \subseteq A.$$

Let $t \in [x]_{f^{-1}(F')}$. Then $t \to x, x \to t \in f^{-1}(F')$. Thus $f(t) \to f(x), f(x) \to f(t) \in F'$. So $f(t) \in [y]_{F'} \subseteq f(A)$. Hence $t \in A$. □

Note 3.18. Let $f : M \to M'$ be a surjective homomorphism and F a filter of M, $R_{f(F)}$ be a binary relation of M' defined by:

$$f(x_1)R_{f(F)}f(x_2) \text{ if and only if } x_1R_Fx_2; \text{ for } x_1, x_2 \in M.$$

Remark 3.19. The relation $R_{f(F)}$ defined above, is a congruence on M'.

From proposition 3.17 we obtain the following Corollary.

Corollary 3.20. *Let $f : M \to M'$ be a bijective homomorphism and A be a non-empty subset of M. Then*

$$f(\overline{R_F}(A)) = \overline{R_{f(F)}}(f(A)) \textit{ and } f(\underline{R_F}(A)) = \underline{R_{f(F)}}(f(A)).$$

Corollary 3.21. *Let $f : M \to M'$ be a homomorphism between two residuated multilattices and F' a filter of M'. If $f^{-1}(F') \subseteq F$ then, $\overline{R_F}(f^{-1}(F'))$ is a filter of M.*

Proof. It is a consequence of the Proposition 3.13

□

4. Conclusion and future work

In this paper we have explored approximation spaces where, the universe is a residuated multilattice and, the equivalence relation is a congruence induced by a filter. We have established some connections between approximation spaces, special filters, residuated full sub-multilatices and homomorphisms. On the other hand, Formal Concept Analysis is a method of relation data analysis identifying interesting clusters (formal concepts) in a collection of objects and their attributes. In 2023, Koguep et al. [6] have used residuated multilattices to evaluate objects and attributes in formal concept analysis setting, and showed that the set of all concepts forms a residuated multilattice. Then how to define the notion of filters and congruences in this residuated multilattice and use the results of this paper to approximate concepts? We will focus on these problems in future.

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Data availability

No data were used to support this study.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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