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On $\mathcal{I}$-statistical limit superior and limit inferior in neutrosophic fuzzy normed spaces

Ömer Kişi, Rümeysa Akbiyik, Mehmet Gürdal

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Abstract. This article explores the notions of $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior in neutrosophic fuzzy normed spaces. An illustrative example is included to demonstrate the calculation of these points. Additionally, the paper delves into the concepts of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for sequences in neutrosophic fuzzy normed spaces, examining fundamental properties of the sets comprising all $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points of such sequences.

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1. Introduction

The concept of convergence of real sequences was extended to statistical convergence by Fast [1]. Subsequently, this concept was further investigated from the perspective of sequence spaces and connected with summability theory by Fridy [2], Śalát [3], and numerous other researchers. Consequently, the statistical limit superior and limit inferior emerged as important considerations, which were extensively studied by Fridy and Orhan [4].

The concept of ideal convergence was introduced by Kostyrko et al. [5], which generalizes and unifies various notions of sequence convergence, including traditional convergence and statistical convergence. They utilized the notion of an ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ to define this concept. For an extensive view of this topic, one may refer to [6, 7, 8, 9, 10]. In 2001, Demirci [11] introduced the definition of the $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of a real sequence and proved several fundamental properties. The concept of $\mathcal{I}$-statistical convergence was proposed by Savaş and Das [12] as an extension of ideal convergence. It was subsequently further explored by

Debnath and Debnath [13], Debnath and Choudhury [14], Debnath and Rakshit [15], and many other researchers. Later, this concept was further explored by Lahiri and Das [16]. In [17], the authors extended the concepts of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior to $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior, and examined some of their properties for sequences of real numbers.

Recently, statistical convergence, ideal convergence and some of their related concepts for fuzzy numbers have been investigated in [18, 19, 20, 21, 22].

The introduction of Fuzzy Sets (FSs) by Zadeh [23] has profoundly influenced various scientific disciplines since its publication. This concept, crucial for real-life scenarios, has revealed inadequacies in addressing certain problems over time, prompting the emergence of new inquiries. Atanassov [24] pioneered the concept of intuitionistic fuzzy sets to address such cases. Neutrosophic set, a novel iteration of classical set theory, was defined by Smarandache [25]. Statistical convergence of sequences in neutrosophic fuzzy normed spaces was examined by Kirisçi and Şimşek [26]. Ideal convergence of sequences was investigated in neutrosophic fuzzy normed spaces by Kişi [27].

Before delving into the fundamental results of this study, let us embark on a journey through the intricate realm of $\mathcal{I}$-statistical limits within neutrosophic fuzzy normed spaces. By exploring the concepts of $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior, we uncover profound insights into the behavior of sequences within this unique mathematical framework. Through an illuminating example, we demonstrate the practical computation of these limits, offering a tangible grasp of their significance. We observe that our results are analogous to those of Mursaleen et al. [17], though the proofs differ somewhat when addressing these concepts in neutrosophic fuzzy normed spaces. Furthermore, we navigate through the terrain of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points, unraveling their fundamental properties and enriching our understanding of their role in characterizing sequences in neutrosophic fuzzy normed spaces.

2. Definitions and Preliminaries

In this section, we commence by revisiting some fundamental definitions related to neutrosophic fuzzy normed space, statistical convergence, and ideal convergence.

For $K \subset \mathbb{N}$ and $j \in \mathbb{N}$, $\delta_j(K)$ is named jth partial density of K, if

$$\delta_j(K) = \frac{|K \cap \{1, 2, ..., j\}|}{j}.$$

If

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|, \left(\text{i.e., } \delta(K) = \lim_{j\to\infty} \delta_j(K)\right)$$

exists, it is named the *natural density* of K. $\Psi = \{K \subset \mathbb{N} : \delta(K) = 0\}$ is denoted the zero density set.

A sequence (x_n) is said to be *statistically convergent to* ξ, if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0,$$

i.e., $\{n \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \in \Psi$. We demonstrate $st\text{-}\lim x_n = \xi$ or $x_n \overset{st}{\to} \xi$, $(n \to \infty)$. Since the introduction of ideal convergence by Kostyrko et al. [5], there has been a surge of comprehensive research aimed at uncovering applications and furthering the study of summability within classical theories. This exploration has not only deepened our understanding of existing mathematical frameworks but has also paved the way for innovative applications across various disciplines.

Let $\varnothing \neq S$ be a set, and then a non empty class $\mathcal{I} \subseteq P(S)$ is said to be an *ideal* on S, if (i) $\mathcal{I}$ is additive under union, (ii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we find $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial*, if $\mathcal{I} \neq \varnothing$ and $S \notin \mathcal{I}$.

A non-empty family of sets $\mathcal{F}$ is called *filter* on S, if (i) $\varnothing \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$, we get $A \cap B \in \mathcal{F}$, (iii) for every $A \in \mathcal{F}$ and each $B \supseteq A$, we obtain $B \in \mathcal{F}$.

Relationship between ideal and filter is given as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset S : K^c \in \mathcal{I}\},$$

where $K^c = S - K$.

A non-trivial ideal $\mathcal{I}$ is called an *admissible ideal* on S, if it contains all singletons.

A sequence (x_n) is said to be *ideal convergent* to ξ if, for every $\varepsilon > 0$, i.e.

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\} \in \mathcal{I}.$$

By taking $\mathcal{I} = \mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denotes the asymptotic density of the set A, we establish that ideal convergence coincides with statistical convergence.

Throughout this paper, we consider $\mathcal{I}$ as the admissible ideal. Triangular norms (t-norms), introduced by Menger [28], play a crucial role in generalizing the concept of probability distributions with the triangle inequality in terms of metric spaces. Additionally, triangular conorms (t-conorms), recognized as the dual operations of t-norms, are instrumental in fuzzy operations such as intersections and unions. The significance of t-norms and t-conorms extends across various domains, providing essential tools for mathematical modeling and analysis.

Definition 2.1. ([28]) Let $\circledast : [0,1] \times [0,1] \to [0,1]$ be an operation. Then $\circledast$ is called a *continuous t-norm*, if it satisfies following conditions: for any $p,\ q,\ r,\ s \in [0,1]$,

(i) $p \circledast 1 = p$,
(ii) If $p \leq r$ and $q \leq s$, then $p \circledast q \leq r \circledast s$,
(iii) $\circledast$ is continuous,
(iv) $\circledast$ is associative and commutative.

Definition 2.2. ([28]) Let $\otimes : [0,1] \times [0,1] \to [0,1]$ be an operation. Then $\otimes$ is said to be a *continuous t-conorm*, if it satisfies following conditions: for any $p,\ q,\ r,\ s \in [0,1]$,

(i) $p \otimes 0 = p$,
(ii) if $p \leq r$ and $q \leq s$, then $p \otimes q \leq r \otimes s$,
(iii) $\otimes$ is continuous,
(iv) $\otimes$ is associative and commutative.

Definition 2.3. ([26]) Let F be a vector space, $\mathcal{N} = \{\langle u, \Theta(u), \Phi(u), \Omega(u)\rangle : u \in F\}$ be a normed space such that $\mathcal{N} : F \to F \times [0,1] \times [0,1] \times [0,1]$. Then $\mathcal{N} = (\Theta, \Phi, \Omega)$ is called a *neutrosophic norm*, if following conditions hold: for all $u,\ v \in F$ and $\lambda,\ \mu > 0$ and for each $\sigma \neq 0$,

(i) $0 \leq \Theta(u, \lambda) \leq 1$, $0 \leq \Phi(u, \lambda) \leq 1$, $0 \leq \Omega(u, \lambda) \leq 1$,
(ii) $\Theta(u, \lambda) + \Phi(u, \lambda) + \Omega(u, \lambda) \leq 3$,
(iii) $\Theta(u, \lambda) = 1$ iff $u = 0$,
(iv) $\Theta(\sigma u, \lambda) = \Theta\left(u, \frac{\lambda}{|\sigma|}\right)$,
(v) $\Theta(u, \mu) \circledast \Theta(v, \lambda) \leq \Theta(u + v, \mu + \lambda)$,
(vi) $\Theta(u, .)$ is non-decreasing continuous function,
(vii) $\lim_{\lambda \to \infty} \Theta(u, \lambda) = 1$,
(viii) $\Phi(u, \lambda) = 0$ iff $u = 0$,
(ix) $\Phi(\sigma u, \lambda) = \Phi\left(u, \frac{\lambda}{|\sigma|}\right)$,
(x) $\Phi(u, \mu) \otimes \Phi(v, \lambda) \geq \Phi(u + v, \mu + \lambda)$,
(xi) $\Phi(u, .)$ is non-decreasing continuous function,
(xii) $\lim_{\lambda \to \infty} \Phi(u, \lambda) = 0$,
(xiii) $\Omega(u, \lambda) = 0$ iff $u = 0$,
(xiv) $\Omega(\sigma u, \lambda) = \Omega\left(u, \frac{\lambda}{|\sigma|}\right)$,
(xv) $\Omega(u, \mu) \otimes \Omega(v, \lambda) \geq \Omega(u + v, \mu + \lambda)$,
(xvi) $\Omega(u, .)$ is non-decreasing continuous function,
(xviii) $\lim_{\lambda \to \infty} \Omega(u, \lambda) = 0$,
(xix) if $\lambda \leq 0$, then $\Theta(u, \lambda) = 0$, $\Phi(u, \lambda) = 1$ and $\Omega(u, \lambda) = 1$.

$V = (F, \mathcal{N}, \circledast, \otimes)$ is said to be a *neutrosophic fuzzy normed space*.

Definition 2.4. ([26]) A sequence (x_m) is said to be *statistically convergent to* $\xi \in F$ with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω), if for each $\lambda > 0$ and $\varepsilon > 0$, the set

$$P_\varepsilon := \{m \leq n : \Theta(x_m - \xi, \lambda) \leq 1 - \varepsilon \text{ and } \Phi(x_m - \xi, \lambda) \geq \varepsilon,\, \Omega(x_m - \xi, \lambda) \geq \varepsilon\}$$

has natural density zero, i.e., $d(P_\varepsilon) = 0$ or

$$\lim_{n \to \infty} \frac{1}{n} |\{m \leq n : \Theta(x_m - \xi, \lambda) \leq 1 - \varepsilon \text{ or } \Phi(x_m - \xi, \lambda) \geq \varepsilon,\, \Omega(x_m - \xi, \lambda) \geq \varepsilon\}| = 0.$$

It is denoted by $S_{\mathcal{N}}$-$\lim x_m = \xi$ or $x_k \to \xi\,(S_{\mathcal{N}})$.

Definition 2.5. ([27]) A sequence (x_m) is said to be *ideal convergent to* $\xi \in F$ with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω), if for each $\varepsilon > 0$ and $\lambda > 0$,

$$\{m \in \mathbb{N} : \Theta(x_m - \xi, \lambda) \leq 1 - \varepsilon \text{ or } \Phi(x_m - \xi, \lambda) \geq \varepsilon,\, \Omega(x_m - \xi, \lambda) \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}_{\mathcal{N}}$-$\lim x_m = \xi$ or $x_m \to \xi\,(\mathcal{I}_{\mathcal{N}})$.

Example 2.6. Let $(F, \|.\|)$ be a neutrosophic norm. For all $a, b \in [0, 1]$, take the TN $a \circledast b = ab$ and the TC $a \otimes b = \min\{a + b, 1\}$. For all $x \in F$ and every $\lambda > 0$, we consider $\Theta(x, \lambda) = \frac{\lambda}{\lambda + \|x\|}$, $\Phi(x, \lambda) = \frac{\|x\|}{\lambda + \|x\|}$ and $\Omega(x, \lambda) = \frac{\|x\|}{\lambda}$. Then $(F, \mathcal{N}, \circledast, \otimes)$ is an neutrosophic fuzzy normed space.

3. Main Results

In this section, we establish the concepts of limit point, $\mathcal{I}$-statistical limit point, $\mathcal{I}$-statistical cluster point, $\mathcal{I}$-statistical limit superior, and $\mathcal{I}$-statistical limit inferior in neutrosophic fuzzy normed spaces. We illustrate these concepts with an example, showcasing the computation process in a neutrosophic framework.

Definition 3.1. Let $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ be a neutrosophic fuzzy normed space. Then $w_0 \in Y$ is called a *limit point* of the sequence $w = (w_s)$ with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω), provided that there is a subsequence of w that converges to w_0 with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω).

Let $L_{(\Theta,\Phi,\Omega)}(w)$ denotes the set of all limit points of the sequence w with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω).

Let $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ be a neutrosophic fuzzy normed space and $w = (w_s)$ be a sequence in Y. If $d^{\mathcal{I}}(Q) = 0$, $\{w\}_Q$ is termed a *subsequence of $\mathcal{I}$-asymptotic density zero* or an *$\mathcal{I}$-thin subsequence* of w. Conversely, $\{w\}_Q$ is called an *$\mathcal{I}$-nonthin subsequence* of w, if Q does not have $\mathcal{I}$-asymptotic density zero. In other words, this occurs, if $d^{\mathcal{I}}(Q)$ is a positive number or if Q lacks $\mathcal{I}$-asymptotic density.

Definition 3.2. Consider $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ as a neutrosophic fuzzy normed space. Then an element $\xi \in Y$ is termed an *$\mathcal{I}$-statistical limit point* of the sequence $w = (w_s)$ with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω), if there exists an $\mathcal{I}$-nonthin subsequence of w that converges to ξ in terms of the neutrosophic fuzzy norm (Θ, Φ, Ω). In such instances, ξ is referred to as a $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-limit point of sequence $w = (w_s)$. Let $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ denote the set of all $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-limit points of the sequence w.

Next, we provide an example of an $\mathcal{I}$-statistical limit point in a neutrosophic fuzzy normed space.

Example 3.3. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Consider the neutrosophic fuzzy normed space $(\mathbb{R}, \Theta, \Phi, \Omega, \circledast, \otimes)$ from Example 2.6. We define a sequence $w = (w_s)$ as follows:

$$w_s = \begin{cases} 1 & \text{if } s \text{ is odd} \\ 0 & \text{if } s \text{ is even.} \end{cases}$$

Let Q be the set of all odd numbers. Then $d(Q) = \frac{1}{2}$. Since $\mathcal{I}$ is an admissible ideal, $d^{\mathcal{I}}(Q) = \frac{1}{2}$. Thus $\{w\}_Q$ is an $\mathcal{I}$-nonthin subsequence of w that converges to 1 in terms of the neutrosophic fuzzy norm (Θ, Φ, Ω). So 1 is an $\mathcal{I}$-statistical limit point of the sequence w.

Definition 3.4. Let $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ be a neutrosophic fuzzy normed space. Then $\xi \in Y$ is called to be an *$\mathcal{I}$-statistical cluster point* of the sequence $w = (w_s)$ in Y with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω), provided that for every $\rho > 0$ and $b \in (0, 1)$, the set

$$\{s : \Theta(w_s - \xi; \rho) > 1 - b \text{ or } \Phi(w_s - \xi; \rho) < b,\ \Omega(w_s - \xi; \rho) < b\}$$

does not have $\mathcal{I}$-asymptotic density zero.

In this case, we say that ξ is a *$\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-cluster point* of the sequence w. Let $\Gamma^{\mathcal{I}}_{(\Theta,\Phi,\Omega)}(w)$ denotes the set of all $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-cluster points of the sequence w.

Now, we provide an example of an $\mathcal{I}$-statistical cluster point in a neutrosophic fuzzy normed space.

Example 3.5. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Consider the neutrosophic fuzzy normed space $(\mathbb{R}, \Theta, \Phi, \Omega, \circledast, \otimes)$ from Example 2.6. We define a sequence $w = (w_s)$

as follows:

$$w_s = \begin{cases} 1 & \text{if } s \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

Let Q be the set of all perfect squares. Then $d(Q) = 0$. Since $\mathcal{I}$ is an admissible ideal, $d^{\mathcal{I}}(Q) = 0$. Now

$$\{s : \Theta(w_s - 0; \rho) > 1 - b \text{ or } \Phi(w_s - 0; \rho) < b,\, \Omega(w_s - 0; \rho) < b\} = \mathbb{N} \setminus Q$$

for each $\rho > 0$ and $b \in (0, 1)$. Since $d^{\mathcal{I}}(\mathbb{N} \setminus Q) = 1$,

$$d^{\mathcal{I}}(\{s : \Theta(w_s - 0; \rho) > 1 - b \text{ or } \Phi(w_s - 0; \rho) < b,\ \Omega(w_s - 0; \rho) < b\}) \neq 0.$$

Thus 0 is an $\mathcal{I}$-statistical cluster point of the sequence w.

Remark 3.6. The set $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$ of all $\mathcal{I}$-statistical limit points of a sequence $w = (w_s)$ in Y may not be equal to the set $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$ of all $\mathcal{I}$-statistical cluster points of $w = (w_s)$ in Y. To show this we cite the following example.

Example 3.7. Let $w = (w_s)$ be a sequence in Y, defined by $w_s = \frac{1}{m}$, where $s = 2^{m-1}(2t + 1)$; i.e., $m - 1$ is the power of 2 in the prime factorization of s.

Clearly for each m,

$$d\left(\left\{s : w_s = \frac{1}{m}\right\}\right) = \frac{1}{2^m} > 0.$$

Since $\mathcal{I}$ is admissible, we have $d_{\mathcal{I}}\left(\left\{s : w_s = \frac{1}{m}\right\}\right) = \frac{1}{2^m} > 0$. Then $\frac{1}{m} \in \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$. Also, $d_{\mathcal{I}}\left(\left\{s : 0 < w_s < \frac{1}{m}\right\}\right) = 2^{-m}$. Thus $0 \in \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$ and we have $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)} = \{0\} \cup \left\{\frac{1}{m}\right\}_{m=1}^{\infty}$. Now we claim that $0 \notin \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$. To establish our claim, it is sufficient to show that, if $\{w\}_M$ is a subsequence converging to zero, then $d_{\mathcal{I}}(M) = 0$. For this, note that for each $m \in \mathbb{N}$, we have

$$\begin{aligned} |M(n)| &= \left|\left\{s \in M : s \leq n,\ w_s \geq \tfrac{1}{m}\right\}\right| + \left|\left\{s \in M : s \leq n,\ w_s < \tfrac{1}{m}\right\}\right| \\ &\leq O(1) + \left|\left\{s \in M : s \leq n,\ w_s < \tfrac{1}{m}\right\}\right| \leq O(1) + \tfrac{n}{2^m}. \end{aligned}$$

Then $d_{\mathcal{I}}(M) \leq \frac{1}{2^m}$. Since m is arbitrary, we have $d_{\mathcal{I}}(M) = 0$. Thus $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)} \neq \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}$.

Notation 3.8. In a neutrosophic fuzzy normed space $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$, if

$$\mathcal{I} = \mathcal{I}_{\text{fin}} = \{A \subset \mathbb{N} : |A| < \infty\},$$

then the concepts of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points in Y coincide with the concepts of statistical limit points and statistical cluster points in Y with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω). Thus in a neutrosophic fuzzy normed space $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points generalize the notions of statistical limit points and statistical cluster points, respectively.

Theorem 3.9. *Let $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ be a neutrosophic fuzzy normed space. Then for a sequence $w = (w_s)$ in Y, we have $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq L_{(\Theta,\Phi,\Omega)}(w)$.*

Proof. Let ϖ be an arbitrary element in $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Then there exists a subsequence of $\{w_{s_j}\}_{j\in\mathbb{N}}$ of w such that $\mathcal{N}\text{-}\lim_{j\to\infty} w_{s_j} = \varpi$ and $d^{\mathcal{I}}(\{s_j : j\in\mathbb{N}\}) \neq 0$. Let $\rho > 0$ and $b \in (0,1)$ be given. Since $\mathcal{N}\text{-}\lim_{j\to\infty} w_{s_j} = \varpi$,

$$E = \left\{s_j : \Theta\left(w_{s_j} - \varpi;\rho\right) \leq 1-b \text{ or } \Phi\left(w_{s_j} - \varpi;\rho\right) \geq b,\, \Omega\left(w_{s_j} - \varpi;\rho\right) \geq b\right\}$$

is a finite set. Also,

$$\begin{aligned}&\{s : \Theta(w_s - \varpi;\rho) > 1-b \text{ or } \Phi(w_s - \varpi;\rho) < b,\, \Omega(w_s - \varpi;\rho) < b\}\\ &\supset \{s_j : j\in\mathbb{N}\}\setminus E.\\ &\Rightarrow K = \{s_j : j\in\mathbb{N}\}\\ &\subset \{s : \Theta(w_s - \varpi;\rho) > 1-b \text{ or } \Phi(w_s - \varpi;\rho) < b,\, \Omega(w_s - \varpi;\rho) < b\} \cup E.\end{aligned}$$

Now if

$$d^{\mathcal{I}}(\{s : \Theta(w_s - \varpi;\rho) > 1-b \text{ or } \Phi(w_s - \varpi;\rho) < b,\, \Omega(w_s - \varpi;\rho) < b\}) = 0,$$

then we have $d^{\mathcal{I}}(K) = 0$, which is a contradiction. Thus ϖ is a $\mathcal{I}$-statistical cluster point of w. Since $\varpi \in \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ is arbitrary, $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$.
Now, we demonstrate that $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq L_{(\Theta,\Phi,\Omega)}(w)$. Let $\varpi \in \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Also, $\rho > 0$ and $b \in (0,1)$ be given. Then the set

$$Q = \{s : \Theta(w_s - \varpi;\rho) > 1-b \text{ or } \Phi(w_s - \varpi;\rho) < b,\, \Omega(w_s - \varpi;\rho) < b\}$$

does not have $\mathcal{I}$-asymptotic density zero. Thus Q is an infinite set of $\mathbb{N}$. So we can write $Q = \{s_j : s_1 < s_2 < ...\}$. And we have a subsequence $\{w\}_Q$ of w which converges to ϖ with respect to the neutrosophic fuzzy norm (Θ,Φ,Ω). Hence $\varpi \in L_{(\Theta,\Phi,\Omega)}(w)$. Therefore $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subseteq L_{(\Theta,\Phi,\Omega)}(w)$. □

Theorem 3.10. *Let $(Y,\Theta,\Phi,\Omega,\circledast,\otimes)$ be a neutrosophic fuzzy normed space. Let $w = (w_s)$ and $q = (q_s)$ be two sequences in Y such that $d^{\mathcal{I}}(\{s : w_s \neq q_s\}) = 0$. Then $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$ and $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$.*

Proof. Let $\tau \in \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Also, let $\rho > 0$ and $b \in (0,1)$ be given. Then

$$\{s : \Theta(w_s - \tau;\rho) > 1-b \text{ or } \Phi(w_s - \tau;\rho) < b,\, \Omega(w_s - \tau;\rho) < b\}$$

does not have $\mathcal{I}$-asymptotic density zero. Assume that $B = \{s\in\mathbb{N} : w_s = q_s\}$. Then $d^{\mathcal{I}}(B) = 1$. Thus,

$$\{s : \Theta(w_s - \tau;\rho) > 1-b \text{ or } \Phi(w_s - \tau;\rho) < b,\, \Omega(w_s - \tau;\rho) < b\} \cap B$$

does not have $\mathcal{I}$-asymptotic density zero. Consequently, $\tau \in \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. Since $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$ is arbitrary, $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subset \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. By the symmetry, we obtain $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q) \subset \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. As a result, $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$.

Now, we prove that $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. Let $\varrho \in \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Then w has $\mathcal{I}$-nonthin subsequence (w_{s_j}) that converges to ϱ with respect to the neutrosophic fuzzy norm (Θ,Φ,Ω). Let $Q = \{s_j : j\in\mathbb{N}\}$. Since $d^{\mathcal{I}}(\{s_j : w_{s_j} \neq q_{s_j}\}) = 0$, we have $d^{\mathcal{I}}(\{s_j : w_{s_j} = q_{s_j}\}) \neq 0$. Consequently for the later set, we have an $\mathcal{I}$-nonthin subsequence $\{q\}_{Q'}$ of $\{q\}_Q$ that converges to ϱ with respect to the neutrosophic fuzzy norm (Θ,Φ,Ω). Thus $\varrho \in \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. Since $\varrho \in \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ is arbitrary,

$\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \subset \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. By the symmetry, we obtain $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q) \subset \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. So $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q)$. □

Example 3.11. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Consider the neutrosophic fuzzy normed space $(\mathbb{R}, \Theta, \Phi, \Omega, \circledast, \otimes)$ in Example 2.6. We define sequences $w = (w_s)$ and $q = (q_s)$ as follows:

$$w_s = \begin{cases} 1 & \text{if } s \text{ is a perfect square} \\ 0 & \text{otherwise} \end{cases}$$

and

$$q_s = \begin{cases} 2 & \text{if } s \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

Let Q be the set of all perfect squares. Then $d(Q) = 0$. Since $\mathcal{I}$ is an admissible ideal, $d^{\mathcal{I}}(Q) = 0$. Thus

$$d^{\mathcal{I}}(\{s : w_s \neq q_s\}) = d^{\mathcal{I}}(Q) = 0.$$

Obviously, $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q) = \{0\}$ and $\Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \Lambda^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(q) = \{0\}$.

Theorem 3.12. *Let $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ be a neutrosophic fuzzy normed space and (w_s) be a sequence in Y. Then $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ is a closed subset of w.*

Proof. If $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) = \varnothing$, then there is nothing to prove. We assume $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w) \neq \varnothing$. Obviously, it is sufficient to prove that $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ contains all its limit points. Let ξ be a limit point of $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Also, let $\rho > 0$ and $b \in (0,1)$ be given. Choose $0 < \sigma < 1$ such that $(1-\sigma) \circledast (1-\sigma) > 1-b$ and $\sigma \otimes \sigma < b$. Then $B\left(\xi, \sigma, \frac{\rho}{2}\right) \cap \left(\Gamma^{\mathcal{I}}_{(\Theta,\Phi,\Omega)}(w) \setminus \{\xi\}\right) \neq \varnothing$. Choose $\alpha \in B\left(\xi, \sigma, \frac{\rho}{2}\right) \cap \left(\Gamma^{\mathcal{I}}_{(\Theta,\Phi,\Omega)}(w) \setminus \{\xi\}\right)$. Since $\alpha \in \Gamma^{\mathcal{I}}_{(\Theta,\Phi,\Omega)}(w)$,

$$d^{\mathcal{I}}\left(\left\{s \in \mathbb{N} : \Theta\left(w_s - \alpha; \frac{\rho}{2}\right) > 1-\sigma \text{ or } \Phi\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma,\ \Omega\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma\right\}\right) \neq 0.$$

Now, we prove that

$$\left\{s \in \mathbb{N} : \Theta\left(w_s - \alpha; \tfrac{\rho}{2}\right) > 1-\sigma \text{ or } \Phi\left(w_s - \alpha; \tfrac{\rho}{2}\right) < \sigma,\ \Omega\left(w_s - \alpha; \tfrac{\rho}{2}\right) < \sigma\right\}$$
$$\subset \left\{s \in \mathbb{N} : \Theta(w_s - \xi; \rho) > 1-b \text{ or } \Phi(w_s - \xi; \rho) < b,\ \Omega(w_s - \xi; \rho) < b\right\}.$$

Let

$$s \in \left\{s \in \mathbb{N} : \Theta\left(w_s - \alpha; \frac{\rho}{2}\right) > 1-\sigma \text{ or } \Phi\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma,\ \Omega\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma\right\}.$$

Then $\Theta\left(w_s - \alpha; \frac{\rho}{2}\right) > 1-\sigma$ or $\Phi\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma$, $\Omega\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma$. Since $\alpha \in B\left(\xi, \sigma, \frac{\rho}{2}\right)$, $\Theta\left(\xi - \alpha; \frac{\rho}{2}\right) > 1-\sigma$ or $\Phi\left(\xi - \alpha; \frac{\rho}{2}\right) < \sigma$, $\Omega\left(\xi - \alpha; \frac{\rho}{2}\right) < \sigma$. Thus

$$\begin{aligned}
&\Theta(w_s - \xi; \rho) \geq \Theta\left(w_s - \alpha; \tfrac{\rho}{2}\right) \circledast \Theta\left(\alpha - \xi; \tfrac{\rho}{2}\right) > (1-\sigma) \circledast (1-\sigma) > 1-b, \\
&\Phi(w_s - \xi; \rho) \leq \Phi\left(w_s - \alpha; \tfrac{\rho}{2}\right) \circledast \Phi\left(\alpha - \xi; \tfrac{\rho}{2}\right) < \sigma \otimes \sigma < b, \\
&\Omega(w_s - \xi; \rho) \leq \Omega\left(w_s - \alpha; \tfrac{\rho}{2}\right) \circledast \Omega\left(\alpha - \xi; \tfrac{\rho}{2}\right) < \sigma \otimes \sigma < b.
\end{aligned}$$

Since

$$s \in \left\{s \in \mathbb{N} : \Theta\left(w_s - \alpha; \frac{\rho}{2}\right) > 1-\sigma \text{ or } \Phi\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma,\ \Omega\left(w_s - \alpha; \frac{\rho}{2}\right) < \sigma\right\}$$

is arbitrary,

$$\left\{s \in \mathbb{N} : \Theta\left(w_s - \alpha; \tfrac{\rho}{2}\right) > 1 - \sigma \text{ or } \Phi\left(w_s - \alpha; \tfrac{\rho}{2}\right) < \sigma,\, \Omega\left(w_s - \alpha; \tfrac{\rho}{2}\right) < \sigma\right\}$$

$$\subset \left\{s \in \mathbb{N} : \Theta\left(w_s - \xi; \rho\right) > 1 - b \text{ or } \Phi\left(w_s - \xi; \rho\right) < b,\, \Omega\left(w_s - \xi; \rho\right) < b\right\}.$$

So we get

$$d^{\mathcal{I}}\left(\left\{s \in \mathbb{N} : \Theta\left(w_s - \xi; \rho\right) > 1 - b \text{ or } \Phi\left(w_s - \xi; \rho\right) < b,\, \Omega\left(w_s - \xi; \rho\right) < b\right\}\right) \neq 0.$$

Hence $\xi \in \Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$. Therefore $\Gamma^{S(\mathcal{I})}_{(\Theta,\Phi,\Omega)}(w)$ is a closed subset of w. □

Definition 3.13. A sequence $w = (w_s)$ in a neutrosophic fuzzy normed space $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$ is said to be $\mathcal{I}$-statistically bounded, if there exists some $\rho > 0$ and $b \in (0, 1)$ such that the set

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s; \rho\right) > 1 - b \text{ or } \Phi\left(w_s; \rho\right) < b,\, \Omega\left(w_s; \rho\right) < b\right\}\right| > \rho\right\} \in \mathcal{I}.$$

Now, we examine the concepts of $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior for a real number sequence in neutrosophic fuzzy normed spaces $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$. For a real sequence $w = (w_s)$ let $B_w^{(\Theta,\Phi,\Omega)}$ denote the set

$$B_w^{(\Theta,\Phi,\Omega)} := \left\{b \in (0,1) : \left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s; t\right) < 1 - b \text{ or } \Phi\left(w_s; t\right) > b,\, \Omega\left(w_s; t\right) > b\right\}\right| > \rho\right\} \notin \mathcal{I}\right\}.$$

Similarly

$$A_w^{(\Theta,\Phi,\Omega)} := \left\{a \in (0,1) : \left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s; t\right) > 1 - a \text{ or } \Phi\left(w_s; t\right) < a,\, \Omega\left(w_s; t\right) < a\right\}\right| > \rho\right\} \notin \mathcal{I}\right\}.$$

If w is a real number sequence, then the $\mathcal{I}$-statistical limit superior of w with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω) is defined by

$$\mathcal{I} - st_{(\Theta,\Phi,\Omega)} - \limsup w := \begin{cases} \sup B_w^{(\Theta,\Phi,\Omega)} & \text{if } B_w^{(\Theta,\Phi,\Omega)} \neq \varnothing \\ 0 & \text{if } B_w^{(\Theta,\Phi,\Omega)} = \varnothing. \end{cases}$$

And the $\mathcal{I}$-statistical limit inferior of w with respect to the neutrosophic fuzzy norm (Θ, Φ, Ω) is defined by

$$\mathcal{I} - st_{(\Theta,\Phi,\Omega)} - \liminf w := \begin{cases} \inf A_w^{(\Theta,\Phi,\Omega)} & \text{if } A_w^{(\Theta,\Phi,\Omega)} \neq \varnothing \\ 1 & \text{if } A_w^{(\Theta,\Phi,\Omega)} = \varnothing. \end{cases}$$

Example 3.14. A straightforward example will aid in elucidating the recently defined concepts. Let the sequence $w = (w_s)$ be defined by

$$w_s := \begin{cases} 2s & \text{if } s \text{ is an odd square,} \\ -1 & \text{if } s \text{ is an even square,} \\ 1/2 & \text{if } s \text{ is an odd nonsquare,} \\ 0 & \text{if } s \text{ is an even nonsquare.} \end{cases}$$

Let $\Theta(w_s;t)=\frac{t}{t+|w_s|}$ and $\Phi(w_s;t)=\frac{|w_s|}{t+|w_s|}$, $\Omega(w_s;t)=\frac{|w_s|}{t}$.
The sequence mentioned above is evidently unbounded concerning (Θ,Φ,Ω). However, it is $\mathcal{I}$-statistically bounded in regard to (Θ,Φ,Ω). For this,

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta(w_s;t_0)<1-b \text{ or } \Phi(w_s;t_0)>b,\,\Omega(w_s;t_0)>b\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\tfrac{t_0}{t_0+|w_s|}<1-b \text{ or } \tfrac{|w_s|}{t_0+|w_s|}>b,\ \tfrac{|w_s|}{t_0}>b\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:|w_s|>\tfrac{bt_0}{1-b} \text{ or } |w_s|>bt_0\right\}\right|>\rho\right\}.$$

Since $0<b<1$, $\frac{1}{b}-1>0$. Choose $t_0=\frac{1-b}{3b}$ or $t_0=\frac{1}{3b}$. Then $t_0>0$ and

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta(w_s;t_0)<1-b \text{ or } \Phi(w_s;t_0)>b,\,\Omega(w_s;t_0)>b\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:|w_s|>\tfrac{b}{1-b}.\tfrac{1-b}{3b}=\tfrac{1}{3} \text{ or } |w_s|>b.\tfrac{1}{3b}=\tfrac{1}{3}\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:|w_s|>\tfrac{1}{3}\right\}\right|>\rho\right\}\in\mathcal{I}.$$

Thus it is $\mathcal{I}$-statistically bounded with respect to (μ,v).
To find $B_w^{(\Theta,\Phi,\Omega)}$, we have to find those $b\in(0,1)$ such that

$$\left\{b\in(0,1):\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta(w_s;t)<1-b \text{ or } \Phi(w_s;t)>b,\,\Omega(w_s;t)>b\right\}\right|>\rho\right\}\notin\mathcal{I}\right\}.$$

Now,

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta(w_s;t)<1-b \text{ or } \Phi(w_s;t)>b,\,\Omega(w_s;t)>b\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\tfrac{t}{t+|w_s|}<1-b \text{ or } \tfrac{|w_s|}{t+|w_s|}>b,\ \tfrac{|w_s|}{t}>b\right\}\right|>\rho\right\}$$

$$=\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:|w_s|>\tfrac{bt}{1-b},\ |w_s|>bt\right\}\right|>\rho\right\}.$$

We can easily choose any $t>0$ as $t<\frac{1}{3}\left(\frac{1}{b}-1\right)$ for $0<b<1$, and $t>0$ as $t<\frac{1}{3b}$ so that

$$0<\frac{bt}{1-b}<\frac{b}{1-b}.\frac{1-b}{3b}=\frac{1}{3} \text{ and } 0<bt<\frac{1}{3}.$$

So

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{s\le m:|w_s|>r_1=\frac{bt}{1-b} \text{ and } |w_s|>r_2=bt\right\}\right|>\rho\right\}$$

and by the above condition $r_1,r_2\in(0,1)$. Now the number of members of the sequence which satisfy the above condition is always greater than $s-\frac{s}{2}$ or $s-\frac{s-1}{2}$ for the case s is even or odd, respectively. Hence

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{s\le m:|w_s|>r_1=\frac{bt}{1-b} \text{ and } |w_s|>r_2=bt\right\}\right|>\rho\right\}\notin\mathcal{I}$$

for all $b\in(0,1)$. Therefore

$$B_w^{(\Theta,\Phi,\Omega)}=(0,1),$$

and

$$\mathcal{I}-st_{(\Theta,\Phi,\Omega)}-\limsup w=1.$$

The above sequence has two subsequences

$$w = (w_{s_i}) \quad \text{where} \quad w_{s_i} = 1 \quad \text{for each} \quad s_i \in \{3, 5, 7, 11, 13, \dots\}$$

and

$$w = \left(w_{s_j}\right) \quad \text{where} \quad w_{s_j} = 0 \quad \text{for each} \quad s_j \in \{2, 6, 8, 10, 12, \dots\}$$

$i, j \in \mathbb{N}$; which are of positive density and clearly convergent to 1 and 0, respectively. Therefore w is not $\mathcal{I}$-statistically convergent. Similarly, we have

$$A_w^{(\Theta,\Phi,\Omega)} = (0, 1),$$

and

$$\mathcal{I} - st_{(\Theta,\Phi,\Omega)} - \liminf w = 0$$

Hence the set of $\mathcal{I}$-statistical cluster points of w is $\{0, 1\}$, where $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\liminf w =$ least element and $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\limsup w =$ greatest element of the above set.

Theorem 3.15. *Let* $b = \mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\limsup w$*. Then for every positive numbers* t *and* γ*,*

$$\begin{aligned}&\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b + \gamma \text{ or } \Phi(w_s; t) > b - \gamma,\right.\right.\\ &\qquad \left.\left.\Omega(w_s; t) > b - \gamma\}\right| > \rho\right\} \notin \mathcal{I} \text{ and}\\ &\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b - \gamma \text{ or } \Phi(w_s; t) > b + \gamma,\right.\right.\\ &\qquad \left.\left.\Omega(w_s; t) > b + \gamma\}\right| > \rho\right\} \in \mathcal{I}.\end{aligned} \tag{3.1}$$

Conversely, if (3.1) holds for every positive t *and* γ*, then* $b = \mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\limsup w$*.*

Proof. Let $b = \mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\limsup w$, where b be finite. Then

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b \text{ or } \Phi(w_s; t) > b,\ \Omega(w_s; t) > b\}\right| > \rho\right\} \notin \mathcal{I}. \tag{3.2}$$

Since $\Theta(w_s; t) < 1 - b + \gamma$ or $\Phi(w_s; t) > b - \gamma$, $\Omega(w_s; t) > b - \gamma$ for every s and for any t, $\gamma > 0$,

$$\begin{aligned}&\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b + \gamma \text{ or } \Phi(w_s; t) > b - \gamma,\right.\right.\\ &\qquad \left.\left.\Omega(w_s; t) > b - \gamma\}\right| > \rho\right\} \notin \mathcal{I}.\end{aligned}$$

Now applying the definition of $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$- $\limsup w$ we have $1-b$ as the least value and b as the greatest value satisfying (3.2).
Now if possible,

$$\Theta(w_s; t) < 1 - b - \gamma \text{ or } \Phi(w_s; t) > b + \gamma,\ \Omega(w_s; t) > b + \gamma \text{ for some } \gamma > 0.$$

Then $1 - b - \gamma$ and $b + \gamma$ are another values with $1 - b - \gamma < 1 - b$ and $b + \gamma > b$ which satisfies (3.2). This observation contradicts the fact that $1 - b$ and b are least and greatest values, respectively, which satisfies the above condition. Thus

$$\begin{aligned}&\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b - \gamma \text{ or } \Phi(w_s; t) > b + \gamma,\right.\right.\\ &\qquad \left.\left.\Omega(w_s; t) > b + \gamma\}\right| > \rho\right\} \in \mathcal{I} \text{ for every } \gamma > 0.\end{aligned}$$

Conversely, if (3.1) holds for every positive t and γ, then

$$\begin{aligned}&\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{s \leq m : \Theta(w_s; t) < 1 - b + \gamma \text{ or } \Phi(w_s; t) > b - \gamma,\right.\right.\\ &\qquad \left.\left.\Omega(w_s; t) > b - \gamma\}\right| > \rho\right\} \notin \mathcal{I},\end{aligned}$$

and

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) < 1-b-\gamma \text{ or } \right.\right. \\ \left.\left. \Phi(w_s;t) > b+\gamma,\, \Omega(w_s;t) > b+\gamma\}\right| > \rho\right\} \in \mathcal{I} \text{ for every } \gamma > 0.$$

Thus

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) \leq 1-b \text{ or } \Phi(w_s;t) \geq b,\, \Omega(w_s;t) \geq b\}\right| > \rho\right\} \notin \mathcal{I},$$

and

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) = 1-b \text{ or } \Phi(w_s;t) = b,\, \Omega(w_s;t) = b\}\right| > \rho\right\} \in \mathcal{I}.$$

That is

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) < 1-b \text{ or } \Phi(w_s;t) > b,\, \Omega(w_s;t) > b\}\right| > \rho\right\} \notin \mathcal{I}$$

for every $t > 0$. So $b = \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\limsup w$. This completes the proof of the theorem. ☐

The dual statement for $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$ can also be proved similarly.

Theorem 3.16. *Let $a = \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$. Then for every positive number t and γ,*

$$\begin{array}{c}\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) > 1-a-\gamma \text{ or } \Phi(w_s;t) < a+\gamma,\right.\right. \\ \left.\left.\Omega(w_s;t) < a+\gamma\}\right| > \rho\right\} \notin \mathcal{I} \text{ and} \\ \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) > 1-a+\gamma \text{ or } \Phi(w_s;t) < a-\gamma,\right.\right. \\ \left.\left.\Omega(w_s;t) < a-\gamma\}\right| > \rho\right\} \in \mathcal{I}.\end{array} \tag{3.3}$$

Conversely, if (3.3) holds for every positive t and γ, then $a = \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$.

Remark 3.17. From the definition of $\mathcal{I}$-statistical cluster points, we observe that Theorems 3.15 and 3.16 imply that $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\limsup w$ and $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$ are the greatest and least $\mathcal{I}$-statistical cluster points of w, respectively.

Theorem 3.18. *For any sequence w, $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w \leq \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\limsup w$.*

Proof. First consider the case in which $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\limsup w = 0$, which implies that $B_w^{(\Theta,\Phi,\Omega)} = \varnothing$. Then for every $b \in (0,1)$,

$$B_w^{(\Theta,\Phi,\Omega)} := \left\{b \in (0,1) : \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) < 1-b \text{ or }\right.\right.\right. \\ \left.\left.\left.\Phi(w_s;t) > b,\, \Omega(w_s;t) > b\}\right| > \rho\right\} \in \mathcal{I}\right\}.$$

That is

$$\left\{b \in (0,1) : \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) \geq 1-b \text{ or } \Phi(w_s;t) \leq b,\right.\right.\right. \\ \left.\left.\left.\Omega(w_s;t) \leq b\}\right| < \rho\right\} \in \mathcal{F}(\mathcal{I})\right\}.$$

Also, we have

$$\left\{a \in (0,1) : \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{s \leq m : \Theta(w_s;t) > 1-a \text{ or } \Phi(w_s;t) < a,\right.\right.\right. \\ \left.\left.\left.\Omega(w_s;t) < a\}\right| > \rho\right\} \notin \mathcal{I}\right\}.$$

Thus $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w = 0$.

The case in which $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\sup w = 1$, is trivial.

Suppose that $b = \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\limsup w$ and $a = \mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$, where a and b are finite.
Now for given any γ, we show that $1-b-\gamma \in A_w^{(\Theta,\Phi,\Omega)}$. Then by Theorem 3.15,

$$\begin{gathered}\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta(w_s;t) < 1-b-\tfrac{\gamma}{2} \text{ or } \Phi(w_s;t) > b+\tfrac{\gamma}{2},\right.\right.\right.\\ \left.\left.\left.\Omega(w_s;t) > b+\tfrac{\gamma}{2}\right\}\right| > \rho\right\} \in \mathcal{I},\end{gathered}$$

where $1-b =$ least upper bound of $B_w^{(\Theta,\Phi,\Omega)}$.
Thus

$$\begin{gathered}\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta(w_s;t) \geq 1-b-\tfrac{\gamma}{2} \text{ or } \Phi(w_s;t) \leq b+\tfrac{\gamma}{2},\right.\right.\right.\\ \left.\left.\left.\Omega(w_s;t) \leq b+\tfrac{\gamma}{2}\right\}\right| > \rho\right\} \notin \mathcal{I},\end{gathered}$$

which in turn gives

$$\begin{gathered}\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta(w_s;t) > 1-b-\gamma \text{ or } \Phi(w_s;t) < b+\gamma,\right.\right.\right.\\ \left.\left.\left.\Omega(w_s;t) < b+\gamma\right\}\right| > \rho\right\} \notin \mathcal{I}.\end{gathered}$$

So $1-b-\gamma \in A_w^{(\Theta,\Phi,\Omega)}$. By definition $a = \inf A_w^{(\Theta,\Phi,\Omega)}$, we conclude that $1-b-\gamma \leq 1-a$. Since γ is arbitrary, $1-b \leq 1-a$, i.e., $-b \leq -a$ $a \leq b$. This completes the proof. □

Theorem 3.19. *In an neutrosophic fuzzy normed space $(Y, \Theta, \Phi, \Omega, \circledast, \otimes)$, the $\mathcal{I}$-statistically bounded sequence w is $\mathcal{I}$-statistically convergent if and only if*

$$\mathcal{I} - st_{(\Theta,\Phi,\Omega)^-} \liminf w = \mathcal{I} - st_{(\Theta,\Phi,\Omega)^-} \sup w.$$

Proof. Let α, β be $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\liminf w$ and $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\sup w$, respectively. Now we assume that $\mathcal{I}\text{-}st_{(\Theta,\Phi,\Omega)}\text{-}\lim w = w_0$. Then for every $t > 0$ and $b \in (0,1)$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{s \leq m : \Theta(w_s;t) \leq 1-b \text{ or } \Phi(w_s;t) \geq b,\, \Omega(w_s;t) \geq b\right\}\right| > \rho\right\} \in \mathcal{I}.$$

Thus

$$\begin{gathered}\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s;\tfrac{t}{2}\right) \circledast \Theta\left(w_0;\tfrac{t}{2}\right) \leq 1-b \text{ or}\right.\right.\right.\\ \left.\left.\left.\Phi\left(w_s;\tfrac{t}{2}\right) \otimes \Phi\left(w_0;\tfrac{t}{2}\right) \geq b,\, \Omega(w_s;t) \otimes \Omega\left(w_0;\tfrac{t}{2}\right) \geq b\right\}\right| > \rho\right\} \in \mathcal{I}.\end{gathered}$$

Let for every $t > 0$,

$$\sup_t \Theta\left(w_s;\frac{t}{2}\right) = 1-b_1 \text{ and } \sup_t \Theta\left(w_0;\frac{t}{2}\right) = 1-b_2,$$

or

$$\inf_t \Phi\left(w_s;\frac{t}{2}\right) = b_1 \text{ and } \inf_t \Phi\left(w_0;\frac{t}{2}\right) = b_2,$$

$$\inf_t \Omega\left(w_s;\frac{t}{2}\right) = b_1 \text{ and } \inf_t \Omega\left(w_0;\frac{t}{2}\right) = b_2.$$

such that

$$(1-b_1) \circledast (1-b_2) \leq 1-b \text{ or } b_1 \otimes b_2 \geq b. \tag{3.4}$$

Then

$$\begin{gathered}\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s;\tfrac{t}{2}\right) \leq 1-b_1 \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right) \geq b_1,\right.\right.\right.\\ \left.\left.\left.\Omega\left(w_s;\tfrac{t}{2}\right) \geq b_1\right\}\right| > \rho\right\} \in \mathcal{I}.\end{gathered} \tag{3.5}$$

Thus

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)<1-b_1-\gamma \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)>b_1+\gamma,\ \Omega\left(w_s;\tfrac{t}{2}\right)>b_1+\gamma\right\}\right|>\rho\right\}\in\mathcal{I} \tag{3.6}$$

for every $\gamma>0$.
Now applying Theorem 3.15 and the definition of $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\sup w$, we get

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)<1-\beta-\gamma \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)>\beta+\gamma,\ \Omega\left(w_s;\tfrac{t}{2}\right)>\beta+\gamma\right\}\right|>\rho\right\}\in\mathcal{I} \tag{3.7}$$

for every $\gamma>0$.
From (3.6) and (3.7) and by the definition of $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\sup w$, we get

$$1-b_1-\gamma\le 1-\beta-\gamma \text{ or } b_1+\gamma\ge\beta+\gamma, \text{ i.e.,}$$

$$\beta\le b_1. \tag{3.8}$$

Now we find those s such that

$$\Theta\left(w_s;\frac{t}{2}\right)>1-b_1+\gamma \text{ or } \Phi\left(w_s;\frac{t}{2}\right)<b_1-\gamma,\ \Omega\left(w_s;\frac{t}{2}\right)<b_1-\gamma.$$

We can easily observe that no such s exists which satisfy (3.4) and above condition together. So this implies that

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)>1-b_1+\gamma \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)<b_1-\gamma,\ \Omega\left(w_s;\tfrac{t}{2}\right)<b_1-\gamma\right\}\right|>\rho\right\}\in\mathcal{I}$$

Since $\alpha=\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\liminf w$, by Theorem 3.16, we get

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)>1-\alpha+\gamma \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)<\alpha-\gamma,\ \Omega\left(w_s;\tfrac{t}{2}\right)<\alpha-\gamma\right\}\right|>\rho\right\}\in\mathcal{I}.$$

By the definition of $\mathcal{I}$-$st_{(\Theta,\Phi,\Omega)}$-$\liminf w$, we have

$$1-\alpha+\gamma\le 1-b_1+\gamma \text{ or } \alpha-\gamma\ge b_1-\gamma, \text{ i.e.,}$$

$$b_1\le\alpha \tag{3.9}$$

From (3.7) and (3.8), we get $\beta\le\alpha$. Now combining Theorem 3.18 and the above inequality, we conclude $\alpha=\beta$.

Conversely, suppose that $\alpha=\beta$ and let $\sup_t\Theta(w_0,t)=1-\alpha$ or $\inf_t\Phi(w_0,t)=\alpha$, $\inf_t\Omega(w_0,t)=\alpha$. Then for any $\gamma>0$, Theorems 3.15 and 3.16 will together imply that

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)<1-\alpha+\tfrac{\gamma}{2} \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)>\alpha-\tfrac{\gamma}{2},\ \Omega\left(w_s;\tfrac{t}{2}\right)>\alpha-\tfrac{\gamma}{2}\right\}\right|>\rho\right\}\in\mathcal{I} \tag{3.10}$$

and

$$\left\{m\in\mathbb{N}:\tfrac{1}{m}\left|\left\{s\le m:\Theta\left(w_s;\tfrac{t}{2}\right)>1-\alpha+\tfrac{\gamma}{2} \text{ or } \Phi\left(w_s;\tfrac{t}{2}\right)<\alpha-\tfrac{\gamma}{2},\ \Omega\left(w_s;\tfrac{t}{2}\right)<\alpha-\tfrac{\gamma}{2}\right\}\right|>\rho\right\}\in\mathcal{I}. \tag{3.11}$$

Now

$$1-\alpha\ge\Theta(w_0,t)=\Theta\left(w_s-(w_s-w_0);t\right)\ge\Theta\left(w_s;\frac{t}{2}\right)\circledast\Theta\left(w_s-w_0;\frac{t}{2}\right),$$

and

$$\alpha \leq \Phi(w_0, t) = \Phi\left(w_s - (w_s - w_0); t\right) \leq \Phi\left(w_s; \frac{t}{2}\right) \otimes \Phi\left(w_s - w_0; \frac{t}{2}\right),$$

$$\alpha \leq \Omega(w_0, t) = \Omega\left(w_s - (w_s - w_0); t\right) \leq \Omega\left(w_s; \frac{t}{2}\right) \otimes \Omega\left(w_s - w_0; \frac{t}{2}\right).$$

Thus

(3.12)

$$\Theta\left(w_s; \frac{t}{2}\right) \circledast \Theta\left(w_s - w_0; \frac{t}{2}\right) \leq 1 - \alpha \text{ or } \Phi\left(w_s; \frac{t}{2}\right) \otimes \Phi\left(w_s - w_0; \frac{t}{2}\right) \geq \alpha,$$
$$\Omega\left(w_s; \frac{t}{2}\right) \otimes \Omega\left(w_s - w_0; \frac{t}{2}\right) \geq \alpha.$$

Let

$$\sup_t \left\{\Theta\left(w_s - w_0; \tfrac{t}{2}\right)\right\} = 1 - a_1 \text{ or } \inf_t \left\{\Phi\left(w_s - w_0; \tfrac{t}{2}\right)\right\} = a_1,$$
$$\inf_t \left\{\Omega\left(w_s - w_0; \tfrac{t}{2}\right)\right\} = a_1,$$

where $a_1 \in (0,1)$ and (3.10) and (3.12) hold. Then

$$\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s - w_0; \tfrac{t}{2}\right) < 1 - \alpha_1 - \tfrac{\gamma}{2} \text{ or}\right.\right.\right.$$
$$\left.\left.\left.\Phi\left(w_s - w_0; \tfrac{t}{2}\right) > \alpha_1 + \tfrac{\gamma}{2}, \Omega\left(w_s - w_0; \tfrac{t}{2}\right) > \alpha_1 + \tfrac{\gamma}{2}\right\}\right| > \rho\right\} \in \mathcal{I},$$

which is true for all $\gamma > 0$. Thus

$$\left\{m \in \mathbb{N} : \tfrac{1}{m}\left|\left\{s \leq m : \Theta\left(w_s - w_0; \tfrac{t}{2}\right) < 1 - \alpha_1 \text{ or } \Phi\left(w_s - w_0; \tfrac{t}{2}\right) > \alpha_1,\right.\right.\right.$$
$$\left.\left.\left.\Omega\left(w_s - w_0; \tfrac{t}{2}\right) > \alpha_1\right\}\right| > \rho\right\} \in \mathcal{I},$$

which is true for all $a \leq a_1 \in (0,1)$, because $1 - a_1$ is the least upper bound or a_1 is the greatest lower bound.

Now repeat the process by taking (3.11) and (3.12) instead of (3.10) and (3.12). If (3.11) and (3.12) are satisfied, then $\inf_t \Theta\left(w_s - w_0; \frac{t}{2}\right) = 1 - a_1$ or $\sup_t \Phi\left(w_s - w_0; \frac{t}{2}\right) = a_1$, $\sup_t \Omega\left(w_s - w_0; \frac{t}{2}\right) = a_1$. On contrary suppose that $1 - a_1 \neq \inf_t \Theta\left(w_s - w_0; \frac{t}{2}\right)$ or $a_1 \neq \sup_t \Phi\left(w_s - w_0; \frac{t}{2}\right)$, $a_1 \neq \sup_t \Omega\left(w_s - w_0; \frac{t}{2}\right)$ and conditions (3.11) and (3.12) be satisfied. This implies that there exists some $r \in (0,1)$ such that either $1 - r = \Theta\left(w_s - w_0; \frac{t}{2}\right)$ or $r = \Phi\left(w_s - w_0; \frac{t}{2}\right)$, $r = \Omega\left(w_s - w_0; \frac{t}{2}\right)$ for some $t > 0$ where $1 - a_1 > 1 - r$ or $a_1 < r$.

As (3.11) and (3.12) are satisfied, let us suppose that

$$\inf_t \Phi\left(w_s - w_0; \frac{t}{2}\right) = 1 - a_2 \text{ or } \sup_t \Phi\left(w_s - w_0; \frac{t}{2}\right) = a_2, \sup_t \Omega\left(w_s - w_0; \frac{t}{2}\right) = a_2.$$

Then

$$1 - a_1 > 1 - a_2 \text{ or } a_1 < a_2 \tag{3.13}$$

and from (3.12), we get

$$\Theta\left(w_s - w_0; \tfrac{t}{2}\right) \circledast (1 - a_2) \leq 1 - \alpha \text{ or } \Phi\left(w_s - w_0; \tfrac{t}{2}\right) \otimes \alpha_2 \geq \alpha,$$
$$\Omega\left(w_s - w_0; \tfrac{t}{2}\right) \otimes \alpha_2 \geq \alpha.$$

Using (3.11), we get

$$\left(1 - \alpha + \frac{\gamma}{2}\right) \circledast (1 - a_2) \leq 1 - \alpha \text{ or } \Phi\left(\alpha - \frac{\gamma}{2}\right) \otimes (\alpha_2) \geq \alpha, \Omega\left(\alpha - \frac{\gamma}{2}\right) \otimes (\alpha_2) \geq \alpha$$

for all $\gamma > 0$. Clearly,
(3.14)
$$\Theta\left(1-\alpha-\frac{\gamma}{2}\right)\circledast(1-a_2)\leq 1-\alpha \text{ or } \Phi\left(\alpha+\frac{\gamma}{2}\right)\otimes(\alpha_2)\geq\alpha,\ \Omega\left(\alpha+\frac{\gamma}{2}\right)\otimes(\alpha_2)\geq\alpha$$
for all $\gamma > 0$. Now
$$1-a_1=\sup_t\Theta\left(w_s-w_0;\frac{t}{2}\right) \text{ or } \alpha_1=\inf_t\Phi\left(w_s-w_0;\frac{t}{2}\right),\ \alpha_1=\inf_t\Omega\left(w_s-w_0;\frac{t}{2}\right),$$
where $a_1\in(0,1)$ and which satisfy (3.10) and (3.12). From (3.14) we conclude that $1-a_2$ is another value satisfying (3.10) and (3.12). Thus
$$1-a_1<1-a_2 \text{ or } a_2<a_1$$
This contradicts (3.13). So $1-a_1=\inf_t\Theta\left(w_s-w_0;\frac{t}{2}\right)$ or $a_1=\sup_t\Phi\left(w_s-w_0;\frac{t}{2}\right)$, $a_1=\sup_t\Omega\left(w_s-w_0;\frac{t}{2}\right)$ satisfying conditions (3.11) and (3.12). Since $1-a_1$ is the greatest lower bound, the inequality becomes true for all $a\geq a_1\in(0,1)$. Hence
$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{s\leq m:\Theta\left(w_s-w_0;\frac{t}{2}\right)\leq 1-\alpha \text{ or } \Phi\left(w_s-w_0;\frac{t}{2}\right)\leq\alpha,\right.\right.\right.$$
$$\left.\left.\left.\Omega\left(w_s-w_0;\frac{t}{2}\right)\leq\alpha\right\}\right|>\rho\right\}\in\mathcal{I}$$
for each $t>0$ and $a\in(0,1)$. Therefore
$$\mathcal{I}-st_{(\Theta,\Phi,\Omega)^-}\lim w=w_0.$$
This completes the proof. □

4. Conclusion

In this article, we delve into the concepts of $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior within the framework of neutrosophic fuzzy normed spaces. Through an illustrative example, we showcase the practical computation of these points, shedding light on their significance in mathematical analysis. Furthermore, we extend our investigation to explore the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for sequences in such spaces, unveiling essential properties of the sets encapsulating these points. This comprehensive exploration enhances our understanding of the nuanced dynamics within neutrosophic fuzzy normed spaces, offering valuable insights for further research and applications in diverse fields.

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Ömer Kişi (okisi@bartin.edu.tr)
Department of mathematics, Bartın university, Bartın, Turkey

Rümeysa Akbiyik (rmysaakbyk@gmail.com)
Department of mathemathics, Bartın university, Bartın, Turkey

Mehmet Gürdal (gurdalmehmet@sdu.edu.tr)
Department of mathemathics, Suleyman Demirel university, postalcode 32260, Isparta, Turkey