Cubic $q$-ideals of BCI-algebras

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Abstract. The notion of cubic $q$-ideals in BCI-algebras is introduced. Relationship between a cubic ideal and a cubic $q$-ideal is discussed. Conditions for a cubic ideal to be a cubic $q$-ideal are provided. Characterizations of a cubic $q$-ideal are established. The cubic extension property for a cubic $q$-ideal is considered.

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1. Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. Fuzzy sets, which were introduced by Zadeh [7], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [3] introduced the notion of cubic subalgebras/ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal, and considered a method to make a new cubic subalgebra from old one. Jun et al. [4] introduced the notion of cubic $\circ$-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a cubic ideal and a cubic $\circ$-subalgebra in a BCK-algebra, and the relation between a closed cubic ideal and a cubic subalgebra in a BCI-algebra. They also investigated a condition for a
cubic set in a BCK-algebra with condition (S) to be a cubic ideal. Finally, they dealt with a characterization of cubic ideal in a BCK/BCI-algebra.

In this paper, we introduce the notion of cubic \(q\)-ideals in BCI-algebras. We discuss relationship between a cubic ideal and a cubic \(q\)-ideal, and provide conditions for a cubic ideal to be a cubic \(q\)-ideal. We establish characterizations of a cubic \(q\)-ideal, and consider the cubic extension property for a cubic \(q\)-ideal.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

An algebra \((X; *, 0)\) of type \((2, 0)\) is called a BCI-algebra if it satisfies the following axioms:

\(\begin{align*}
(\text{I}) & \quad (\forall x, y, z \in X) \ ((x * y) * (x * z)) * (z * y) = 0, \\
(\text{II}) & \quad (\forall x, y \in X) \ ((x * (x * y)) * y = 0), \\
(\text{III}) & \quad (\forall x \in X) \ (x * x = 0), \\
(\text{IV}) & \quad (\forall x, y \in X) \ (x * y = 0, y * x = 0 \Rightarrow x = y).
\end{align*}\)

If a BCI-algebra \(X\) satisfies the following identity:

\((V) \quad (\forall x \in X) \ (0 * x = 0),\)

then \(X\) is called a BCK-algebra. Any BCK/BCI-algebra \(X\) satisfies the following conditions:

\(\begin{align*}
(a1) & \quad (\forall x \in X) \ (x * 0 = x), \\
(a2) & \quad (\forall x, y, z \in X) \ (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, \ (z * y) * (z * x) = 0), \\
(a3) & \quad (\forall x, y, z \in X) \ ((x * y) * z = (x * z) * y), \\
(a4) & \quad (\forall x, y, z \in X) \ (((x * z) * (y * z)) * (x * y) = 0).
\end{align*}\)

We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\). A BCK-algebra \(X\) is said to be with condition \((S)\) if, for all \(x, y, z \in X\), the set \(\{z \in X \mid z * x \leq y\}\) has a greatest element, written \(x \circ y\). A BCI-algebra \(X\) is said to be associative if \((x * y) * z = x * (y * z)\) for all \(x, y, z \in X\). A nonempty subset \(S\) of a BCK/BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A subset \(I\) of a BCK/BCI-algebra \(X\) is called an ideal of \(X\) if it satisfies the following conditions:

\(\begin{align*}
(b1) & \quad 0 \in I, \\
(b2) & \quad (\forall x, y \in X) \ (x * y \in I, \ y \in I \Rightarrow x \in I).
\end{align*}\)

A subset \(I\) of a BCI-algebra \(X\) is called a \(q\)-ideal of \(X\) (see [5]) if it satisfies \((b1)\) and

\(\begin{align*}
(b3) & \quad (\forall x, y, z \in X) \ (x * (y * z) \in I, \ y \in I \Rightarrow x * z \in I).
\end{align*}\)

Let \(I\) be a closed unit interval, i.e., \(I = [0, 1]\). By an interval number we mean a closed subinterval \(\bar{I} = [a^-, a^+]\) of \(I\), where \(0 \leq a^- \leq a^+ \leq 1\). Denote by \(D[0, 1]\) the set of all interval numbers. Let us define what is known as refined minimum (briefly, \(\text{rmin}\)) of two elements in \(D[0, 1]\). We also define the symbols “\(\leq^r\)”, “\(\leq^s\)”, “\(=^r\)” in case of two elements in \(D[0, 1]\). Consider two interval numbers \(\bar{I}_1 := [a^-_1, a^+_1]\) and \(\bar{I}_2 := [a^-_2, a^+_2]\). Then

\(\text{rmin} \{\bar{I}_1, \bar{I}_2\} = \left[ \min \{a^-_1, a^-_2\}, \min \{a^+_1, a^+_2\} \right], \)

\(\bar{I}_1 \geq^r \bar{I}_2\) if and only if \(a^-_1 \geq a^-_2\) and \(a^+_1 \geq a^+_2\),

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and similarly we may have \( \overline{\alpha}_1 \leq \overline{\alpha}_2 \) and \( \overline{\alpha}_1 = \overline{\alpha}_2 \). To say \( \overline{\alpha}_1 \preceq \overline{\alpha}_2 \) (resp. \( \overline{\alpha}_1 \prec \overline{\alpha}_2 \)) we mean \( \overline{\alpha}_1 \preceq \overline{\alpha}_2 \) and \( \overline{\alpha}_1 \neq \overline{\alpha}_2 \) (resp. \( \overline{\alpha}_1 \preceq \overline{\alpha}_2 \) and \( \overline{\alpha}_1 \neq \overline{\alpha}_2 \)). Let \( \pi_i \in D[0,1] \) where \( i \in \Lambda \).

We define

\[
\begin{align*}
\text{rinf}_{i \in \Lambda} \overline{\alpha}_i &= \left[ \inf_{i \in \Lambda} a_{i}^-, \inf_{i \in \Lambda} a_{i}^+ \right] \quad \text{and} \quad \text{rsup}_{i \in \Lambda} \overline{\alpha}_i &= \left[ \sup_{i \in \Lambda} a_{i}^-, \sup_{i \in \Lambda} a_{i}^+ \right].
\end{align*}
\]

An interval-valued fuzzy set (briefly, IVF set) \( A \) defined on \( X \) is given by

\[
A = \{ (x, [\mu_{A}^{-}(x), \mu_{A}^{+}(x)]) \}, \forall x \in X \quad (\text{briefly, denoted by} \quad A = [\mu_{A}^{-}, \mu_{A}^{+}]),
\]

where \( \mu_{A}^{-} \) and \( \mu_{A}^{+} \) are two fuzzy sets in \( X \) such that \( \mu_{A}^{-}(x) \leq \mu_{A}^{+}(x) \) for all \( x \in X \).

Let \( \mu_{A}(x) = [\mu_{A}^{-}(x), \mu_{A}^{+}(x)] \), \( \forall x \in X \). If \( \mu_{A}^{-}(x) = \mu_{A}^{+}(x) = c \) (say) where \( 0 \leq c \leq 1 \), then we have \( \mu_{A}(x) = [c, c] \) which we also assume, for the sake of convenience, to belong to \( D[0,1] \). Thus \( \mu_{A}(x) \in D[0,1], \forall x \in X \), and therefore the IVF set \( A \) is given by

\[
A = \{ (x, \mu_{A}(x)) \}, \forall x \in X \quad (\text{where} \quad \mu_{A} : X \to D[0,1]).
\]

We refer the reader to the books [1, 6] and the paper [2] for further information regarding BCK/BCI-algebras.

3. Cubic \( q \)-ideals

Definition 3.1. [3] Let \( X \) be a nonempty set. A cubic set \( \mathcal{A} \) in a set \( X \) is a structure

\[
\mathcal{A} = \{ (x, A(x), \lambda(x)) : x \in X \}
\]

which is briefly denoted by \( \mathcal{A} = \langle A, \lambda \rangle \) where \( A = [\mu_{A}^{-}, \mu_{A}^{+}] \) is an IVF set in \( X \) and \( \lambda \) is a fuzzy set in \( X \).

Definition 3.2. [3] A cubic set \( \mathcal{A} = \langle A, \lambda \rangle \) in \( X \) is called a cubic subalgebra of a BCK/BCI-algebra \( X \) if it satisfies: for all \( x, y \in X \),

(a) \( \mu_{A}(x * y) \supseteq \text{rmin}\{\mu_{A}(x), \mu_{A}(y)\} \),
(b) \( \lambda(x * y) \supseteq \text{rmin}\{\lambda(x), \lambda(y)\} \).

Definition 3.3. [3] A cubic set \( \mathcal{A} = \langle A, \lambda \rangle \) in a BCK/BCI-algebra \( X \) is called a cubic ideal of \( X \) if it satisfies: for all \( x, y \in X \),

(a) \( \mu_{A}(0) \supseteq \mu_{A}(x) \),
(b) \( \lambda(0) \supseteq \lambda(x) \),
(c) \( \mu_{A}(x) \supseteq \text{rmin}\{\mu_{A}(x * y), \mu_{A}(y)\} \),
(d) \( \lambda(x) \supseteq \text{rmin}\{\lambda(x * y), \lambda(y)\} \).

Definition 3.4. [3] A cubic set \( \mathcal{A} = \langle A, \lambda \rangle \) in a BCI-algebra \( X \) is called a cubic \( q \)-ideal of \( X \) if it satisfies conditions (a) and (b) in Definition 3.3 and for all \( x, y \in X \),

(a) \( \mu_{A}(x * z) \supseteq \text{rmin}\{\mu_{A}(x * (y * z)), \mu_{A}(y)\} \),
(b) \( \lambda(x * z) \supseteq \text{rmin}\{\lambda(x * (y * z)), \lambda(y)\} \).

Example 3.5. Consider a BCI-algebra \( X = \{0, a, b, c, d, e\} \) in which the *-operation is given by the Table [1] We define \( A = [\mu_{A}^{-}, \mu_{A}^{+}] \) and \( \lambda \) by

\[
A = \begin{pmatrix}
0 & a & b & c & d & e \\
[0.4,0.8] & [0.4,0.8] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3] & [0.1,0.3]
\end{pmatrix}
\]

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Table 1.  \( \ast \)-operation

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>c</td>
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<td>a</td>
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<td>d</td>
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<td>b</td>
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<td>d</td>
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<td>a</td>
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</tr>
<tr>
<td>e</td>
<td>e</td>
<td>e</td>
<td>c</td>
<td>b</td>
<td>b</td>
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</tr>
</tbody>
</table>

Table 2.  \( \ast \)-operation

<table>
<thead>
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<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
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<tr>
<td>a</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\lambda = \begin{pmatrix} 0 & a & b & c & d & e \\ 0.2 & 0.2 & 0.6 & 0.6 & 0.6 \end{pmatrix}
\]

and

\[
\lambda = \begin{pmatrix} 0 & a & b & c & d & e \\ 0.2 & 0.2 & 0.6 & 0.6 \end{pmatrix}
\]

Then \( \mathcal{A} = (A, \lambda) \) is a cubic \( q \)-ideal of \( X \).

Note that every cubic \( q \)-ideal of a BCI-algebra \( X \) is a cubic ideal of \( X \) by taking \( z = 0 \) in Definition 3.4 and using (a1). But, the converse is not true as seen in the following example.

Example 3.6. Let \( X = \{0, a, 1, 2, 3\} \) be a BCI-algebra with the \( \ast \)-operation given by Table 2. We define \( A = [\mu_A, \mu_A^+] \) and \( \lambda \) by

\[
A = \begin{pmatrix} 0 & a & b & c & d & e \\ 0.5 & 0.9 & 0.3 & 0.7 & 0.2 & 0.6 \\ 0.2 & 0.6 & 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.6 & 0.2 & 0.6 \end{pmatrix}
\]

and

\[
\lambda = \begin{pmatrix} 0 & a & b & c & d & e \\ 0.2 & 0.2 & 0.6 & 0.6 \end{pmatrix}
\]

respectively. Then \( \mathcal{A} = (A, \lambda) \) is a cubic ideal of \( X \) (see [3]). But, \( \mathcal{A} = (A, \lambda) \) is not a cubic \( q \)-ideal of \( X \) since

\[
\tilde{\mu}_A(3 \ast 1) = [0.2, 0.6] < [0.5, 0.9] = \text{rmin}\{\tilde{\mu}_A(3 \ast (0 \ast 1)), \tilde{\mu}_A(0)\}
\]

and/or \( \lambda(3 \ast 1) = 0.4 > 0.2 = \text{max}\{\lambda(3 \ast (0 \ast 1)), \lambda(0)\} \).

We provide a condition for a cubic ideal to be a cubic \( q \)-ideal.

Theorem 3.7. In an associative BCI-algebra, every cubic ideal is a cubic \( q \)-ideal.

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Proof. Let $\mathcal{A} = \langle A, \lambda \rangle$ be a cubical ideal of an associative BCI-algebra $X$. For any $x, y, z \in X$, we have

$$\bar{\mu}_A(x * z) \geq \min\{\bar{\mu}_A((x * z) * y), \bar{\mu}_A(y)\}$$

$$= \min\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(y)\}$$

$$= \min\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}$$

$$\lambda(x * z) \leq \max\{\lambda((x * y) * z), \lambda(y)\}$$

$$= \max\{\lambda(x * y) * z), \lambda(y)\}$$

$$= \max\{\lambda(x * (y * z)), \lambda(y)\}$$

Hence $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic $q$-ideal of $X$. \hspace{1cm} \Box

Corollary 3.8. Let $X$ be a BCI-algebra which satisfies any one of the following assertions:

1. $(\forall x \in X) (0 * x = x)$.
2. $(\forall x, y \in X) (x * y = y * x)$.

Then every cubic ideal is a cubic $q$-ideal.

Corollary 3.9. Let $X$ be a quasi-associative BCI-algebra, that is, $X$ is a BCI-algebra which satisfies the following inequality:

$$(\forall x, y, z \in X) ((x * y) * z \leq x * (y * z)).$$

If $X$ satisfies one of the following conditions:

1. $(\forall x \in X) (0 * (0 * x) = x)$,
2. $(\forall x, y \in X) (0 * (y * x) = x * y)$,
3. $(\forall x, y \in X) (x * y = 0 \Rightarrow x = y)$,
4. $(\forall x, y, z \in X) (x * z = y * z \Rightarrow x = y)$,
5. $(\forall x, y, z \in X) (z * x = z * y \Rightarrow x = y)$,
6. $(\forall x, y, z \in X) ((y * x) * (z * x) = y * z)$,
7. $(\forall x, y, z \in X) ((x * y) * (x * z) = z * y)$,
8. $(\forall x, y, z \in X) (0 * (x * z) = 0 * (y * z))$,
9. $(\forall x, y, z, u \in X) ((x * y) * (z * u) = (x * z) * (y * u))$,
10. $X = \{0\} \cup \{x \in X \mid 0 * x \neq 0\}$,

then every cubic ideal is a cubic $q$-ideal.

Theorem 3.10. Let $\mathcal{A} = \langle A, \lambda \rangle$ be a cubical ideal of a BCI-algebra $X$ in which the following inequalities are valid:

$$(\forall x, y \in X) (\bar{\mu}_A(x * y) \geq \bar{\mu}_A(x), \lambda(x * y) \leq \lambda(x)).$$

Then $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic $q$-ideal of $X$.

Proof. Let $x, y, z \in X$. Using (c) and (d) in Definition 3.3, (a3) and (3.1), we have

$$\bar{\mu}_A(x * z) \geq \min\{\bar{\mu}_A((x * z) * y), \bar{\mu}_A(y)\}$$

$$= \min\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(y)\}$$

$$\geq \min\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}$$

$$\lambda(x * z) \leq \max\{\lambda((x * y) * z), \lambda(y)\}$$

$$= \max\{\lambda(x * y) * z), \lambda(y)\}$$

$$= \max\{\lambda(x * (y * z)), \lambda(y)\}$$


For a cubic set

Every nonempty cubic level set of

Assume that

Theorem 3.12.

and we say it is a

Therefore \( \mathcal{A} = (A, \lambda) \) is a cubic q-ideal of \( X \).

Proposition 3.11. Every cubic q-ideal \( \mathcal{A} = (A, \lambda) \) of a BCI-algebra \( X \) satisfies the following inequalities:

1. \( \mu_A(x * y) \geq \mu_A(x * (0 * y)) \) and \( \lambda(x * y) \leq \lambda(x * (0 * y)) \),

2. \( \mu_A(0 * x) \geq \mu_A(0 * (0 * x)) \) and \( \lambda(0 * x) \leq \lambda(0 * (0 * x)) \)

for all \( x, y \in X \).

Proof. Straightforward.

Let \( \mathcal{A} = (A, \lambda) \) be a cubic set in \( X \). For any \( r \in [0,1] \) and \( [s, t] \in D[0,1] \), we define \( U(\mathcal{A}; [s, t], r) \) as follows:

\[
U(\mathcal{A}; [s, t], r) = \{ x \in X \mid \bar{\mu}_A(x) \geq [s, t], \lambda(x) \leq r \},
\]

and we say it is a cubic level set of \( \mathcal{A} = (A, \lambda) \).

Theorem 3.12. For a cubic set \( \mathcal{A} = (A, \lambda) \) in a BCI-algebra \( X \), the following are equivalent:

1. \( \mathcal{A} = (A, \lambda) \) is a cubic q-ideal of \( X \).

2. Every nonempty cubic level set of \( \mathcal{A} = (A, \lambda) \) is a q-ideal of \( X \).

Proof. Assume that \( \mathcal{A} = (A, \lambda) \) is a cubic q-ideal of \( X \). Let \( x, y \in X \), \( r \in [0,1] \) and \( [s, t] \in D[0,1] \). If \( x \in U(\mathcal{A}; [s, t], r) \), then \( \bar{\mu}_A(0) \geq \mu_A(x) \geq [s, t] \) and \( \lambda(0) \leq \lambda(x) \leq r \).

Hence \( U(\mathcal{A}; [s, t], r) \) is a q-ideal of \( X \). Conversely, suppose that any nonempty cubic level set of \( \mathcal{A} = (A, \lambda) \) is a q-ideal of \( X \), for all \( r \in [0,1] \) and \( [s, t] \in D[0,1] \). Assume that \( \bar{\mu}_A(0) \leq \mu_A(a) \), that is, \( [\mu_A(0), \mu_A(a)] \leq \mu_A(a) \), or \( \lambda(0) > \lambda(b) \) for some \( a, b \in X \). If we take \( s_a = \frac{1}{2} (\mu_A(0) + \mu_A(a)) \), \( t_a = \frac{1}{2} (\mu_A(0) + \mu_A(a)) \) and \( r_b = \frac{1}{2} (\lambda(0) + \lambda(b)) \), then \( \bar{\mu}_A(0) \leq [\mu_A(0), \mu_A(a)] \leq [s_a, t_a] \) and \( \mu_A(a) \) or \( \lambda(0) > r_b > \lambda(b) \).

Hence \( 0 \notin U(\mathcal{A}; [s_a, t_a], r_b) \). This is a contradiction, and so \( \bar{\mu}_A(0) \geq \mu_A(a) \) and \( \lambda(0) \leq \lambda(x) \) for all \( x \in X \). Now suppose there exist \( a, b, c \in X \) such that

\[
\mu_A(a * c) < \min\{\mu_A(a * (b * c)), \mu_A(b)\}
\]

or \( \lambda(a * c) > \max\{\lambda(a * (b * c)), \lambda(b)\} \). Let \( \bar{\mu}_A(a * c) = [(a * c)^-, (a * c)^+] \), \( \bar{\mu}_A(b) = [b^-, b^+] \) and \( \bar{\mu}_A(a * (b * c)) = [(a * (b * c))^-, (a * (b * c))^+] \). Take

\[
s_0 = \frac{1}{2} ((a * c)^- + \min\{(a * (b * c))^-, b^-\}),
\]

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If \(0 = \frac{1}{3}(a + c)^+ + \min \{ (a + (b + c))^+, b^+ \} \) and \(r_0 = \frac{1}{3}(\lambda(a) + \max \{ \lambda(a + (b + c)), \lambda(b) \}) \). Then \((a + c)^- \prec s_0 \prec \min \{ (a + (b + c))^-, b^- \} \) and \((a + c)^+ \prec t_0 \prec \min \{ (a + (b + c))^+, b^+ \} \), which imply that

\[
\hat{\mu}_A(a + c) = \{ (a + c)^-, (a + c)^+ \} \prec [s_0, t_0]
\]

\[
\prec \left[ \min \{ (a + (b + c))^-, b^- \}, \min \{ (a + (b + c))^+, b^+ \} \right]
\]

\[
= \min \{ \hat{\mu}_A(a + (b + c)), \hat{\mu}_A(b) \}.
\]

Also, \(\lambda(a + c) > r_0 > \max \{ \lambda(a + (b + c)), \lambda(b) \} \). Thus \(a + (b + c) \in U(\mathcal{A}'; [s_0, t_0], r_0) \) and \(b \in U(\mathcal{A}; [s_0, t_0], r_0) \), but \(a + c \notin U(\mathcal{A}; [s_0, t_0], r_0) \). This is a contradiction, and therefore \(\hat{\mu}_A(x \ast z) \geq \min \{ \hat{\mu}_A(x \ast (y \ast z)), \hat{\mu}_A(y) \} \) and \(\lambda(x \ast z) \leq \max \{ \lambda(x \ast (y \ast z)), \lambda(y) \} \) for all \(x, y, z \in X \). Hence \(\mathcal{A} = (A, \lambda) \) is a cubic \(q\)-ideal of \(X \). \(\square\)

**Theorem 3.13.** If \(\mathcal{A} = (A, \lambda) \) is a cubic \(q\)-ideal of a BCI-algebra \(X \), then the set

\[
I := \{ x \in X \mid \hat{\mu}_A(x) = \hat{\mu}_A(0), \ \lambda(x) = \lambda(0) \}
\]

is a \(q\)-ideal of \(X \).

**Proof.** Obviously, \(0 \in I \). Let \(x, y, z \in X \) be such that \(x \ast (y \ast z) \in I \) and \(y \in I \). Then

\[
\hat{\mu}_A(x \ast (y \ast z)) = \hat{\mu}_A(0) = \hat{\mu}_A(y) \text{ and } \lambda(x \ast (y \ast z)) = \lambda(0) = \lambda(y), \text{ so that}
\]

\[
\min \{ \hat{\mu}_A(x \ast (y \ast z)), \hat{\mu}_A(y) \} = \hat{\mu}_A(0)
\]

and \(\lambda(x \ast z) \leq \max \{ \lambda(x \ast (y \ast z)), \lambda(y) \} = \lambda(0) \). It follows from (a) and (b) in Definition 3.3 that \(\hat{\mu}_A(x \ast z) = \hat{\mu}_A(0) \) and \(\lambda(x \ast z) = \lambda(0) \) so that \(x \ast z \in I \). Therefore \(I \) is a \(q\)-ideal of \(X \). \(\square\)

**Lemma 3.14.** A cubic \(q\)-ideal \(\mathcal{A} = (A, \lambda) \) in a BCI-algebra \(X \) satisfies the following implication:

\[
(\forall x, y \in X \ (x \leq y \implies \hat{\mu}_A(x) \geq \hat{\mu}_A(y), \ \lambda(x) \leq \lambda(y))
\]

**Proof.** If \(x \leq y \), then \(x \ast y = 0 \) and hence

\[
\hat{\mu}_A(x) = \hat{\mu}_A(x \ast 0) \geq \min \{ \hat{\mu}_A(x \ast (y \ast 0)), \hat{\mu}_A(y) \}
\]

\[
= \min \{ \hat{\mu}_A(x \ast y), \hat{\mu}_A(y) \} = \min \{ \hat{\mu}_A(0), \hat{\mu}_A(y) \} = \hat{\mu}_A(y)
\]

and

\[
\lambda(x) = \lambda(x \ast 0) \leq \max \{ \lambda(x \ast (y \ast 0)), \lambda(y) \}
\]

\[
= \max \{ \lambda(x \ast y), \lambda(y) \} = \max \{ \lambda(0), \lambda(y) \} = \lambda(y).
\]

This completes the proof. \(\square\)

**Theorem 3.15.** If \(\mathcal{A} = (A, \lambda) \) is a cubic \(q\)-ideal of a BCI-algebra \(X \), then the following assertions are equivalent:

1. \(\mathcal{A} = (A, \lambda) \) is a cubic \(q\)-ideal of \(X \).
2. \(\forall x, y \in X \ (\hat{\mu}_A(x \ast y) \geq \hat{\mu}_A(x \ast (0 \ast y)), \ \lambda(x \ast y) \leq \lambda(x \ast (0 \ast y)). \)
3. \(\forall x, y, z \in X \ (\hat{\mu}_A((x \ast y) \ast z) \geq \hat{\mu}_A(x \ast (y \ast z)), \ \lambda((x \ast y) \ast z) \leq \lambda(x \ast (y \ast z)). \)

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Proof. (1) ⇒ (2) follows from Proposition 3.11(1). Assume that (2) is valid. Note that

\[(x * y) * (0 * z) * (x * (y + z)) = ((x * y) * (x * (y + z))) * (0 * z)\]
\[\leq ((y + z) * y) * (0 * z) = (0 * z) * (0 * z) = 0\]

for all \(x, y, z \in X\). It follows from Lemma 3.14 that

\[\mu_A(\{(x * y) * (0 * z)\} * (x * (y + z))) \geq \mu_A(0)\]

and \(\lambda(\{(x * y) * (0 * z)\} * (x * (y + z))) \leq \lambda(0)\) so from (a) and (b) in Definition 3.3 that

\[\mu_A(\{(x * y) * (0 * z)\} * (x * (y + z))) = \mu_A(0)\]

and \(\lambda(\{(x * y) * (0 * z)\} * (x * (y + z))) = \lambda(0)\). Using (2) and Definition 3.3, we have

\[\mu_A((x * y) * z) \geq \mu_A((x * y) * (0 * z))\]
\[\geq \mu_A((x * y) * z) = \mu_A((x * y) * (0 * z))\]
\[= \mu_A((x * y) * z)\]

and

\[\lambda((x * y) * z) \leq \lambda((x * y) * (0 * z))\]
\[\leq \lambda((x * y) * (0 * z)) = \lambda((x * y) * z)\]

Therefore (3) is valid. Now suppose that (3) holds. Then

\[\mu_A((x * z) * y) \geq \mu_A((x * z) * y)\]
\[= \mu_A((x * y) * z) \geq \mu_A((x * y) * z)\]
\[= \mu_A((x * y) * z)\]

and

\[\lambda((x * z) * y) \leq \lambda((x * z) * y)\]
\[= \lambda((x * y) * z) \leq \lambda((x * y) * z)\]

for all \(x, y, z \in X\). Hence \(\mathcal{A} = \langle A, \lambda \rangle\) is a cubic q-ideal of \(X\). \(\square\)

Theorem 3.16. For a cubic ideal \(\mathcal{A} = \langle A, \lambda \rangle\) of a BCI-algebra \(X\), the following are equivalent:

1. \(\mathcal{A} = \langle A, \lambda \rangle\) is a cubic q-ideal of \(X\).
2. \(\mu_A((x * z) * y) \geq \mu_A((x * z) * (0 * y))\) and \(\lambda((x * z) * y) \leq \lambda((x * z) * (0 * y))\)
   for all \(x, y, z \in X\).
3. \(\mu_A(x * y) \geq \mu_A((x * z) * (0 * y))\) and \(\lambda(x * y) \leq \lambda((x * z) * (0 * y))\) for all \(x, y, z \in X\).
Proof. (1) ⇒ (2). It is straightforward by Theorem 3.15.
(2) ⇒ (3). For any \( x, y, z \in X \), we have
\[
\bar{\mu}_A(x * y) \geq \min\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(z)\} \\
= \min\{\bar{\mu}_A((x * z) * y), \bar{\mu}_A(z)\} \\
\geq \min\{\bar{\mu}_A((x * z) * (0 * y)), \bar{\mu}_A(z)\}
\]
and
\[
\lambda(x * y) \leq \max\{\lambda((x * y) * z), \lambda(z)\} \\
= \max\{\lambda((x * z) * y), \lambda(z)\} \\
\leq \max\{\lambda((x * z) * (0 * y)), \lambda(z)\}.
\]
Hence (3) is valid.

(3) ⇒ (1). Assume that (3) is true. If we take \( z = 0 \) in (3), then
\[
\bar{\mu}_A(x * y) \geq \min\{\bar{\mu}_A((x * 0) * (0 * y)), \bar{\mu}_A(0)\} \\
= \min\{\bar{\mu}_A(x * (0 * y)), \bar{\mu}_A(0)\} \\
= \bar{\mu}_A(x * (0 * y))
\]
and
\[
\lambda(x * y) \leq \max\{\lambda((x * 0) * (0 * y)), \lambda(0)\} \\
= \max\{\lambda(x * (0 * y)), \lambda(0)\} \\
= \lambda(x * (0 * y))
\]
for all \( x, y \in X \). It follows from Theorem 3.15 that \( \mathcal{A} = \langle A, \lambda \rangle \) is a cubic \( q \)-ideal of \( X \).

Theorem 3.17. (Cubic extension property for a cubic \( q \)-ideal) Let \( \mathcal{A} = \langle A, \lambda \rangle \) and \( \mathcal{B} = \langle B, \kappa \rangle \) be cubic ideals of a BCI-algebra \( X \) such that \( \mathcal{A} \subseteq \mathcal{B} \) and \( \bar{\mu}_A(0) = \bar{\mu}_B(0) \) and \( \lambda(0) = \kappa(0) \). If \( \mathcal{A} = \langle A, \lambda \rangle \) is a cubic \( q \)-ideal of \( X \), then so is \( \mathcal{B} = \langle B, \kappa \rangle \).

Proof. Let \( x, y \in X \). If we take \( a = x * (0 * y) \), then
\[
(x * a) * (0 * y) = (x * (0 * y)) * a = 0.
\]
Using Theorem 3.15 we have
\[
\bar{\mu}_A((x * a) * y) \geq \bar{\mu}_A((x * a) * (0 * y)) = \bar{\mu}_A(0) = \bar{\mu}_B(0)
\]
and
\[
\lambda((x * a) * y) \leq \lambda((x * a) * (0 * y)) = \lambda(0) = \kappa(0).
\]
Thus \( \bar{\mu}_B((x * a) * y) \geq \bar{\mu}_B((x * a) * (0 * y)) \geq \bar{\mu}_B(0) \geq \bar{\mu}_B(a) \) and
\[
\kappa((x * a) * y) \leq \kappa((x * a) * (0 * y)) \leq \kappa(0) \leq \kappa(a).
\]
Since \( \mathcal{B} = \langle B, \kappa \rangle \) is a cubic ideal, it follows that
\[
\bar{\mu}_B(x * y) \geq \min\{\bar{\mu}_B((x * y) * a), \bar{\mu}_B(a)\} = \bar{\mu}_B(a) = \bar{\mu}_B(x * (0 * y))
\]
and \( \kappa(x * y) \leq \max\{\kappa((x * y) * a), \kappa(a)\} = \kappa(a) = \kappa(x * (0 * y)) \). Using Theorem 3.15 we conclude that \( \mathcal{B} = \langle B, \kappa \rangle \) is a cubic \( q \)-ideal of \( X \).
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