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s-b-open sets and its applications in fuzzy setting

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Abstract. In this paper we first introduce fuzzy s-b-open set, the class of which is strictly larger than that of fuzzy open, fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open set. Using this newly defined fuzzy set here we introduce a weak form of fuzzy regularity, a strong form of fuzzy compactness and fuzzy T_2-space. Afterwards, we introduce three different types of fuzzy continuous-like functions and establish the mutual relationships of these newly defined functions with fuzzy continuity. Lastly several applications of these functions are established here.

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Keywords: Fuzzy s-b-open set, Fuzzy regular open set, Fuzzy s-b-r-continuous function, Fuzzy s-b-continuity, Fuzzy almost s-b-continuity, Fuzzy extremally disconnected space.

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1. Introduction

In [1], fuzzy topology was introduced. Afterwards many mathematicians have engaged themselves to introduce different types of fuzzy-open like sets. In [2], fuzzy semiopen set was introduced. Using fuzzy semiopen set as a basic tool, here we introduce fuzzy s-b-open set. It is shown that the intersection of any two fuzzy s-b-open set need not be so and hence the collection of all fuzzy s-b-open sets on a non-empty set does not form a fuzzy topology.
Recently, new types of fuzzy sets, viz., fuzzy soft set and fuzzy octahedron set are introduced and studied. A new branch in fuzzy system is developed using these types of fuzzy sets. In this context we have to mention [3, 4, 5, 6, 7, 8].

2. Preliminaries

Throughout the paper, (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [1]. In [9], Zadeh introduced fuzzy set as follows : A *fuzzy set* A in an fts X is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The *support* of a fuzzy set A, denoted by $suppA$, is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The *complement* of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$ for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$ for all $x \in X$ while AqB means A is *quasi-coincident* (q-coincident, for short) [10] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ stand for the *fuzzy closure* and the *fuzzy interior* of A in X [1]. $A \in I^X$ is said to be *fuzzy regular open* [2] [resp., *fuzzy semiopen* [2], *fuzzy preopen* [11], *fuzzy α-open* [12], *fuzzy β-open* [13]], if $A = int(clA)$ [resp., $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$. The complement of fuzzy regular open [resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open] set is said to be *fuzzy regular closed* [resp., *fuzzy semiclosed*, *fuzzy preclosed*, *fuzzy α-closed*, *fuzzy β-closed*] set. The smallest fuzzy semiclosed [resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed] set containing a fuzzy set A in X is called the *fuzzy semiclosure* [resp., the *fuzzy preclosure*, the *fuzzy α-closure*, the *fuzzy β-closure*] of A, denoted by $sclA$ [resp., $pclA$, αclA, βclA]. It is obvious that $A \in I^X$ is fuzzy semiclosed [resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed] if and only if $A = sclA$ [resp., $A = pclA$, $A = \alpha clA$, $A = \beta clA$]. The collection of all fuzzy regular open [resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open] sets in X is denoted by $FRO(X)$ [rssp., $FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $F\beta O(X)$] and the collection of all fuzzy regular closed [resp., fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed] sets in X is denoted by $FRC(X)$ [rssp., $FSC(X)$, $FPC(X)$, $F\alpha C(X)$, $F\beta C(X)$]. For a fuzzy open set A in X, $sclA = int(clA)$ [14].

3. Fuzzy s-b-Open Set : Some properties

In this section fuzzy s-b-open set is introduced and studied, the class of which is strictly larger than that of fuzzy open, fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open sets. Some basic properties of fuzzy s-b-open sets are discussed here. First we recall some definitions from [15] for ready references.

Definition 3.1 ([15])**.** Let (X, τ) be an fts and $A \in I^X$. A fuzzy point x_α in X is said to be *fuzzy θ-semicluster point* of A, if $clUqA$ for all $U \in FSO(X)$ with $x_\alpha qU$. The union of all fuzzy θ-semicluster points of A is called the *fuzzy θ-semiclosure* of A and is denoted by θ-$sclA$. It is clear that $A(\in I^X)$ is fuzzy θ-semiclosed if and only if $A = \theta$-$sclA$.

Definition 3.2 ([15])**.** Let (X, τ) be an fts and $A \in I^X$. Then *r-kernel* of A, denoted by r-Ker$A$, is defined as follows :

$$r\text{-Ker}A = \bigwedge\{U : U \in FRO(X), A \leq U\}.$$

Let us now introduce the following concept.

Definition 3.3. A fuzzy set A in an fts (X, τ) is said to be *fuzzy s-b-open*, if $A \leq cl(sint(clA))$. The complement of a fuzzy s-b-open set is said to be *fuzzy s-b-closed.* The collection of all fuzzy s-b-open [resp., fuzzy s-b-closed] sets in an fts X is denoted by $FsbO(X)$ [resp., $FsbC(X)$].

Remark 3.4. Union of any two fuzzy s-b-open sets is also so. But the intersection of any two fuzzy s-b-open sets may not be so, as it seen from the following example.

Example 3.5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider two fuzzy sets B, C defined by $B(a) = 0.6, B(b) = 0.4, C(a) = 0.4, C(b) = 0.7$. Then clearly $B, C \in FsbO(X)$. Let $D = B \wedge C$. Then $D(a) = D(b) = 0.4$. Thus $cl(sint(clD)) = 0_X \not\geq D$. So $D \notin FsbO(X)$.
Hence we can conclude that the set of all fuzzy s-b-open sets in an fts X does not form a fuzzy topology.

Remark 3.6. It is clear from definitions that fuzzy open set, fuzzy regular open set, fuzzy semiopen set, fuzzy preopen set, fuzzy α-open set, fuzzy β-open set imply fuzzy s-b-open set, but the reverse implications are not necessarily true follow from the following example.

Example 3.7. Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U\}$ where $A \leq U \leq 1_X \setminus A$. Consider a fuzzy set B defined by $B(a) = B(b) = 0.5$. Clearly $B \notin \tau$, $B \notin FRO(X)$, $B \notin FPO(X)$. But $cl(sint(clB)) = 1_X \setminus A \geq B \Rightarrow B \in FsbO(X)$.
Next consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0$. Then clearly $C \notin FSO(X)$, but $cl(sint(clC)) = 1_X \setminus A \geq C \Rightarrow C \in FsbO(X)$.
Again $int(cl(intC)) = 0_X \not\geq C \Rightarrow C \notin F\alpha O(X)$.

Theorem 3.8. *Let (X, τ) be an fts. Then the union of any collection of fuzzy s-b-open sets in X is fuzzy s-b-open in X.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy s-b-open sets in X. Then for any $\alpha \in \Lambda$, $G_\alpha \leq cl(sint(clG_\alpha))$. Also, $G_\alpha \leq \bigvee_{\alpha \in \Lambda} G_\alpha$. Then $clG_\alpha \leq cl(\bigvee_{\alpha \in \Lambda} G_\alpha)$ implies that $G_\alpha \leq cl(sint(clG_\alpha)) \leq cl(sint(cl(\bigvee_{\alpha \in \Lambda} G_\alpha)))$ and this is true for all $\alpha \in \Lambda$. Thus $\bigvee_{\alpha \in \Lambda} G_\alpha \leq cl(sint(cl(\bigvee_{\alpha \in \Lambda} G_\alpha)))$. So $\bigvee_{\alpha \in \Lambda} G_\alpha$ is a fuzzy s-b-open in X. □

Let us now introduce a new type of closure-like operator.

Definition 3.9. Let (X, τ) be an fts and $A \in I^X$. Then the *fuzzy s-b-closure* of A, denoted by $sbclA$, is defined by

$$sbclA = \bigwedge\{U \in I^X : A \leq U, U \in FsbC(X)\}$$

and the *fuzzy s-b-interior* of A, denoted by $sbintA$, is defined by

$$sbintA = \bigvee\{G : G \leq A, G \in FsbO(X)\}.$$

Note 3.10. By Remark 3.4, we can conclude that for any fuzzy set A in an fts X, $sbclA$ is fuzzy s-b-closed and $sbintA$ is fuzzy s-b-open. Again, if $A \in FsbC(X)$ [resp.,$A \in FsbO(X)$], then $A = sbclA$ [resp.,$A = sbintA$].

Result 3.11. *Let (X, τ) be an fts. Then the following statements are true:*

(1) *for any fuzzy point x_t in X and any $U \in I^X$, $x_t \in sbclU$ and so for any $V \in FsbO(X)$ with $x_t qV$, VqU,*

(2) *for any two fuzzy sets U, V, where $V \in FsbO(X)$, if $U \not{q} V$, then $sbclU \not{q} V$.*

Proof. (1) Let $x_t \in sbclU$ and $V \in FsbO(X)$ with $x_t qV$. Then $x_t \notin 1_X \setminus V \in FsbC(X)$. Thus $U \not\leq 1_X \setminus V$. So UqV.

(2) Suppose $U \not{q} V$ and assume that $sbclUqV$. Then there exists $x \in X$ such that $(sbclU)(x) + V(x) > 1$. Thus $V(x) + t > 1$, where $t = (sbclU)(x)$. So $x_t \in sbclU$, where $x_t qV, V \in FsbO(X)$. By (1), VqU, a contradiction. □

Result 3.12. *Let (X, τ) be an fts and $A \in I^X$. Then the following statements are true:*

(1) *$sbcl(1_X \setminus A) = 1_X \setminus sbintA$,*

(2) *$1_X \setminus sbclA = sbintA(1_X \setminus A)$.*

Proof. (1) Let $x_t \in sbcl(1_X \setminus A)$ and assume that $x_t \notin 1_X \setminus sbintA$. Then $x_t qsbintA$. Thus there exists $U \in FsbO(X)$ with $U \leq A$ such that $x_t qU$. Since $x_t \in sbcl(1_X \setminus A)$, by Result 3.11(1), $Uq(1_X \setminus A)$. So $Aq(1_X \setminus A)$, a contradiction. Hence we have

$$sbcl(1_X \setminus A) \leq 1_X \setminus sbintA. \tag{3.1}$$

Conversely, let $x_t \in 1_X \setminus sbintA$. Then $1 - sbintA(x) \geq t$. Thus $x_t \not{q} sbintA$. So we get

$$x_t \not{q} U, \text{ where } U \in FsbO(X) \text{ with } U \leq A. \tag{3.2}$$

Let $V \in FsbC(X)$ with $1_X \setminus A \leq V$. Then $1_X \setminus V \leq A$, where $1_X \setminus V \in FsbO(X)$. By (3.2), $x_t \not{q}(1_X \setminus V) \Rightarrow x_t \in V \Rightarrow x_t \in sbcl(1_X \setminus A)$. Thus we have

$$1_X \setminus sbintA \leq sbcl(1_X \setminus A). \tag{3.3}$$

Combining (3.1) and (3.3), (1) holds.

(2) Writing $1_X \setminus A$ for A in (1), we get the proof. □

Lemma 3.13 ([15])**.** *Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold:*

(1) *for any $A \in FRO(X)$, θ-$sclA = A$,*

(2) *for any $A \in F\beta O(X), clA = \alpha clA$,*

(3) *for any $A \in FSO(X)$, $clA = pclA$,*

(4) *for any $A \in \tau, sclA = \theta$-$sclA$.*

4. Fuzzy s-b-r-continuous, s-b-continuous and almost s-b-continuous functions

In this section a new type of fuzzy continuous-like function is introduced which is an independent concept of fuzzy continuity [1]. Also we characterize this newly defined function in several ways. Next we introduce two more functions and finally establish the mutual relationships of these functions among themselves.
Let us now introduce the following concept.

Definition 4.1. Let (X, τ) and (Y, τ_1) be two fts's. Then $f : X \to Y$ is said to be *fuzzy s-b-r-continuous function*, if $f^{-1}(A) \in FsbC(X)$ for all $A \in FRO(Y)$.

Theorem 4.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then the following statements are equivalent:*
(1) *f is fuzzy s-b-r-continuous,*
(2) *$f^{-1}(A) \in FsbO(X)$ for all $A \in FRC(Y)$,*
(3) *$f(sbcl_\tau U) \leq r\text{-}ker(f(U))$ for all $U \in I^X$,*
(4) *$sbcl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-}ker(A))$ for all $A \in I^Y$,*
(5) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R))$ for all $R \in \tau_1$,*
(6) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(scl_{\tau_1} R))$ for all $R \in \tau_1$,*
(7) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1} R))$ for all $R \in \tau_1$,*
(8) *$f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) \in FsbC(X)$ for all $A \in \tau_1$,*
(9) *$f^{-1}(cl_{\tau_1}(int_{\tau_1} F)) \in FsbO(X)$ for all $F \in \tau_1^c$,*
(10) *$f^{-1}(cl_{\tau_1} U) \in FsbO(X)$ for all $U \in F\beta O(Y)$,*
(11) *$f^{-1}(cl_{\tau_1} U) \in FsbO(X)$ for all $U \in FSO(Y)$,*
(12) *$f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in FsbC(X)$ for all $U \in FPO(Y)$,*
(13) *$f^{-1}(\alpha cl_{\tau_1} U) \in FsbO(X)$ for all $U \in F\beta O(Y)$,*
(14) *$f^{-1}(pcl_{\tau_1} U) \in FsbO(X)$ for all $U \in FSO(Y)$.*

Proof. (1)$\Leftrightarrow$ (2) Obvious.

(2)$\Rightarrow$ (3) Suppose (2) holds and let $U \in I^X$ and suppose that y_t be a fuzzy point in Y with $y_t \notin r$-ker$(f(U))$. Then there exists $V \in FRO(Y)$ such that $f(U) \leq V$ and $y_t \notin V \Rightarrow V(y) < t$. Thus $y_t q(1_Y \setminus V) \in FRC(Y)$ and $1_Y \setminus f(U) \geq 1_Y \setminus V$. So $f(U) \not{q}(1_Y \setminus V)$ implies that $U \not{q} f^{-1}(1_Y \setminus V)$. By (2), $f^{-1}(1_Y \setminus V) = 1_X \setminus f^{-1}(V) \in FsbO(X)$. By Result 3.11(2), $sbcl_\tau U \not{q}(1_X \setminus f^{-1}(V))$. Then $sbcl_\tau U \leq f^{-1}(V)$. Thus $f(sbcl_\tau U) \leq V$ implies that $1_Y \setminus f(sbcl_\tau U) \geq 1_Y \setminus V$. So $1 - f(sbcl_\tau U)(y) \geq 1 - V(y) > 1 - t$. Hence $t > f(sbcl_\tau U)(y)$. Then $y_t \notin f(sbcl_\tau U)$. Therefore $f(sbcl_\tau U) \leq r$-ker$(f(U))$.

(3) $\Rightarrow$ (4) Suppose (3) holds and let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (3), $f(sbcl_\tau f^{-1}(A)) \leq r$-ker$(f(f^{-1}(A))) \leq r$-ker$(A)$. Thus $sbcl_\tau(f^{-1}(A)) \leq f^{-1}(r$-ker$(A))$.

(4) $\Rightarrow$ (1) Suppose (4) holds and let $A \in FRO(Y)$. By (4), $sbcl_\tau(f^{-1}(A)) \leq f^{-1}(r$-ker$(A)) = f^{-1}(A)$. But $f^{-1}(A) \leq sbcl_\tau(f^{-1}(A))$ and thus $f^{-1}(A) = sbcl_\tau(f^{-1}(A))$. So $f^{-1}(A) \in FsbC(X)$. Hence f is fuzzy s-b-r-continuous function.

(5) $\Leftrightarrow$ (4) The proof follows from Lemma 3.13(4).

(6) $\Leftrightarrow$ (7) Obvious.

(7) $\Rightarrow$ (1) Suppose (7) holds and let $A \in FRO(Y)$. By (7), $sbcl_\tau(f^{-1}(A)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) = f^{-1}(A)$. Thus $f^{-1}(A) \in FsbC(X)$. So f is fuzzy s-b-r-continuous function.

(1) $\Rightarrow$ (7) Suppose (1) holds and let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1} A) \in FRO(Y)$. By (1), $f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) \in FsbC(X)$. Thus $sbcl_\tau(f^{-1}(A)) \leq sbcl_\tau(f^{-1}(int_{\tau_1}(cl_{\tau_1} A))) = f^{-1}(int_{\tau_1}(cl_{\tau_1} A))$.

(1)$\Rightarrow$ (8) Suppose (1) holds and let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1} A) \in FRO(Y)$. Thus by (1), $f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) \in FsbC(X)$.

(8) $\Rightarrow$ (1) Suppose (8) holds and let $A \in FRO(Y)$. Then $A \in \tau_1$. Thus by (8), $f^{-1}(A) = f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) \in FsbC(X)$.

(2) $\Rightarrow$ (9) Suppose (2) holds and let $F \in \tau_1^c$. Then $cl_{\tau_1} int_{\tau_1} F \in FRC(Y)$. Thus by (2), $f^{-1}(cl_{\tau_1}(int_{\tau_1} F)) \in FsbO(X)$.

(9) $\Rightarrow$ (2) Suppose (9) holds and let $F \in FRC(Y)$. By (9), $f^{-1}(F) = f^{-1}(cl_{\tau_1}(int_{\tau_1} F)) \in FsbO(X)$.

(2) $\Rightarrow$ (10) Suppose (2) holds and let $U \in F\beta O(Y)$. Then $U \leq cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U)) \leq cl_{\tau_1} U$. Thus $cl_{\tau_1} U \leq cl_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U))) = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U)) \leq cl_{\tau_1}(cl_{\tau_1} U) = cl_{\tau_1} U$. So $cl_{\tau_1} U = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U))$. Hence $cl_{\tau_1} U \in FRC(Y)$. Therefore by (2), $f^{-1}(cl_{\tau_1} U) \in FsbO(X)$.

(10) $\Rightarrow$ (9) Suppose (10) holds. Since $FSO(Y) \subseteq F\beta O(Y)$, by (10), $f^{-1}(cl_{\tau_1} U) \in FsbO(X)$ for all $U \in FSO(Y)$.

(11) $\Rightarrow$ (12) Suppose (11) holds and let $U \in FPO(Y)$. Then $U \leq int_{\tau_1}(cl_{\tau_1} U)$. We claim that $int_{\tau_1}(cl_{\tau_1} U) \in FRO(Y)$. Indeed, $int_{\tau_1}(cl_{\tau_1} U) \leq int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U))) \leq int_{\tau_1}(cl_{\tau_1} U)$ implies that $int_{\tau_1}(cl_{\tau_1} U) = int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U)))$. Thus $1_Y \setminus int_{\tau_1}(cl_{\tau_1} U) \in FRC(Y)$. So $1_Y \setminus int_{\tau_1}(cl_{\tau_1} U) \in FSO(Y)$. By (11), $f^{-1}(cl_{\tau_1}(1_Y \setminus int_{\tau_1}(cl_{\tau_1} U))) \in FsbO(X)$. Hence $1_X \setminus f^{-1}(int_{\tau_1}(int_{\tau_1}(cl_{\tau_1} U))) = 1_X \setminus f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in FsbO(X)$. Therefore $f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in FsbC(X)$.

(12) $\Rightarrow$ (1) Suppose (12) holds and let $U \in FRO(Y)$. Then $U \in FPO(Y)$. Thus by (12), $f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in FsbC(X)$. So $f^{-1}(U) = f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in FsbC(X)$. Hence (1) follows.

(10) $\Leftrightarrow$ (13) The proof follows from Lemma 3.13(2).

(11) $\Leftrightarrow$ (14) The proof follow from Lemma 3.13(3). □

Theorem 4.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Let us consider the following statements:*

(1) *for each fuzzy point x_t in X and each $A \in FSO(Y)$ with $f(x_t)qA$, there exists $U \in FsbO(X)$ with $x_t qU$ and $f(U) \leq cl_{\tau_1} A$,*

(2) *$f(sbcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$ for all $P \in I^X$,*

(3) *for each fuzzy point x_t in X and each $A \in FSO(Y)$ with $f(x_t) \in A$, there exists $U \in FsbO(X)$ such that $x_t \in U$ and $f(U) \leq cl_{\tau_1} A$,*

(4) *$f^{-1}(A) \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$ for all $A \in FSO(Y)$,*

(5) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R))$ for all $R \in I^Y$,*

(6) *f is fuzzy s-b-r-continuous function. Then* (1), (2), (3), (4) *and* (5) *are equivalent, and* (5) *implies* (6).

Proof. (1) $\Rightarrow$ (2) Suppose (1) holds and let $P \in I^X$ and x_t be any fuzzy point in X such that $x_t \in sbcl_\tau P$ and $G \in FSO(Y)$ with $f(x_t)qG$. By (1), there exists $U \in FsbO(X)$ with $x_t qU$, $f(U) \leq cl_{\tau_1} G$. As $x_t \in sbcl_\tau P$, by Result 3.11(1), UqP. Then $f(U)qf(P)$. Thus $f(P)qcl_{\tau_1} G$. So $f(x_t) \in \theta\text{-}scl_{\tau_1}(f(P))$. Hence $f(sbcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$.

(2) $\Rightarrow$ (5) Suppose (2) holds and let $R \in I^Y$. Then $f^{-1}(R) \in I^X$. Thus by (2), $f(sbcl_\tau(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}(f(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1} R$. So $sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R)$.

(5) $\Rightarrow$ (1) Suppose (5) holds and let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t)qA$. Since $cl_{\tau_1} A \not q (1_Y \setminus cl_{\tau_1} A)$, by definition, $f(x_t) \notin \theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1} A)$. Then $x_t \notin f^{-1}(\theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1} A))$. By (5), $x_t \notin sbcl_\tau(f^{-1}(1_Y \setminus cl_{\tau_1} A))$. Thus there exists $U \in FsbO(X)$ such that $x_t qU$, $U \not q f^{-1}(1_Y \setminus cl_{\tau_1} A)$. So $f(U) \not q (1_Y \setminus cl_{\tau_1} A)$. Hence $f(U) \leq cl_{\tau_1} A$.

(1) ⇒ (4) Suppose (1) holds and let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_t q f^{-1}(A)$. Then $f(x_t) q A$. Thus by (1), there exists $U \in FsbO(X)$ such that $x_t q U$, $f(U) \leq cl_{\tau_1} A \Rightarrow x_t q U \leq f^{-1}(cl_{\tau_1} A)$. So $x_t q U = sbint_\tau U \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$. Hence $x_t q sbint_\tau(f^{-1}(cl_{\tau_1} A))$ since $sbint_\tau(f^{-1}(cl_{\tau_1} A))$ is the union of all fuzzy s-b-open sets in X contained in $f^{-1}(cl_{\tau_1} A)$. Therefore $f^{-1}(A) \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$.

(4) ⇒ (1) Suppose (4) holds and let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t) q A$. Then by (4), $x_t q f^{-1}(A) \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$. Thus there exists $U \in FsbO(X)$ with $x_t q U$, $U \leq f^{-1}(cl_{\tau_1} A)$. So $f(U) \leq cl_{\tau_1} A$.

(3) ⇒ (4) Suppose (3) holds and let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_t \in f^{-1}(A)$. Then $f(x_t) \in A$. By (3), there exists $U \in FsbO(X)$ with $x_t \in U$ and $f(U) \leq cl_{\tau_1} A$. Thus $U \leq f^{-1}(cl_{\tau_1} A)$. So $x_t \in U = sbint_\tau U \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$. Hence $f^{-1}(A) \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$.

(4) ⇒ (3) Suppose (4) holds and let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t) \in A$. Then by (4), $x_t \in f^{-1}(A) \leq sbint_\tau(f^{-1}(cl_{\tau_1} A))$. Thus there exists $U \in FsbO(X)$ with $x_t \in U$ and $U \leq f^{-1}(cl_{\tau_1} A)$. So $f(U) \leq cl_{\tau_1} A$.

(5) ⇒ (6) Suppose (5) holds and let $A \in FRO(Y)$. Then by (5), $sbcl_\tau(f^{-1}(A)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} A) = f^{-1}(A)$. Thus by Lemma 3.13(1), $f^{-1}(A) \in FsbC(X)$. So f is fuzzy s-b-r-continuous function. □

Theorem 4.4. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function satisfying $sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R)$ for all $R \in I^Y$. Then the following statements hold:*

(1) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R)$ for all $R \in FSO(Y)$,*
(2) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R)$ for all $R \in FPO(Y)$,*
(3) *$sbcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1} R)$, for all $R \in F\beta O(Y)$.*

Proof. Obvious. □

Definition 4.5. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be:

(i) *fuzzy s-b-continuous*, if $f^{-1}(A) \in FsbO(X)$ for all $A \in \tau_1$,
(ii) *fuzzy almost s-b-continuous*, if $f^{-1}(A) \in FsbO(X)$ for all $A \in FRO(Y)$.

Let us now recall the following definition from [1] for ready references.

Definition 4.6 ([1])**.** Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be *fuzzy continuous function*, if $f^{-1}(U) \in \tau$ for all $U \in \tau_1$.

Remark 4.7. It is clear from definitions that

(1) fuzzy continuity ⇒ fuzzy s-b-continuity ⇒ fuzzy almost s-b-continuity, but reverse implications are not necessarily true, in general, follow from the next examples,

(2) fuzzy s-b-r-continuity is an independent concept of fuzzy continuity, fuzzy s-b-continuity and fuzzy almost s-b-continuity, follow from the next examples.

Example 4.8. Fuzzy continuity, fuzzy s-b-continuity and fuzzy almost s-b-continuity $\not\Rightarrow$ fuzzy s-b-r-continuity.
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, B\}$, where $A(a) = A(b) =$

$0.5, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy continuous and thus fuzzy s-b-continuous as well as fuzzy almost s-b-continuous function. Now $B \in FRO(X, \tau_2)$. $i^{-1}(B) = B$. Then $int_{\tau_1}(scl_{\tau_1}(int_{\tau_1}(B))) = A \not\geq B \Rightarrow B \notin FsbC(X, \tau_1) \Rightarrow i$ is not fuzzy s-b-r-continuous function.

Example 4.9. Fuzzy s-b-r-continuity, fuzzy almost s-b-continuity $\not\Rightarrow$ fuzzy s-b-continuity, fuzzy continuity.
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$, where $A(a) = A(b) = 0.4$, $B(a) = B(b) = 0.5, C(a) = 0.5, C(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fuzzy s-b-r-continuous and fuzzy almost s-b-continuous function, but not fuzzy continuous function. Now $C \in \tau_2$, but $i^{-1}(C) = C \not\leq cl_{\tau_1}(sint_{\tau_1}(cl_{\tau_1}C)) = B \Rightarrow C \notin FsO(X, \tau_1) \Rightarrow i$ is not fuzzy s-b-continuous function.

Example 4.10. Fuzzy s-b-continuity $\not\Rightarrow$ fuzzy continuity.
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$, where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is not fuzzy continuous function. Now $B \in \tau_2$, $i^{-1}(B) = B \leq cl_{\tau_1}(sint_{\tau_1}(cl_{\tau_1}B)) = A \Rightarrow B \in FsbO(X, \tau_1) \Rightarrow i$ fuzzy s-b-continuous function.

Example 4.11. Fuzzy s-b-r-continuity $\not\Rightarrow$ fuzzy almost s-b-continuity.
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in FRO(X, \tau_2)$, $i^{-1}(B) = B$. Then $int_{\tau_1}(scl_{\tau_1}(int_{\tau_1}B)) = 0_X \leq B \Rightarrow B \in FsbC(X, \tau_1) \Rightarrow i$ is fuzzy s-b-r-continuous function. But $cl_{\tau_1}(sint_{\tau_1}(cl_{\tau_1}B)) = 0_X \not\geq B \Rightarrow B \notin FsbO(X, \tau_1) \Rightarrow i$ is not fuzzy almost s-b-continuous function.

Definition 4.12 ([16]). An fts (X, τ) is said to be *fuzzy extremally disconnected*, if the closure of ever fuzzy open set in X is fuzzy open in X.

Theorem 4.13. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Let (Y, τ_1) be fuzzy extremally disconnected space. Then f is fuzzy s-b-r-continuous if and only if f is fuzzy almost s-b-continuous function.*

Proof. Suppose that f is fuzzy s-b-r-continuous function and let $U \in FRO(Y)$. Then $U = int_{\tau_1}(cl_{\tau_1}U)$. As Y is fuzzy extremally disconnected, $cl_{\tau_1}U \in \tau_1$. Thus $U = int_{\tau_1}cl_{\tau_1}U = cl_{\tau_1}U = cl_{\tau_1}int_{\tau_1}U$. So $U \in FRC(Y)$. By the hypothesis, $f^{-1}(U) \in FsbO(X)$. Hence f is fuzzy almost s-b-continuous function.

Conversely, suppose f is fuzzy almost s-b-continuous function and let $U \in FRC(Y)$. As Y is fuzzy extremally disconnected, $U \in FRO(Y)$. Then by the hypothesis, $f^{-1}(U) \in FsbO(X)$. Thus f is fuzzy s-b-r-continuous function. □

Remark 4.14. Composition of two fuzzy s-b-r-continuous (resp., fuzzy s-b-continuous and fuzzy almost s-b-continuous) functions need not be so, as it seen from the following examples.

Example 4.15. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$, where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1),

(X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$, $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy s-b-r-continuous functions. Let $i_3 = i_2 \circ i_1$. Now $B \in FRO(X, \tau_3)$, $i_3^{-1}(B) = B$. Then $int_{\tau_1}(scl_{\tau_1}(int_{\tau_1}(B))) = A \not\geq B \Rightarrow B \notin FsbC(X, \tau_1) \Rightarrow i_3$ is not fuzzy s-b-r-continuous function.

Example 4.16. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy s-b-continuous and hence fuzzy almost s-b-continuous functions. Let $i_3 = i_2 \circ i_1$. Bow $B \in \tau_3$ as well as $B \in FRO(X, \tau_3)$. $i_3^{-1}(B) = B$. Now $cl_{\tau_1}(sint_{\tau_1}(cl_{\tau_1}B)) = 0_X \not\geq B \Rightarrow B \notin FsbO(X, \tau_1) \Rightarrow i_3$ is not fuzzy s-b-continuous and also fuzzy almost s-b-continuous functions.

5. Fuzzy s-b-regular, s-b-compact and s-b-T_2-spaces

In this section new types of separation axioms and compactness are introduced and studied. Then the mutual relationships of these spaces with the spaces defined in [1, 17] are established.

Definition 5.1. An fts (X, τ) is called a *fuzzy s-b-regular space*, if for each fuzzy point x_t in X and each fuzzy s-b-closed set F with $x_t \notin F$, there exist a fuzzy open set U and a fuzzy s-b-open set V in X such that $x_t q U$, $F \leq V$ and $U \not{q} V$.

Theorem 5.2. *For an fts (X, τ), the following statements are equivalent:*
(1) *X is fuzzy s-b-regular,*
(2) *for each fuzzy point x_t in X and each fuzzy s-b-open set U in X with $x_t q U$, there exists a fuzzy open set V in X such that $x_t q V \leq sbclV \leq U$,*
(3) *for each fuzzy s-b-closed set F in X, $\bigwedge\{clV : F \leq V, V \in FsbO(X)\} = F$,*
(4) *for each fuzzy set G in X and each fuzzy s-b-open set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $sbclV \leq U$.*

Proof. (1)⇒(2) Suppose (1) holds and Let x_t be a fuzzy point in X and U, a fuzzy s-b-open set in X with $x_t q U$. Then $x_t \notin 1_X \setminus U \in FsbC(X)$. Thus by (1), there exist a fuzzy open set V and a fuzzy s-b-open set W in X such that $x_t q V$, $1_X \setminus U \leq W$, $V \not{q} W$. So $x_t q V \leq 1_X \setminus W \leq U$. Hence $x_t q V \leq sbclV \leq sbcl(1_X \setminus W) = 1_X \setminus W \leq U$.

(2)⇒(1) Suppose (2) holds and Let F be a fuzzy s-b-closed set in X and x_t be a fuzzy point in X with $x_t \notin F$. Then $x_t q(1_X \setminus F) \in FsbO(X)$. Thus by (2), there exists a fuzzy open set V in X such that $x_t q V \leq sbclV \leq 1_X \setminus F$. So $U \in FsbO(X)$ and $x_t q V$, $F \leq U$ and $U \not{q} V$, where $U = 1_X \setminus sbclV$.

(2)⇒(3) Suppose (2) holds and let F be fuzzy s-b-closed set in X. Then we have

$$F \leq \bigwedge\{clV : F \leq V, V \in FsbO(X)\}.$$

Conversely, let $x_t \notin F \in FsbC(X)$. Then $F(x) < t$. Thus $x_t q(1_X \setminus F)$, where $1_X \setminus F \in FsbO(X)$. By (2), there exists a fuzzy open set U in X such that $x_t q U \leq sbclU \leq 1_X \setminus F$. Put $V = 1_X \setminus sbclU$. Then $F \leq V$ and $U \not{q} V$. Thus $x_t \notin clV$. So we get

$$\bigwedge\{clV : F \leq V, V \in FsbO(X)\} \leq F.$$

Hence $\bigwedge\{clV : F \leq V, V \in FsbO(X)\} = F$.

(3)⇒(2) Suppose (3) holds and let V be any fuzzy s-b-open set in X and x_t any fuzzy point in X with $x_t q V$. Then $V(x)+t > 1$. Thus $x_t \notin (1_X \setminus V)$, where $1_X \setminus V \in FsbC(X)$. By (3), there exists $G \in FsbO(X)$ such that $1_X \setminus V \leq G$ and $x_t \notin clG$. So there exists a fuzzy open set U in X with $x_t q U$, $U \not{q} G \Rightarrow U \leq 1_X \setminus G \leq V$ $\Rightarrow x_t q U \leq sbclU \leq sbcl(1_X \setminus G) = 1_X \setminus G \leq V$.

(3)⇒(4) Suppose (3) holds and let G be any fuzzy set in X and U any fuzzy s-b-open set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = t$. Then $x_t q U$. Thus $x_t \notin 1_X \setminus U$, where $1_X \setminus U \in FsbC(X)$. By (3), there exists $W \in FsbO(X)$ such that $1_X \setminus U \leq W$ and $x_t \notin clW \Rightarrow (clW)(x) < t \Rightarrow$ $x_t q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x)+t > 1 \Rightarrow$ $V(x)+G(x) > 1 \Rightarrow VqG$ and $sbclV = sbcl(1_X \setminus clW) \leq sbcl(1_X \setminus W) = 1_X \setminus W \leq U$.

(4)⇒(2) Obvious. □

Note 5.3. It is clear from Theorem 5.2 that in a fuzzy s-b-regular space, every fuzzy s-b-closed set is fuzzy closed and hence every fuzzy s-b-open set is fuzzy open. As a result, in a fuzzy s-b-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy s-b-closed (resp., fuzzy s-b-open) sets coincide.

Definition 5.4. Let A be a fuzzy set in X. A collection $\mathcal{U}$ of fuzzy sets in X is called a *fuzzy cover* of A, if $sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$ (See [18]). In particular, if $A = 1_X$, we get the definition of fuzzy cover of X (See [1]).

Definition 5.5. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to *have a finite subcover* $\mathcal{U}_0$, if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A (See [18]). In particular, if $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$ (See [1]).

Definition 5.6. A fuzzy set A in an fts (X, τ) is said to be *fuzzy compact* (See [18]), if every fuzzy covering $\mathcal{U}$ of A by fuzzy open sets in X has a finite subcovering $\mathcal{U}_0$ of $\mathcal{U}$. In particular, if $A = 1_X$, we get the definition of fuzzy compact space (See [1]).

Definition 5.7. An fts (X, τ) is said to be *fuzzy s-closed* [19] [resp., *fuzzy nearly compact* [16]], if every fuzzy covering of X by fuzzy regular closed [resp., fuzzy regular open] sets of X contains a finite subcovering.

Let us now introduce the following concept.

Definition 5.8. A fuzzy set A in an fts (X, τ) is called *fuzzy s-b-compact*, if every fuzzy covering of A by fuzzy s-b-open sets of X has a finite subcovering. In particular, if $A = 1_X$, we get the definition of fuzzy s-b-compact space.

Remark 5.9. It is clear from above discussion that fuzzy s-b-compact space is fuzzy compact. But the converse is not necessarily true follows from the next example.

Example 5.10. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. The clearly (X, τ) is a fuzzy compact space. Here every fuzzy set is fuzzy s-b-open set in X. Consider the fuzzy cover $\mathcal{U} = \{U_n(a) : n \in \mathbb{N}\}$ where $U_n(a) = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is a fuzzy s-b-open cover of X. But it does not have any subcovering of X. Thus X is not fuzzy s-b-compact space.

Theorem 5.11. *Every fuzzy s-b-closed set A in a fuzzy s-b-compact space X is fuzzy s-b-compact.*

Proof. Let A be a fuzzy s-b-closed set in a fuzzy s-b-compact space X. Let $\mathcal{U}$ be a fuzzy covering of A by fuzzy s-b-open sets in X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy s-b-open covering of X. By the hypothesis, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcovering of A. Consequently, A is fuzzy s-b-compact. □

Let us now recall the following definition from [17] for ready references.

Definition 5.12. [17] Let (X, τ) be an fts. Then X is said to be a *fuzzy T_2-space*, if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Now we introduce the following concept.

Definition 5.13. Let (X, τ) be an fts. Then X is said to be a *fuzzy s-b-T_2-space,* if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy s-b-open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy s-b-open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Let us now recall the following definition from [17] for ready references.

Definition 5.14. [17] An fts (X, τ) is said to be a *fuzzy regular space*, if for any fuzzy point x_t in X and any fuzzy closed set F in X with $x_t \notin F$, there exist fuzzy open sets U, V in X such that $x_t q U, F \leq V$ and $U \not{q} V$.

Remark 5.15. It is clear from Note 5.3 that fuzzy s-b-regular space is fuzzy regular and fuzzy T_2-space is fuzzy s-b-T_2-space. But the reverse implications are not necessarily true, follow from the next example.

Example 5.16. Consider Example 5.10. It is clear that (X, τ) is fuzzy regular and fuzzy s-b-T_2-space (as every fuzzy set is fuzzy s-b-open set as well as fuzzy s-b-closed set). Now consider the fuzzy point $a_{0.4}$ and a fuzzy set A defined by $A(a) = 0.3$. Then $a_{0.4} \notin A \in FsbC(X)$. But there do not exist any fuzzy open set U and a fuzzy s-b-open set V in X such that $a_{0.4} q U, A \leq V$ and $U \not{q} V$ (because 1_X is the only fuzzy open set in X with $a_{0.4} q 1_X$ and $1_X q V$ for all fuzzy set $V (\neq 0_X)$ in X). Then X is not fuzzy s-b-regular space. Consider two fuzzy points $a_{0.4}$ and $a_{0.5}$ in X. But there do not exist fuzzy open sets U, V in X such that $a_{0.4} \in U, a_{0.5} q V$ and $U \not{q} V$. Thus X is not fuzzy T_2-space.

6. Applications of fuzzy s-b-r-continuous, s-b-continuous and almost s-b-continuous functions

In this section the applications of the functions introduced in this paper are established.

First we recall the following definition from [20] for ready references.

Definition 6.1 ([20])**.** A function $f : X \to Y$ is said to be a *fuzzy open function*, if $f(U)$ is a fuzzy open set in Y for every fuzzy open set U in X.

Theorem 6.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ surjective, fuzzy s-b-r-continuous function. If X is fuzzy s-b-compact space, then Y is fuzzy s-closed space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of Y by fuzzy regular closed sets of Y. As f is fuzzy s-b-r-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ covers X by fuzzy s-b-open sets of X. As X is fuzzy s-b-compact space, there exists a finite subset Λ_0 of Λ such that $1_X = \bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)$. Then we have

$$1_Y = f(\bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)) = \bigvee_{\alpha \in \Lambda_0} f(f^{-1}(U_\alpha)) \leq \bigvee_{\alpha \in \Lambda_0} U_\alpha.$$

Thus Y is fuzzy s-closed space. □

Theorem 6.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ fuzzy s-b-continuous function. If A is fuzzy s-b-compact set relative to X, then the image $f(A)$ is fuzzy compact relative to Y.*

Proof. Let A be fuzzy s-b-compact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ a fuzzy covering of $f(A)$ by fuzzy open sets of Y, i.e, $f(A) \leq \bigvee_{\alpha \in \Lambda} U_\alpha$. Then $A \leq f^{-1}(\bigvee_{\alpha \in \Lambda} U_\alpha) = \bigvee_{\alpha \in \Lambda} f^{-1}(U_\alpha)$. Thus $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy s-b-open sets in X. As A is fuzzy s-b-compact set relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. So

$$f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \leq \bigvee_{i=1}^{n} U_{\alpha_i}.$$

Hence $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Therefore $f(A)$ is fuzzy compact relative to Y. □

Theorem 6.4. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ fuzzy almost s-b-continuous function. If A is fuzzy s-b-compact relative to X, then the image $f(A)$ is fuzzy nearly compact relative to Y.*

Proof. Let A be fuzzy s-b-compact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ a fuzzy covering of $f(A)$ by fuzzy regular open sets of Y, i.e, $f(A) \leq \bigvee_{\alpha \in \Lambda} U_\alpha$. Then $A \leq f^{-1}(\bigvee_{\alpha \in \Lambda} U_\alpha) = \bigvee_{\alpha \in \Lambda} f^{-1}(U_\alpha)$. Thus $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy s-b-open sets in X (since f is fuzzy almost s-b-continuous function). As A is fuzzy s-b-compact set relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. So we have

$$f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \leq \bigvee_{i=1}^{n} U_{\alpha_i}.$$

Hence $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Therefore $f(A)$ is fuzzy nearly compact relative to Y. □

Theorem 6.5. *Let (X,τ) and (Y,τ_1) be two fts's and $f : X \to Y$ be injective, fuzzy s-b-continuous function and Y is fuzzy T_2-space. Then X is fuzzy s-b-T_2-space.*

Proof. Let x_α and y_β be two distinct fuzzy points in X, where $x \neq y$. As f is injective, $f(x_\alpha) \neq f(y_\beta)$. As Y is a fuzzy T_2-space, there exist fuzzy open sets $U_1,\ U_2,\ V_1,\ V_2$ in Y such that $f(x_\alpha) \in U_1, f(y_\beta)qV_1$ and $U_1\ \not{q}V_1$ and $f(x_\alpha)qU_2$, $f(y_\beta) \in V_2$ and $U_2\ \not{q}V_2$. Then $x_\alpha \in f^{-1}(U_1),\ y_\beta q f^{-1}(V_1)$ and $f^{-1}(U_1)\ \not{q} f^{-1}(V_1)$. Indeed, $f^{-1}(U_1)qf^{-1}(V_1) \Rightarrow$ there exists $z \in X$ such that $f^{-1}(U_1)(z)+f^{-1}(V_1)(z) > 1 \Rightarrow U_1(f(z)) + V_1(f(z)) > 1 \Rightarrow U_1qV_1$. This is a contradiction. Also, we get

$$x_\alpha q f^{-1}(U_2),\ y_\beta \in f^{-1}(V_2) \text{ and } f^{-1}(U_2)\ \not{q} f^{-1}(V_2),$$

where $f^{-1}(U_1),\ f^{-1}(V_1),\ f^{-1}(U_2),\ f^{-1}(V_2) \in FsbO(X,\tau_1)$.
Similarly, when $x = y$, $\alpha < \beta$ (say), there exist $U_1,\ U_2 \in \tau_1$ such that $f(x_\alpha) \in U_1, f(y_\beta)qU_2$ and $U_1\ \not{q}U_2$. Then $x_\alpha \in f^{-1}(U_1), y_\beta q f^{-1}(U_2)$ and $f^{-1}(U_1)\ \not{q} f^{-1}(U_2)$ where $f^{-1}(U_1), f^{-1}(U_2) \in FsbO(X,\tau_1)$. Hence X is fuzzy s-b-T_2-space. □

Theorem 6.6. *If a bijective function $h : X \to Y$ is fuzzy s-b-continuous, fuzzy open function from a fuzzy s-b-regular space X onto an fts Y, then Y is fuzzy regular space.*

Proof. Let y_t be a fuzzy point in Y and F, a fuzzy closed set in Y with $y_t \notin F$. As h is injective, there exists $x \in X$ such that $h(x) = y$. Then $h(x_t) \notin F$. As h is fuzzy s-b-continuous function, $x_t \notin h^{-1}(F) \in FsbC(X)$. As X is fuzzy s-b-regular space, there exist a fuzzy open set U and a fuzzy s-b-open set V in X such that $x_tqU,\ h^{-1}(F) \leq V$ and $U\ \not{q}V$. Since X is fuzzy s-b-regular, by Note 5.3, V is also fuzzy open set in X. As h is fuzzy open function, we have $h(x_\alpha)qh(U), F \leq h(V)$ and $h(U)\ \not{q}h(V)$, where $h(U), h(V)$ are fuzzy open sets in Y. Thus Y is fuzzy regular space. □

7. Conclusions

By introducing a larger class of fuzzy open-like sets here we introduce a weaker form of fuzzy regularity. But in this new type of fuzzy regular space, fuzzy open set and this new type of fuzzy set coincide. Our next goal is to find interrelations between these types of fuzzy open-like sets defined earlier.

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