A note on gradation of RS-compactness and S*-closed spaces in L-topological spaces

A. Haydar Eş

Received 28 June 2012; Revised 11 July 2012; Accepted 6 August 2012

Abstract. In this paper we define gradation of RS-compactness and S*-closedness spaces in L-topological spaces. We generalize the graduations of RS-compactness and S*-closedness to the L-topological space and give the positions of them under weak forms of L-continuity.

2010 AMS Classification: 54A40, 06D72

Keywords: Fuzzy topology, Fuzzy RS-compactness, Fuzzy S-closedness, Fuzzy continuity.

1. Introduction

Zadeh in [11] introduced the fundamental concept of a fuzzy set. The notions of compactness plays an important role in topological spaces. As in defined in [5, 7, 8, 10], it is a natural problem to consider some degree of compactness in L-topological spaces. In [2, 8] some weaker forms of gradation of compactness in L-topological spaces. The present paper studies the graduations of RS-compactness and S*-closedness notions based on the quadruple $M = (L, \leq, \otimes, *)$ where $(L, \leq)$, $\otimes$ and $*$ respectively denote a complete lattice and binary operations on $L$, was introduced by Höhle and Sostak [6, 9, 10].

2. Preliminaries

The notions L-co topological space and L-closure operator are given from Demirci [6] in his paper as a dual form of the binary operation $\otimes$ on $L$. In this study, we always assume that $(L, \leq)$ is a complete lattice, where $\wedge, \vee, \top, \bot$ respectively denote the meet operation, join operation, the greatest element of $L$ and the least element of $L$. 
The quadruple \( M = (L, \leq, \otimes, *) \) consists of an integral, commutative cl-monoid \((L, \leq, *)\) and a cl-quasi-monoid \(M = (L, \leq, \otimes)\). In any integral, commutative cl-monoid \(M = (L, \leq, \otimes)\), there exists a further binary operation \(\rightarrow\) on \(L\), called the residuum operation on \(L\), such that
\[
\alpha \ast \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma, \forall \alpha, \beta, \gamma \in L.
\]
The residuum operation \(\rightarrow\) is explicitly given by the formula
\[
\alpha \rightarrow \beta = \lor\{\lambda \in L : \alpha \ast \lambda \leq \beta\}, \forall \alpha, \beta \in L.
\]
A mapping \(f : X \mapsto L\) is called an \(L\)-fuzzy set of \(X\). The set of all \(L\)-fuzzy sets of \(X\) is denoted by \(L^X\). In any integral, commutative cl-monoid \((L, \leq, \ast)\), the negation is defined as an unary operation in the sense of \(\neg : L \mapsto L\) by \(\neg(\alpha) = \alpha \rightarrow \bot, \forall \alpha \in L\).

**Definition 2.1.** A subset \(\tau\) of \(L^X\), is called an \(L\)-topology on \(X\) iff \(\tau\) satisfies the following conditions:

L01. \(1_X, 1_\varnothing \in \tau\),

L02. \(f, g \in \tau \Rightarrow f \otimes g \in \tau\), for each \(f, g \in L^X\),

L03. \(\{f_i : i \in I\} \subseteq \tau \Rightarrow \forall_{i \in I} f_i \in \tau, \forall\{f_i : i \in I\} \subseteq L^X\).

For a given \(L\)-topology \(\tau\) on \(X\), the pair \((X, \tau)\) is called an \(L\)-topological space.

**Definition 2.2.** A map \(I : L^X \mapsto L^X\) is called an \(L\)-interior operator on \(X\) iff \(I\) satisfies the next conditions:

I1. \(I(1_X) = 1_X\),

I2. \(f \leq g \Rightarrow I(f) \leq I(g), \forall f, g \in L^X\),

I3. \(I(f) \otimes I(g) \leq I(f \otimes g), \forall f, g \in L^X\),

I4. \(I(f) \leq f, \forall f \in L^X\),

I5. \(I(f) \leq I(f), \forall f \in L^X\).

**Remark 2.3.** Each \(L\)-topology \(\tau\) on \(X\) induces an \(L\)-interior operator \(I_\tau\) by \(I_\tau(f) = \lor\{g \in \tau : g \leq f\}\) for each \(f \in L^X\). Conversely, each \(L\)-interior operator \(I\) induces \(L\)-topology \(\tau_I\) by the \(\tau_I = \{g \in L^X : g \leq I(g)\}\).

**Definition 2.4.** Let \((X, \tau)\) be an \(L\)-topological space and \(f \in L^X\). Then

(i). \(f\) is said to be \(\tau\)-closed iff \(\neg(\neg(f)) \in \tau\)

(ii). \(\tau\)-closure \(\bar{f}\) of \(f\) is defined by \(\bar{f} = \land\{h \in L^X : \neg(h) \in \tau, f \leq h\}\).
Definition 2.5. [2] A L-fuzzy set \( f \in L^X \) in an L-topological space \((X, \tau)\) is said to be

(i) regular L-open iff \( f = (\bar{f})^0 \),

(ii) regular L-closed iff \( f = (\bar{f})^0 \)

Definition 2.6. [6] The map \( \bar{\psi} : L^X \times L^X \rightarrow L \) defined by \( \bar{\psi}(f, g) = \land_{x \in X} f(x) \rightarrow g(x) \), for each \( f, g \in L^X \), is called the L-fuzzy inclusion relation on \( L^X \). The element \( \bar{\psi}(f, g) \) of \( L \) can be conceived as the degree for which \( f \) is included by \( g \).

Definition 2.7. \([1, 2, 3]\) Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces. A function \( \Phi : (X, \tau) \rightarrow (Y, \delta) \) is called L-continuous iff for each \( g \in \delta, \Phi^{-1}(g) \in \tau \)

Definition 2.8. \([1, 2]\) Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces. A function \( \Phi : (X, \tau) \rightarrow (Y, \delta) \) is called almost L-continuous iff for each \( (f)^0 \in \delta, \Phi^{-1}((f)^0) \in \tau \)

Definition 2.9. \([2]\) Let \((X, \tau)\) be an L-topological space.

(i)\((X, \tau)\) is L-compact iff for every family \( \{f_i : i \in I\} \) of \( \tau \) such that \( \forall_{i \in I} f_i(x) = \bar{T}, \forall x \in X \), there exists a finite subset \( I_0 \subseteq I \) such that \( \forall_{i \in I_0} f_i(x) = \bar{T}, \forall x \in X \).

(ii) Let \( f \in L^X \). The L-fuzzy set \( f \) is said to be L-compact iff for every family \( \{f_i : i \in I\} \) of \( \tau \) such that \( f(x) \leq \forall_{i \in I} f_i(x), \forall x \in X \), there exists a finite subset \( I_0 \subseteq I \) such that \( f(x) \leq \forall_{i \in I_0} f_i(x), \forall x \in X \).

Definition 2.10. \([2]\) Let \((X, \tau)\) be an L-topological space and \( f \) be an L-fuzzy set in \( X \). The element \( c(f) \) of \( L \) is defined by \( c(f) = \land \{ \bar{\psi}(f, \vartheta) \rightarrow [\forall \{ \bar{\psi}(f, \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \}] : \vartheta \subseteq \tau \} \) is called the degree of compactness of \( f \), where \( \vartheta_0 \subseteq \vartheta \) means that \( \vartheta_0 \) is a finite subfamily of \( \vartheta \).

Definition 2.11. \([2]\) Let \((X, \tau)\) be an L-topological space and \( f \) be an L-fuzzy set in \( X \).

(i) \( f \) is said to be almost L-compact iff for every family \( \{f_i : i \in I\} \) of \( \tau \) such that \( f \leq \forall_{i \in I} f_i \), there exists a finite subset \( I_0 \subseteq I \) such that \( f \leq \forall_{i \in I_0} f_i \), where ‘\( \bar{\cdot} \)’ denotes the closure of \( f \) in \((X, \tau)\).

(ii) \( f \) is said to be nearly L-compact iff for every family \( \{f_i : i \in I\} \) of \( \tau \) such that \( f \leq \forall_{i \in I} f_i \), there exists a finite subset \( I_0 \subseteq I \) such that \( f \leq \forall_{i \in I_0} f_i \), where ‘\( \bar{\cdot} \)’ denotes the interior operation in \((X, \tau)\).

Definition 2.12. \([2]\) Let \((X, \tau)\) be an L-topological space and \( f \) be an L-fuzzy set in \( X \).

(i) The element \( ac(f) = \land \{ \bar{\psi}(f, \vartheta) \rightarrow [\forall \{ \bar{\psi}(f, \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \}] : \vartheta \subseteq \tau \} \) is called the degree of almost compactness of \( f \).

(ii) The element \( nc(f) = \land \{ \bar{\psi}(f, \vartheta) \rightarrow [\forall \{ \bar{\psi}(f, \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \}] : \vartheta \subseteq \tau \} \) is called the degree of near compactness of \( f \).
Proposition 2.13. [2] Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). Then \(c(f) \leq nc(f) \leq ac(f)\).

Proposition 2.14. [2] Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces and \(\Phi : (X, \tau) \longrightarrow (Y, \delta)\) be a function. If \(\Phi\) is L-continuous, then \(c(f) \leq c(\Phi(f))\).

Definition 2.15. [1] Let \((X, \tau)\) be a L-topological space. Then \(f \in L^X\) is called semiclosed if there exists \(g \in \tau\) such that \(g \leq f \leq (g)\) \((f\) is semiclosed if \(\neg(f)\) is semiopen).

Definition 2.16. [1] Let \((X, \tau)\) be a L-topological space. Then \((X, \tau)\) is called regular iff each fuzzy open set of \(X\) is a union of fuzzy open sets of \(X\) such that \(f_i \leq f\), for each \(i\).

Definition 2.17. [4] A L-fuzzy set \(f \in L^X\) in an L-topological space \((X, \tau)\) is said to be

(i) regular semiopen if \(f = sIntsClf\),

(ii) regular semiclosed if \(f = sClsIntf\).

Let \((X, \tau)\) be an L-topological space and \(f \in L^X\). Then \(IntClf \leq sIntsClf \leq ClIntsClf\), and \(IntClf \leq sClf\) [4].

Definition 2.18. [2] Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). \(f\) is said to be strong L-compact iff for every subfamily \(\vartheta\) of \(\tau\) such that \(f \leq \vartheta\), there exists a finite subfamily \(\text{Pre-L-open } \vartheta_0 \subseteq \vartheta\) such that \(f \leq \bigvee_{\vartheta \in \vartheta_0} \vartheta\).

3. GRADATION OF RS-COMPACTNESS IN L-TOPOLOGICAL SPACES

In this section we shall generalize the degrization of RS-compactness in L-topological spaces.

Definition 3.1. Let \((X, \tau)\) be an L-topological space.

(i) \((X, \tau)\) is RS L-compact iff for every regular semiopen family \(\{f_i : i \in I\}\) of L-fuzzy sets with \(\forall i \in I \ f_i = T\), there exists a finite subset \(I_0 \subset I\) such that \(\forall i \in I_0 \ f_i = T\) for each \(x \in X\), where ‘\(\forall\)’ denotes the interior of \(f\) in \((X, \tau)\).

(ii) Let \(f \in L^X\). The L-fuzzy set \(f\) is said to be RS L-compact iff for every family \(\{f_i : i \in I\}\) of regular semiopen L-fuzzy sets with \(f(x) \leq \bigvee_{i \in I} f_i(x), \forall x \in X\), there exist a finite subset \(I_0 \subset I\) such that \(f(x) \leq \bigvee_{i \in I_0} f_i(x), \forall x \in X\).

(iii) \((X, \tau)\) is weakly RS L-compact iff for every regular semiopen family \(\{f_i : i \in I\}\) of L-fuzzy sets with \(\forall i \in I \ f_i = T\), there exists a finite subset \(I_0 \subset I\) such that \(\forall i \in I_0 \ f_i(x) = T, \forall x \in X\).

(iv) Let \(f \in L^X\). The L-fuzzy set \(f\) is said to be weakly RS L-compact iff for every family \(\{f_i : i \in I\}\) of regular semiopen L-fuzzy sets with \(f(x) \leq \bigvee_{i \in I} f_i(x), \forall x \in X\), there exists a finite subset \(I_0 \subset I\) such that \(f(x) \leq \bigvee_{i \in I_0} f_i(x), \forall x \in X\).
Obviously every RS L-compact set is WRS L-compact set.

**Definition 3.2.** Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then

(i) The element $\text{rsc}(f)$ of $L$ is defined by

$$\text{rsc}(f) = \land \{\tilde{\epsilon}(f, \lor \vartheta_0) : \vartheta_0 \subseteq \vartheta \} : \vartheta \in \text{RSO}(X)$$

is called the degree of RS-compactness of $f$, where $\vartheta_0 \subseteq \vartheta$ means that $\vartheta_0$ is a finite subfamily of $\vartheta$ and $\vartheta \in \text{RSO}(X)$ means that $\vartheta$ is a regular semiopen family of $L^X$.

(ii) The element $\text{wrsc}(f)$ of $L$ is defined by

$$\text{wrsc}(f) = \land \{\tilde{\epsilon}(f, \lor \vartheta) : \vartheta \subseteq \vartheta_0 \} : \vartheta \in \text{RSO}(X)$$

is called the degree of weakly RS-compactness of $f$, where $\vartheta_0 \subseteq \vartheta$ means that $\vartheta_0$ is a finite subfamily of $\vartheta$ and $\vartheta \in \text{RSO}(X)$ means that $\vartheta$ is a regular semiopen family of $L^X$.

**Proposition 3.3.** Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then the following implications are valid:

(i) $f$ is RS L-compact $\Rightarrow$ $\text{rsc}(f) = \top$,

(ii) $f$ is WRS L-compact $\Rightarrow$ $\text{wrsc}(f) = \top$.

**Proof.** The proofs of (i) and (ii) follow immediately from the Definition 3.2. $\Box$

**Proposition 3.4.** Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then $\text{rsc}(f) \leq \text{wrsc}(f)$.

**Proof.** Let $\vartheta$ be regular semiopen family of $L^X$. Since $\lor_{\vartheta_0 \subseteq \vartheta} \vartheta_0 \leq \lor_{\vartheta_0 \subseteq \vartheta_0} \vartheta_0$ and from the isotonity of the residuum operation, we have

$$f \rightarrow \lor_{\vartheta_0 \subseteq \vartheta} \vartheta_0 \leq f \rightarrow \lor_{\vartheta_0 \subseteq \vartheta_0} \vartheta_0.$$ 

Moreover we obtain $\land(f \rightarrow \lor_{\vartheta_0 \subseteq \vartheta_0} \vartheta_0) \leq \land(f \rightarrow \lor_{\vartheta_0 \subseteq \vartheta_0} \vartheta_0).$

Thus $\tilde{\epsilon}(f, \lor \vartheta_0) \leq \tilde{\epsilon}(f, \lor \vartheta_0)$, for every $\vartheta_0 \subseteq \vartheta$.

Hence $[\lor_{\vartheta_0 \subseteq \vartheta} \vartheta_0 : \vartheta_0 \subseteq \vartheta] \subseteq [\lor_{\vartheta_0 \subseteq \vartheta_0} \vartheta_0 : \vartheta_0 \subseteq \vartheta_0]$.

We conclude that $\text{rsc}(f) \leq \text{wrsc}(f)$.

**Definition 3.5.** (i). Let $(X, \tau)$ be an L-topological space. $(X, \tau)$ is SL-closed iff for every family $\{f_i : i \in I\}$ of semiopen L-fuzzy sets with $\lor_{i \in I} f_i = \top$, there exist a finite subset $I_0 \subseteq I$ such that $\lor_{i \in I} f_i = \top$, for each $x \in X$.

(ii). Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. $f$ is said to be SL-closed if for every family $\{f_i : i \in I\}$ of semiopen L-fuzzy sets with $f(x) \leq \lor_{i \in I} f_i(x)$, there exist a finite subset $I_0 \subseteq I$ such that $f(x) \leq \lor_{i \in I} f_i(x)$ for each $x \in X$.  

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Definition 3.6. Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then the element $s-cl(f)$ of L is defined by

\[ s-cl(f) = \bigwedge \{ \tilde{\zeta}(f, \forall \vartheta) \rightarrow \left[ \forall \{ \tilde{\zeta}(f, \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} : \vartheta \in SO(X) \right] \] is called the degree of S-closedness of $f$, where $\vartheta \in SO(X)$ means that $\vartheta$ is a semiopen family of $L^X$.

Theorem 3.7. Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then the following implications are valid:

\[ f \text{ is nearly L-compact} \quad \right\downarrow \quad f \text{ is L-compact} \rightarrow f \text{ is almost L-compact} \quad \left\uparrow \right\downarrow \quad f \text{ is RS L-compact} \quad f \text{ is RS L-compact} \]

Proof. It is clear from Definition 3.2 Definition 3.5 and Definition 2.11.

Theorem 3.8. Let $(X, \tau)$ and $(Y, \delta)$ be L-topological spaces and $\Phi : (X, \tau) \mapsto (Y, \delta)$ be a function. If $\Phi$ is almost L-continuous, then $\text{wrsc}(f) \leq \text{wrsc}(\Phi(f))$.

Proof. Let $\text{wrsc}(\Phi(f)) = \bigwedge \{ \tilde{\zeta}(\Phi(f), \forall \vartheta) \rightarrow \left[ \forall \{ \tilde{\zeta}(\Phi(f), \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} : \vartheta \in \text{RSO}(X) \} \}$. Conversely, we suppose that $\text{wrsc}(f) > \text{wrsc}(\Phi(f))$. Then there exists $\vartheta \in \text{RSO}(X)$ such that $\tilde{\zeta}(\Phi(f), \forall \vartheta) \rightarrow \left[ \forall \{ \tilde{\zeta}(\Phi(f), \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} \right] < \text{wrsc}(f)$. Furthermore, from the almost L-continuity of $\Phi$ given us $\Phi^{-1}(\vartheta) \in \tau$. From the definition weakly RS-compactness degree of $f$ is $\text{wrsc}(f) = \bigwedge \{ \tilde{\zeta}(f, \forall \vartheta) \rightarrow \left[ \forall \{ \tilde{\zeta}(f, \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} : \vartheta \subseteq \tau \} \}$. Now we take the family $\nu = \varphi^{-1}(\vartheta)$. Hence $[\forall \{ \tilde{\zeta}(\Phi(f), \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} : \vartheta \subseteq \tau \}$. It follows that $s-cl(f) = \bigwedge \{ \tilde{\zeta}(f, \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} < \text{wrsc}(f, \forall \vartheta \in \vartheta_0 \vartheta) = \bigwedge \{ \tilde{\zeta}(f, \forall \vartheta \in \vartheta_0 \vartheta) : \vartheta_0 \subseteq \vartheta \} \leq \text{wrsc}(f, \forall \vartheta \in \vartheta_0 \vartheta)$. Hence we have $\text{wrsc}(f) \leq \text{wrsc}(\Phi(f))$.

Proposition 3.9. Let $(X, \tau)$ and $(Y, \delta)$ be L-topological spaces and $\Phi : (X, \tau) \mapsto (Y, \delta)$ be a function. If $\Phi$ is weakly L-continuous, then $\text{rsc}(f) \leq \text{rsc}(\Phi(f))$.

Proof. It can be proved to the previous theorem, similarly.

Proposition 3.10. Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then $f$ is SL-closed $\Rightarrow s-cl(f) = \tau$.

Proof. Trivial.

Proposition 3.11. Let $(X, \tau)$ be an L-topological space and $f$ be an L-fuzzy set in $X$. Then $\text{rsc}(f) \leq \text{wrsc}(f) \leq s-cl(f) \leq \text{ac}(f)$.

Proof. This follows a similar procedure to the Proposition 3.4.
Theorem 3.12. Let \((X, \tau)\) be an L-topological space and let \(X\) be an extremally disconnected space (i.e. \(f \in \tau\) for every \(f \in \tau\)). Then the following conditions are equivalent:

(i). \(f \in L^X\) is almost L-compact,

(ii). \(f \in L^X\) is nearly L-compact,

(iii). \(f \in L^X\) is SL-closed,

(iv). \(f \in L^X\) is weakly RS L-compact.

Proof. It is clear from the previous definitions.

Proposition 3.13. Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces and \(\Phi : (X, \tau) \rightarrow (Y, \delta)\) be a function. If \(\Phi\) is weakly L-continuous and almost L-open mapping, then \(s-cl(f) \leq s-cl(\Phi(f))\).

Proof. It can be proved to the Proposition 3.4, similarly.

Proposition 3.14. Let \((X, \tau)\) be an extremally disconnected L-topological space and L-semiregular topological space. Then \(c(f) = s-cl(f)\).

Proof. Similar to Proposition 3.4.

Corollary 3.15. Let \((X, \tau)\) be an extremally disconnected L-topological space and L-semiregular topological space. Then \(c(f) = nc(f) = ac(f) = s-cl(f) = wrsc(f)\).

Proof. This result follows from Proposition 3.11 and Proposition 3.14.

4. Gradation of \(S^*-\)closed spaces in L-topological spaces

In this section we shall generalize the degradation of \(S^*-\)closedness in L-topological spaces.

Definition 4.1. Let \((X, \tau)\) be an L-topological space.

(i). \((X, \tau)\) is SL-compact iff for every semiopen family \(\{f_i : i \in I\}\) of L-fuzzy sets with \(\bigvee_{i \in I} f_i = \top\), there exists a finite subset \(I_0 \subseteq I\) such that \(\bigvee_{i \in I_0} f_i(x) = \top, \forall x \in X\).

(ii). Let \(f \in L^X\). The L-fuzzy set \(f\) is said to be SL-compact iff for every family \(\{f_i : i \in I\}\) of semiopen L-fuzzy sets with \(f(x) \leq \bigvee_{i \in I} f_i(x), \forall x \in X\), there exist a finite subset \(I_0 \subseteq I\) such that \(f(x) \leq \bigvee_{i \in I_0} f_i(x), \forall x \in X\).

Obviously every SL-compact L-topological space is SL-closed.

Definition 4.2. Let \((X, \tau)\) be an L-topological space.

(i). \((X, \tau)\) is \(S^*L\)-closed iff for every semiopen family \(\{f_i : i \in I\}\) of L-fuzzy sets with \(\bigvee_{i \in I} f_i = \top\), there exists a finite subset \(I_0 \subseteq I\) such that \(\bigvee_{i \in I_0} s-cl f_i(x) = \top, \forall x \in X\).
Definition 4.3. Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). Then

(i). The element \(sc(f)\) of \(L\) is defined by

\[
sc(f) = \land\{\underline{\land}(f, \forall \vartheta) \rightarrow [\lor\{(\underline{\lor}(f, \forall \vartheta \in \vartheta_0, \vartheta): \vartheta_0 \subseteq \vartheta') : \vartheta_0 \in SO(X)\}]
\]

is called the degree of S-compactness of \(f\), where \(\vartheta_0 \subseteq \vartheta'\) means that \(\vartheta_0\) is a finite subfamily of \(\vartheta\) and \(\vartheta \in SO(X)\) means that \(\vartheta\) is a semiopen family of \(L^X\).

(ii). The element \(s'c\ell(f)\) of \(L\) is defined by

\[
s'c\ell(f) = \land\{\underline{\land}(f, \forall \vartheta) \rightarrow [\lor\{(\underline{\lor}(f, \forall \vartheta \in \vartheta_0, s\ell\vartheta) : \vartheta_0 \subseteq \vartheta') : \vartheta_0 \in SO(X)\}]
\]

is called the degree of \(S^*\)-closedness of \(f\), where \(\vartheta_0 \subseteq \vartheta'\) means that \(\vartheta_0\) is a finite subfamily of \(\vartheta\) and \(\vartheta \in SO(X)\) means that \(\vartheta\) is a semiopen family of \(L^X\).

Proposition 4.4. Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). Then

(i). \(f\) is \(S^*\)-closed \(\Rightarrow\) \(S^*\ell(f) = \top\),

(ii). \(f\) is \(SL\)-compact \(\Rightarrow\) \(sc(f) = \top\).

Proof. It is clear from the previous definitions. \(\square\)

Proposition 4.5. Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). Then

\[
nc(f) \leq S^*\ell(f).
\]

Proof. Let \(\vartheta\) be open family of \(L^X\). Since \(\lor_{\vartheta \in \vartheta_0} \text{IntCl}(\vartheta) \leq \lor_{\vartheta \in \vartheta_0} s\ell\text{Cl}(\vartheta)\) and from the isotonity of the residum operation, we have

\[
f \rightarrow \lor_{\vartheta \in \vartheta_0} \text{IntCl}(\vartheta) \leq f \rightarrow \lor_{\vartheta \in \vartheta_0} s\ell\text{Cl}(\vartheta).
\]

Moreover we obtain \(\land(f \rightarrow \lor_{\vartheta \in \vartheta_0} \text{IntCl}(\vartheta)) \leq \land(f \rightarrow \lor_{\vartheta \in \vartheta_0} s\ell\text{Cl}(\vartheta))\).

Thus \(\underline{\land}(f, \lor \text{IntCl}(\vartheta_0)) \leq \underline{\land}(f, \lor \text{sCl}(\vartheta_0))\), for every \(\vartheta_0 \subseteq \vartheta'\).

Hence \(\lor\{(\underline{\land}(f, \lor \text{IntCl}(\vartheta_0)) : \vartheta_0 \subseteq \vartheta') \subseteq \lor\{(\underline{\land}(f, \lor \text{sCl}(\vartheta_0)) : \vartheta_0 \subseteq \vartheta')\}\\)

We conclude that \(nc(f) \leq S^*\ell(f)\). \(\square\)

Theorem 4.6. Let \((X, \tau)\) be an L-topological space and \(f\) be an L-fuzzy set in \(X\). Then the following implications are valid:

- \(f\) is nearly L-compact \(\Rightarrow\) \(f\) is \(S^*\)-closed \(\Rightarrow\) \(f\) is almost L-compact.

Proof. This immediately follows from Definition 4.2 and Definition 2.11. \(\square\)
Proposition 4.7. Let $(X, \tau)$ be an extremally disconnected topological space and $f$ be an $L$-fuzzy set in $X$. Then $nc(f) = S^*cl(f)$.

Proof. Similar to Proposition 4.5. □

Proposition 4.8. Let $(X, \tau)$ be an $L$-topological space and $f$ be an $L$-fuzzy set in $X$. Then

\[ f \text{ is strongly } L\text{-compact} \implies f \text{ is } L\text{-compact}. \]

Proof. It is clear from the previous definitions. □

Corollary 4.9. Let $(X, \tau)$ be an $L$-topological space and let $X$ be an extremally disconnected space (i.e. $f \in \tau$ for every $f \in \tau$). Then the following conditions are equivalent:

(i). $f \in L^X$ is almost $L$-compact,

(ii). $f \in L^X$ is nearly $L$-compact,

(iii). $f \in L^X$ is $SL$-closed,

(iv). $f \in L^X$ is weakly RS $L$-compact,

(v). $f \in L^X$ is $S^\ast L$-closed,

(vi). $f \in L^X$ is strongly $L$-compact.

Proof. It is clear from the previous definitions. □

Definition 4.10. Let $(X, \tau)$ and $(Y, \delta)$ be $L$-topological spaces. A function $\Phi : (X, \tau) \rightarrow (Y, \delta)$ is called $L$-semicontinuous iff for each $f \in \delta$, $\Phi^{-1}(f)$ is a fuzzy semiopen set of $L^X$.

Theorem 4.11. Let $(X, \tau)$ and $(Y, \delta)$ be $L$-topological spaces and $\Phi : (X, \tau) \rightarrow (Y, \delta)$ be a function. If $\Phi$ is $L$-semicontinuous, then $sc(f) \leq c(\Phi(f))$.

Proof. Let $sc(f) = \wedge \{ \tilde{\mu}(f, v, \vartheta) : v \in SO(X) \}$. Conversely, we suppose that $sc(f) > c(\Phi(f))$. Then there exists $\vartheta \subseteq \delta$ such that $\tilde{\mu}(\Phi(f), v, \vartheta) > \tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta)$. Furthermore, from the $L$-semicontinuity of $\Phi$, $\Phi^{-1}(f)$ is a fuzzy semiopen set of $L^X$. From the definition of $S$-compactness degree of $f$ is $sc(f) = \wedge \{ \tilde{\mu}(f, v, \vartheta) : v \in SO(X) \}$ we take the family $v = \Phi(\vartheta)$. Hence $\tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta) < \tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta)$. It follows that there exists $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \vartheta$ such that $\tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta) < \tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta_0 \subseteq \vartheta)$. Thus $\tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta_0 \subseteq \vartheta) = \tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta) < \tilde{\mu}(\Phi(f), v, \vartheta_0 \subseteq \vartheta_0 \subseteq \vartheta)$. Hence we have $sc(f) < c(\Phi(f))$. □

Definition 4.12. Let $(X, \tau)$ and $(Y, \delta)$ be $L$-topological spaces. A function $\Phi : (X, \tau) \rightarrow (Y, \delta)$ is called $L$-irresolute iff for each semiopen set of $f \in L^Y$, $\Phi^{-1}(f)$ is a fuzzy semiopen set of $L^X$. 9
Theorem 4.13. Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces and \(\Phi : (X, \tau) \mapsto (Y, \delta)\) be a function. If \(\Phi\) is L-irresolute, then \(S^*\text{cl}(f) \leq S^*\text{cl}(\Phi(f))\).

Proof. It can be proved to the Theorem 4.11, similarly. \(\square\)

Theorem 4.14. Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces and \(\Phi : (X, \tau) \mapsto (Y, \delta)\) be a function. If \(\Phi\) is L-semicontinuous, then \(s^*\text{cl}(f) \leq ac(\Phi(f))\).

Proof. It can be proved to the Theorem 4.11, similarly. \(\square\)

Definition 4.15. Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces. A function \(\Phi : (X, \tau) \mapsto (Y, \delta)\) is called semi weakly L-continuous iff for each \(f \in L^Y\) semiopen set, we have \(\Phi^{-1}(f) \leq s\text{int}[\Phi^{-1}(s\text{Cl}(f))]\).

Theorem 4.16. Let \((X, \tau)\) and \((Y, \delta)\) be L-topological spaces and \(\Phi : (X, \tau) \mapsto (Y, \delta)\) be a function. If \(\Phi\) is semi weakly L-continuous, then \(sc(f) \leq s^*\text{cl}(\Phi(f))\).

Proof. This is analogous to the proof of Theorem 4.11. \(\square\)

Acknowledgement. The author is very grateful to the editor and the reviewers for their valuable suggestions.

References

A. Haydar Es (haydares@baskent.edu.tr) – Department of Mathematics Education, Başkent University, Bağlica, 06490 Ankara,Turkey