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Soft set theory applied to residuated multilattices

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Abstract. Dealing with uncertainty, imprecision and fuzziness, *Soft set theory* is one of many mathematical tools; it is a classical generalization of fuzzy set theory. Many researchers study the case of a universe set endowed with an algebraic structure. This paper uses residuated multilattices as the universe set for soft set theory. The notions of *f-soft residuated multilattice, r-soft residuated multilattice, soft filter, filteristic residuated multilattice* are defined, and several related properties are investigated.

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1. Introduction

Uncertainties occur in all areas of science and must therefore be managed rather than avoided. Theories such as probability, fuzzy sets [1], vague sets [2] and rough sets [3] are mathematical tools for handling uncertainties. As pointed out in [4], the major drawback of all these theories is the lack of consideration of parameters. That is why, soft set theory [4] was proposed by Molodtsov in 1999 to deal with uncertainty in a parametric manner. A soft set is defined as a parameterized family of sets which can be considered as an approximated description of an object, precisely consisting of two parts, namely, predicate and approximate value set. Because of its great potential, soft set theory has been rapidly developed with several applications. In [5, 6], a theoretical study is carried out on soft sets. A survey on some of the main developments of applications of soft sets theory in decision-making problems is given in [4, 7, 8]. Applications of soft set theory to some algebraic structures such as groups [9], rings [10, 11], lattices [12, 13], BCK/BCI-algebras [14], BL-algebras [15] and to general algebras in [16].

One of the most widely used algebraic theories is lattice theory. A lattice is a poset where every pair of elements has a least upper bound and a greatest lower bound. However, there are posets in which the set of upper (resp. lower) bounds of a pair of elements has no smallest (resp. greatest) element, but minimal (resp. maximal) elements. Benado has formalized this in [17] by extending the notion of lattice to that of multilattice by weakening the uniqueness condition regarding upper and lower bounds to the existence of minimal upper bounds and maximal lower bounds. Residuation plays an important role in the algebraic study of logical systems; this lead to logical algebraic structures, i.e., algebraic structures modeling logical systems. The most widely used and important of these are residuated lattices. Focusing on the link between the theory of multilattices and residuation, Cabrera et al. introduce residuated multilattices [18]. Some authors have used multilattices and residuated multilattices in the framework of fuzzy sets theory [19, 20].

We have made in [21] application of soft set theory to multilattices. Now, we focus on residuated multilattices in the framework of soft sets. We defined the notions of *f-soft residuated multilattice, r-soft residuated multilattice, soft filter, filteristic soft residuated multilattice* and studied their related properties. The paper is organized as follows: in Section 2, we set some preliminaries on soft sets and residuated multilattices. In Section 3, we defined the notions of f-soft residuated multilattice and r-soft residuated multilattice and study their properties related to operations on soft sets. We focus on soft filter of an f-soft residuated multilattice in Section 4 while Section 5 deals with filteristic soft residuated multilattice. Finally, we draw some conclusions and prospects for future work in Section 6.

2. Preliminaries

Given an initial universe set U and a set E of parameters, Molodtsov [4] defines the notion on *soft set* over U as a pair (f, E) such that f is a mapping from E to the power set $P(U)$ of U. Thus, a soft set over a universe U is a parameterized family of subsets of U. To define a soft set over a universe U means, for each parameter, to point out the set of all elements of U that possess the given parameter. Of course, this set can be empty for some parameters. For a soft set (f, E), the set of parameters with a non-empty image subset of U is called the *support* of (f, E) and denoted by *supp(f,E)*. Then *supp(f,E)* $= \{p \in E : f(p) \neq \varnothing\}$. It is often the practice to define a soft set with a part of the set of parameters rather than the whole parameter's set. In what follows, we denote a soft set (f, E) with support A by f_A. If the support A is empty, then $f_\varnothing$ is the *null soft set* and will be denoted by Φ.

Let us recall some basics operators on soft sets (one can refer to [6]). Let f_A and h_B be two soft sets over a universe U with the set of parameters E. Then f_A is called a *soft subset* of h_B, if $A \subseteq B$ and $f(p) \subseteq h(p)$ for all $p \in A$. In this case, h_B is a *soft supper set* of f_A. f_A and h_B are said to be *soft equal*, if f_A is a soft subset and a soft supper set of h_B. The *intersection* of f_A and h_B denoted by $f_A \cap h_B$ is the soft set (t, E), where $t(p) = f(p) \cap h(p)$ for all $p \in E$. Clearly, if $A \cap B = \varnothing$, then $f_A \cap h_B$ is the null soft set. If $f_A \cap h_B$ is non-null, then its support is the set $\{p \in A \cap B : f(p) \cap h(p) \neq \varnothing\}$. The *union* of the soft sets f_A and h_B is the soft set (t, E), where $t(p) = f(p) \cup h(p)$ for all $p \in E$. It is possible [5] to restrict the parameter set of the union or extend that of the intersection of two soft sets and get

a *restricted union* $\cup_r$ by considering the intersection of the sets of parameters or an *extended intersection* $\cap_e$ by considering the union of the sets of parameters. Clearly, the restricted union of soft sets f_A and h_B over a common universe U denoted by $f_A \cup_r h_B$ is the soft set $t_{A\cap B}$ such that $t(p) = f(p) \cup h(p)$ if $p \in A \cap B$ and the extended intersection of f_A and h_B denoted by $f_A \cap_e h_B$ is the soft set $t_{A\cup B}$ such that for all $p \in A \cup B$, $t(p) = \begin{cases} f(p) & \text{if } p \in A \setminus B \\ h(p) & \text{if } p \in B \setminus A, \\ f(p) \cap h(p) & \text{else} \end{cases}$
and the restricted union of f_A and h_B with $A \cap B \neq \varnothing$ is the soft set $t_{A\cap B}$ such that for all $p \in A \cap B$, $t(p) = f(p) \cup h(p)$.

Given two soft sets f_A and h_B over a common universe parameterized by a set E, one can define some soft sets with the set of parameters being a subset of $E \times E$. The soft set"f_A AND h_B", denoted by $f_A \wedge h_B$ is defined by $f_A \wedge h_B = (t, A \times B)$, where $t(p, q) = f(p) \cap h(q)$ for all $(p, q) \in E \times E$. While "f_A OR h_B", denoted by $f_A \vee h_B$ is defined by $f_A \vee h_B = (s, A \times B)$, with $s(p, q) = f(p) \cup h(q)$, for all $(p, q) \in A \times B$.

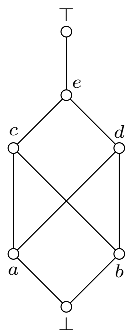
Additional operations such as difference, restricted difference, symmetric difference and restricted symmetric difference of two soft sets can be found in [5, 21].

Now, let us give some preliminaries on residuated multilattices (see [17, 22]). Multilattices are natural generalization of lattices. Given a partially ordered set (poset) $(P, \leq)$, x and y two elements of P such that $x \leq y$, we say that x is *below* y or y is *above* x. A poset $(P, \leq)$ is called a *multilattice*, if for any finite subset X of P, each upper bound of X is above a minimal upper bound of X and each lower bound of X is below a maximal lower bound of X. We denote $x \sqcap y$ (resp. $x \sqcup y$) the set of minimal upper bounds (resp. maximal lower bounds) of $\{x, y\}$. We will sometimes write $x \sqcap y = a$ instead of $x \sqcap y = \{a\}$ and $x \sqcup y = b$ instead of $x \sqcup y = \{b\}$ when $x \sqcap y$ or $x \sqcup y$ is a singleton. A multilattice is said to be *full*, if $x \sqcap y$ and $x \sqcup y$ are non-empty for all x, y. A map $\varphi : M \to N$ between two multilattices is said to be a *homomorphism* [17] if $\varphi(x \sqcap y) \subseteq \varphi(x) \sqcap \varphi(y)$ and $\varphi(x \sqcup y) \subseteq \varphi(x) \sqcup \varphi(y)$ for all $x, y \in M$. The authors in [18] showed that when the initial multilattice is full, the notion of homomorphism can be characterized in terms of equalities as follows: let $\varphi : M \to N$ be a map between multilattices such that M is full. Then φ is a homomorphism iff $\varphi(x \sqcap y) = (\varphi(x) \sqcap \varphi(y)) \cap \varphi(M)$ and $\varphi(x \sqcup y) = (\varphi(x) \sqcup \varphi(y)) \cap \varphi(M)$ for all $x, y \in M$.

A structure $\mathcal{P} := (P, \leq, \top, \odot, \to)$, is said to be *pocrim* (partially ordered commutative residuated integral monoid), if $(P, \leq, \top)$ is a poset with a greatest element $\top$ and $(P, \odot, \top)$ is a commutative monoid such that

$$x \odot y \leq z \Leftrightarrow x \leq y \to z \quad \text{for all } x, y, z \in P. \tag{2.1}$$

Then the pair $(\odot, \to)$ is called the *residuation*. If a pocrim has a least element, usually denoted by $\perp$, it is said to be *bounded*. A *residuated multilattice* ($\mathcal{RML}$ for short) is a multilattice endowed with a residuation. It combines a structure of pocrim and a structure of multilattice. Quite simply, it is a pocrim whose underling poset is a multilattice [18]. When the underling poset is a lattice, the pocrim is called a *residuated lattice*. Clearly, a residuated lattice is an $\mathcal{RML}$. When an $\mathcal{RML}$ is not a residuated lattice, it is said to be *pure*. The authors in [23] show that every



$\odot$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	a	a	a
b	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	b
c	$\bot$	a	$\bot$	a	a	a	c
d	$\bot$	a	$\bot$	a	a	a	d
e	$\bot$	a	$\bot$	a	a	a	e
$\top$	$\bot$	a	b	c	d	e	$\top$

$\to$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	$\top$	$\top$	$\top$	$\top$
b	e	e	$\top$	$\top$	$\top$	$\top$	$\top$
c	b	e	b	$\top$	e	$\top$	$\top$
d	b	e	b	e	$\top$	$\top$	$\top$
e	b	e	b	e	e	$\top$	$\top$
$\top$	$\bot$	a	b	c	d	e	$\top$

FIGURE 1. A residuated multilattice with seven elements: RML-7.

bounded pure $\mathcal{RML}$ has at least seven elements. Figure 1 gives the $\mathcal{RML}$ with seven elements RML-7 from [23] that we will use throughout the paper.

Let $\mathcal{P}$ be an $\mathcal{RML}$ and X be a subset of P. X is called a *full residuated submultilattice* (f**-sub-**$\mathcal{RML}$), if X contains $\top$, $x \odot y, x \to y$, and $x \sqcap y, x \sqcup y \subseteq X$ for all $x, y \in X$. Similarly, X is called a *restricted residuated submultilattice* (r**-sub-**$\mathcal{RML}$), if X contains $\top$, $x \odot y, x \to y$ and $(x \sqcap y) \cap X \neq \varnothing$, $(x \sqcup y) \cap X \neq \varnothing$ for all $x, y \in X$. For example, $\{a, c, e, \top\}$ is an f-sub-$\mathcal{RML}$ and an r-sub-$\mathcal{RML}$ of RML-7. $\{a, c, d, e, \top\}$ is an r-sub-$\mathcal{RML}$ of RML-7 which is not an f-sub-$\mathcal{RML}$, since $c \sqcap d = \{a, b\}$ and $b \notin \{a, c, d, e\top\}$. One can remark that, as for Multilattices, an f-sub-$\mathcal{RML}$ of an $\mathcal{RML}$ is an $\mathcal{RML}$ on its own. Moreover, any intersection of two f-sub-$\mathcal{RML}s$ is also an f-sub-$\mathcal{RML}$, but it is not the case for r-sub-$\mathcal{RML}s$. Indeed, $X = \{\bot, a, b, c, e, \top\}$ and $Y = \{\bot, a, b, d, e, \top\}$ are r-sub-$\mathcal{RML}s$, but $X \cap Y = \{\bot, a, b, e, \top\}$ is not an r-sub-$\mathcal{RML}$, since $a \sqcup b = \{c, d\}$ and $\{c, d\} \cap (X \cap Y) = \varnothing$.

We end this section by recalling some useful properties that hold in a pocrim in general and in an $\mathcal{RML}$ in particular. We consider a pocrim $\mathcal{P}$ which is an $\mathcal{RML}$. Let $x, y, z \in P$.

P1 $x \to x = \top$ and $\top \to x = x$,
P2 $x \odot y \leq x$ and $x \odot y \leq y$, then $x \leq y \to x$,
P3 $x \odot y \leq x \to y$,
P4 $x \leq y \Leftrightarrow x \to y = \top$,
P5 $x \to y = y \to x = \top \Leftrightarrow x = y$.

3. f-SOFT RESIDUATED MULTILATTICE AND r-SOFT RESIDUATED MULTILATTICE

Throughout this section and the rest of the study, we will consider an $\mathcal{RML}$ $\mathcal{M} := (M, \leq, \top, \odot, \to)$ as our universe set and E a set of parameters. To simplify, we will write M to denote the support or the whole residuated multilattice.

Definition 3.1. Let M be an $\mathcal{RML}$ and f_A be a non-null soft set over M. Then f_A said to be:

(i) *full soft residuated multilattice* (briefly, f-soft-$\mathcal{RML}$) over M, if $f(p)$ is an f-sub-$\mathcal{RML}$ of M for all $p \in A$,

(ii) *restricted soft residuated multilattice* (briefly, r-soft-$\mathcal{RML}$) over M, if $f(p)$ is an r-sub-$\mathcal{RML}$ of M for all $p \in A$.

Remark 3.2. Let M be an $\mathcal{RML}$ and $x \in M$. The following statements hold:

(1) $\{\top\}$ is an f-sub-$\mathcal{RML}$ of M,
(2) $\{x, \top\}$ is an f-sub-$\mathcal{RML}$ of M iff $x \odot x = x$.

Example 3.3. Let consider RML-7 of Figure 1 and $E = \{p_1, p_2, p_3\}$. Let f_A and h_B be the soft sets over RML-7 defined by

$$f_A = \{(p_1, \{\bot, \top\}), (p_2, \{a, \top\})\},$$
$$h_B = \{(p_1, \{\bot, a, b, c, e, \top\}), (p_2, \{\bot, b, d, e, \top\})\},$$
$$t_C = \{(p_1, \{\bot, \top\}), (p_2, \{\bot, a, \top\})\}.$$

Then we have $f(p_1) = \{\bot, \top\}$, $f(p_2) = \{a, \top\}$. By Remark 3.2, $f(p_1)$ and $f(p_2)$ are f-sub-$\mathcal{RML}s$ of RML-7. Thus f_A is an f-soft-$\mathcal{RML}$ over RML-7. However $h(p_1) = \{\bot, a, b, c, e, \top\}$ is not an f-sub-$\mathcal{RML}$ of RML-7, as $a \sqcup b = \{c, d\}$ and we have $\{c, d\} \nsubseteq \{\bot, a, b, c, e, \top\}$. So h_B is not an f-soft-$\mathcal{RML}$ over RML-7. Nevertheless, h_B is an r-soft-$\mathcal{RML}$ over RML-7.

t_C is neither an f-soft-$\mathcal{RML}$ nor an r-soft-$\mathcal{RML}$ because $t(p_2) = \{\bot, a, \top\}$ and $a \to \bot = b \notin t(p_2)$.

Remark 3.4. Let M be an $\mathcal{RML}$ and f_A be an f-soft-$\mathcal{RML}$ (resp. r-soft-$\mathcal{RML}$) over M. If $\varnothing \neq B \subseteq A$, then f_B is an f-soft-$\mathcal{RML}$ (resp. r-soft-$\mathcal{RML}$) over M.

Knowing that the intersection of f-sub-$\mathcal{RML}s$ of an $\mathcal{RML}$ M is also an f-sub-$\mathcal{RML}$ of M, we have the following result concerning the intersection of two f-soft-$\mathcal{RML}s$ over an $\mathcal{RML}$ M, whose proof is straightforward.

Proposition 3.5. *Let M be an $\mathcal{RML}$, f_A and h_B be two non-null f-soft-$\mathcal{RML}s$ over M. Then $f_A \cap h_B$ and $f_A \wedge h_B$ are f-soft-$\mathcal{RML}s$ over M if they are non-null.*

We have shown in Section 2 that the intersection of r-sub-$\mathcal{RML}s$ of an $\mathcal{RML}$ M is not always an r-sub-$\mathcal{RML}$. This is also the case for the intersection of r-soft-$\mathcal{RML}s$. Indeed, by considering the $\mathcal{RML}$ in Figure 1, $f_A = \{(p_1, \{\bot, a, b, c, e, \top\})\}$ and $h_B = \{(p_1, \{\bot, a, b, d, e, \top\})\}$, clearly, f_A and h_B are r-soft-$\mathcal{RML}s$ over RML-7 but $f_A \cap h_B$ is not since $f(p_1) \cap h(p_1) = \{\bot, a, b, e, \top\}$ is not an r-sub-$\mathcal{RML}$ of RML-7. Moreover, $f_A \wedge h_B = \{(p_1, p_1), \{\bot, a, b, e, \top\})\}$ is not an r-soft-$\mathcal{RML}$.

Proposition 3.6. *Let M be an $\mathcal{RML}$, f_A and h_B be two non-null f-soft-$\mathcal{RML}s$* (resp. r-soft-$\mathcal{RML}s$) *over M. If $A \cap B = \varnothing$, then $f_A \cup h_B$ is an f-soft-$\mathcal{RML}$* (resp. r-soft-$\mathcal{RML}$) *over M.*

Proof. We know that $f_A \cup h_B$ is the soft set (t, E), where $t(p) = f(p) \cup h(p)$ for all $p \in A \cup B$ and $t(p) = \varnothing$ if $p \notin A \cup B$. Suppose that f_A and h_B are f-soft-$\mathcal{RML}s$ over M and $A \cap B = \varnothing$. Then for all $p \in A \cup B$, either $t(p) = f(p)$ or $t(p) = h(p)$, as $A \cup B = (A \setminus B) \cup (B \setminus A)$. As f_A and h_B are f-soft-$\mathcal{RML}s$ over M, if $p \in A$, then $f(p)$ is an f-sub-$\mathcal{RML}$ of M and if $p \in B$, then $h(p)$ is an f-sub-$\mathcal{RML}$ of M. That is $t(p)$ is an f-sub-$\mathcal{RML}$ of M for all $p \in A \cup B$. This implies that $f_A \cup h_B$ is an f-soft-$\mathcal{RML}$ over M as required.

The proof for r-soft-$\mathcal{RML}$ can be obtained similarly. □

If $A \cap B \neq \varnothing$, then $f_A \cup h_B$ may not be an f-soft-$\mathcal{RML}$. Let consider, for example $f_A = \{(p_1, \{\bot, \top\})\}$ and $h_B = \{(p_1, \{a, \top\})\}$ which are f-soft-$\mathcal{RML}s$ over RML-7. We have $f_A \cup h_B = \{(p_1, \{\bot, a, \top\})\}$ which is not an f-soft-$\mathcal{RML}$ because $\{\bot, a, \top\}$ is not an f-sub-$\mathcal{RML}$ of RML-7, since $a \to \bot = b \notin \{\bot, a, \top\}$. However, $f_A \cup h_B$ can be an f-soft-$\mathcal{RML}$ even if $A \cap B \neq \varnothing$ as explained below in Proposition 3.7, whose proof is similar to Proposition 3.6.

Proposition 3.7. *Let M be an $\mathcal{RML}$, f_A and h_B be two non-null f-soft-$\mathcal{RML}s$ (resp. r-soft-$\mathcal{RML}s$) over M such that for all $p \in E$, $f(p) \subseteq h(p)$ or $h(p) \subseteq f(p)$. Then $f_A \cup h_B$ is an f-soft-$\mathcal{RML}$* (resp. r-soft-$\mathcal{RML}$) *over M.*

If f_A and h_B are two non-null f-soft-$\mathcal{RML}s$ over an $\mathcal{RML}$ M, $f_A \vee h_B$ may not be an f-soft-$\mathcal{RML}$ even if $A \cap B = \varnothing$. Indeed, from the $\mathcal{RML}$ of Figure 1, let consider $f_A = \{(p_1, \{\bot, \top\})\}$ and $h_B = \{(p_2, \{a, \top\})\}$. By Remark 3.2, f_A and h_B are f-soft-$\mathcal{RML}s$ over RML-7. However, $f_A \vee h_B = \{((p_1, p_2), \{\bot, a, \top\})\}$ is not an f-soft-$\mathcal{RML}$. One can also observe that f_A and h_B are r-soft-$\mathcal{RML}s$ but $f_A \vee h_B$ is not.

Proposition 3.8. *Let M be an $\mathcal{RML}$, f_A and h_B be two non-null f-soft-$\mathcal{RML}s$ over M. Then $f_A \cap_{\mathfrak{g}} h_B$ is an f-soft-$\mathcal{RML}$ over M if it is non-null.*

Proof. If $f_A \cap_{\mathfrak{g}} h_B$ is non-null, then $f_A \cap_{\mathfrak{g}} h_B$ is the soft set (t, E) such that for all $p \in A \cup B$, $t(p) = \begin{cases} f(p) & \text{if } p \in A \setminus B \\ h(p) & \text{if } p \in B \setminus A \\ f(p) \cap h(p) & \text{else .} \end{cases}$
In each case, $t(p)$ is an f-sub-$\mathcal{RML}$ of M. We conclude that $f_A \cap_{\mathfrak{g}} h_B$ is an f-soft-$\mathcal{RML}$ over M. □

4. Soft filter of an f-soft residuated multilattice

The notion of filter, originated in topology, is very useful in various algebraic structures endowed with a (partial) order. As stated above, the structure of residuated multilattice is a combination of two structures, namely the structure of pocrim and the structure of multilattice. In what follows, we will consider the notion of filter in the structure of pocrim that we will call p-filter, the notion of filter in a multilattice that we will call m-filter and we will call filter the one that combines the structures of pocrim and multilattice. In this section, we extend this notion to the framework of soft set theory.

Let recall from [18] the definition and some properties on filters in residuated multilattices.

Definition 4.1. Let M be an $\mathcal{RML}$. A non-empty subset $F \subseteq M$ is said to be a *p-filter* of M, if the following conditions hold:

(i) if $x, y \in F$, then $x \odot y \in F$,
(ii) if $x \in F$ and $x \leq y$, then $y \in F$.

Note that if F is a p-filter of an $\mathcal{RML}$ M, then $\top \in F$. Moreover, if $x, x \to y \in F$, then $y \in F$. A subset $D \subseteq M$ containing $\top$ and satisfying $(x, x \to y \in D)$ imply $y \in D$ is called a *deductive system*. Thus a p-filter is a deductive system. The converse is also true.

Definition 4.2. Let M be an $\mathcal{RML}$. A non-empty set $F \subseteq M$ is said to be an *m-filter*, if the following conditions hold:

(i) for all $x, y \in F$, $\varnothing \neq x \sqcap y \subseteq F$,
(ii) for all $x \in F$ and $y \in M$, $x \sqcup y \subseteq F$,
(iii) for all $x, y \in M$, if $(x \sqcup y) \cap F \neq \varnothing$, then $x \sqcup y \subseteq F$.

Cabrera et al. [18] have proposed a notion of filter in the $\mathcal{RML}$ structure which combines pocrim and multilattice structures.

Definition 4.3. Let M be an $\mathcal{RML}$. A non-empty subset $F \subseteq M$ is said to be a *filter*, if it is a deductive system and the following condition holds:

$$x \to y \in F \text{ implies } (x \sqcup y) \to y \subseteq F \text{ and } x \to (x \sqcap y) \subseteq F.$$

It is known that a filter of an $\mathcal{RML}$ is an m-filter and a p-filter but the converse is not true [18]. One can easily observe that:

Remark 4.4. Let M be an $\mathcal{RML}$. The following conditions hold:

(1) $\{\top\}$ and M are filters of M,
(2) if F is a filter of M, then F is an f-sub-$\mathcal{RML}$ of M.

Proposition 4.5. *Let M be an $\mathcal{RML}$, F and G be two filters of M. Then $F \cap G$ is a filter of M.*

One can easily observe that if X is an f-sub-$\mathcal{RML}$ of an $\mathcal{RML}$ M, then X is an $\mathcal{RML}$ on its own right. Then we introduce the notion of soft filter of an f-soft-$\mathcal{RML}$ as follows:

Definition 4.6. Let h_F be a non-null soft set over an $\mathcal{RML}$ M and f_A be an f-soft-$\mathcal{RML}$ over M. Then h_F is called a *soft filter* of f_A, if the following conditions hold:

(i) $F \subseteq A$,
(ii) for all $p \in F$, $h(p)$ is a filter of $f(p)$.

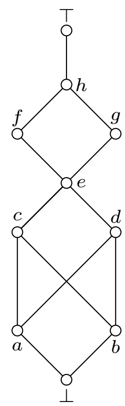
Example 4.7. Let M be the $\mathcal{RML}$ described in Figure 2. The soft set f_A over M defined by $f_A = \{(p_1, \{g, h, \top\}), (p_2, \{f, \top\}), (p_3, \{\bot, \top\})\}$ is clearly an f-soft-$\mathcal{RML}$ over M. $h_F = \{(p_1, \{h, \top\}), (p_2, \{\top\})\}$ is a soft filter of f_A.

Using Remark 4.4, one can observe that any soft filter is an f-soft-$\mathcal{RML}$ over M. Let us now give some properties of soft filters.

Proposition 4.8. *Let f_A be an f-soft-$\mathcal{RML}$ over an $\mathcal{RML}$ M. Let h_F and t_G be two soft filters of f_A. Then $h_F \cap t_G$ is a soft filter of f_A if it is non-null.*

Proof. Suppose $h_F \cap t_G$ is non-null. Then $h_F \cap t_G = u_C$ with $C = \{p \in F \cap G : h(p) \cap t(p) \neq \varnothing\}$ and $u(p) = h(p) \cap t(p)$ for all $p \in C$. As h_F and t_G are soft filters of f_A, we have $F \subseteq A$ and $G \subseteq A$, that is, $C \subseteq A$. If $p \in C$, then $p \in F$ and $p \in G$. This implies that $h(p)$ and $t(p)$ are filters of $f(p)$. By Proposition 4.5, we deduce that $u(p) = h(p) \cap t(p)$ is a filter of $f(p)$. It follows that u_C is a soft filter of f_A. □

The union of two soft filters of an f-soft-$\mathcal{RML}$ is not always a soft filter. Indeed, let consider M as in Example 4.7 and define $f_A = \{(p_1, M)\}$ which is an f-soft-$\mathcal{RML}$ over M. The soft sets $h_F = \{(p_1, \{g, h, \top\})\}$ and $t_G = \{(p_1, \{f, h, \top\})\}$ are



$\odot$	$\bot$	a	b	c	d	e	f	g	h	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	a
b	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	b
c	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	c
d	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	d
e	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	e
f	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	f	$\bot$	f	f
g	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	g	g	g
h	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	f	g	h	h
$\top$	$\bot$	a	b	c	d	e	f	g	h	$\top$

$\to$	$\bot$	a	b	c	d	e	f	g	h	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	h	$\top$	h	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
b	h	h	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
c	h	h	h	$\top$	h	$\top$	$\top$	$\top$	$\top$	$\top$
d	h	h	h	h	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
e	h	h	h	h	h	$\top$	$\top$	$\top$	$\top$	$\top$
f	g	g	g	g	g	g	$\top$	g	$\top$	$\top$
g	f	f	f	f	f	f	f	$\top$	$\top$	$\top$
h	e	e	e	e	e	e	f	g	$\top$	$\top$
$\top$	$\bot$	a	b	c	d	e	f	g	h	$\top$

Figure 2. Residuated multilattice with 10 elements

soft filters of f_A but $h_F \cup t_G = \{(p_1, \{f, g, h, \top\})\}$ is not a soft filter of f_A because $\{f, g, h, \top\}$ is not a filter of M, as $f \sqcap g = e \notin \{f, g, h, \top\}$. However, the following result gives a sufficient condition for the union of two soft filters of an f-soft-$\mathcal{RML}$ to be a soft filter.

Proposition 4.9. *Let f_A be an f-soft-$\mathcal{RML}$ over an $\mathcal{RML}$ M. Let h_F and t_G be two soft filters of f_A. If $F \cap G = \varnothing$, then $h_F \cup t_G$ is a soft filter of f_A.*

Proof. We know that $h_F \cup t_G$ is the soft set (u, E), where $u(p) = h(p) \cup t(p)$ for all $p \in F \cup G$ and $u(p) = \varnothing$ if $p \notin F \cup G$. As h_F and t_G are soft filters of f_A, we have $F \subseteq A$ and $G \subseteq A$, which implies that $F \cup G \subseteq A$. If $p \in F \cup G$, then either $p \in F \setminus G$ or $p \in G \setminus F$. That is, $u(p) = h(p)$ or $u(p) = t(p)$. In both cases, $t(p)$ is a filter of $f(p)$, as h_F and t_G are soft filters of f_A. We can conclude that $h_F \cup t_G$ is a soft filter of f_A. $\square$

Proposition 4.10. *Let f_A be an f-soft-$\mathcal{RML}$ over an $\mathcal{RML}$ M. Let h_F and t_G be two soft filters of f_A. Then $h_F \cap_{\mathfrak{g}} t_G$ is a soft filter of f_A if it is non-null.*

Proof. Suppose $h_F \cap_{\mathfrak{g}} t_G$ is non-null. Then $h_F \cap_{\mathfrak{g}} t_G$ is the soft set (u, E), where for all $p \in F \cup G$,

$$u(p) = \begin{cases} h(p) & \text{if } p \in F \setminus G \\ t(p) & \text{if } p \in G \setminus F \\ h(p) \cap t(p) & \text{else.} \end{cases}$$

As h_F and t_G are soft filters of f_A, $F \subseteq A$ and $G \subseteq A$. That is, $F \cup G \subseteq A$. Knowing that the intersection of two filters of an $\mathcal{RML}$ is also a filter (Proposition 4.5), $u(p) = h(p) \cap t(p)$ is a filter of $f(p)$, if $p \in F \cap G$. For the others two cases, $u(p)$ is evidently a filter of $f(p)$, as h_F and t_G are soft filters of f_A. It follows that $h_F \cap_{\mathfrak{g}} t_G$ is a soft filter of f_A. □

5. Filteristic soft residuated multilattice

In the previous section, we have presented the notion of soft filter, which is related to an f-soft-$\mathcal{RML}$. In this section, we present the notion of filteristic soft residuated multilattice, which is related to the universe. We also look about the link between homomorphisms and filters.

Definition 5.1. A non-null soft set f_A over an $\mathcal{RML}$ M is called a *filteristic soft residuated multilattice* over M, if $f(p)$ is a filter of M for all $p \in A$.

Example 5.2. Let M be the $\mathcal{RML}$ of Figure 2.
$f_A = \{(p_1, \{f, h, \top\}), (p_2, \{g, h, \top\}), (p_3, \{h, \top\})\}$ is a filteristic soft residuated multilattice over M.

Remark 5.3. A soft filter of an f-soft-$\mathcal{RML}$ over an $\mathcal{RML}$ M is not always a filteristic soft residuated multilattice over M. Indeed, let M be the $\mathcal{RML}$ of Figure 2. The soft set $f_A = \{(p_1, M \setminus \{f, g\})\}$ is an f-soft-$\mathcal{RML}$ over M and $h_F = \{(p_1, \{e, h\top\})\}$ is a soft filter of f_A but it is not a filteristic soft residuated multilattice because $h(p_1) = \{e, h, \top\}$ is not a filter of M.

Proposition 5.4. *Let f_A and h_B be two filteristic soft residuated multilattices over an $\mathcal{RML}$ M. Then:*

1. *$f_A \cap h_B$ is a filteristic soft residuated multilattice over M if it is non-null,*
2. *if $A \cap B = \varnothing$, then $f_A \cup h_B$ is a filteristic soft residuated multilattice over M,*
3. *$f_A \wedge h_B$ is a filteristic soft residuated multilattice over M if it is non-null,*
4. *$f_A \cap_{\mathfrak{g}} g_B$ is a filteristic soft residuated multilattice over M if it is non-null.*

Proof. (1) The proof is similar to Proposition 4.8.
(2) The proof is similar to Proposition 4.9.
(3) The proof follows from Proposition 4.5.
(4) The proof is similar to Proposition 4.10. □

Given an $\mathcal{RML}$ M, we know that $\{\top\}$ and M are filters of M. They are known as trivial filter and whole filter. We set a trivial and whole filteristic soft residuated multilattices to be as follows:

Definition 5.5. Let M be an $\mathcal{RML}$. Then a filteristic soft residuated multilattice f_A over M is said to be:

(i) *trivial*,if $f(p) = \{\top\}$ for all $p \in A$.
(ii) *whole*, if $f(p) = M$ for all $p \in A$.

In general, a homomorphism between residuated multilattices is a homomorphism of multilattice which preserves the multiplication ($\odot$) and the implication ($\to$). In what follows, we investigate the interaction between filters and homomorphisms.

According to the fact that an $\mathcal{RML}$ is always full (as multilattice), we have the following definition:

Definition 5.6 ([18])**.** Let M and N be two $\mathcal{RML}s$. A map $\varphi : M \to N$ is said to be an *homomorphism*, if the following conditions hold: for all $x, y \in M$,
(i) $\varphi(x \sqcap y) = (\varphi(x) \sqcap \varphi(y)) \cap \varphi(M)$ and $\varphi(x \sqcup y) = (\varphi(x) \sqcup \varphi(y)) \cap \varphi(M)$,
(ii) $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$ and $\varphi(x \to y) = \varphi(x) \to \varphi(y)$.

One can easily observe that $\varphi(\top) = \top$ for all homomorphism φ between residuated multilattices.

Proposition 5.7. [18]*Let $\varphi : M \to N$ be an homomorphism between residuated multilattices. Then:*
(1) *$\varphi^{-1}(\top)$ is a filter of M,*
(2i) *if F is a filter of N, then $\varphi^{-1}(F)$ is a filter of M.*

Proposition 5.8. *Let $\varphi : M \to N$ be an homomorphism between residuated multilattices. If F is a filter of M and φ bijective, then $\varphi(F)$ is a filter of N.*

Proof. Suppose F is a filter of M. Note that as φ is surjective, we deduce from Definition 5.6 that $\varphi(x \sqcap y) = \varphi(x) \sqcap \varphi(y)$ and $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$ for all $x, y \in M$. Moreover, if $x \leq y$, then $\varphi(x) = \varphi(x \sqcap y) = \varphi(x) \sqcap \varphi(y)$, which implies that $\varphi(x) \leq \varphi(y)$. Then φ is an isotone function. Thus φ^{-1} is an isotone function.

(i) Let $z, t \in \varphi(F)$. Then there exist $x, y \in F$ such that $z = \varphi(x)$ and $t = \varphi(y)$. Thus we have $z \odot t = \varphi(x) \odot \varphi(y) = \varphi(x \odot y)$. As F is a filter of M, $x \odot y \in F$. That is, $z \odot t = \varphi(x \odot y) \in \varphi(F)$.
(ii) Let $z \in \varphi(F)$ and $t \in N$ such that $z \leq t$. Then there exist $x \in F$ such that $\varphi(x) = z$ and by the surjectivity of φ, an element $y \in M$ such that $\varphi(y) = t$. We have to show that $t \in \varphi(F)$. As φ^{-1} is isotone, we have from $z \leq t$ that $\varphi^{-1}(z) \leq \varphi^{-1}(t)$. Thus $\varphi^{-1}(\varphi(x)) \leq \varphi^{-1}(\varphi(y))$, i.e., $x \leq y$. As F is a filter of M and $x \in F$, we deduce that $y \in F$, which implies that $t = \varphi(y) \in \varphi(F)$.
(iii) Let $z, t \in N$ such that $z \to t \in \varphi(F)$. We will show that $(z \sqcup t) \to t \subseteq \varphi(F)$ and $z \to (z \sqcap t) \subseteq \varphi(F)$. As φ is surjective, there exist $x, y \in M$ such that $z = \varphi(x)$ and $t = \varphi(y)$. From $z \to t \in \varphi(F)$, we have $\varphi(x) \to \varphi(y) \in \varphi(F)$, i.e., $\varphi(x \to y) \in \varphi(F)$, which implies by the bijectivity of φ that $x \to y \in F$. Now, we have $(z \sqcup t) \to t = (\varphi(x) \sqcup \varphi(y)) \to \varphi(y) = \varphi((x \sqcup y) \to y)$. As $x \to y \in F$, we have $(x \sqcup y) \to y \subseteq F$, which implies that $(z \sqcup t) \to t = \varphi((x \sqcup y) \to y) \subseteq \varphi(F)$. Similarly, we obtain $z \to (z \sqcap t) \subseteq \varphi(F)$.

□

Let f_A be a soft set over an $\mathcal{RML}$ M and $\varphi : M \to N$ be a map between $\mathcal{RML}s$. Let consider the mapping $\tilde{\varphi} : P(M) \to P(N)$ such that $\tilde{\varphi}(X) = \varphi(X)$ (the direct image), for all $X \in P(M)$. We can see that $(\tilde{\varphi} \circ f, E)$ is a soft set over N. If $p \in A$, then $f(p) \neq \varnothing$, i.e., $\varphi(f(p)) \neq \varnothing$. If rather $f(p) = \varnothing$ ($p \notin A$), then $\varphi(f(p)) = \varnothing$. Thus f and $(\tilde{\varphi} \circ f, E)$ have the same support A; we will denoted $(\tilde{\varphi} \circ f, E)$ by φf_A in the sequel.

Proposition 5.9. *Let M and N be two $\mathcal{RML}$s and $\varphi : M \to N$ be a surjective homomorphism. If f_A is an f-soft-$\mathcal{RML}$ over M, then φf_A is an f-soft-$\mathcal{RML}$ over N.*

Proof. Suppose f_A is an f-soft-$\mathcal{RML}$ over M and let $p \in A$. We have to show that $\varphi f(p)$ is an f-sub-$\mathcal{RML}$ of N. As f_A is an f-soft-$\mathcal{RML}$, we have $\top \in f(p)$, which implies $\top = \varphi(\top) \in \varphi f(p)$. Now, let $z, t \in \varphi f(p)$. Then there exists $x, y \in f(p)$ such that $z = \varphi(x)$ and $t = \varphi(y)$. We have to show that $z \sqcup t \subseteq \varphi(f(p))$ and $z \sqcap t \subseteq \varphi(f(p))$. As f_A is an f-soft residuated multilattice over M, $f(p)$ is an f-sub-$\mathcal{RML}$ of M, $x \sqcup y \subseteq f(p)$ and $x \sqcap y \subseteq f(p)$ which implies that

$$\varphi(x \sqcup y) \subseteq \varphi(f(p)) \text{ and } \varphi(x \sqcap y) \subseteq \varphi(f(p)).$$

As φ is surjective, $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$ and $\varphi(x \sqcap y) = \varphi(x) \sqcap \varphi(y)$. Thus we have $\varphi(x) \sqcup \varphi(y) \subseteq \varphi(f(p))$ and $\varphi(x) \sqcap \varphi(y) \subseteq \varphi(f(p))$ which implies that $z \sqcup t \subseteq \varphi(f(p))$ and $z \sqcup t \subseteq \varphi(f(p))$ as required. It remains to show that $z \odot t, z \to t \in \varphi f(p)$. Since $f(p)$ is an f-sub-$\mathcal{RML}$ of M, $x \odot y \in f(p)$ and $x \to y \in f(p)$, i.e., $\varphi(x) \odot \varphi(y) = \varphi(x \odot y) \in \varphi f(p)$ and $\varphi(x) \to \varphi(y) = \varphi(x \to y) \in \varphi f(p)$. It follows that $z \odot t \in \varphi f(p)$ and $z \to t \in \varphi f(p)$. □

Proposition 5.10. *Let $\varphi : M \to N$ be a bijective homomorphism of residuated multilattices and f_A be an f-soft-$\mathcal{RML}$ over M. If h_F is a soft filter of f_A, then φh_F is a soft filter of φf_A.*

Proof. Suppose h_F is a soft filter of f_A. Clearly, φf_A is an f-soft-$\mathcal{RML}$ over N according to Proposition 5.9. $F \subseteq A$ stems from the fact that h_F is a soft filter of f_A. It remains to show that $\varphi(h(p))$ is a filter of $\varphi(f(p))$ for all $p \in F$. In fact, if $p \in F$, then $h(p)$ is a filter of $f(p)$. Since $f(p)$ is an $\mathcal{RML}$ on it own, by applying Proposition 5.8, $\varphi(h(p))$ is a filter of $\varphi(f(p))$ as required. □

Proposition 5.11. *Let $\varphi : M \to N$ be a bijective homomorphism of residuated multilattices. If f_A is a filteristic soft residuated multilattice over M, then φf_A is a filteristic soft residuated multilattice over N.*

Proof. Let φ be a bijective homomorphism and suppose f_A is a filteristic soft residuated multilattice. Then f_A is non-null, which implies that φf_A is a non-null soft set over N. For all $p \in A$, $f(p)$ is a filter of M, as f_A is a filteristic soft residuated multilattice. As φ is bijective, we deduce from Proposition 5.8 that $\varphi(f(p))$ is a filter of N. It follows that φf_A is a filteristic soft residuated multilattice over N. □

Proposition 5.12. *Let f_A be a filteristic soft residuated multilattice over an $\mathcal{RML}$ M and $\varphi : M \to N$ be a bijective homomorphism of residuated multilattices.*

(1) *If $f(p) = \varphi^{-1}(\top)$ for all $p \in A$, then φf_A is a trivial filteristic soft residuated multilattice over N.*
(2) *If f_A is whole, then φf_A is a whole filteristic soft residuated multilattice over N.*

Proof. Let f_A be a filteristic soft residuated multilattice over M and φ a bijective homomorphism.

(1) Suppose $f(p) = \varphi^{-1}(\top)$ for all $p \in A$. Then $\varphi(f(p)) = \varphi(\varphi^{-1}(\top)) = \top$. Thus φf_A is a trivial filteristic soft residuated multilattice.

(2) Suppose f_A is whole. Then for all $p \in A$, $f(p) = M$, that is $\varphi(f(p)) = \varphi(M) = N$. Thus φf_A is a whole filteristic soft residuated multilattice over M. □

6. Conclusion

We have used residuated multilattices as universe sets within the framework of soft set theory, studying the notions of f-soft residuated multilattices and r-soft residuated multilattices. We introduced some related notions such as soft filters and filteristic soft residuated multilattices with illustrative examples. Some properties of these concepts are proven. We plan to investigate some applications of this study to formal concept analysis.

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