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Fuzzy derivations (m, n)-fold BCC-ideals on BCC-algebras

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Abstract. Fuzzy derivation concepts have been given on some algebra types and their basic properties have been examined. In this paper, we observe that, the notions of derivation (m, n)-fold BCC-ideals and fuzzy derivation (m, n)-fold BCC-ideals, for each positive integers m, n, are indeed the natural generalization of BCC-ideals and fuzzy BCC-ideals, respectively. A characterization of derivation (m, n)-fold BCC-ideals and fuzzy derivation (m, n)-fold BCC-ideals is given, and conditions for which an ideal (respectively fuzzy ideal) is an derivation (m, n)-fold BCC-ideal (respectively fuzzy derivation (m, n)-fold BCC-ideal) are studied. We also establish extension properties for derivation (m, n)-fold BCC -ideals and fuzzy derivation (m, n)-fold BCC-ideals. Furthermore, this paper focuses on

• application on fuzzy left (right) derivation (m, n)-fold ideals in BCC-algebras,
• left-right derivation (m, n)-fold BCC-ideals of a BCC-algebra,
• the homomorphic image and the pre-image of left-right derivation (mn)-fold BCC-ideals of a BCC-algebra,
• the Cartesian product left-right derivation (m, n)-fold BCC-ideals of a BCC-algebra.

There are applications of this work in the field of medicine, engineering, industry, statistics, etc.

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1. Introduction

In 1966, Iséki and Tanaka [1, 2] introduced the notion of BCK-algebras. Iséki [3] and Huang [4] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. A BCC-algebra is an important class of logical algebras introduced by Komori [5] and was extensively investigated by many researchers (See [6, 7, 8, 9, 10, 11]). The concept of fuzzy sets was introduced by Zadeh [12]. Xi [13] applied the concept of fuzzy sets to BCK-algebras. In the theory of rings, the properties of derivations are important. There are many authors discussed derivations on rings and near rings (See [14, 15, 16]). After that, Jun and Xin [17] introduced the idea of derivation on BCI-algebras and got new results. Furthermore, the new concept presented by Them on derivations of BCI-algebra called a *regular derivation*. They investigated some of its properties, defined a d-derivation ideal and gave conditions for an ideal to be d-derivation. Two years later, Hamza and Al-Shehri [18, 19] studied derivation in BCK-algebras, a left derivation in BCI-algebras and investigated a regular left derivation of BCI-algebras. Prabpayak and Leerawat [20] applied the notion of a regular derivation to BCC-algebras and investigated some related properties. Recently, Yılmaz [21] considered fuzzy derivations of some ideals on d-algebras, investigated fuzzy (right, left) derivations of fuzzy d-ideals on d-algebras and studied some of their properties.

We believe that the concept of fuzzy left (right) derivation (m, n)-fold BCC-ideals in BCC-algebras that we will propose can be applied not only to topology but also to traffic and transportation and medicine. Therefore, we would like to proceed with our research as follows. First, we introduce the concept of fuzzy left (right) derivation (m, n)-fold BCC-ideals in BCC-algebras and investigate some of its properties. Second, the concepts of the image and the pre-image of fuzzy left (right) derivation (m, n)-fold BCC-ideals under homomorphism of BCC-algebras is given and studies some its properties. Finally, the Cartesian product of fuzzy left (right) derivation (m, n)-fold BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1 ([5])**.** A *BCC-algebra* $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms: for any $x,\ y,\ z \in X$,

(BCC_1) $[(x * y) * (z * y)] * (x * z)] = 0$,
(BCC_2) $x * 0 = x$,
(BCC_3) $x * x = 0$,
(BCC_4) $0 * x = 0$,
(BCC_5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Definition 2.2 ([9])**.** Let $(X, *, 0)$ be a BCC-algebra. Then we can define a binary relation $\leq$ on X as follows: for any $x,\ y \in X$,

$$x \leq y \Leftrightarrow x * y = 0.$$

It is clear that $(X, \leq)$ is a partially ordered set.

Result 2.3 (See Proposition 2, [9])**.** *Let $(X, *, 0)$ be a BCC-algebra. Then the followings hold: for any $x,\ y,\ z \in X$,*
(1) $(x * y) * x = 0$, *i.e.,* $x * y \leq x$,
(2) $x \leq y$ *implies* $x * z \leq y * z,\ z * y \leq z * x$,
(3) $(x * y) * (z * y) \leq x * z$.

For any element $x,\ y$ of a BCC-algebra X, we denote $x \wedge y = y * (y * x)$.

Result 2.4 (See Remark 3, [6])**.** *Let $(X, *, 0)$ of type $(2, 0)$ be a BCC-algebra. Then if for any $x,\ y \in X$,*
(1) $0 \wedge x = 0$,
(2) $x \wedge y \leq y$.

Definition 2.5 ([11])**.** Let $(X, *, 0)$ be a BCC-algebra and S a nonempty subset of X. Then S is called a *BCC-subalgebra*, if

$$x * y \in S, \text{ for any } x,\ y \in S.$$

Definition 2.6 ([6])**.** A nonempty subset A of a BCC-algebra X is called an *ideal* of X, if it satisfies the following conditions : for any $x,\ y \in X$,
(I_1) $0 \in A$,
(I_2) $x * y,\ y \in I$ imply $x \in A$.

Definition 2.7 ([11])**.** A nonempty subset A of a BCC-algebra X is called a *BCC-ideal* of X, if it satisfies the following conditions : for any $x,\ y \in X$,
(BCCI_1) $0 \in A$,
(BCCI_2) $(x * y) * z,\ y \in A$ imply $x * z \in A$.

For a nonempty set X, a mapping $A : X \to I$ is called a *fuzzy set* in X, where $I = [0, 1]$ (See [12]). The *fuzzy whole* [resp. *empty*] *set* in X, denoted by $\mathbf{1}$ [resp. $\mathbf{0}$] is a fuzzy set in X defined as follows: for each $x \in X$,

$$\mathbf{1}(x) = 1 \text{ [resp. } \mathbf{0}(x) = 0].$$

We will denote the set of all fuzzy sets by I^X.

For any $a,\ b \in I$, we denote $a \vee b = max\{a, b\}$ and $a \wedge b = min\{a, b\}$.

Definition 2.8 ([10])**.** Let $A \in I^X$. Then A is called a *fuzzy subalgebra* of X, if for any $x,\ y \in X$,

$$A(x * y) \geq A(x) \wedge A(y).$$

Definition 2.9 ([10])**.** Let $A \in I^X$. Then I is called a *fuzzy BCC-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(FBCCI_1) $A(0) \geq A(x)$,
(FBCCI_2) $A(x * z) \geq A((x * y) * z) \wedge A(y)$.

Definition 2.10 ([20]). Let $(X, *, 0)$ be a BCC-algebra. Then a mapping $d : X \to X$ is called a:

(i) *left-right derivation* (briefly, (l, r)-derivation) of X, if

$$d(x * y) = (d(x) * y) \wedge (x * d(y)) \text{ for any } x,\ y \in X,$$

(ii) *right-left derivation* (briefly, (r, l)-derivation) of X, if

$$d(x * y) = (x * d(y)) \wedge (d(x) * y) \text{ for any } x,\ y \in X,$$

(iii) *derivation* of X, if it is both a (l, r)-derivation and a (r, l)-derivation of X.

Example 2.11. Let $X = \{0, 1, 2, 3\}$ be the BCC-algebra with the operation $*$ defined by the table 2.1(See [20]):

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Table 2.1

(1) Consider the mapping $d : X \to X$ defined by: for each $x \in X$,

$$d(x) = \begin{cases} 0 & \text{if } x \in \{0, 1, 3\} \\ 2 & \text{if } x = 2, \end{cases} \tag{2.1}$$

Then it is clear that d is a derivation of X.

(2) Consider the mapping $d : X \to X$ defined by: for each $x \in X$,

$$d(x) = \begin{cases} x & \text{if } x \in \{1, 3\} \\ 0 & \text{if } x \in \{0, 2\}. \end{cases} \tag{2.2}$$

Then d is an (r, l)-derivation but not an (l, r)-derivation of X.

Definition 2.12 ([20]). Let $(X, *, 0)$ be a BCC-algebra. Then a mapping $d : X \to X$ is said to be *regular*, if $d(0) = 0$.

Result 2.13 (See Theorem and Corollary, [20]). (1) *A (r, l)-derivation d of a BCC-algebra X is regular.*

(2) *A derivation d of BCC-algebra X is regular.*

Result 2.14 (See Proposition, [20]). *Let $(X, *, 0)$ be a BCC-algebra and d a derivation of X Then the followings hold: for any $x,\ y \in X$,*

(1) $d(x) \leq x$,
(2) $d(x * y) \leq d(x) * y$,
(3) $d(x * y) \leq x * d(y)$,
(4) $d(d(x)) \leq x$,
(5) $d(x * d(x)) = 0$,
(6) $d^{-1}(0) = \{x \in X : d(x) = 0\}$ *is a subalgebra of X.*

3. Fuzzy derivation (n, m)-fold BCC-ideals of BCC-algebras

In this section, we will discuss and investigate a new notion is called fuzzy left (right) derivation BCC-ideals on BCC-algebras and study several basic properties which are related to fuzzy left (right) derivation BCC-ideals.

In what follows, let X denote a BCC-algebra unless otherwise specified, we begin with the following definition. For any elements and of a BCC-algebra X, we use the following abbreviated notation: for any positive integers $n,\ m,$

$$x^n * y = (x * (x * (x * \cdots (x * y)) \cdots),$$
$$x * y^m = (x * y) * y) * y) \cdots y))),$$

where x occurs n times and y occurs m times.

Definition 3.1. Let d be a derivation of X and A a nonempty subset of X. Then A is called a:

(i) *left derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(DI$_1$) $0 \in A,$

(LDI$_2$) $d(x * y) * z^m,\ d(y) \in A$ imply $d(x * z^n) \in A,$

(ii) *right derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(DI$_1$) $0 \in A,$

(RDI$_2$) $(x * y) * d(z^m),\ d(y) \in A$ imply $d(x * z^n) \in A,$

(iii) *derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(DI$_1$) $0 \in A,$

(DI$_2$) $d((x * y) * z^m),\ d(y) \in A$ imply $d(x * z^n) \in A.$

Example 3.2. Let d be the derivation of X given in Example 2.11 (1) and consider the subset $J = \{0, 1, 2\}$. For any positive integers $m,\ n,$ let us take $x = y = z = 2$ Then by the conditions (BCC$_3$) and (BCC$_4$) we have

$$d(2 * 2) * 2^m = d(0) * 2^m = 0 * z^m = 0 \in J \text{ and } d(2) = 2 \in J.$$

Moreover, we can easily check that $d(2 * 2^n) \in J$. Thus or all the possibilities that $x,\ y,$ and z can take, J satisfies the conditions (DI$_1$) and (LDI$_2$). So J is a left (m, n)-fold BCC-ideal of X.

Definition 3.3. Let d be a derivation of X and $A \in I^X$. Then A is called a:

(i) *fuzzy left derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(FDI$_1$) $A(0) \geq A(d(x)),$

(FLDI$_2$) $A(d(x * z^n)) \geq A(d(x * y) * z^m) \wedge A(d(y)),$

(ii) *fuzzy right derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(FDI$_1$) $A(0) \geq A(d(x)),$

(FRDI$_2$) $A(d(x * z^n)) \geq A((x * y) * d(z^m)) \wedge A(d(y)),$

(iii) *fuzzy derivation (m, n)-fold BCC-ideal with respect to d* of X, if the following conditions hold: for any $x,\ y,\ z \in X$ and any positive integers $m,\ n,$

(FDI$_1$) $A(0) \geq A(d(x)),$

(FDI$_2$) $A(d(x * z^n)) \geq A(d((x * y) * z^m) \wedge A(d(y))$.

Remark 3.4. (1) If d is an identity mapping, then Definition 3.1 gives the definition of BCC-ideal.

(2) If d is an identity mapping, then Definition 3.3 gives the definition of fuzzy BCC-ideal.

Example 3.5. (1) Let d be the self mapping given in Example 2.11 (1) and consider the fuzzy set $A : X \to I$ defined by:

$$A(d(0)) = t_0,\ A(d(1)) = t_1,\ A(d(2)) = A(d(3)) = t_2,$$

where $t_0 > t_1 > t_2$. Then by routine calculations, A is not fuzzy left (right)-derivation (m, n)-fold BCC-ideal of X.

(2) Let $X = \{0, 1, 2, 3, 4, 5\}$ be the BCC-algebra in which the operation $*$ is defined by the table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Table 3.1

Let $d : X \to X$ be the self mapping defined by: for any $x \in X$,

$$d(x) = \begin{cases} 0 & \text{if } x \in \{0, 1, 3, 4\} \\ 5 & \text{if } x = 5, \end{cases} \tag{3.1}$$

Consider the fuzzy set $A : X \to I$ in X defined as follows:

$$A(d(0)) = t_0,\ A(d(1)) = t_1,\ A(d(2)) = A(d(3))A(d(4)) = A(d(5)) = t_2,$$

where $t_0 > t_1 > t_2$. Then by routine calculations, A is a fuzzy left (right)-derivation (m, n)-fold BCC-ideal of X.

Proposition 3.6. *Let A be a fuzzy left* [resp. right] *derivation (m, n)-fold BCC-ideal of X.*

(1) *If $x \leq d(y)$ for any $x,\ y \in X$, then $A(d(x)) \geq A(d(y))$.*

(2) *If $x * y) \leq d(x)$ for any $x,\ y \in X$, then $A(d(x * y)) \geq A(d(x))$.*

(3) *If $(x * y) * (z^m * y) \leq d(x * z^n)$ for any $x,\ y,\ z \in X$ and any positive integers $m,\ n$, then $A(d((x * y) * (z^m * y)) \geq A(d(x * z^n))$.*

(4) *If $A(d(x * y))) = A(d(0))$ for any $x,\ y \in X$, then $A(d(x)) \geq A(d(y))$.*

Proof. (1) Suppose $x \leq d(y)$ for any $x,\ y \in X$. By Result 2.14 (1), $d(y) \leq y$. Then $x \leq y$, i.e., $x * y = 0$. Thus we have

$$\begin{aligned} A(d(x)) &= A(d(x * 0)) \text{ [By the condition (BCC}_2\text{)]} \\ &\geq A(d(x * y) * 0)) \wedge A(d(y)) \text{ [By (FLDI}_2\text{)]} \\ &= A(d(x * y)) \wedge A(d(y)) \text{ [By the condition (BCC}_2\text{)]} \end{aligned}$$

$= A(d(0)) \wedge A(d(y))$ [Since $x * y = 0$]
$= A(0)) \wedge A(d(y))$ [By Result 2.13]
$= A(d(y))$. [By (DI_1)]

(2) The proof follows from (1).
(3) The proof follows from (1).
(4) Suppose $A(d(x * y))) = A(d(0))$ for any $x,\ y \in X$. Then we get

$$\begin{aligned} A(d(x)) &= A(d(x * 0)) \text{ [By the condition (BCC}_2\text{)]} \\ &\geq A(d(x * y) * 0)) \wedge A(d(y)) \text{ [By (FLDI}_2\text{)]} \\ &= A(d(x * y)) \wedge A(d(y)) \text{ [By the condition (BCC}_2\text{)]} \\ &= A(d(0)) \wedge A(d(y)) \text{ [By the hypothesis]} \\ &= A(d(y)). \text{ [By (FDI}_1\text{)]} \end{aligned}$$

□

Proposition 3.7. *Let $(A_j)_{j \in J}$ be a family of fuzzy left* [resp. *right*] *derivation (m, n)-fold BCC-ideal of X. Then $\bigcap_{j \in J} A_j$ is a fuzzy left* [resp. *right*] *derivation (m, n)-fold BCC-ideal of X.*

Proof. Let $x \in X$. Then we have

$$\begin{aligned} \left(\bigcap_{j \in J} A_j\right)(d(x)) &= \bigwedge_{j \in J} A_j(d(x)) \\ &\leq \bigwedge_{j \in J} A_j(0) \text{ [By (FDI}_1\text{)]} \\ &= \left(\bigcap_{j \in J} A_j\right)(0). \end{aligned}$$

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} \left(\bigcap_{j \in J} A_j\right)(d(x * z^n)) &= \bigwedge_{j \in J} A_j(d(x * z^n)) \\ &\geq \bigwedge_{j \in J}[A_j(d(x * y) * z^m)) \wedge A_j(d(y))] \text{ [By (FLDI}_2\text{)]} \\ &= \left(\bigwedge_{j \in J} A_j(d(x * y) * z^m))\right) \wedge \left(\bigwedge_{j \in J} A_j(d(y))\right) \\ &= \left(\bigcap_{j \in J} A_j\right)(d(x * y) * z^m)) \wedge \left(\bigcap_{j \in J} A_j\right)((d(y)). \end{aligned}$$

Thus $\bigcap_{j \in J} A_j$ satisfies the conditions (FDI_1) and ($FLDI_2$). So $\bigcap_{j \in J} A_j$ is a fuzzy left derivation (m, n)-fold BCC-ideal of X.

The proof of the second part is similar. □

For a fuzzy set A in a nonempty set X, the *t-level set* $[A]_t$ of A (See [22]) is a subset of X defined by:

$$[A]_t = \{x \in X : A(x) \geq t\},$$

where $t \in I$.

We obtain a relationship between fuzzy left (m, n)-fold BCC-ideals and BCC-ideals of a BCC-algebra X.

Theorem 3.8. *Let A be a fuzzy set in X. Then A is a fuzzy* [resp. *left and right*] *derivation (m, n)-fold BCC-ideal of X if and only if either $[A]_t = \varnothing$ or $[A]_t$ is a* [resp. *left and right*] *derivation (m, n)-fold BCC-ideal of X for each $t \in I$.*

In this case, $[A]_t$ is called a [resp. *left and right*] *derivation (m, n)-fold BCC-ideal* of X.

Proof. Suppose A is a fuzzy left derivation (m, n)-fold BCC-ideal of X. Without loss of generality, we can assume that $[A]_t \neq \varnothing$ for each $t \in I$ and let $x \in [A]_t$. Then by the condition (FDI_1), $A(0) \geq A(x) \geq t$. Thus $0 \in [A]_t$. So $[A]_t$ satisfies the condition (DI_1).

Now suppose $d(x*y)*z^m,\ d(y)\in[A]_t$ for any $x,\ y,\ z\in X$. Then clearly,

$$A(d(x*y)*z^m)\geq t,\ \ A(d(y))\geq t.$$

Thus by (FLDI$_2$),

$$A(d(x*z^n))\geq A(d(x*y)*z^m))\wedge A(d(y))\geq t.$$

So $d(x*z^n)\in[A]_t$. Hence $[A]_t$ satisfies the condition (LDI$_2$). Therefore $[A]_t$ is a left derivation (m,n)-fold BCC-ideal of X.

Conversely, suppose the necessary condition holds. Assume that the condition (FDI$_1$) does not hold. Then there is $x\in X$ such that $A(0)<A(x)$. Let us take $t_0\in I$ such that

$$t_0=\frac{1}{2}(A(0)+A(x)).$$

Then clearly, $A(0)<t_0\leq A(x)$. Thus $0\notin[A]_{t_0}$ but $x\in[A]_{t_0}$. This is a contradiction. So A satisfies the condition (FDI$_1$).

Now assume that he condition (FLDI$_2$) does not hold. Then there are $x,\ y,\ z\in X$ such that

$$A(d(x*z^n))<A(d(x*y)*z^m))\wedge A(d(y)).$$

Let us take $t\in I$ such that

$$t=\frac{1}{2}(A(d(x*z^n))+A(d(x*y)*z^m))\wedge A(d(y))).$$

Then $A(d(x*z^n))<t<A(d(x*y)*z^m))\wedge A(d(y))$. Thus $d(x*z^n)\in[A]_t$ but $d(x*y)*z^m),\ d(y)\in[A]_t$. This is a contradiction. So A satisfies the condition (FLDI$_2$). Hence A is a fuzzy left derivation (m,n)-fold BCC-ideal of X.

The proof of remainders parts are similar. □

4. The image (pre-image) fuzzy derivation (m,n)-fold BCC-ideals under a homomorphism

In this section, we introduce the concepts of the image and the pre-image of fuzzy left and right derivations BCC-ideals in BCC-algebras under homomorphism of BCC-algebras

Definition 4.1 (See [12]). Let $X,\ Y$ be sets, $f:X\to Y$ a mapping and $A\in I^X,\ B\in I^Y$. Then

(i) the *image of A under f*, denoted by $f(A)$, is a fuzzy set in Y defined by: for each $y\in Y$,

$$f(A)(y)=\begin{cases}\bigvee_{x\in f^{-1}(y)}A(x) & \text{if } f^{-1}(y)\neq\varnothing\\ 0 & \text{othewise,}\end{cases}\tag{4.1}$$

(ii) the *pre-image of B under f*, denoted by $f^{-1}(B)$, is a fuzzy set in X defined by: for each $x\in X$,

$$f^{-1}(B)(x)=B(f(x)).$$

Definition 4.2. Let $f : X \to Y$ be a homomorphism of BCC-algebras and d a self mapping. Then the *self mapping* $\overline{d}$ *of* Y *induced by* f *and* d is defined as follows: for each $y \in Y$,

$$\overline{d}(y) = \begin{cases} f(d(x)) & \text{if } x \in f^{-1}(y) \neq \varnothing \\ 0 & \text{othewise.} \end{cases} \tag{4.2}$$

It is obvious that if $x \in f^{-1}(y)$, then $\overline{d}(y) = \overline{d}(f(x)) = f(d(x))$.

Example 4.3. Let $(X, *, 0) = (\{0, 1, 2, 3\}, *, 0)$, $(Y, *^{'}, 0) = (\{0, 1, 2, 3, 4, 5\}, *^{'}, 0)$ be the BCC-algebras given in Examples 2.11 (1) and 3.5 (2), respectively. Let d be the self mapping defined in Example 2.11 (1). Consider the mapping $f : X \to Y$ defined by: for each $x \in X$,

$$f(x) = x *^{'} 1.$$

Then by routine calculation, we can see that f is a homomorphism of BCC-algebras. Moreover, we can obtain the self mapping $\overline{d}$ of Y induced by f and d:

$$\overline{d}(y) = \begin{cases} 0 & \text{if } y \in \{0, 1, 3, 4, 5\} \\ 2 & \text{if } y = 2. \end{cases} \tag{4.3}$$

Proposition 4.4. *Let $f : X \to Y$ be a homomorphism of BCC-algebras and d a self mapping of X. If d is an (l, r)* [resp. (r, l)]*-derivation, then so is $\overline{d}$.*

Proof. The proof is straightforward. □

Proposition 4.5. *Let $f : X \to Y$ be a homomorphism of BCC-algebras. If B is a fuzzy right* [resp. $left$] *derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y, then $f^{-1}(B)$ is a fuzzy right* [resp. $left$] *derivation (m, n)-fold BCC-ideal with respect to d of X.*

Proof. Suppose B is a fuzzy right derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y and let $x \in X$. Then we have

$$f^{-1}(B)(d(x)) = B(f(d(x))) = B(\overline{d}(f(x))) \leq B(0) = B(f(0)) = f^{-1}(B)(0).$$

Thus $f^{-1}(B)$ satisfies the condition (FDI$_1$).

Now let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} f^{-1}(B)(d(x * z^n)) &= B(f(d(x * z^n))) \text{ [By Definition 4.1 (i)]} \\ &= B(\overline{d}(f(x * z^n))) \text{ [By Proposition 4.4]} \\ &= B(\overline{d}(f(x) * f(z)^n))) \text{ [Since } f \text{ is a homomorphism]} \\ &\geq B((f(x) * f(y)) * \overline{d}(f(z)^m))) \wedge B(\overline{d}(f(y))) \\ &\quad \text{[By the condition (FRDI}_2\text{)]} \\ &= B(f(x * y) * \overline{d}(f(z^m))) \wedge B(\overline{d}(f(y))) \\ &\quad \text{[Since } f \text{ is a homomorphism]} \\ &= B(f(x * y) * f(d(z^m))) \wedge B(f(d(y))) \\ &= B(f((x * y) * d(z^m))) \wedge B(f(d(y))) \\ &= f^{-1}(B)((x * y) * d(z^m)) \wedge f^{-1}(B)(d(y)). \\ &\quad \text{[By Definition 4.1 (i)]} \end{aligned}$$

Thus $f^{-1}(B)$ satisfies the condition (FRDI$_2$). So $f^{-1}(B)$ is a fuzzy right derivation (m, n)-fold BCC-ideal with respect to d of X.

The proof of the second part is similar. □

Proposition 4.6. *Let $f : X \to Y$ be an epimorphism of BCC-algebras and $B \in I^Y$. If $f^{-1}(B)$ is a fuzzy right* [resp. *left*] *derivation (m, n)-fold BCC-ideal with respect to d of X, then B is a fuzzy right* [resp. *left*] *derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y.*

Proof. Suppose If $f^{-1}(B)$ is a fuzzy right derivation (m, n)-fold BCC-ideal with respect to d of X and $x \in Y$. Since f is surjective, there is $a \in X$ such that $x = f(a)$. Then we have

$$B(\overline{d}(x)) = B(\overline{d}f(a)) = B(f(d(a))) = f^{-1}(B(d(a))) \leq f^{-1}(B)(0) = B(f(0)) = B(0).$$

Thus B satisfies the condition (FDI_1).

Now let $x,\ y,\ z \in Y$. Since f is surjective, there are $a,\ b,\ c \in X$ such that $x = f(a)$, $y = f(b)$, $z = f(c)$. Then we get

$$\begin{aligned} B(\overline{d}((x * z^n)) &= B(d(f(a) * f(c^n))) \\ &= B(\overline{d}(f(a * c^n)) \text{ [Since } f \text{ is a homomorphism]} \\ &= B(f(d(a * c^n)) \\ &= f^{-1}(B)(d(a * c^n)) \\ &\geq f^{-1}(B)((a * b) * d(c^m))) \text{ [By the condition (FRDI}_2\text{)]} \\ &= B(f((a * b) * d(c^m))) \\ &= B((f(a) * f(b)) * f(d(c^m))) \\ &= B((f(a) * f(b)) * \overline{d}(f(c^m))) \\ &= B((x * y) * \overline{d}(z^m)). \end{aligned}$$

Thus B satisfies the condition ($FRDI_2$). So B is a fuzzy right derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y.

The proof of the second part is similar. □

Definition 4.7 ([23])**.** Let X be a nonempty set and $A \in I^X$. Then we say that A *has the sup-property*, if for each $T \in 2^X$, there is $t_0 \in T$ such that $A(t_0) = \bigvee_{t \in T} A(t)$.

Proposition 4.8. *Let $f : X \to Y$ be a homomorphism of BCC-algebras and A a fuzzy left* [resp. *right*] *derivation (m, n)-fold BCC-ideal with respect to d of X. If A has the sup-property, then $f(A)$ is a fuzzy left* [resp. *right*] *derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y.*

Proof. Without loss of generality, we can assume that f is an epimorphism, suppose A has the sup-property and let $y \in Y$. Then we get

$$\begin{aligned} f(A)(0) &= \textstyle\bigvee_{x \in f^{-1}(0)} A(x) \\ &= A(0) \text{ [Since } 0 \in f^{-1}(0)] \\ &\geq A(d(t_0)) \text{ for some } d(t_0) \in f^{-1}(\overline{d}(y)) \\ &= \textstyle\bigvee_{d(t) \in f^{-1}(y)} A(t) \text{ [Since } A \text{ has the sup-property]} \\ &= f(A)(\overline{d}(y)). \end{aligned}$$

Thus $f(A)$ satisfies the condition (FDI_1).

Now, for any $x,\ y,\ z \in Y$ and positive integers $m,\ n$, let $d(a_0) \in f^{-1}(\overline{d}(x))$, $d(b_0) \in f^{-1}(\overline{d}(y))$, $d(c_0^m) \in f^{-1}(\overline{d}(z^m))$, $d(c_0^n) \in f^{-1}(\overline{d}(z^n))$ such that

$$A(d(a_0 * c_0^n)) = \bigvee_{d(t) \in f^{-1}(\overline{d}(x * z^n))} A(d(t)), \tag{4.4}$$

$$A(d(b_0)) = \bigvee_{d(t)\in f^{-1}(\overline{d}(y))} A(d(t)), \tag{4.5}$$

$$A(d((a_0 * b_0) * c_0^m))) = \bigvee_{d(t)\in f^{-1}(\overline{d}((x*y)*z^m)))} A(d(t)). \tag{4.6}$$

Then we have

$$\begin{aligned}
&f(A)(\overline{d}(x * z^n))\\
&= \textstyle\bigvee_{d(t)\in f^{-1}(\overline{d}(x*z^n))} A(d(t)) \text{ [By (4.1)]}\\
&= A(d(a_0 * c_0^n)) \text{ [By (4.4)]}\\
&\geq A(d((a_0 * b_0) * c_0^m))) \wedge A(d(b_0)) \text{ [By (FLDI}_2\text{]}\\
&= \left(\textstyle\bigvee_{d(t)\in f^{-1}(\overline{d}((x*y)*z^m)))} A(d(t))\right) \wedge \left(\textstyle\bigvee_{d(t)\in f^{-1}(\overline{d}(y))} A(d(t))\right)\\
&\quad \text{[By (4.5) and (4.6)]}\\
&= f(A)(\overline{d}(x * y) * z^m)) \wedge f(A)(\overline{d}(y)).
\end{aligned}$$

Thus $f(A)$ satisfies the condition ($FLDI_2$). So $f(A)$ is a fuzzy left derivation (m, n)-fold BCC-ideal with respect to $\overline{d}$ of Y.

The proof of the second part is similar. □

5. The Cartesian product of fuzzy left (right derivation (m, n)-fold BCC-ideals

Definition 5.1 ([23])**.** Let X be a nonempty set. Then a mapping $R : X \times X \to I$ is called a *fuzzy relation on* X.

Definition 5.2 ([23])**.** Let X be a nonempty set and $A \in I^X$. Then R_A is called a fuzzy relation on A, if $R_A(x, y) \leq A(x) \wedge A(y)$ for any $x,\ y \in X$.

Definition 5.3 ([23])**.** Let X be a nonempty set and $A,\ B \in I^X$. Then the *Cartesian product* of A and B, denoted by $A \times B$, is a fuzzy set in $X \times X$ defined as follows: for any $x,\ y \in X$,

$$(A \times B)(x, y) = A(x) \wedge B(y).$$

Note that if d is a self mapping of X, then $d \times d$ is a self mapping of $X \times X$, where

$$(d \times d)(x, y) = (d(x), d(y)) \text{ for each } (x, y) \in X \times X.$$

Definition 5.4. Let X be a nonempty set, d a self mapping of X and $A \in I^X$. Then the *fuzzy derivation set of* A *with respect to* d, denoted by A_d, is a fuzzy set in X defined by: for each $x \in X$,

$$A_d(x) = A(d(x)).$$

Example 5.5. Let $X = \{0, a, b, c\}$ be a universal set and d the self mapping of X defined by: for each $x \in X$,

$$d(x) = \begin{cases} x & \text{if } x \in \{0, a, c\} \\ c & \text{if } x = b. \end{cases} \tag{5.1}$$

Consider the fuzzy set A in X defined by:

$$A(0) = 0.7,\ A(a) = 0.4,\ A(b) = 0.3,\ A(c) = 0.1.$$

Then the fuzzy derivation set A_d of A with respect to d is given as follows:

$$A_d(0) = 0.7,\ A_d(a) = 0.7,\ A_d(b) = 0.1,\ A_d(c) = 0.7.$$

Definition 5.6. Let X be a nonempty set and d a self mapping of X. Let A_d be a fuzzy derivation set of A with respect to d in X and R a fuzzy relation on X. Then $R_{d\times d}$ is called a *fuzzy derivation relation of R with respect to $d \times d$ on X*, if it satisfies the the following condition:

$$R_{d\times d}(x, y) = R((d \times d)(x, y)) \leq A(d(x)) \wedge A(d(y)) \text{ for any } x,\ y \in X.$$

Definition 5.7. Let X be a nonempty set, and $A,\ B$ be fuzzy derivation sets of A_d and B_d with respect to d in X, respectively. Then the *Cartesian product* of A and B, denoted by $A_d \times B_d = (A \times B)_{d\times d}$, is a fuzzy relation on X defined by: for any $x,\ y \in X$,

$$(A \times B)((d \times d)(x, y)) = (A_d \times B_d)(x, y) = A_d(x) \wedge B_d(y) = A(d(x)) \wedge B(d(y)).$$

It is obvious that $A_d \times B_d$ is a fuzzy derivation relation of $A \times B$ with respect to $d \times d$ on X.

Example 5.8. Let $X = \{0, a, b, c\}$ and d the self mapping of X defined as follows: for each $x \in X$ Consider two fuzzy sets $A,\ B$ in X defined as follows:

$$d(x) = \begin{cases} x & \text{if } x \in \{0, a, c\} \\ b & \text{if } x = b. \end{cases} \tag{5.2}$$

$$A(0) = 0.7,\ A(a) = 0.4,\ A(b) = 0.3,\ A(c) = 0.1,$$
$$B(0) = 0.8,\ B(a) = 0.6,\ B(b) = 0.5,\ B(c) = 0.3.$$

Then clearly, we have

$$A_d(0) = 0.7,\ A_d(a) = 0.7,\ A_d(b) = 0.3,\ A_d(c) = 0.7,$$
$$B_d(0) = 0.8,\ B_d(a) = 0.8,\ B_d(b) = 0.5,\ B_d(c) = 0.8.$$

Thus $A_d \times B_d$ is the fuzzy derivation relation of $A \times B$ with respect to $d \times d$ on X defined by the table:

$(A_d \times B_d)(x, y)$	0	a	b	c
0	0.7	0.7	0.5	0.7
a	0.7	0.7	0.5	0.7
b	0.3	0.3	0.3	0.3
c	0.7	0.7	0.5	0.7

Table 5.1

Definition 5.9. Let X be a nonempty set and A_d a fuzzy derivation set of A with respect to d in X. Then $R_{A,d\times d}$ is called the *strongest fuzzy derivation relation of R_A with respect to $d\times d$ on A*, if it satisfies the following condition: for any $x,\ y \in X$,

$$R_{A,d\times d}(x, y) = R_A((d \times d)(x, y)) = A(d(x)) \wedge A(d(y)).$$

Definition 5.10. Let $X,\ Y$ be BCC-algebras. We define the operation $*$ on $X \times Y$ by: for any $(x, y),\ (u, v) \in X \times Y$,

$$(x, y) * (u, v) = (x * u, y * v).$$

It is clear that $(X \times Y, *, (0, 0))$ is a BCC-algebra.

Remark 5.11. Let $X,\ Y$ be nonempty sets, d_1 and d_2 self mappings of X and Y, respectively. Then the mapping $d = d_1 \times d_2 : X \times Y \to X \times Y$ defined by: for each $(x, y) \in X \times Y$,

$$d(x, y) = (d_1(x), d_2(y))$$

is a self mapping of $X \times Y$.

From Definition 2.10, we can easily see that if $d_1,\ d_2$ are (l, r) [resp. (r, l)]-derivations, then d is an (l, r) [resp. (r, l)]-derivation. Furthermore, we can show that $d_1,\ d_2$ are regular if and only if d is regular.

Proposition 5.12. *Let A be a fuzzy derivation set with respect to d in X and R_A the strongest fuzzy derivation relation with respect to $d \times d$ on A. If R_A is a fuzzy derivation (m, n)-fold BCC-ideal of $X \times X$, then $A(d(x)) \leq A(0)$ for each $x \in X$.*

Proof. Suppose R_A is a fuzzy derivation (m, n)-fold BCC-ideal of $X \times X$ and let $x \in X$. Then we have

$$\begin{aligned} A(d(x)) &= A(d(x)) \wedge A(d(x)) \\ &= R_A((d \times d)(x, x)) \text{ [By Definition 5.9]} \\ &\leq R_A((d \times d)(0, 0)) \text{ [By (FDI}_1\text{)]} \\ &= A(d(0)) \wedge A(d(0)) \\ &= A(d(0)) \\ &= A(0). \end{aligned}$$

Thus $A(d(x)) \leq A(0)$. □

Proposition 5.13. *Let $X,\ Y$ be BCC-algebras and $d_1,\ d_2$ self mappings of X and Y, respectively. If A and B are fuzzy derivation (m, n)-fold BCC-ideals with respect to $d_1,\ d_2$, of X and Y, respectively, then $A \times B$ is a fuzzy derivation (m, n)-fold BCC-ideal with respect to $d = d_1 \times d_2$ of $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$. Then we get

$$\begin{aligned} (A \times B)(0, 0) &= A(0) \wedge B(0) \\ &\geq A(d(x)) \wedge B(d(y)) \text{ [By the condition (FDI}_1\text{)]} \\ &= (A \times B)(d \times d)(x, y)). \end{aligned}$$

Thus $A \times B$ satisfies the condition (FDI$_1$).

Now let $(x_1, x_2),\ (y_1, y_2), (z_1, z_2) \in X \times Y$. Then we have

$$\begin{aligned} &(A \times B)(d(x_1 * z_1^n, x_2 * z_2^n)) \\ &= A(d_1(x_1 * z_1^n)) \wedge B(d_2(x_2 * z_2^n)) \\ &\geq [A(d_1((x_1 * y_1) * z_1^m)) \wedge A(d_1(y_1))] \wedge [B(d_2((x_2 * y_2) * z_2^m)) \wedge B(d_2(y_2))] \\ &\quad \text{[By the condition (FDI}_2\text{)]} \\ &= [A(d_1((x_1 * y_1) * z_1^m)) \wedge B(d_2((x_2 * y_2) * z_2^m))] \wedge [A(d_1(y_1)) \wedge B(d_2(y_2))] \\ &= (A \times B)(d((x_1 * y_1) * z_1^m), (x_2 * y_2) * z_2^m)) \wedge (A \times B)(d(y_1, y_2)). \end{aligned}$$

Thus $A \times B$ satisfies the condition (FDI$_2$). So $A \times B$ is a fuzzy derivation (m, n)-fold BCC-ideal with respect to d of $X \times Y$. □

Analogous to Theorem 2.2 in [24], we have a similar result for fuzzy derivation (m,n)- fold BCC-ideal, which can be proved in similar manner. Then we state the result without proof.

Proposition 5.14. *Let X, Y be BCC-algebras and d_1, d_2 self mappings of X and Y, respectively. Let $A \times B$ is a fuzzy derivation (m,n)-fold BCC-ideal with respect to d of $X \times Y$. Then the followings hold:*
(1) *either $A(d_1(x)) \leq A(d_1(0))$ or $B(d_2(x)) \leq B(d_2(0))$ for each $x \in X$,*
(2) *if $A(d_1(x)) \leq A(d_1(0))$ for each $x \in X$, then either $A(d_1(x)) \leq B(d_2(0))$ or $B(d_2(x)) \leq B(d_2(0))$,*
(3) *if $B(d_2(x)) \leq B(d_2(0))$ for each $x \in X$, then either $A(d_1(x)) \leq A(d_1(0))$ or $B(d_2(x) \leq A(d_1(0))$,*
(4) *either A or B is a fuzzy derivation (m,n)-fold BCC-ideal of X or Y.*

Theorem 5.15. *Let A_d be a fuzzy derivation set of A with respect to d in X and $R_{A,d\times d}$ the strongest fuzzy derivation relation of R_A with respect to $d\times d$ on A. Then A is a fuzzy derivation (m,n)-fold BCC-ideal with respect to d of X if and only if R_A is a fuzzy derivation (m,n)-fold BCC-ideal with respect to $d \times d$ of $X \times X$.*

Proof. Suppose A is a fuzzy derivation (m,n)-fold BCC-ideal with respect to d of X and let $(x,y) \in X\times X$. Since d is a derivation, by Remark 5.11, $d\times d$ is a derivation. By Result 2.13, $d \times d$ is regular, i.e., $(d \times d)(0,0) = (0,0)$. Then we get

$$\begin{aligned} R_A(0,0) &= R_A((d \times d)(0,0)) \\ &= A(d(0)) \wedge A(d(0)) \text{ [By Definition 5.9]} \\ &= A(0) \wedge A(0) \text{ [By Result 2.13]} \\ &\geq A(d(x)) \wedge A(d(y)) \text{ [By the condition (FDI}_1\text{)]} \\ &= R_A((d \times d)(x,y)). \end{aligned}$$

Thus R_A satisfies the condition (FDI$_1$).

Now let $(x_1,x_2),\ (y_1,y_2),\ (z_1,z_2) \in X \times X$. Then

$$\begin{aligned} &R_A((d \times d)(x_1 * z_1^n, x_2 * z_2^n)) \\ &= A(d(x_1 * z_1^n)) \wedge A(d(x_2 * z_2^n)) \\ &\geq [A(d((x_1 * y_1) * z_1^m))) \wedge A(d(y_1))] \wedge [A(d((x_2 * y_2) * z_2^m))) \wedge A(d(y_2))] \\ &\quad \text{[By the condition (FDI}_2\text{)]} \\ &= [A(d((x_1 * y_1) * z_1^m))) \wedge A(d((x_2 * y_2) * z_2^m)))] \wedge [A(d(y_1)) \wedge A(d(y_2))] \\ &= R_A((d \times d)(((x_1 * y_1) * z_1^m), (x_2 * y_2) * z_2^m))) \wedge R_A((d \times d)(y_1, y_2)). \end{aligned}$$

Thus R_A satisfies the condition (FDI$_2$). So R_A is a fuzzy derivation (m,n)-fold BCC-ideal with respect to $d \times d$ of $X \times X$.

Conversely, suppose R_A is a fuzzy derivation (m,n)-fold BCC-ideal with respect to $d \times d$ of $X \times X$ and let $(x,y) \in X \times X$. Since $d \times d$ is a derivation, by Remark 5.11, d is a derivation. By Result 2.13, d is regular, i.e., $d(0) = 0$. Then we have

$$\begin{aligned} A(0) \wedge A(0) &= A(d(0)) \wedge A(d(0)) \\ &= R_A((d \times d)(0,0)) \\ &\geq R_A((d \times d)(x,y)) \text{ [By the condition (FDI}_1\text{)]} \\ &= A(d((x)) \wedge A(d(y)). \end{aligned}$$

Thus $A(0) \geq A(d(x))$. So A satisfies the condition (FDI$_1$).

Now let $(x,0),\ (y,0),\ (z,0) \in X \times X$. Then we get

$$\begin{aligned} &A(d(x * z^n)) \\ &= A(d(x * z^n)) \wedge A(d(0 * 0^n)) \text{ [Since } d \text{ is regular and } A \text{ satisfies (FDI}_1\text{)]} \end{aligned}$$

$= R_A((d \times d)(x * z^n, 0 * 0^n))$
$\geq R_A((d \times d)((x * y) * z^m), (0 * 0) * 0^m)) \wedge R_A((d \times d)(y, 0))$
[By the condition (FDI$_2$)]
$= [A(d((x * y) * z^m)) \wedge A(d((0 * 0) * 0^m))] \wedge [A(d(y)) \wedge A(d(0)]$
$= [A(d((x * y) * z^m)) \wedge A(d(y))] \wedge [A(d((0 * 0) * 0^m)) \wedge A(d(0))]$
$= A(d((x * y) * z^m)) \wedge A(d(y))$. [Since d is regular and A satisfies (FDI$_1$)]

Thus A satisfies the condition (FDI$_2$). So A is a fuzzy derivation (m, n)-fold BCC-ideal with respect to d of X. □

6. Conclusions

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, the notion of fuzzy left and right derivation BCC-ideal in BCC-algebra are introduced and investigated the useful properties of fuzzy left and right derivation BCC-ideals in BCC-algebras.

In connection with the notion of homomorphism, the authors study how the image and the pre-image of fuzzy left (right) derivation (m, n)-fold BCC-ideals under homomorphism of BCC-algebras become fuzzy left (right) derivation (m, n)-fold BCC-ideals. Furthermore, the Cartesian product of fuzzy left (right) derivation (m, n)-fold BCC-ideals in Cartesian product of BCC-algebras is introduced and investigated some related properties.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCI-algebra, BCH-algebra, Hilbert algebra, BF-algebra, J-algebra, WS-algebra, CI-algebra, SU-algebra, BCL-algebra, BP-algebra, BO-algebra, PU-algebras and so forth. The main purpose of our future work is to investigate:

(1) The interval valued, bipolar and intuitionist fuzzy left and right derivations BCC-ideal in BCC-algebra.

(2) To consider the cubic structure left and right derivations BCC-ideal in BCC-algebra.

We hope the fuzzy left and right derivations BCC-ideals in BCC-algebras, have applications in different branches of theoretical physics and computer science.

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