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On (σ, τ)-single valued neutrosophic soft sub-implicative ideals of KU-algebras

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Abstract. We propose the concept of (σ, τ)-single valued neutrosophic soft sub-implicative ideals in KU-algebras and investigate some of its properties. We give conditions for (σ, τ)-single valued neutrosophic soft ideal to be a single valued neutrosophic soft sub-implicative ideal. We show that any (σ, τ)-single valued neutrosophic soft sub-implicative ideal is a (σ, τ)-single valued neutrosophic soft ideal, but the converse is not true. Using a level set of a single valued neutrosophic soft set in a KU-algebra, we give a characterization of (σ, τ)-single valued neutrosophic soft sub-implicative ideals. We study how to deal with the homomorphic image (pre-image) of a (σ, τ)-single valued neutrosophic soft ideal of KU-algebra. Moreover, we introduce the notion of Cartesian product of (σ, τ)-single valued neutrosophic soft ideals and study some of its properties.

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1. Introduction

Zadeh [1] laid the foundation of fuzzy set theory in 1965. The basic component of a fuzzy set is a membership function expressed as truth or T. Fuzzy set theory is now a well established area. Currently, there are many extensions of fuzzy sets. Among these, one generalization was proposed by Atanassov [2] in 1983, viz., intuitionistic fuzzy set, which is characterized by a membership function (truth or T) and a nonmembership function (falsity or F). In real life, there are cases in which data are fuzzy with different levels of hesitation. Such cases can be modeled by using intuitionistic fuzzy set theory. Fuzzy and intuitionistic fuzzy set theories span a wide range of applications ranging from industrial process control to medical diagnosis and group decision processes. After then, Smarandache 1998 [3, 4], gave the notion of neutrosophic sets which generalizes fuzzy set and intuitionistic fuzzy set. This new concept is difficult to apply in the real application. It is a set in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). Recently, Wang et al. [5] introduced the notion of single valued neutrosophic sets which is a sub class of neutrosophic set and then a generalization of fuzzy sets and intuitionistic fuzzy sets. The algebraic structures and concepts in classical mathematical areas such as algebra, topology, rings, field, modules, graph theory, Lie algebra and $BCH/BCI/BCK/QS/KU/PU$-algebras with rapid growth in the fuzzy setting have also been considered by many researchers. For the general development of $BCH/BCI/BCK/QS/KU/PU$-algebras, the ideal theory plays an important role. Thus much research emphasis has been on the ideal theory of $BCH/BCI/BCK/QS/KU/PU$-algebras. Prabpayak and Leerawat [6, 7] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. Also, they defined a homomorphism of KU-algebras and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Mostafa et al. [8, 9, 10] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Mostafa et al. [11] proposed the concepts of KU-sub implicative KU- positive implicative and KU-sub commutative ideals in KU-algebras and discussed some their related properties (See [12, 13, 14, 15, 16] for the further research). Recently, Almuhaimeed [17] introduced the concept of interior KU-algebras and studied its various properties. Baek et al. [18] defined positive implicative [resp. implicative and commutative] Γ-KU-algebras as a generalization of classical positive implicative [resp. implicative and commutative] KU-algebra, and obtain their some properties (including characterizations) respectively and some relationships among them. Also, Ansari and Koam [19] proposed the concept of modules for KU-algebras and dealt with some of its properties.

This paper expands on the foundational studies of fuzzy set theory, intuitionistic fuzzy sets, and neutrosophic sets to explore the complex and nuanced structure of (σ, τ)-single valued neutrosophic soft ideals in the context of KU-algebras. The evolution of fuzzy logic into more sophisticated systems like neutrosophic logic represents a crucial development in addressing the complexities of real-world data and decision-making, where uncertainties and contradictory information often coexist. By delving into the characteristics and relationships of these algebraic ideals, this study aims to bridge theoretical mathematics with practical applications in fields where uncertainty and partial knowledge play critical roles.

Also, we give conditions for a (σ, τ)-single valued neutrosophic soft ideal to be a single valued neutrosophic soft sub implicative ideal. We show that any single valued neutrosophic soft sub implicative ideal is a (σ, τ)-single valued neutrosophic ideal but the converse is not true. Using a level set of a single valued neutrosophic soft set in a KU-algebra, we give a characterization of (σ, τ)-single valued neutrosophic soft sub implicative ideals. We investigate how to deal with the homomorphic image (preimage) of a (σ, τ)-single valued neutrosophic soft ideal of KU-algebras. Moreover, we introduce the notion of Cartesian product of (σ, τ)-single valued neutrosophic soft ideals and then we study some related properties

2. Preliminaries

Now, we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this section.

Definition 2.1 ([6, 7])**.** An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *KU-algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(KU_1) $(x * y) * [(y * z) * (x * z)] = 0$,
(KU_2) $x * 0 = 0$,
(KU_3) $0 * x = x$,
(KU_4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

It is well-known that in KU-algebra X, the following equality (See [6]):

$$x * x = 0 \text{ for each } x \in X. \tag{2.1}$$

On a KU-algebra X, we can define a binary relation $\leq$ on X by: for any $x,\ y \in X$,

$$x \leq y \Leftrightarrow y * x = 0.$$

Then we obtain the following consequence.
For any KU-algebra X and any $x,\ y,\ z \in X$,
(1) $(y * z) * (x * z) \leq x * y$,
(2) $0 \leq x$,
(3) $x \leq y,\ y \leq x$ imply $x = y$,
(4) $y * x \leq x$.

For any elements x and y of a KU-algebra X and each positive integer n, $y * x^n$ defines by:

$$y * x^n = ((y * x) * x) * \cdots * x \ (n \text{ times}).$$

Result 2.2 (See Lemmas 2.5, 2.6 and 2.7, [9]; Theorem 2.2, [20])**.** *In a KU-algebra X, the followings hold: for any $x,\ y,\ z \in X$,*
(1) $x \leq y$ *implies* $y * z \leq x * z$,
(2) $x * (y * z) = y * (x * z)$,
(3) $[(y * x) * x] \leq y$,
(4) $y * x^3 = y * x$.

We will refer to is a KU-algebra unless otherwise indicated.

Definition 2.3 ([6])**.** Let I be a nonempty subset of X. Then I is called an *ideal* of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(I_1) $0 \in I$,
(I_2) $y * z,\ y \in I$ imply $z \in I$.

Definition 2.4 ([6])**.** Let I be a nonempty subset of X. Then I is called an *KU-ideal* of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(KUI_1) $0 \in I$,
(KUI_2) $x * (y * z),\ y \in I$ imply $x * z \in I$.

Definition 2.5 ([11])**.** Let A be a nonempty subset of X. Then A is called a *KU-sub implicative ideal* (briefly KUSII) of X, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,

(KUSII$_1$) $0 \in A$,

(KUSII$_2$) $z * [(x * y) * (y * x^2)],\ z \in A$ imply $x * y^2 \in A$.

For a nonempty set X, a mapping $A : X \to [0,1]$ is called a *fuzzy set* in X (See [1]) and $[0,1]^X$ denotes the set of all fuzzy sets. For any $A,\ B \in [0,1]^X$, $A \cap B$ and $A \cup B$ be defined as follows: for each $x \in X$,

$$(A \cap B)(x) = A(x) \wedge B(x) = min\{A(x), B(x)\}$$

and

$$(A \cup B)(x) = A(x) \vee B(x) = max\{A(x), B(x)\}.$$

3. (σ,τ)-single valued neutrosophic soft sub-implicative Ideals of KU-algebras

First of all, we recall the concepts of single valued neutrosophic sets, soft sets and fuzzy soft sets. Next, we introduce the notion of (σ,τ)-single valued neutrosophic soft sets and obtain some of its properties. Finally, we discuss some properties of (σ,τ)-single valued neutrosophic soft ideals of a KU-algebra.

For an initial universe U, a set of parameters E and $A \subset E$, a mapping $F_{A} : A \to P(U)$ is called a *soft set over* U (See [24]), where $P(U)$ denotes the set of all subsets of U (See [25] for further operations on soft sets and some of their properties).

For an initial universe U, a set of parameters E and $A \subset E$, a mapping $\widetilde{F}_{A} : A \to [0,1]^U$ is called a *fuzzy soft set over* U (briefly FSS) over U (briefly FSS) (See [26]). We will denote the set of all FSSs over U as $FSS(U)$.

It is easy to see that every (classical) soft set may be considered as a fuzzy soft set. For each $e \in A$, since $\widetilde{F}_{A}(e) \in [0,1]^U$, $\widetilde{F}_{A}(e)$ may be considered as the set of e-approximate elements of the fuzzy soft set $\widetilde{F}_{A}$.

For a nonempty set X, a mapping $\overline{A} = \left\langle A^T, A^I, A^F \right\rangle : X \to [0,1] \times [0,1] \times [0,1]$ is called a *single valued neutrosophic set* (briefly SVNS) in X (See [5]). We will denote the set of all SVNSs in X as $SVNS(X)$.

Definition 3.1. Let X be a nonempty set, let E be a set of parameters and let $A \subset E$. Then a mapping $\overline{\widetilde{F}}_{A} = \left\langle \widetilde{F}^T_{A}, \widetilde{F}^I_{A}, \widetilde{F}^F_{A} \right\rangle : A \to [0,1]^X \times [0,1]^X \times [0,1]^X$ is called a *single valued neutrosophic soft set over* X (briefly SVNSS).

We will denote the set of all SVNSSs over X as $SVNSS_E(X)$. For each $e \in A$, $\overline{\widetilde{F}}_{A}(e)$ may be considered as the set of e-approximate elements of the SVNSS $\overline{\widetilde{F}_{A}}$. In fact, $\overline{\widetilde{F}}_{A}(e) = \left\langle \widetilde{F}^T_{A}(e), \widetilde{F}^I_{A}(e), \widetilde{F}^F_{A}(e) \right\rangle : X \to [0,1] \times [0,1] \times [0,1]$, i.e., $\overline{\widetilde{F}}_{A}(e)$ is a SVNS in X. In this case, $\widetilde{F}^T_{A}$ [resp. $\widetilde{F}^I_{A}$ and $\widetilde{F}^F_{A}$] is called the *truth-membership* [resp. the *indeterminacy-membership* and the *falsity-membership*] *function* of the SVNSS $\overline{\widetilde{F}}_{A}$.

For two SVNSSs $\overline{\widetilde{F}}_{A}$ and $\overline{\widetilde{G}}_{A}$, $\overline{\widetilde{F}}_{A} \leq \overline{\widetilde{G}}_{A}$ is defined by: for each $x \in X$ and each $e \in A$,

$$\widetilde{F}^T_{A}(e)(x) \leq \widetilde{G}^T_{A}(e)(x),\ \widetilde{F}^I_{A}(e)(x) \leq \widetilde{G}^I_{A}(e)(x),\ \widetilde{F}^F_{A}(e)(x) \geq \widetilde{G}^F_{A}(e)(x).$$

Definition 3.2. Let X be a KU-algebra, let E be a set of parameters, let $A \subset E$ and let $0 \leq \sigma < \tau \leq 1$. Then a $\overline{\overline{F}}_A \in SVNSS_E(X)$ is called a (σ, τ)-*single valued soft ideal* (briefly (σ, τ)-SVNSI) of X, if it satisfies the following conditions: for any $x,\ y \in X$, and each $e \in A$,

(SVNSI$_1$) $\widetilde{F}^T_A(e)(0) \vee \sigma \geq \widetilde{F}^T_A(e)(x) \wedge \tau$,
$\widetilde{F}^I_A(e)(0) \vee \sigma \geq \widetilde{F}^I_A(e)(x) \wedge \tau,\ \widetilde{F}^F_A(e)(0) \wedge \sigma \leq \widetilde{F}^F_A(e)(x) \vee \tau$,
(SVNSI$_2$) $\widetilde{F}^T_A(e)(y) \vee \sigma \geq [\widetilde{F}^T_A(e)(x * y) \wedge \widetilde{F}^T_A(e)(x)] \wedge \tau$,
(SVNSI$_3$) $\widetilde{F}^I_A(e)(y) \vee \sigma \geq [\widetilde{F}^I_A(e)(x * y) \wedge \widetilde{F}^I_A(e)(x)] \wedge \tau$,
(SVNSI$_4$) $\widetilde{F}^F_A(e)(y) \wedge \sigma \leq [\widetilde{F}^F_A(e)(x * y) \vee \widetilde{F}^F_A(e)(x)] \vee \tau$,

Definition 3.3. Let X be a KU-algebra, let E be a set of parameters, let $A \subset E$ and let $0 \leq \sigma < \tau \leq 1$. Then a $\overline{\overline{F}}_A \in SVNSS_E(X)$ is called a (σ, τ)-*single valued soft sub-implicative ideal* (briefly (σ, τ)-SVNSSII) of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$, and each $e \in A$,

(SVNSI$_1$) $\widetilde{F}^T_A(e)(0) \vee \sigma \geq \widetilde{F}^T_A(e)(x) \wedge \tau$,
$\widetilde{F}^I_A(e)(0) \vee \sigma \geq \widetilde{F}^I_A(e)(x) \wedge \tau,\ \widetilde{F}^F_A(e)(0) \wedge \sigma \leq \widetilde{F}^F_A(e)(x) \vee \tau$,
(SVNSSII$_1$) $\widetilde{F}^T_A(e)(x * y^2) \vee \sigma \geq [\widetilde{F}^T_A(e)(z * ((x * y) * (y * x^2))) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$,
(SVNSSII$_2$) $\widetilde{F}^I_A(e)(x * y^2) \vee \sigma \geq [\widetilde{F}^I_A(e)(z * ((x * y) * (y * x^2))) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$,
(SVNSSII$_3$) $\widetilde{F}^F_A(e)(x * y^2) \wedge \sigma \leq [\widetilde{F}^F_A(e)(z * ((x * y) * (y * x^2))) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$.

Remark 3.4. It is obvious that if $\sigma = 0,\ \tau = 1$, then we obtain the results in [28].

Example 3.5. Let $X = \{x_0, x_1, x_2, x_3\}$ be a set with the binary operation $*$ defined by the following table

$*$	x_0	x_1	x_2	x_3
x_0	x_0	x_1	x_2	x_3
x_1	x_0	x_0	x_0	x_2
x_2	x_0	x_2	x_0	x_1
x_3	x_0	x_0	x_0	x_0

Table 3.1

Then clearly, X is a KU-algebra. Let E be a set of parameters and $A = \{e_1, e_2\} \subset E$. Consider the SVNSS $\overline{\overline{F}}_A = \left\langle \widetilde{F}^T_A, \widetilde{F}^I_A, \widetilde{F}^F_A \right\rangle$ over X defined as follows:

$\widetilde{F}^T_A(e_1)(x_0) = \widetilde{F}^T_A(e_1)(x_2) = 0.7,\ \widetilde{F}^T_A(e_1)(x_1) = \widetilde{F}^T_A(e_1)(x_3) = 0.2,$
$\widetilde{F}^I_A(e_1)(x_0) = \widetilde{F}^I_A(e_1)(x_2) = 0.6,\ \widetilde{F}^I_A(e_1)(x_1) = \widetilde{F}^I_A(e_1)(x_3) = 0.2,$
$\widetilde{F}^F_A(e_1)(x_0) = \widetilde{F}^F_A(e_1)(x_2) = 0,\ \widetilde{F}^F_A(e_1)(x_1) = \widetilde{F}^F_A(e_1)(x_3) = 0.3,$
$\widetilde{F}^T_A(e_2)(x_0) = \widetilde{F}^T_A(e_2)(x_2) = 0.8,\ \widetilde{F}^T_A(e_2)(x_1) = \widetilde{F}^T_A(e_2)(x_3) = 0.3,$
$\widetilde{F}^I_A(e_2)(x_0) = \widetilde{F}^I_A(e_2)(x_2) = 0.5,\ \widetilde{F}^I_A(e_2)(x_1) = \widetilde{F}^I_A(e_2)(x_3) = 0.1,$
$\widetilde{F}^F_A(e_2)(x_0) = \widetilde{F}^F_A(e_2)(x_2) = 0,\ \widetilde{F}^F_A(e_2)(x_1) = \widetilde{F}^F_A(e_2)(x_3) = 0.2.$

Let $\sigma = 0.1,\ \tau = 0.3$. Then the (σ, τ)-SVNSS $\overline{\overline{F}}_A$ over X is described as the following table:

Thus by routine calculations, we can see that $\overline{\overline{F}}_A$ is a (σ, τ)-SVNSSII of X.

X	x_0	x_1	x_2	x_3
$\widetilde{F}^T_A(e_1)$	0.7	0.2	0.7	0.2
$\widetilde{F}^I_A(e_1)$	0.6	0.2	0.6	0.2
$\widetilde{F}^F_A(e_1)$	0	0.3	0	0.3
$\widetilde{F}^T_A(e_2)$	0.8	0.3	0.8	0.3
$\widetilde{F}^I_A(e_2)$	0.5	0.1	0.5	0.1
$\widetilde{F}^F_A(e_2)$	0	0.2	0	0.2

Table 3.2

Proposition 3.6. *Let E be a set of parameters and le $A \subset E$. Then every (σ, τ)-SVNSSII of a KU-algebra X is order reversing, i.e., the following inequalities hold: for any $x,\ z \in X$ such that $x \leq z$ and each $e \in A$,*

$$\widetilde{F}^T_A(e)(x) \vee \sigma \geq \widetilde{F}^T_A(e)(z)] \wedge \tau,\ \ \widetilde{F}^I_A(e)(x) \vee \sigma \geq \widetilde{F}^I_A(e)(z)] \wedge \tau,$$

$$\widetilde{F}^F_A(e)(x) \wedge \sigma \leq \widetilde{F}^F_A(e)(z)] \vee \tau.$$

Proof. Let $\widetilde{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X, let $x,\ z \in X$ such that $x \leq z$ and let $e \in A$. Then clearly, $z * x = 0$. Thus we have

$\widetilde{F}^T_A(e)(x) \vee \sigma \geq [\widetilde{F}^T_A(e)(z * x) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [By the axiom (SVNSSII_1)]
$= [\widetilde{F}^T_A(e)(0) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [Since $z * x = 0$]
$= \widetilde{F}^T_A(e)(z)] \wedge \tau$, [By the axiom ($\text{SVNSI}_1$)]
$\widetilde{F}^I_A(e)(x) \vee \sigma \geq [\widetilde{F}^I_A(e)(z * x) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$ [By the axiom (SVNSSII_2)]
$= [\widetilde{F}^I_A(e)(0) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$
$= \widetilde{F}^I_A(e)(z)] \wedge \tau$,
$\widetilde{F}^F_A(e)(x) \wedge \sigma \leq [\widetilde{F}^F_A(e)(z * x) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$ [By the axiom (SVNSSII_3)]
$= [\widetilde{F}^F_A(e)(0) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$
$= \widetilde{F}^F_A(e)(z)] \vee \tau$. □

Lemma 3.7. *Let E be a set of parameters, let $A \subset E$ and let $\widetilde{\widetilde{F}}_A$ be a (σ, τ)-SVNSSII of a KU-algebra X. If $y * x \leq z$ for any $x,\ y,\ z \in X$, then for each $e \in A$,*

$$\widetilde{F}^T_A(e)(y) \vee \sigma \geq (\widetilde{F}^T_A(e)(x) \wedge \widetilde{F}^T_A(e)(z)) \wedge \tau,$$

$$\widetilde{F}^I_A(e)(y) \vee \sigma \geq (\widetilde{F}^I_A(e)(x) \wedge \widetilde{F}^I_A(e)(z)) \wedge \tau,$$

$$\widetilde{F}^F_A(e)(y) \vee \sigma \leq (\widetilde{F}^F_A(e)(x) \vee \widetilde{F}^F_A(e)(z)) \vee \tau.$$

Proof. $\widetilde{\widetilde{F}}_A$ be a (σ, τ)-SVNSSII of X and let $e \in A$. Suppose $y * x \leq z$ for any $x,\ y,\ z \in X$. Then clearly, $z * (y * x) = 0$. By Result 2.2 (2), $y * (z * x) = 0$, i.e., $z * x \leq y$. By Proposition 3.6, we get

$$\widetilde{F}^T_A(e)(z * x) \vee \sigma \geq \widetilde{F}^T_A(e)(y) \wedge \tau, \tag{3.1}$$

$$\widetilde{F}^I_A(e)(z * x) \vee \sigma \geq \widetilde{F}^I_A(e)(y) \wedge \tau, \tag{3.2}$$

$$\widetilde{F}^F_A(e)(z * x) \wedge \sigma \leq \widetilde{F}^F_A(e)(y) \vee \tau. \tag{3.3}$$

Thus we have

$\widetilde{F}^T_A(e)(x) \vee \sigma$
$= \widetilde{F}^T_A(e)(x * x^2) \vee \sigma$ [By the axioms (KU$_4$) and (KU$_3$)]
$\geq [\widetilde{F}^T_A(e)(z * ((x * x) * (x * x^2))) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [By the axiom (SVNSSII$_1$)]
$= [\widetilde{F}^T_A(e)(z * x) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [By the axioms (KU$_4$) and (KU$_3$)]
$\geq [\widetilde{F}^T_A(e)(y) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$, [By (3.1)]

$\widetilde{F}^I_A(e)(x) \vee \sigma$
$= \widetilde{F}^I_A(e)(x * x^2) \vee \sigma$
$\geq [\widetilde{F}^I_A(e)(z * ((x * x) * (x * x^2))) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$
$= [\widetilde{F}^I_A(e)(z * x) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$
$\geq [\widetilde{F}^I_A(e)(y) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$, [By (3.2)]

$\widetilde{F}^F_A(e)(x) \wedge \sigma$
$= \widetilde{F}^F_A(e)(x * x^2) \wedge \sigma$
$\leq [\widetilde{F}^F_A(e)(z * ((x * x) * (x * x^2))) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$
$= [\widetilde{F}^F_A(e)(z * x) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$
$\leq [\widetilde{F}^F_A(e)(y) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$. [By (3.3)] □

Lemma 3.8. *Let E be a set of parameters, let $A \subset E$ and let X be an implicative KU-algebra. Then every (σ, τ)-SVNSI of X is a (σ, τ)-SVNSSII of X.*

Proof. Let $\overline{\widetilde{F}}_A$ be a (σ, τ)-SVNSI of X, let $e \in A$ and let $x,\ y,\ z \in X$. Then we have

$\widetilde{F}^T_A(e)(x * y^2) \vee \sigma$
$\geq [\widetilde{F}^T_A(e)(z * (x * y^2)) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [By the axiom (SVNSI$_2$)]
$= [\widetilde{F}^T_A(e)(z * ((x * y) * (x * y^2))) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$
[Since $x * y^2 = (x * y) * (x * y^2)$ by the hypothesis]

$\widetilde{F}^I_A(e)(x * y^2) \vee \sigma$
$\geq [\widetilde{F}^I_A(e)(z * (x * y^2)) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$ [By the axiom (SVNSI$_3$)]
$= [\widetilde{F}^I_A(e)(z * ((x * y) * (x * y^2))) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$

$\widetilde{F}^F_A(e)(x * y^2) \wedge \sigma$
$\leq [\widetilde{F}^F_A(e)(z * (x * y^2)) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$ [By the axiom (SVNSI$_4$)]
$= [\widetilde{F}^F_A(e)(z * ((x * y) * (x * y^2))) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$.

Thus $\overline{\widetilde{F}}_A$ satisfies the axioms (SVNSSII$_1$), (SVNSSII$_2$) and (SVNSSII$_3$). So $\overline{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X. □

Proposition 3.9. *Let E be a set of parameters, let $A \subset E$ and let $\overline{\widetilde{F}}_A$ be a SVNSS in a KU-algebra X. If $\overline{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X, then it satisfies the following inequalities:*

(SVNSSII$_4$) $\widetilde{F}^T_A(e)(x * y^2) \vee \geq \widetilde{F}^T_A(e)((x * y) * (y * x^2)) \wedge \tau$,
(SVNSSII$_5$) $\widetilde{F}^I_A(e)(x * y^2) \vee \geq \widetilde{F}^I_A(e)((x * y) * (y * x^2)) \wedge \tau$,

(SVNSSII$_6$) $\widetilde{F}^F_A(e)(x*y^2)\wedge \leq \widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee\tau$.

Proof. Suppose $\widetilde{\widetilde{F}}_A$ is a (σ,τ)-SVNSSII of X and let $x,\ y\in X$, $e\in A$. Then we get

$$\begin{aligned}
&\widetilde{F}^T_A(e)(x*y^2)\vee\sigma\\
&\geq [\widetilde{F}^T_A(e)(0*((x*y)*(y*x^2)))\wedge \widetilde{F}^T_A(e)(0)]\wedge\tau \text{ [By the axiom (SVNSSII}_1)]\\
&= [\widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^T_A(e)(0)]\wedge\tau \text{ [By the axiom (KU}_3)]\\
&= [\widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^T_A(e)(0)]\wedge\tau\\
&= \widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge\tau,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{F}^I_A(e)(x*y^2)\vee\sigma\\
&\geq [\widetilde{F}^I_A(e)(0*((x*y)*(y*x^2)))\wedge \widetilde{F}^I_A(e)(0)]\wedge\tau \text{ [By the axiom (SVNSSII}_2)]\\
&= [\widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^I_A(e)(0)]\wedge\tau\\
&= [\widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^I_A(e)(0)]\wedge\tau\\
&= \widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge\tau,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{F}^F_A(e)(x*y^2)\wedge\sigma\\
&\leq [\widetilde{F}^F_A(e)(0*((x*y)*(y*x^2)))\vee \widetilde{F}^F_A(e)(0)]\vee\tau \text{ [By the axiom (SVNSSII}_3)]\\
&= [\widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee \widetilde{F}^F_A(e)(0)]\vee\tau\\
&= [\widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee \widetilde{F}^F_A(e)(0)]\vee\tau\\
&= \widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee\tau.
\end{aligned}$$

□

Proposition 3.10. *Let E be a set of parameters, let $A\subset E$ and let X be a KU-algebra. Then every (σ,τ)-SVNSSII of X is a (σ,τ)-SVNSI of X but the converse does not hold.*

Proof. Let $\widetilde{\widetilde{F}}_A$ be a (σ,τ)-SVNSSII of X and let $x,\ z\in X$, $e\in A$. Then we have

$$\begin{aligned}
&\widetilde{F}^T_A(e)(x)\vee\sigma\\
&= \widetilde{F}^T_A(e)(x*x^2)\vee\sigma \text{ [By (2.1) and the axiom (KU}_3)]\\
&\geq [\widetilde{F}^T_A(e)(z*((x*x)*(x*x^2)))\wedge \widetilde{F}^T_A(e)(z)]\wedge\tau \text{ [By the axiom (SVNSSII}_1)]\\
&= [\widetilde{F}^T_A(e)(z*(0*x)))\wedge \widetilde{F}^T_A(e)(z)]\wedge\tau \text{ [By (2.1) and the axiom (KU}_3)]\\
&= [\widetilde{F}^T_A(e)(z*x)))\wedge \widetilde{F}^T_A(e)(z)]\wedge\tau \text{ [By (2.1)]},
\end{aligned}$$

$$\begin{aligned}
&\widetilde{F}^I_A(e)(x)\vee\sigma\\
&= \widetilde{F}^I_A(e)(x*x^2)\vee\sigma\\
&\geq [\widetilde{F}^I_A(e)(z*((x*x)*(x*x^2)))\wedge \widetilde{F}^I_A(e)(z)]\wedge\tau \text{ [By the axiom (SVNSSII}_2)]\\
&= [\widetilde{F}^I_A(e)(z*(0*x)))\wedge \widetilde{F}^I_A(e)(z)]\wedge\tau\\
&= [\widetilde{F}^I_A(e)(z*x)))\wedge \widetilde{F}^I_A(e)(z)]\wedge\tau,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{F}^F_A(e)(x)\wedge\sigma\\
&= \widetilde{F}^F_A(e)(x*x^2)\wedge\sigma\\
&\geq [\widetilde{F}^F_A(e)(z*((x*x)*(x*x^2)))\vee \widetilde{F}^F_A(e)(z)]\vee\tau \text{ [By the axiom (SVNSSII}_3)]\\
&= [\widetilde{F}^F_A(e)(z*(0*x)))\vee \widetilde{F}^F_A(e)(z)]\vee\tau
\end{aligned}$$

$= [\widetilde{F}_A^F(e)(z * x))) \vee \widetilde{F}_A^F(e)(z)] \vee \tau.$

Thus $\overline{\overline{\widetilde{F}}}_A$ satisfies the conditions (SVNSI$_1$), (SVNSI$_2$), (SVNSI$_3$) and (SVNSI$_4$). So $\overline{\overline{\widetilde{F}}}_A$ is a (σ, τ)-SVNSI of X. $\square$

The following example shows that the converse of Theorem 3.10 may not be true.

Example 3.11. Let E be a set of parameters, let $A \subset E$ and let $X = \{x_0, x_1, x_2, x_3, x_4\}$ be the KU-algebra with the binary operation $*$ defined by the following table

$*$	x_0	x_1	x_2	x_3	x_4
x_0	x_0	x_1	x_2	x_3	x_4
x_1	x_0	x_0	x_1	x_3	x_4
x_2	x_0	x_0	x_0	x_3	x_4
x_3	x_0	x_0	x_0	x_0	x_4
x_4	x_0	x_0	x_0	x_0	x_0

Table 3.3

For $\sigma = 1,\ \tau = 0.5$ and a fixed $e \in A$, consider SVNSS over X as the following table:

X	x_0	x_1	x_2	x_3	x_4
$\widetilde{F}_A^T(e)$	0.7	0.2	0.2	0.2	0.2
$\widetilde{F}_A^I(e)$	0.6	0.2	0.6	0.2	0.3
$\widetilde{F}_A^F(e)$	0	0.3	0	0.3	0.1

Table 3.4

Then we get

L.H.S of the axiom (SVNSSII$_1$)$=\widetilde{F}_A^T(e)(x_1 * x_2^2) \vee 0.1$
$= \widetilde{F}_A^T(e)(x_1 \vee 0.1$
$= 0.2 \vee 0.1 = 0.2,$

R.H.S of the axiom (SVNSSII$_1$)$=[\widetilde{F}_A^T(e)(z*((x_1*x_2)*(x_2*x_1^2))) \wedge \widetilde{F}_A^T(e)(x_0)] \wedge 0.5$
$= \widetilde{F}_A^T(e)(x_0 \vee 0.5$
$= 0.7 \vee 0.5 = 0.5$

Thus $\widetilde{F}_A^T(e)(x_1 * x_2^2) \vee 0.1 \ngeq [\widetilde{F}_A^T(e)(z * ((x_1 * x_2) * (x_2 * x_1^2))) \wedge \widetilde{F}_A^T(e)(x_0)] \wedge 0.5.$

We now give a condition for a (σ, τ)-SVNSI to be a $\sigma, \tau)$-SVNSSII.

Proposition 3.12. *Let E be a set of parameters, let $A \subset E$ and let X be a KU-algebra X. Then every (σ, τ)-SVNSI of X satisfying the conditions* (SVNSSII$_4$), (SVNSSII$_5$) and (SVNSSII$_6$) is a (σ, τ)-SVNSSII of X.

Proof. Let $\overline{\overline{\widetilde{F}}}_A$ be a (σ, τ)-SVNSI of X satisfying the conditions (SVNSSII$_4$), (SVNSSII$_5$) and (SVNSSII$_6$). Let $x,\ y \in X$ and let $e \in A$. Then we get

$$\widetilde{F}_A^T(e)(x * y) \vee \sigma \geq \widetilde{F}_A^T(e)((x * y) * (y * x^2)) \wedge \tau, \tag{3.4}$$

$$\widetilde{F}_A^I(e)(x * y) \vee \sigma \geq \widetilde{F}_A^I(e)((x * y) * (y * x^2)) \wedge \tau, \tag{3.5}$$

$$\widetilde{F}^F_A(e)(x*y)\wedge\sigma \le \widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee\tau. \tag{3.6}$$

Since $\widetilde{\widetilde{F}}_A$ is a (σ,τ)-SVNSI of X, by Definition 3.2, we have

$$\widetilde{F}^T_A(e)((x*y)*(y*x^2))\vee\sigma \ge [\widetilde{F}^T_A(e)(z*((x*y)*(y*x^2)))\wedge \widetilde{F}^T_A(e)(z)]\wedge\tau, \tag{3.7}$$

$$\widetilde{F}^I_A(e)((x*y)*(y*x^2))\vee\sigma \ge [\widetilde{F}^I_A(e)(z*((x*y)*(y*x^2)))\wedge \widetilde{F}^I_A(e)(z)]\wedge\tau, \tag{3.8}$$

$$\widetilde{F}^F_A(e)((x*y)*(y*x^2))\wedge\sigma \le [\widetilde{F}^F_A(e)(z*((x*y)*(y*x^2)))\vee \widetilde{F}^F_A(e)(z)]\vee\tau. \tag{3.9}$$

Since $0\le\sigma<\tau\le 1$, from (3.4) and (3.7), (3.5) and (3.8), (3.6) and (3.9), we obtain respectively the following inequalities:

$$\widetilde{F}^T_A(e)(x*y)\vee\sigma \ge [\widetilde{F}^T_A(e)(z*((x*y)*(y*x^2)))\wedge \widetilde{F}^T_A(e)(z)]\wedge\tau,$$
$$\widetilde{F}^I_A(e)(x*y)\vee\sigma \ge [\widetilde{F}^I_A(e)(z*((x*y)*(y*x^2)))\wedge \widetilde{F}^I_A(e)(z)]\wedge\tau,$$
$$\widetilde{F}^F_A(e)(x*y)\wedge\sigma \le [\widetilde{F}^F_A(e)(z*((x*y)*(y*x^2)))\vee \widetilde{F}^F_A(e)(z)]\vee\tau.$$

Thus $\widetilde{\widetilde{F}}_A$ satisfies the conditions (SVNSI_1), (SVNSSII_1), (SVNSSII_2) and (SVNSSII_3). So $\widetilde{\widetilde{F}}_A$ is a (σ,τ)-SVNSSII of X. □

Theorem 3.13. *Let E be a set of parameters, let $A\subset E$ and let $\widetilde{\widetilde{F}}_A$ be a (σ,τ)-SVNSI of a KU-algebra X. Then the followings are equivalent:*

(1) *$\widetilde{\widetilde{F}}_A$ is a (σ,τ)-SVNSSII of X,*

(2) *for any $x,\ y\in X$ and each $e\in A$, the following inequalities:*

$$\widetilde{F}^T_A(e)(x*y^2)\vee\sigma \ge \widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge\tau,$$
$$\widetilde{F}^I_A(e)(x*y^2)\vee\sigma \ge \widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge\tau,$$
$$\widetilde{F}^F_A(e)(x*y^2)\wedge\sigma \le \widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee\tau.$$

Proof. (1)⇒(2) Suppose $\widetilde{\widetilde{F}}_A$ is a (σ,τ)-SVNSSII of X. Let $x,\ y\in X$ and let $e\in A$. Then we have

$\widetilde{F}^T_A(e)(x*y^2)\vee\sigma$
$\ge [\widetilde{F}^T_A(e)(0*((x*y)*(y*x^2)))\wedge \widetilde{F}^T_A(e)(0)]\wedge\tau$ [By (SVNSSII_1)]
$= [\widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^T_A(e)(0)]\wedge\tau$ [By the axiom (KU_3)]
$= \widetilde{F}^T_A(e)((x*y)*(y*x^2))\wedge\tau$, [By ($\text{SVNSI}_1$)]

$\widetilde{F}^I_A(e)(x*y^2)\vee\sigma$
$\ge [\widetilde{F}^I_A(e)(0*((x*y)*(y*x^2)))\wedge \widetilde{F}^I_A(e)(0)]\wedge\tau$ [By (SVNSSII_2)]
$= [\widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge \widetilde{F}^I_A(e)(0)]\wedge\tau$
$= \widetilde{F}^I_A(e)((x*y)*(y*x^2))\wedge\tau$,

$\widetilde{F}^F_A(e)(x*y^2)\wedge\sigma$
$\ge [\widetilde{F}^F_A(e)(0*((x*y)*(y*x^2)))\vee \widetilde{F}^F_A(e)(0)]\vee\tau$ [By (SVNSSII_3)]
$= [\widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee \widetilde{F}^F_A(e)(0)]\vee\tau$
$= \widetilde{F}^F_A(e)((x*y)*(y*x^2))\vee\tau.$

Thus (2) holds.

(2)⇒(1) Suppose (2) holds, let $x,\ y,\ z \in X$ and let $e \in A$. Then we get

$$\begin{aligned}
&[(z*((x*y)*(y*x^2))]*[(x*y)*(y*x^2)] \\
&= [(x*y)*(z*(y*x^2))]*[(x*y)*(y*x^2)] \text{ [By Result 2.2 (2)]} \\
&\leq (z*(y*x^2))*(y*x^2) \text{ [By the axiom (KU}_1\text{)]} \\
&= [(z*(y*x^2))]*[0*(y*x^2)] \text{ [By the axiom (KU}_3\text{)]} \\
&\leq 0*z \text{ [By the axiom (KU}_1\text{)]} \\
&= z. \text{ [By the axiom (KU}_3\text{)]}
\end{aligned}$$

Thus by Lemma 3.7, we have

$$\widetilde{F}^T_A(e)((x*y)*(y*x^2)) \vee \sigma \geq [\widetilde{F}^T_A(e)((x*y)*(y*x^2)) \wedge \widetilde{F_A}^T(e)(z)] \wedge \tau,$$
$$\widetilde{F}^I_A(e)((x*y)*(y*x^2)) \vee \sigma \geq [\widetilde{F}^I_A(e)((x*y)*(y*x^2)) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau,$$
$$\widetilde{F}^F_A(e)((x*y)*(y*x^2)) \wedge \sigma \leq [\widetilde{F}^F_A(e)((x*y)*(y*x^2)) \vee \widetilde{F}^F_A(e)(z)] \vee \tau.$$

So by the hypothesis, we get

$$\widetilde{F}^T_A(e)(x*y^2) \vee \sigma \geq [\widetilde{F}^T_A(e)(z*((x*y)*(y*x^2))) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau,$$
$$\widetilde{F}^I_A(e)(x*y^2) \vee \sigma \geq [\widetilde{F}^I_A(e)(z*((x*y)*(y*x^2))) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau,$$
$$\widetilde{F}^F_A(e)(x*y^2) \wedge \sigma \leq [\widetilde{F}^F_A(e)(z*((x*y)*(y*x^2))) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$$

Hence $\overline{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X. □

Definition 3.14. Let E be a set of parameters, let $A \subset E$ and let $\overline{\widetilde{F}_A}$ be a SVNSS over a nonempty set X. For any $t,\ i,\ f \in (0, 1]$ and each $e \in A$, the subset $[\overline{\widetilde{F}}_A(e)]_{\langle t,i,f\rangle}$ of X defined by

$$[\overline{\widetilde{F}}_A(e)]_{\langle t,i,f\rangle} = \{x \in X : \widetilde{F}^T_A(e)(x) \vee \sigma \geq t,\ \widetilde{F}^I_A(e)(x) \vee \sigma \geq i,\ \widetilde{F}^F_A(e)(x) \wedge \sigma \leq f\}$$

is called a *level subset* of $\overline{\widetilde{F}}_A$ for e, where $0 \leq \sigma < \tau \leq 1$.

Theorem 3.15. *Let E be a set of parameters, let $A \subset E$ and let $\overline{\widetilde{F}}_A$ be a (σ, τ)-SVNSI of a KU-algebra X. Then for each $e \in A$, $\overline{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X if and only if for any $t,\ i,\ f \in (0, 1]$, $[\overline{\widetilde{F}}_A]_{\langle t\wedge\tau,i\wedge\tau,f\vee\tau\rangle} \neq \varnothing$ is a sub-implicative ideal of X*

Proof. Suppose $\overline{\widetilde{F}}_A$ is a (σ, τ)-SVNSSII of X and let $[\overline{\widetilde{F}}_A(e)]_{\langle t\wedge\tau,i\wedge\tau,f\vee\tau\rangle} \neq \varnothing$ for each $e \in A$. Then there is $x \in [\overline{\widetilde{F}}_A(e)]_{\langle t\wedge\tau,i\wedge\tau,f\vee\tau\rangle}$. Thus we have

$$\widetilde{F}^T_A(e)(x) \vee \sigma \geq t \wedge \tau,\ \widetilde{F}^I_A(e)(x) \vee \sigma \geq i \wedge \tau,\ \widetilde{F}^F_A(e)(x) \wedge \sigma \leq f \vee \tau.$$

By the axiom (SVNSI$_1$), we get

$$\widetilde{F}^T_A(e)(0) \vee \sigma \geq \widetilde{F}^T_A(e)(x) \wedge \tau,$$
$$\widetilde{F}^I_A(e)(x) \vee \sigma \geq \widetilde{F}^I_A(e)(x) \wedge \tau,$$
$$\widetilde{F}^F_A(e)(x) \wedge \sigma \leq \widetilde{F}^F_A(e)(x) \vee \tau.$$

Since $0 \leq \sigma < \tau \leq 1$, we have

$$\widetilde{F}^T_A(e)(0) \vee \sigma \geq t \wedge \tau,$$

$$\widetilde{F}^I_A(e)(0) \vee \sigma \geq i \wedge \tau,$$
$$\widetilde{F}^F_A(e)(0) \wedge \sigma \leq f \vee \tau.$$

So $0 \in [\overline{\widetilde{F}}_A(e)]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$.

Suppose $z * ((x * y) * (y * x^2)),\ z \in [\overline{\widetilde{F}}_A(e)]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$ for any $x,\ y,\ z \in X$. Then clearly,

$$\widetilde{F}^T_A(e)(z * ((x * y) * (y * x^2))) \vee \sigma \geq t \wedge \tau,\ \widetilde{F}^T_A(e)(z) \vee \sigma \geq t \wedge \tau, \tag{3.10}$$

$$\widetilde{F}^I_A(e)(z * ((x * y) * (y * x^2))) \vee \sigma \geq i \wedge \tau,\ \widetilde{F}^I_A(e)(z) \vee \sigma \geq i \wedge \tau, \tag{3.11}$$

$$\widetilde{F}^F_A(e)(z * ((x * y) * (y * x^2))) \wedge \sigma \leq f \vee \tau,\ \widetilde{F}^F_A(e)(z) \wedge \sigma \leq f \vee \tau. \tag{3.12}$$

Thus we have

$\widetilde{F}^T_A(e)(x * y^2) \vee \sigma$
$\geq [\widetilde{F}^T_A(e)(z * ((x * y) * (y * x^2))) \wedge \widetilde{F}^T_A(e)(z)] \wedge \tau$ [By the axiom (SVNSSII$_1$)]
$= (\widetilde{F}^T_A(e)(z * ((x * y) * (y * x^2))) \wedge \tau) \wedge (\widetilde{F}^T_A(e)(z) \wedge \tau)$
$\geq (\widetilde{F}^T_A(e)(z * ((x * y) * (y * x^2))) \vee \sigma) \wedge (\widetilde{F}^T_A(e)(z) \vee \sigma)$ [Since $0 \leq \sigma < \tau \leq 1$]
$\geq t \wedge \tau$, [By (3.13)]

$\widetilde{F}^I_A(e)(x * y^2) \vee \sigma$
$\geq [\widetilde{F}^I_A(e)(z * ((x * y) * (y * x^2))) \wedge \widetilde{F}^I_A(e)(z)] \wedge \tau$ [By the axiom (SVNSSII$_2$)]
$= (\widetilde{F}^I_A(e)(z * ((x * y) * (y * x^2))) \wedge \tau) \wedge (\widetilde{F}^I_A(e)(z) \wedge \tau)$
$\geq (\widetilde{F}^I_A(e)(z * ((x * y) * (y * x^2))) \vee \sigma) \wedge (\widetilde{F}^I_A(e)(z) \vee \sigma)$ [Since $0 \leq \sigma < \tau \leq 1$]
$\geq i \wedge \tau$, [By (3.14)]

$\widetilde{F}^F_A(e)(x * y^2) \wedge \sigma$
$\leq [\widetilde{F}^F_A(e)(z * ((x * y) * (y * x^2))) \vee \widetilde{F}^F_A(e)(z)] \vee \tau$ [By the axiom (SVNSSII$_3$)]
$= (\widetilde{F}^F_A(e)(z * ((x * y) * (y * x^2))) \vee \tau) \vee (\widetilde{F}^F_A(e)(z) \vee \tau)$
$\leq (\widetilde{F}^F_A(e)(z * ((x * y) * (y * x^2))) \wedge \sigma) \vee (\widetilde{F}^F_A(e)(z) \wedge \sigma)$ [Since $0 \leq \sigma < \tau \leq 1$]
$\leq f \vee \tau$. [By (3.15)]

So $x * y^2 \in [\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$. Hence $[\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$ is a sub-implicative ideal of X.

Conversely, suppose $[\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle} \neq \varnothing$ is a sub-implicative ideal of X. For each $x \in X$, let $\widetilde{F}^T_A(e)(x) \vee \sigma = t \wedge \tau$, $\widetilde{F}^I_A(e)(x) \vee \sigma = i \wedge \tau$, $\widetilde{F}^F_A(e)(x) \wedge \sigma = f \vee \tau$. Then clearly, $x \in [\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$. Since $0 \in [\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$, we get

$$\widetilde{F}^T_A(e)(0) \vee \sigma \geq t \wedge \tau = \widetilde{F}^T_A(e)(x) \vee \sigma = \widetilde{F}^T_A(e)(x) \wedge \tau,$$
$$\widetilde{F}^I_A(e)(0) \vee \sigma \geq i \wedge \tau = \widetilde{F}^I_A(e)(x) \vee \sigma = \widetilde{F}^I_A(e)(x) \wedge \tau,$$
$$\widetilde{F}^F_A(e)(0) \wedge \sigma \geq f \vee \tau = \widetilde{F}^F_A(e)(x) \wedge \sigma = \widetilde{F}^F_A(e)(x) \vee \tau.$$

Thus $\overline{\widetilde{F}}_A$ satisfies the axiom (SVNSI$_1$).

Now assume that the axioms (SVNSSII$_1$), (SVNSSII$_{12}$) and (SVNSSII$_3$) do not hold. Then there are $a,\ b,\ c \in X$ and $e \in A$ such that

$$\widetilde{F}^T_A(e)(a * b^2) < [\widetilde{F}^T_A(e)(c * ((a * b) * (b * a^2))) \wedge \widetilde{F}^T_A(e)(c)] \wedge \tau,$$

$$\widetilde{F}^I_A(e)(a*b^2) < [\widetilde{F}^I_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^I_A(e)(c)] \wedge \tau,$$
$$\widetilde{F}^F_A(e)(a*b^2) > [\widetilde{F}^F_A(e)(c*((a*b)*(b*a^2))) \vee \widetilde{F}^F_A(e)(c)] \vee \tau.$$

Taking $t_0,\ i_0,\ f_0 \in (0,1]$ as follows:

$$t_0 = \frac{1}{2}\left(\widetilde{F}^T_A(e)(a*b^2) \vee \sigma + [\widetilde{F}^T_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^T_A(e)(c)] \wedge \tau\right),$$
$$i_0 = \frac{1}{2}\left(\widetilde{F}^I_A(e)(a*b^2) \vee \sigma + [\widetilde{F}^I_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^I_A(e)(c)] \wedge \tau\right),$$
$$f_0 = \frac{1}{2}\left(\widetilde{F}^F_A(e)(a*b^2) \wedge \sigma + [\widetilde{F}^F_A(e)(c*((a*b)*(b*a^2))) \vee \widetilde{F}^F_A(e)(c)] \vee \tau\right).$$

Then clearly, we get

$$\widetilde{F}^T_A(e)(a*b^2) \vee \sigma < t_0 < [\widetilde{F}^T_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^T_A(e)(c)] \wedge \tau,$$
$$\widetilde{F}^I_A(e)(a*b^2) \vee \sigma < i_0 < [\widetilde{F}^I_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^I_A(e)(c)] \wedge \tau,$$
$$\widetilde{F}^F_A(e)(a*b^2) \wedge \sigma > f_0 > [\widetilde{F}^F_A(e)(c*((a*b)*(b*a^2))) \wedge \widetilde{F}^F_A(e)(c)] \vee \tau.$$

Thus $c*((a*b)*(b*a^2)),\ c \in [\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$ but $a*b^2 \notin [\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$. So $[\overline{\widetilde{F}}_A]_{\langle t\wedge\tau, i\wedge\tau, f\vee\tau\rangle}$ is not a sub-implicative ideal of X. This is a contradiction. Hence $\overline{\widetilde{F}}_A$ is a (σ,τ)-SVNSSII of X. □

4. The image (pre-image) of a (σ,τ)-single valued neutrosophic soft sub implicative ideal under a homomorphism of KU-algebras

Definition 4.1. Let $(X,*,0)$ and $(Y,*',0')$ be two KU-algebras. Then a mapping $f: X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x)*' f(y)$ for any $x,\ y \in X$. Note that if $f: X \to Y$ is a homomorphism of KU-algebras, then $f(0) = 0'$.

Definition 4.2. Let X and Y be two nonempty sets and let $\overline{A} \in SVNS(X)$, $\overline{B} \in SVNS(Y)$.

(i) The *image* of $\overline{A}$, denoted by $f(\overline{A})$, is a SVNS in Y defined as follows: for each $y \in Y$,

(4.1)
$$f(\overline{A})(y) = \begin{cases} \left\langle \bigvee_{x\in f^{-1}(y)} A^T(x), \bigvee_{x\in f^{-1}(y)} A^I(x), \bigwedge_{x\in f^{-1}(y)} A^F(x)\right\rangle & \text{if } f^{-1}(y) \neq \varnothing \\ \langle 0,0,1\rangle & \text{otherwise.}\end{cases}$$

(ii) The *pre-image* of $\overline{B}$, denoted by $f^{-1}(\overline{B})$, is a SVNS in X defined as follows: for each $x \in X$,

$$f^{-1}(\overline{B})(x) = \left\langle B^T(f(x)), B^I(f(x)), B^F(f(x))\right\rangle. \tag{4.2}$$

Definition 4.3. Let X, Y be two nonempty sets, let E, E' be two sets of parameters, let $A \subset E$, $B \subset E'$ and let $p: A \to B$ be a bijective mapping. Let $\overline{\widetilde{F}}_A \in SVNSS_E(X)$ and let $\overline{\widetilde{F}}_B \in SVNSS_{E'}(Y)$.

(i) The *image* of $\widetilde{\widetilde{F}}_A$, denoted by $f\left(\widetilde{\widetilde{F}}_A\right) = \left\langle f\left(\widetilde{F}^T_A\right), f\left(\widetilde{F}^I_A\right), f\left(\widetilde{F}^F_A\right)\right\rangle$, is a SVNSS over Y defined as follows: for each $y \in Y$ and each $e^{'} \in B$,
(4.3)

$$f\left(\widetilde{\widetilde{F}}_A\right)(e^{'})(y) = \begin{cases} \left\langle f\left(\widetilde{F}^T_A\right)(e^{'})(y), f\left(\widetilde{F}^I_A\right)(e^{'})(y), f\left(\widetilde{F}^F_A\right)(e^{'})(y)\right\rangle & \text{if } f^{-1}(y) \neq \varnothing \\ \langle 0, 0, 1\rangle & \text{otherwise,} \end{cases}$$

where $f\left(\widetilde{F}^T_A\right)(e^{'})(y) = \bigvee_{x \in f^{-1}(y)} \widetilde{F}^T_A(p^{-1}(e^{'}))(x)$,
$f\left(\widetilde{F}^I_A\right)(e^{'})(y) = \bigvee_{x \in f^{-1}(y)} \widetilde{F}^I_A(p^{-1}(e^{'}))(x)$,
$f\left(\widetilde{F}^F_A\right)(e^{'})(y) = \bigwedge_{x \in f^{-1}(y)} \widetilde{F}^F_A(p^{-1}(e^{'}))(x)$.

(ii) The *pre-image* of $\widetilde{\widetilde{F}}_B$, denoted by $f^{-1}\left(\widetilde{\widetilde{F}}_B\right) = \left\langle f^{-1}\left(\widetilde{F}^T_B\right), f^{-1}\left(\widetilde{F}^I_B\right), f^{-1}\left(\widetilde{F}^F_B\right)\right\rangle$, is a SVNSS over X defined as follows: for each $x \in X$ and each $e \in A$,

(4.4) $$f^{-1}\left(\widetilde{\widetilde{F}}_B\right)(e)(x) = \left\langle f^{-1}\left(\widetilde{F}^T_B\right)(e)(x), f^{-1}\left(\widetilde{F}^I_B\right)(e)(x), f^{-1}\left(\widetilde{F}^F_B\right)(e)(x)\right\rangle,$$

where $f^{-1}\left(\widetilde{F}^T_B\right)(e)(x) = \widetilde{F}^T_B(p(e))(f(x))$,
$f^{-1}\left(\widetilde{F}^I_B\right)(e)(x) = \widetilde{F}^I_B(p(e))(f(x))$,
$f^{-1}\left(\widetilde{F}^F_B\right)(e)(x) = \widetilde{F}^F_B(p(e))(f(x))$.

Proposition 4.4. *Let $f : X \to Y$ be a homomorphism of KU-algebras, let E, $E^{'}$ be two sets of parameters, let $A \subset E$, $B \subset E^{'}$ and let $p : A \to B$ be a bijective mapping. If $\widetilde{\widetilde{F}}_B$ is a (σ, τ)-SVNSSII of Y, then $f^{-1}\left(\widetilde{\widetilde{F}}_B\right)$ is a (σ, τ)-SVNSSII of X.*

Proof. Suppose $\widetilde{\widetilde{F}}_B$ is a SVNSSII of Y and let $x \in X$, $e \in A$. Then we have

$$\begin{aligned} f^{-1}\left(\widetilde{F}^T_B\right)(e)(0) \vee \sigma &= \widetilde{F}^T_B(p(e))(f(0)) \vee \sigma \\ &= \widetilde{F}^T_B(p(e))(0) \vee \sigma \text{ [Since } f \text{ is a homomorphism]} \\ &\geq \widetilde{F}^T_B(p(e))(f(x)) \wedge \tau \text{ [By the axiom (SVNSI}_1\text{)} \\ &= f^{-1}\left(\widetilde{F}^T_B\right)(e)(x) \wedge \tau. \end{aligned}$$

Similarly, we get

$$f^{-1}\left(\widetilde{F}^I_B\right)(e)(0) \vee \sigma \geq f^{-1}\left(\widetilde{F}^I_B\right)(e)(x) \wedge \tau,$$

$$f^{-1}\left(\widetilde{F}^F_B\right)(e)(0) \wedge \sigma \leq f^{-1}\left(\widetilde{F}^F_B\right)(e)(x) \vee \tau.$$

Thus $f^{-1}\left(\widetilde{\widetilde{F}}_B\right)$ satisfies the axiom (SVNSI$_1$).

Now let x, y, $z \in X$ and let $e \in A$. Then we have

$$\begin{aligned} & f^{-1}\left(\widetilde{F}^T_B\right)(e)(x * y^2) \vee \sigma \\ &= \widetilde{F}^T_B(p(e))(f(x * y^2)) \vee \sigma \\ &= \widetilde{F}^T_B(p(e))(f(x) * f(y^2)) \vee \sigma \text{ [Since } f \text{ is a homomorphism]} \\ &\geq [\widetilde{F}^T_B(p(e))(f(z) * (f(x) * f(y)) * (f(y) * f(x^2)))) \wedge \widetilde{F}^T_B(p(e))(f(z))] \wedge \tau \end{aligned}$$

[By the axiom (SVNSSII$_1$)]
$= [\widetilde{F}^T_B(p(e))(f(z*((x*y)*(y*x^2)))) \wedge \widetilde{F}^T_B(p(e))(f(z))] \wedge \tau$
$= f^{-1}\left(\widetilde{F}^T_B\right)(e)(z*((x*y)*(y*x^2))) \wedge f^{-1}\left(\widetilde{F}^T_B\right)(e)(z)] \wedge \tau,$

$f^{-1}\left(\widetilde{F}^I_B\right)(e)(x*y^2) \vee \sigma$
$= \widetilde{F}^I_B(p(e))(f(x*y^2)) \vee \sigma$
$= \widetilde{F}^I_B(p(e))(f(x)*f(y^2)) \vee \sigma$
$\geq [\widetilde{F}^I_B(p(e))(f(z)*(f(x)*f(y))*(f(y)*f(x^2)))) \wedge \widetilde{F}^I_B(p(e))(f(z))] \wedge \tau$
[By the axiom (SVNSSII$_2$)]
$= [\widetilde{F}^I_B(p(e))(f(z*((x*y)*(y*x^2)))) \wedge \widetilde{F}^I_B(p(e))(f(z))] \wedge \tau$
$= f^{-1}\left(\widetilde{F}^I_B\right)(e)(z*((x*y)*(y*x^2))) \wedge f^{-1}\left(\widetilde{F}^I_B\right)(e)(z)] \wedge \tau,$

$f^{-1}\left(\widetilde{F}^F_B\right)(e)(x*y^2) \wedge \sigma$
$= \widetilde{F}^F_B(p(e))(f(x*y^2)) \wedge \sigma$
$= \widetilde{F}^F_B(p(e))(f(x)*f(y^2)) \wedge \sigma$
$\leq [\widetilde{F}^F_B(p(e))(f(z)*(f(x)*f(y))*(f(y)*f(x^2)))) \vee \widetilde{F}^F_B(p(e))(f(z))] \vee \tau$
[By the axiom (SVNSSII$_3$)]
$= [\widetilde{F}^F_B(p(e))(f(z*((x*y)*(y*x^2)))) \vee \widetilde{F}^F_B(p(e))(f(z))] \vee \tau$
$= f^{-1}\left(\widetilde{F}^F_B\right)(e)(z*((x*y)*(y*x^2))) \vee f^{-1}\left(\widetilde{F}^F_B\right)(e)(z)] \vee \tau.$

Thus $f^{-1}\left(\overline{\overline{F}}_B\right)$ satisfies the axioms (SVNSSII$_1$), (SVNSSII$_2$) and (SVNSSII$_3$). So $f^{-1}\left(\overline{\overline{F}}_B\right)$ is a (σ,τ)-SVNSII of X. □

Definition 4.5 (See [27])**.** Let X be a nonempty set, let E be a set of parameters, let $A \subset E$ and let $\overline{\overline{F}}_A \in SVNSS_E(X)$. Then we say that $\overline{\overline{F}}_A$ *has sup property*, if for any subset T of X and each $e \in A$, there is $t_0 \in T$ such that

$$\overline{\overline{F}}_A(e)(t_0)\left\langle \bigvee_{t\in T}\widetilde{F}^T_A(e)(t), \bigvee_{t\in T}\widetilde{F}^I_A(e)(t), \bigwedge_{t\in T}\widetilde{F}^F_A(e)(t)\right\rangle.$$

Proposition 4.6. *Let $f: X \to Y$ be an epiorphism of KU-algebras, let E, E' be two sets of parameters, let $A \subset E$, $B \subset E'$ and let $p: A \to B$ be a bijective mapping. If $\overline{\overline{F}}_A$ is a (σ,τ)-SVNSSII of X having the sup property, then $f\left(\overline{\overline{F}}_A\right)$ is a (σ,τ)-SVNSSII of Y. Moreover, even if f is not an epimorphism, $f\left(\overline{\overline{F}}_A\right)$ is a (σ,τ)-SVNSSII of Y.*

Proof. Suppose $\overline{\overline{F}}_A$ is a (σ,τ)-SVNSSII of X having the sup property. Let $y \in Y$ and let $e' \in B$. Then we get

$$\begin{aligned} f\left(\widetilde{F}^T_A\right)(e')(0) \vee \sigma &= \textstyle\bigvee_{t\in f^{-1}(0)} \widetilde{F}^T_A(p^{-1}(e')) \vee \sigma \\ &= \widetilde{F}^T_A(p^{-1}(e'))(0) \vee \sigma \text{ [Since } f \text{ is an epimorphism]} \\ &\geq \widetilde{F}^T_A(p^{-1}(e'))(x) \wedge \tau \text{ for each } x \in X. \end{aligned}$$

[By the axiom (SVNSI$_1$)]

Similarly, we have

$$f\left(\widetilde{F}^I_{A}\right)(e^{'})(0) \vee \sigma \geq \widetilde{F}^I_{A}(p^{-1}(e^{'}))(x) \wedge \tau \text{ for each } x \in X,$$

$$f\left(\widetilde{F}^F_{A}\right)(e^{'})(0) \vee \sigma \leq \widetilde{F}^F_{A}(p^{-1}(e^{'}))(x) \vee \tau \text{ for each } x \in X.$$

Thus we get

$$f\left(\widetilde{F}^T_{A}\right)(e^{'})(0) \vee \sigma \geq \bigvee_{x \in f^{-1}(y)} \widetilde{F}^T_{A}(p^{-1}(e^{'}))(x) \wedge \tau = f\left(\widetilde{F}^T_{A})(e^{'}\right)(y) \wedge \tau,$$

$$f\left(\widetilde{F}^I_{A}\right)(e^{'})(0) \vee \sigma \geq \bigvee_{x \in f^{-1}(y)} \widetilde{F}^I_{A}(p^{-1}(e^{'}))(x) \wedge \tau = f\left(\widetilde{F}^I_{A})(e^{'}\right)(y) \wedge \tau,$$

$$f\left(\widetilde{F}^F_{A}\right)(e^{'})(0) \vee \sigma \leq \bigwedge_{x \in f^{-1}(y)} \widetilde{F}^F_{A}(p^{-1}(e^{'}))(x) \vee \tau = f\left(\widetilde{F}^F_{A}\right)(e^{'})(y) \vee \tau.$$

So $f\left(\overline{\widetilde{F}}_{A}\right)$ satisfies the axiom (SVNSI$_1$).

Now let $x^{'},\ y^{'},\ z^{'} \in Y$ and let $e^{'} \in B$. Since f is surjective and $\overline{\widetilde{F}_{A}}$ has the sup property, we have there are $x_0 \in f^{-1}(x^{'})$, $y_0 \in f^{-1}(y^{'})$, $z_0 \in f^{-1}(z^{'})$ such that

$$\widetilde{F}^T_{A}(p^{-1}(e^{'}))(x_0) = \bigvee_{t \in f^{-1}(x^{'})} \widetilde{F}^T_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^T_{A}(p^{-1}(e^{'}))(y_0) = \bigvee_{t \in f^{-1}(y^{'})} \widetilde{F}^T_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^T_{A}(p^{-1}(e^{'}))(z_0) = \bigvee_{t \in f^{-1}(z^{'})} \widetilde{F}^T_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^I_{A}(p^{-1}(e^{'}))(x_0) = \bigvee_{t \in f^{-1}(x^{'})} \widetilde{F}^I_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^I_{A}(p^{-1}(e^{'}))(y_0) = \bigvee_{t \in f^{-1}(y^{'})} \widetilde{F}^I_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^I_{A}(p^{-1}(e^{'}))(z_0) = \bigvee_{t \in f^{-1}(z^{'})} \widetilde{F}^I_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^F_{A}(p^{-1}(e^{'}))(x_0) = \bigwedge_{t \in f^{-1}(x^{'})} \widetilde{F}^F_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^F_{A}(p^{-1}(e^{'}))(y_0) = \bigwedge_{t \in f^{-1}(y^{'})} \widetilde{F}^F_{A}(p^{-1}(e^{'}))(t),$$

$$\widetilde{F}^F_{A}(p^{-1}(e^{'}))(z_0) = \bigwedge_{t \in f^{-1}(z^{'})} \widetilde{F}^F_{A}(p^{-1}(e^{'}))(t).$$

Thus we have

$$\begin{aligned} & f\left(\widetilde{F}^T_{A}\right)(e^{'})(x^{'} * {y^{'}}^2) \vee \sigma \\ &= \textstyle\bigvee_{t \in f^{-1}(x^{'} * {y^{'}}^2)} \widetilde{F}^T_{A}(p^{-1}(e^{'}))(t) \vee \sigma \\ &= \widetilde{F}^T_{A}(p^{-1}(e^{'}))(x_0 * y_0^2) \vee \sigma \end{aligned}$$

$$\geq [\widetilde{F}^T_A(p^{-1}(e^{'}))(z_0 * ((x_0 * y_0) * (y_0 * x_0^2))) \wedge \widetilde{F}^T_A(p^{-1}(e^{'}))(z_0)] \wedge \tau$$
[By the axiom (SVNSSII$_1$)]
$$= [\bigvee_{t\in f^{-1}(z^{'}*(x^{'}*y^{'})*(y^{'}*x^{'2})))} \widetilde{F}^T_A(p^{-1}(e^{'}))(t) \wedge \bigvee_{t\in f^{-1}(z)\widetilde{F}^T_A(p^{-1}(e^{'}))(t)}] \wedge \tau$$
$$= \left[f\left(\widetilde{F}^T_A\right)(e^{'})(x^{'} * y^{'}) * (y^{'} * x^{'2}))) \wedge f\left(\widetilde{F}^T_A\right)(e^{'})(z^{'})\right] \wedge \tau,$$

$$f\left(\widetilde{F}^I_A\right)(e^{'})(x^{'} * y^{'2}) \vee \sigma$$
$$= \bigvee_{t\in f^{-1}(x^{'}*y^{'2})} \widetilde{F}^I_A(p^{-1}(e^{'}))(t) \vee \sigma$$
$$= \widetilde{F}^I_A(p^{-1}(e^{'}))(x_0 * y_0^2) \vee \sigma$$
$$\geq [\widetilde{F}^I_A(p^{-1}(e^{'}))(z_0 * ((x_0 * y_0) * (y_0 * x_0^2))) \wedge \widetilde{F}^I_A(p^{-1}(e^{'}))(z_0)] \wedge \tau$$
[By the axiom (SVNSSII$_2$)]
$$= [\bigvee_{t\in f^{-1}(z^{'}*(x^{'}*y^{'})*(y^{'}*x^{'2})))} \widetilde{F}^I_A(p^{-1}(e^{'}))(t) \wedge \bigvee_{t\in f^{-1}(z)\widetilde{F}^I_A(p^{-1}(e^{'}))(t)}] \wedge \tau$$
$$= \left[f\left(\widetilde{F}^I_A\right)(e^{'})(x^{'} * y^{'}) * (y^{'} * x^{'2}))) \wedge f\left(\widetilde{F}^I_A\right)(e^{'})(z^{'})\right] \wedge \tau,$$

$$f\left(\widetilde{F}^F_A\right)(e^{'})(x^{'} * y^{'2}) \wedge \sigma$$
$$= \bigwedge_{t\in f^{-1}(x^{'}*y^{'2})} \widetilde{F}^F_A(p^{-1}(e^{'}))(t) \wedge \sigma$$
$$= \widetilde{F}^F_A(p^{-1}(e^{'}))(x_0 * y_0^2) \wedge \sigma$$
$$\leq [\widetilde{F}^F_A(p^{-1}(e^{'}))(z_0 * ((x_0 * y_0) * (y_0 * x_0^2))) \vee \widetilde{F}^F_A(p^{-1}(e^{'}))(z_0)] \vee \tau$$
[By the axiom (SVNSSII$_3$)]
$$= [\bigwedge_{t\in f^{-1}(z^{'}*(x^{'}*y^{'})*(y^{'}*x^{'2})))} \widetilde{F}^F_A(p^{-1}(e^{'}))(t) \vee \bigwedge_{t\in f^{-1}(z)\widetilde{F}^F_A(p^{-1}(e^{'}))(t)}] \vee \tau$$
$$= \left[f\left(\widetilde{F}^F_A\right)(e^{'})(x^{'} * y^{'}) * (y^{'} * x^{'2}))) \vee f\left(\widetilde{F}^F_A\right)(e^{'})(z^{'})\right] \vee \tau.$$

So $f\left(\widetilde{\widetilde{F}}_A\right)$ satisfies the axioms (SVNSSII$_1$), (SVNSSII$_2$) and (SVNSSII$_3$). Hence $f\left(\widetilde{\widetilde{F}}_A\right)$ is a (σ,τ)-SVNSSII of Y.

Finally, suppose f is not epimorphism. For each $x^{'} \in Y$, let $X_{x^{'}} = f^{-1}(x^{'})$. Then clearly, either $X_{x^{'}} = \varnothing$ or $X_{x^{'}} \neq \varnothing$. f is a homomorphism, we have: for any $x^{'}$, $y^{'}$, $z^{'} \in Y$,

$$X_{z^{'}} * ((X_{x^{'}} * X_{y^{'}}) * (X_{y^{'}} * X_{x^{'2}})) \subset X_{z^{'}*((x^{'}*y^{'})*(y^{'}*x^{'2}))}. \tag{4.5}$$

Suppose $z^{'} * ((x^{'} * y^{'}) * (y^{'} * x^{'2})) \notin Im(f) = f(X)$, i.e., $X_{z^{'}*((x^{'}*y^{'})*(y^{'}*x^{'2}))} = \varnothing$. Then by Definition 4.3, we have

$$f\left(\widetilde{F}^T_A\right)(e^{'})(z^{'} * ((x^{'} * y^{'}) * (y^{'} * x^{'2}))) = 0,$$

$$f\left(\widetilde{F}^I_A\right)(e^{'})(z^{'} * ((x^{'} * y^{'}) * (y^{'} * x^{'2}))) = 0,$$

$$f\left(\widetilde{F}^F_A\right)(e^{'})(z^{'} * ((x^{'} * y^{'}) * (y^{'} * x^{'2}))) = 1.$$

Thus by (4.5), at least one of $x^{'}$, $y^{'}$, $z^{'} \in f(X)$, we get
$$f\left(\widetilde{F}^T_A\right)(e^{'})(x^{'} * y^{'2}) \vee \sigma$$
$$\geq 0$$

$= [\left(\widetilde{F}^T_A\right)(e')z' * ((x' * y') * (y' * x'^2))) \wedge \left(\widetilde{F}^T_A\right)(e')(z')] \wedge \tau,$

$f\left(\widetilde{F}^I_A\right)(e')(x' * y'^2) \vee \sigma$
≥ 0
$= [\left(\widetilde{F}^I_A\right)(e')z' * ((x' * y') * (y' * x'^2))) \wedge \left(\widetilde{F}^I_A\right)(e')(z')] \wedge \tau,$

$f\left(\widetilde{F}^F_A\right)(e')(x' * y'^2) \wedge \sigma$
≤ 1
$= [\left(\widetilde{F}^F_A\right)(e')z' * ((x' * y') * (y' * x'^2))) \vee \left(\widetilde{F}^F_A\right)(e')(z')] \vee \tau.$

So $f\left(\overline{\widetilde{F}}_A\right)$ is a (σ, τ)-SVNSSII of Y. □

5. The product of (σ, τ)-single valued soft sub implicative ideals of a KU-algebra

Definition 5.1. Let X a nonempty set and let $\overline{A} \in SVNS(X)$, $\overline{B} \in SVNS(Y)$. Then the *Cartesian product* $\overline{A} \times \overline{B} = \langle A^T \times B^T, A^I \times B^I, A^F \times B^F \rangle$ of $\overline{A}$ and $\overline{B}$ is a SVNS in $X \times X$ defined as follows: for each $(x, y) \in X \times X$,
$(A^T \times B^T)(x, y) = A^T(x) \wedge B^T(y),\ (A^I \times B^I)(x, y) = A^I(x) \wedge B^I(y),$
$(A^F \times B^F)(x, y) = A^F(x) \vee B^F(y)$, where $A^T \times B^T,\ A^I \times B^I,\ A^F \times B^F \in [0, 1]^{X \times X}$.

Definition 5.2. Let X be a nonempty set, let E be set of parameters and let $A \subset E$. Let $\overline{\widetilde{F}}_A,\ \overline{\widetilde{G}}_A \in SVNSS_E(X)$. Then the *Cartesian product* of $\overline{\widetilde{F}_A}$ and $\overline{\widetilde{F}_B}$, denoted by

$$\overline{\widetilde{F}}_A \times \overline{\widetilde{G}}_A = \left\langle \widetilde{F}^T_A \times \widetilde{G}^T_A, \widetilde{F}^I_A \times \widetilde{G}^I_A, \widetilde{F}^F_A \times \widetilde{G}^F_A \right\rangle$$

is a SVNSS over $E \times E$ defined as follows: for each $(x, y) \in X \times X$ and each $(e, f) \in A \times A$,

$$(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e, f)(x, y) = \widetilde{F}^T_A(e)(x) \wedge \widetilde{G}^T_A(f)(y),$$
$$(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e, f)(x, y) = \widetilde{F}^I_A(e)(x) \wedge \widetilde{G}^I_A(f)(y),$$
$$(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e, f)(x, y) = \widetilde{F}^F_A(e)(x) \vee \widetilde{G}^F_A(f)(y).$$

Remark 5.3. Let X and Y be two KU-algebras. We define $*$ on $X \times Y$ by: for any $(x, y),\ (u, v) \in X \times Y$,

$$(x, y) * (u, v) = (x * u, y * v).$$

Then clearly, $(X \times Y, *, (0, 0))$ is a KU-algebra.

Proposition 5.4. *Let X be a KU-algebra, let E be set of parameters and let $A \subset E$. If $\overline{\widetilde{F}}_A$ and $\overline{\widetilde{G}}_A$ are two (σ, τ)-SVNSSII of X, then $\overline{\widetilde{F}}_A \times \overline{\widetilde{G}}_A$ is a (σ, τ)-SVNSII of $X \times X$.*

Proof. Let $(x, y) \in X \times X$ and let $(e, f) \in A \times A$. Then we have
$(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e, f)(0, 0) \vee \sigma$
$= (\widetilde{F}^T_A(e)(0) \wedge \widetilde{G}^T_A(f)(0)) \vee \sigma$
$= (\widetilde{F}^T_A(e)(0) \vee \sigma) \wedge (\widetilde{G}^T_A(f)(0) \vee \sigma)$

$\geq (\widetilde{F}^T_A(e)(x) \wedge \tau) \wedge (\widetilde{G}^T_A(f)(y) \wedge \tau)$ [By the axiom (SVNSI$_1$)]
$= (\widetilde{F}^T_A(e)(x) \wedge \widetilde{G}^T_A(f)(y)) \wedge \tau$
$= (\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)(x,y) \wedge \tau.$

Similarly, we get

$$(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)(0,0) \vee \sigma \geq (\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)(x,y) \wedge \tau,$$

$$(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)(0,0) \wedge \sigma \leq (\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)(x,y) \vee \tau.$$

Thus $\overline{\widetilde{F}}_A \times \overline{\widetilde{G}}_A$ satisfies the axiom (SVNSI$_1$).

Now let $(x_1,x_2),\ (y_1,y_2),\ (z_1,z_2) \in X \times X$ and let $(e,f) \in A \times A$. Then we get

$$\begin{aligned}
&[(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)((z_1,z_2) * (((x_1,x_2) * (y_1,y_2)) * ((y_1,y_2) * (x_1,x_2)^2))))\\
&\quad \wedge(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)(z_1,z_2)] \wedge \tau\\
&= [(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2)), (z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \wedge(\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)(z_1,z_2)] \wedge \tau\\
&= [(\widetilde{F}^T_A(e)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2))) \wedge (\widetilde{G}^T_A(e)(z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \wedge(\widetilde{F}^T_A(e)(z_1) \wedge \widetilde{G}^T_A(f)(z_2))] \wedge \tau\\
&\leq [\widetilde{F}^T_A(e)(x_1 * y_1^2 \wedge \widetilde{G}^T_A(f)(x_2 * y_2^2] \vee \sigma \text{ [By the axiom (SVNSSII}_1\text{)]}\\
&= (\widetilde{F}^T_A \times \widetilde{G}^T_A)(e,f)((x_1 * x_2), (y_1,y_2)^2) \vee \sigma,
\end{aligned}$$

$$\begin{aligned}
&[(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)((z_1,z_2) * (((x_1,x_2) * (y_1,y_2)) * ((y_1,y_2) * (x_1,x_2)^2))))\\
&\quad \wedge(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)(z_1,z_2)] \wedge \tau\\
&= [(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2)), (z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \wedge(\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)(z_1,z_2)] \wedge \tau\\
&= [(\widetilde{F}^I_A(e)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2))) \wedge (\widetilde{G}^I_A(e)(z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \wedge(\widetilde{F}^I_A(e)(z_1) \wedge \widetilde{G}^I_A(f)(z_2))] \wedge \tau\\
&\leq [\widetilde{F}^I_A(e)(x_1 * y_1^2 \wedge \widetilde{G}^I_A(f)(x_2 * y_2^2] \vee \sigma \text{ [By the axiom (SVNSSII}_2\text{)]}\\
&= (\widetilde{F}^I_A \times \widetilde{G}^I_A)(e,f)((x_1 * x_2), (y_1,y_2)^2) \vee \sigma,
\end{aligned}$$

$$\begin{aligned}
&[(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)((z_1,z_2) * (((x_1,x_2) * (y_1,y_2)) * ((y_1,y_2) * (x_1,x_2)^2))))\\
&\quad \vee(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)(z_1,z_2)] \vee \tau\\
&= [(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2)), (z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \vee(\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)(z_1,z_2)] \vee \tau\\
&= [(\widetilde{F}^F_A(e)(z_1 * ((x_1 * y_1) * (y_1 * x_1^2))) \vee (\widetilde{G}^F_A(e)(z_2 * ((x_2 * y_2) * (y_2 * x_2^2)))\\
&\quad \vee(\widetilde{F}^F_A(e)(z_1) \vee \widetilde{G}^F_A(f)(z_2))] \vee \tau\\
&\geq [\widetilde{F}^F_A(e)(x_1 * y_1^2 \vee \widetilde{G}^F_A(f)(x_2 * y_2^2] \wedge \sigma \text{ [By the axiom (SVNSSII}_3\text{)]}\\
&= (\widetilde{F}^F_A \times \widetilde{G}^F_A)(e,f)((x_1 * x_2), (y_1,y_2)^2) \wedge \sigma.
\end{aligned}$$

Thus $\overline{\widetilde{F}}_A \times \overline{\widetilde{G}}_A$ satisfies the axioms (SVNSSII$_1$), (SVNSSII$_2$) and (SVNSSII$_3$). So $\overline{\widetilde{F}}_A \times \overline{\widetilde{G}}_A$ is a (σ,τ)-SVNSII of $X \times X$. □

6. Algorithm for KU-algebras

Input (X: a set, $*$: a binary operation)

Output ("X is a KU-algebra or not")
Begin
If $X = \varnothing$, then go to (1);
EndIf
If $0 \notin X$, then go to (1);
EndIf
Stop: =false;
$i := 1$;
While $i \leq\mid X \mid$ and not (Stop) do
If $x_i * x_j \neq 0$, then
Stop: = true;
EndIf
$j := 1$
While $i \leq\mid X \mid$ and not (Stop) do
If $(y_j * x_i) * x_i \neq 0$, then
Stop: = true;
EndIf
EndIf
$k := 1$
While $k \leq\mid X \mid$ and not (Stop) do
If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \neq 0$, then
Stop: = true;
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
(1) Output ("X is not a KU-algebra")
Else
Output ('X is a KU-algebra")
EndIf
End

7. Algorithm for KU-ideals

Input (X : a KU-algebra, I : a subset of X);
Output ("I is a KU-ideal of X or not")
Begin
If $I \neq \varnothing$, then go to (1);
EndIf
If $0 \notin I$, then go to (1);
EndIf
Stop: =false; $i := 1$
While $i \leq\mid I \mid$ and not (Stop) do
$j := 1$
While $j \leq\mid I \mid$ and not (Stop) do
$k := 1$

While $k \leq | I |$ and not (Stop) do
If $x_i * ((y_j * z_k) \in I$ and $y_j \in I$, then
If $x_i * z_k \notin I$, then
Stop: = true;
EndIf
EndIf EndIf While
EndIf While
EndIf While
If Stop then
Output ("I is a KU-ideal of X")
Else
(1) Output ("I is not a KU-ideal of X")
EndIf
End

8. Algorithm for a KU-sub implicative ideal

Input (X : a KU-algebra, I : a subset of X);
Output ("I is a KU- sub implicative ideal of X or not")
Begin
If $I = \varnothing$. then go to (1);
EndIf
If $0 \notin I$, then go to (1);
EndIf
Stop: =false;
$i := 1$
While $i \leq | I |$ and not (Stop) do
$j := 1$
While $j \leq | I |$ and not (Stop) do
$k := 1$
While $k \leq | I |$ and not (Stop) do
If $z_k * ((x_i * y_j) * (y_j * x_i^2) \in I$ and $z_k \in I$
If $x_i * y_j^2 \notin I$, then
Stop: = true;
EndIf
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output ("I is a KU -sub implicative ideal of X")
Else
(1) Output ("I is not a KU -sub implicative ideal of X")
End If
End.

9. Conclusions

In the present paper, by using the notion of single valued neutrosophic sub-implicative ideal KU-algebras [28], we defined a (σ, τ)-single valued neutrosophic soft sub implicative ideal of a KU-algebras and investigated some of their useful properties. We think that some definitions proposed and main results obtained by us can be similarly extended to some other fuzzy algebraic systems such as hyper groups, hyper semigroups, hyper rings, etc. It is our hope that this work would other foundations for further study of the theory of $BCK/BCI/KU$-algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis.

In our future study of (σ, τ)-single valued neutrosophic soft sub commutative ideals of KU-algebras, may be the following topics should be considered:

(1) To establish (σ, τ)-single valued neutrosophic soft (s-weakÔÇôstrong) hyper KU-ideals in hyper KU-algebras,

(2) To get more results in (σ, τ)-single valued neutrosophic soft ideals hyper KU-algebras and application.

(3) To consider the structure of (σ, τ)-single valued neutrosophic dot hyper KU-ideals of hyper KU-algebras.

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