

A new method for solving fully fuzzy linear programming problems

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ABSTRACT. Several authors have used ranking function for solving fuzzy linear programming problems. In this paper some fuzzy linear programming problems are chosen which can't be solved by using any of the existing methods and a new method is proposed to solve such type of fuzzy linear programming problems. The main advantage of the proposed method over existing methods is that the fuzzy linear programming problems which can be solved by the existing methods can also be solved by the proposed method but there exist several fuzzy linear programming problems which can be solved only by using the proposed method i.e., it is not possible to solve these fuzzy linear programming problems by using the existing methods. To show the advantage of the proposed method over existing methods some fuzzy linear programming problems, which can't be solved by the existing methods, are solved by the proposed method.

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1. INTRODUCTION

Linear programming is one of the most frequently applied operations research techniques. Although it is investigated and expanded for more than six decades by many researchers from the various point of views, it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming. Any linear programming model representing real world situations involves a lot of parameters whose values are assigned by experts, and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However, both experts and the decision makers frequently do not precisely know the value of those parameters. If exact values are suggested these are

only statistical inference from past data and their stability is doubtful, so the parameters of the problem are usually defined by the decision makers in an uncertain way or by means of language statement parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data [21]. The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. [18] in the framework of the fuzzy decision of Bellman and Zadeh [2]. The first formulation of fuzzy linear programming (FLP) problems was proposed by Zimmermann [20]. Afterwards, many authors [3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17] considered various types of the FLP problems and proposed several approaches for solving these problems. Real numbers can be linearly ordered by the relation \geq or \leq , however this type of inequality does not exist in fuzzy numbers. Since fuzzy numbers are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than other. An efficient method for ordering the fuzzy numbers is by the use of a ranking function, which maps each fuzzy number into the real line, where a natural order exists. Jain [9] proposed the concept of ranking function for comparing normal fuzzy numbers. Maleki [14] proposed a method to unify some of the existing approaches which are using different ranking functions for solving fuzzy programming problems. Moreover, they introduced a method for solving linear programming with vagueness in constraints by using linear ranking function. Maleki et al. [16] proposed method for solving fuzzy number linear programming problems using the concept of ranking of fuzzy numbers. Nasseri and Ardil [17] developed simplex method to FLP problems by using certain ranking function. This method uses simplex tableau which is used for solving linear programming problems in crisp environment before. Mahadavi and Nasseri [12, 13] proposed duality results and a dual simplex method to solve FLP problems with trapezoidal fuzzy variables by the use of ranking function. Ganesan and Veeramani [7] dealt with a kind of fuzzy linear programming problem involving symmetric trapezoidal fuzzy numbers. Allahviranloo et al. [1] proposed a new method for solving fully fuzzy linear programming problem by using ranking function taking variables as restricted fuzzy numbers.

In this paper some fuzzy linear programming problems are chosen which can't be solved by using any of the existing methods and a new method is proposed to solve such type of fuzzy linear programming problems. The main advantage of the proposed method over existing methods is that the fuzzy linear programming problems which can be solved by the existing methods can also be solved by the proposed method but there are several fuzzy linear programming problems which can be solved only by using the proposed method i.e., it is not possible to solve these fuzzy linear programming problems by using the existing methods. To show the advantage of the proposed method over existing methods some fuzzy linear programming problems, which can't be solved by using the existing methods, are solved by the proposed method. This paper is organized as follows: In Section 2, some basic definitions, arithmetic operations and Yager's ranking approach [19] for the ranking of fuzzy numbers are reviewed. In Section 3 formulation of FLP problem is discussed. In Section 4, the shortcomings of existing method are pointed out. In Section 5, modified formulation of FLP problems is proposed. In Section 6, applicability of existing method is discussed. In Section 7, the limitations of

existing method are pointed out. In Section 8, a new method is proposed to find the optimal solution of FLP problems. In Section 9, advantages of proposed methods are discussed. In Section 10, results obtained by the existing methods and by proposed method are compared. The conclusion is discussed in Section 11.

2. PRELIMINARIES

In this section some basic definitions, arithmetic operations and Yager’s ranking function approach for the ranking of fuzzy numbers are reviewed.

2.1. Basic definitions. In this section some basic definitions are reviewed [6].

Definition 2.1. A fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be an L - R flat fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & \text{for } x \leq m, \alpha > 0 \\ R\left(\frac{x-n}{\beta}\right), & \text{for } x \geq n, \beta > 0 \\ 1, & \text{otherwise} \end{cases}$$

If $m = n$ then $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ will be converted into $\tilde{A} = (m, \alpha, \beta)_{LR}$ and is said to be an L - R fuzzy number. L and R are called reference functions, which are continuous, non-increasing functions that defining the left and right shapes of $\mu_{\tilde{A}}(x)$ respectively and $L(0) = R(0) = 1$. Two special cases are triangular and trapezoidal fuzzy number, for which $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$, are linear functions. Three commonly used nonlinear reference functions with parameters q , denoted as RF_q , are summarized as follows:

$$\begin{aligned} \text{power: } & RF_q(x) = \text{maximum}(0, 1 - x^q), & q \geq 0, \\ \text{exponential power: } & RF_q(x) = e^{-x^q}, & q \geq 0, \\ \text{rational: } & RF_q(x) = \frac{1}{(1+x^q)}, & q \geq 0. \end{aligned}$$

Definition 2.2. Let $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ be an L - R flat fuzzy number and λ be a real number in the interval $[0, 1]$ then the crisp set

$$A_\lambda = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)],$$

is said to be λ -cut of \tilde{A} .

Definition 2.3. A L - R flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be non-negative L - R flat fuzzy number if $m - \alpha \geq 0$.

2.2. Arithmetic operations. Let $\tilde{A}_1 = (m_1, n_1, \alpha, \beta)_{LR}$ be non-negative L - R flat fuzzy numbers and $\tilde{A}_2 = (m_2, n_2, \alpha', \beta')_{LR}$, $\tilde{A}_3 = (m_3, n_3, \alpha'', \beta'')_{LR}$ be any L - R flat fuzzy numbers then

- (i) $\tilde{A}_1 \otimes \tilde{A}_2 = (m_1 m_2, n_1 n_2, m_1 \alpha' + m_2 \alpha, n_1 \beta' + n_2 \beta)_{LR}$ for $m_1 - \alpha > 0$ and $m_2 - \alpha' > 0$
- (ii) $\tilde{A}_2 \oplus \tilde{A}_3 = (m_2 + m_3, n_2 + n_3, \alpha' + \alpha'', \beta' + \beta'')_{LR}$
- (iii) $\lambda \tilde{A}_2 = \begin{cases} (\lambda m_2, \lambda n_2, \lambda \alpha', \lambda \beta')_{LR}, & \lambda \geq 0; \\ (\lambda m_2, \lambda n_2, -\lambda \beta', -\lambda \alpha')_{RL}, & \lambda \leq 0. \end{cases}$
- (iv) $\tilde{A}_1 \otimes \tilde{A}_2 = (a_1, a_2, a_3, a_4)_{LR}$,

where, $a_1 = \text{minimum}(m_1m_2, n_1n_2)$, $a_2 = \text{maximum}(m_1n_2, n_1m_2)$,

$$a_3 = \text{minimum}(m_1m_2, n_1m_2) - \text{minimum}((m_1 - \alpha)(m_2 - \alpha'), (\beta + n_1)(m_2 - \alpha')),$$

$$a_4 = \text{maximum}(m_1n_2, n_1m_2) - \text{maximum}((m_1 - \alpha)(\beta' + n_2), (\beta + n_1)(\beta' + n_2))$$

and $\text{minimum}((m_1 - \alpha)(\beta' - n_2), (\beta + n_1)(m_2 - \alpha')) = \frac{(m_1 - \alpha)(\beta' + n_2) + (\beta + n_1)(m_2 - \alpha')}{2} - \left| \frac{(m_1 - \alpha)(\beta' + n_2) - (\beta + n_1)(m_2 - \alpha')}{2} \right|$

and $\text{maximum}((\beta + n_1)(\beta' + n_2), (m_1 - \alpha)(m_2 - \alpha')) = \frac{(\beta + n_1)(\beta' + n_2) + (m_1 - \alpha)(m_2 - \alpha')}{2} + \left| \frac{(\beta + n_1)(\beta' + n_2) - (m_1 - \alpha)(m_2 - \alpha')}{2} \right|$

2.3. Yager’s ranking approach. A number of approaches have been proposed for the ranking of fuzzy numbers. A relatively simple computational and easily understandable ranking method proposed by Yager [19] is considered for the ranking of fuzzy numbers in this paper. Yager [19] proposed a procedure for ordering fuzzy sets in which a ranking index $\mathfrak{R}(\tilde{A})$ is calculated for the fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ from its λ -cut $A_\lambda = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)]$ according to the following formula:

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \left(\int_0^1 (m - \alpha L^{-1}(\lambda)) d\lambda + \int_0^1 (n + \beta R^{-1}(\lambda)) d\lambda \right) \quad (1)$$

Let $\tilde{A} = (m, n, \alpha, \beta)$ be a L - R flat fuzzy number, where $L(x) = \text{maximum}(0, (1 - x^4))$ and $R(x) = \text{maximum}(0, (1 - x^2))$ then, by using Yager’s ranking formula (1),

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \left(m + n - \frac{4}{5} \alpha + \frac{2}{3} \beta \right).$$

Let \tilde{A} and \tilde{B} be two fuzzy numbers then

- (i) $\tilde{A} \succ \tilde{B}$ if $\mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B})$,
- (ii) $\tilde{A} \approx \tilde{B}$ if $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$,
- (iii) $\tilde{A} \succeq \tilde{B}$ if $\mathfrak{R}(\tilde{A}) \geq \mathfrak{R}(\tilde{B})$.

Since $\mathfrak{R}(\tilde{A})$ is calculated from the extreme values of λ -cut of \tilde{A} i.e., $m - \alpha L^{-1}(\lambda)$ and $n + \beta R^{-1}(\lambda)$, rather than its membership function, it is not required knowing the explicit form of the membership functions of the fuzzy numbers to be ranked. That is, unlike most of the ranking methods that require the knowledge of the membership functions of all fuzzy numbers to be ranked, the Yager’s ranking index is still applicable even if the explicit form of membership function of the fuzzy number is unknown.

2.3.1. Linearity property of Yager’s ranking index. Let $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{L_1R_1}$ and $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{L_2R_2}$ be two L - R flat fuzzy numbers and k_1, k_2 be two non-negative real numbers then using Definition 2.2, the λ -cut A_λ and B_λ corresponding to \tilde{A} and \tilde{B} are $A_\lambda = [m_1 - \alpha_1 L_1^{-1}(\lambda), n_1 + \beta_1 R_1^{-1}(\lambda)]$ and

$$B_\lambda = [m_2 - \alpha_2 L_2^{-1}(\lambda), n_2 + \beta_2 R_2^{-1}(\lambda)].$$

Using the property, $(\delta_1 A_1 + \delta_2 A_2)_\lambda = \delta_1(A_1)_\lambda + \delta_2(A_2)_\lambda, \forall \delta_1, \delta_2 \in R$ (R is a set of real numbers), the λ -cut $(k_1 A + k_2 B)_\lambda$ corresponding to $(k_1 \tilde{A} \oplus k_2 \tilde{B})$ is

$$(k_1 A + k_2 B)_\lambda = [k_1 m_1 + k_2 m_2 - k_1 \alpha_1 L_1^{-1}(\lambda) - k_2 \alpha_2 L_2^{-1}(\lambda), \\ k_1 n_1 + k_2 n_2 + k_1 \beta_1 R_1^{-1}(\lambda) + k_2 \beta_2 R_2^{-1}(\lambda)]$$

Using (1), the Yager's ranking index $\mathfrak{R}(k_1 \tilde{A} \oplus k_2 \tilde{B})$ corresponding to fuzzy number $(k_1 \tilde{A} \oplus k_2 \tilde{B})$ is

$$\mathfrak{R}(k_1 \tilde{A} \oplus k_2 \tilde{B}) = \frac{1}{2} k_1 \left[\int_0^1 (m_1 - \alpha_1 L_1^{-1}(\lambda)) d\lambda + \int_0^1 (n_1 + \beta_1 R_1^{-1}(\lambda)) d\lambda \right] \\ + \frac{1}{2} k_2 \left[\int_0^1 (m_2 - \alpha_2 L_2^{-1}(\lambda)) d\lambda + \int_0^1 (n_2 + \beta_2 R_2^{-1}(\lambda)) d\lambda \right] \\ = k_1 \mathfrak{R}(\tilde{A}) + k_2 \mathfrak{R}(\tilde{B})$$

Similarly, it can be proved that $\mathfrak{R}(k_1 \tilde{A} \oplus k_2 \tilde{B}) = k_1 \mathfrak{R}(\tilde{A}) + k_2 \mathfrak{R}(\tilde{B})$ for all real numbers k_1 and k_2 .

3. FULLY FUZZY LINEAR PROGRAMMING PROBLEMS

A fully fuzzy linear programming problem with m constraints and n variables may be written as [1]:

$$\text{Maximize (or Minimize) } \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \\ \text{subject to} \tag{P_1} \\ \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ \tilde{x}_j \succeq 0$$

where, $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, y_j, \alpha_j'', \beta_j'')_{LR}$, $\tilde{b}_i = (b_i, g_i, \lambda_i'', \rho_i'')_{LR}$ and $\tilde{c}_j = (t_j, u_j, \alpha_j''', \beta_j''')_{LR}$ are L - R flat fuzzy numbers.

3.1. Fuzzy optimal solution of fully FLP problems. A set of L - R flat fuzzy numbers $\{\tilde{x}_j\}$ is said to be fuzzy optimal solution of P_1 if the following properties are satisfied.

- (i) $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m$
- (ii) $\tilde{x}_j \succeq 0 \quad j = 1, 2, \dots, n$
- (iii) If there exist any set of fuzzy numbers $\{\tilde{y}_j\}$ such that

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j \preceq, \approx, \succeq \tilde{b}_i \text{ and } \tilde{y}_j \succeq 0 \quad j = 1, 2, \dots, n$$

then $(\sum_{j=1}^n \tilde{c}_j \otimes \tilde{x}_j) \succ \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{y}_j)$ (in case of maximization problem) and

$(\sum_{j=1}^n \tilde{c}_j \otimes \tilde{x}_j) \prec \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{y}_j)$ (in case of minimization problem).

4. SHORTCOMINGS OF EXISTING METHODS

Allahviranloo et al. [1] have used the following products for solving the fully FLP problems P_1 .

Let $\tilde{A}_1 = (m_1, \alpha, \beta)_{LR}$, $\tilde{A}_2 = (m_2, \alpha', \beta')_{LR}$ be two L - R type fuzzy numbers and $\tilde{A}_3 = (m_3, \alpha'', \beta'')_{RL}$ be R - L type fuzzy number then,

- (i) $\tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, \alpha + \alpha', \beta + \beta')_{LR}$,
- (ii) $\tilde{A}_2 \ominus \tilde{A}_3 = (m_2 - m_3, \alpha'' + \beta', \beta'' + \alpha')_{LR}$,
- (iii) $\tilde{A}_1 \odot \tilde{A}_2 \simeq (m_1 m_2, m_1 \alpha' + m_2 \alpha, m_1 \beta' + m_2 \beta)_{LR}$, for $m_1 - \alpha > 0$ and $m_2 - \alpha' > 0$,
- (iv) $\tilde{A}_3 \odot \tilde{A}_2 \simeq (m_3 m_2, m_2 \alpha'' - m_3 \beta', m_2 \beta'' - m_3 \alpha')_{RL}$ for $m_3 - \alpha'' < 0$ and $m_2 - \alpha' > 0$,
- (v) $\tilde{A}_1 \odot \tilde{A}_2 \simeq (m_1 m_2, -m_2 \beta - m_1 \beta', -m_2 \alpha - m_1 \alpha')_{RL}$ for $m_1 - \alpha < 0$ and $m_2 - \alpha' < 0$.

The above product rules can be used only if the nature of both the fuzzy numbers are known. For solving all the problems, Allahviranloo et al. [1] have assumed that $\tilde{x} \succeq 0 \Rightarrow m - \alpha \geq 0$. But the assumption $\tilde{x} \succeq 0 \Rightarrow m - \alpha \geq 0$ is not always true e.g, $\tilde{x} = (-1, 1, 5)_{LR}$ is L - R type fuzzy number such that $\Re(\tilde{x}) = 1 > 0$ but $m - \alpha = -2 < 0$.

Due to the assumption $\tilde{x} \succeq 0 \Rightarrow m - \alpha \geq 0$ the results of the numerical problems solved by the existing method [1] are incorrect, e.g., let $\tilde{a} = (-1, 1, 5)_{RL}$ and $\tilde{b} = (2, 1, 1)_{LR}$ be two L - R fuzzy numbers then according to Dubois and Prade [6] $\tilde{a} \odot \tilde{b}$ should be replaced by $(-2, 3, 11)_{RL}$ while according to Allahviranloo et al. [1], since $\Re(\tilde{a}) \geq 0$, so $\tilde{a} \odot \tilde{b}$ should be replaced by $(-2, 1, 9)_{RL}$.

5. MODIFIED FORMULATION

The shortcomings of the existing method [1], pointed out in Section 4, are due to the incorrectness of the existing formulation (P_1). If in the existing formulation the restriction $\tilde{x}_j \succeq 0$, is replaced by the restriction \tilde{x}_j is a non-negative fuzzy numbers then the restriction $\tilde{x}_j \succeq 0$ of the existing formulation will be automatically satisfied and all the shortcomings of the existing methods will be removed. Also by replacing $\tilde{x}_j \succeq 0$ by \tilde{x}_j is a non-negative fuzzy number all the results of the numerical problems, obtained by Allahviranloo et al. [1], will be valid. Due to these reasons, in this section the existing formulation P_1 is modified by P'_1

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \\ &\text{subject to} \tag{P'_1} \\ &\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ &\tilde{x}_j \text{ is a non-negative fuzzy number} \end{aligned}$$

where, $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{i,j}, \beta_{ij})_{LR}$, $\tilde{x}_j = (m_j, n_j, \alpha'_j, \beta'_j)_{LR}$, $\tilde{b}_i = (p_i, q_i, \alpha''_i, \beta''_i)_{LR}$ and $\tilde{c}_j = (t_j, u_j, \alpha'''_j, \beta'''_j)_{LR}$ are L - R flat fuzzy numbers.

5.1. Fuzzy optimal solution of modified formulation. A set of fuzzy numbers $\{\tilde{x}_j\}$ is said to be fuzzy optimal solution of P_1 if the following properties are satisfied.

- (i) $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i$
- (ii) \tilde{x}_j is non-negative fuzzy number.
- (iii) If there exist $\{\tilde{y}_j\}$ such that $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j \preceq, \approx, \succeq \tilde{b}_i$ then $(\sum_{j=1}^n \tilde{c}_j \otimes \tilde{x}_j) \succ \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{y}_j)$ (in case of maximization problem) and $\sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \prec \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{y}_j)$ (in case of minimization problem).

6. APPLICABILITY OF EXISTING METHODS

In this section, the different types of FLP problems, which can be solved by existing methods, are discussed.

(i) The existing method [17] may be used only for solving the following type of FLP problems:

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^n (\tilde{c}_j x_j) \\ &\text{subject to} \tag{P_2} \\ &\quad \sum_{j=1}^n a_{ij} x_j \leq, = b_i, \quad i = 1, 2, \dots, m \\ &\quad x_j \geq 0 \end{aligned}$$

where, a_{ij}, b_i, x_j are real numbers, $\tilde{c}_j = (t_j, u_j, \alpha_j'', \beta_j'')_{LR}$ and $L(x) = R(x) = \text{maximum } \{0, 1 - x\}$.

Example 6.1 ([17]). Maximize $((5, 8, 2, 5)_{LR})x_1 \oplus ((6, 10, 2, 6)_{LR})x_2$
subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ 5x_1 + 4x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

where, $L(x) = R(x) = \text{maximum } \{0, 1 - x\}$.

(ii) The existing method [12] may be used only for solving the following type of FLP problems:

$$\begin{aligned} &\text{Maximize (or Minimize) } \sum_{j=1}^n (c_j \tilde{x}_j) \\ &\text{subject to} \tag{P_3} \\ &\quad \sum_{j=1}^n a_{ij} \tilde{x}_j \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ &\quad \tilde{x}_j \succeq 0 \end{aligned}$$

where, c_j, a_{ij} are real numbers, $\tilde{x}_j = (x_j, y_j, \alpha_j'', \beta_j'')_{LR}$, $\tilde{b}_i = (b_i, g_i, \lambda_i'', \rho_i'')_{LR}$ and $L(x) = R(x) = \text{maximum } \{0, 1 - x\}$.

Example 6.2 ([12]). Maximize $(6\tilde{x}_1 \oplus 10\tilde{x}_2)$
subject to

$$\begin{aligned} 2\tilde{x}_1 \oplus 5\tilde{x}_2 &\succeq (5, 8, 2, 5)_{LR} \\ 3\tilde{x}_1 \oplus 4\tilde{x}_2 &\succeq (6, 10, 2, 6)_{LR} \\ \tilde{x}_1, \tilde{x}_2 &\succeq 0 \end{aligned}$$

where, $\tilde{x}_1 = (x_1, y_1, \alpha_1'', \beta_1'')_{LR}$, $\tilde{x}_2 = (x_2, y_2, \alpha_2'', \beta_2'')_{LR}$ and

$$L(x) = R(x) = \text{maximum}\{0, 1 - x\}.$$

(iii) The existing method [7] may be used for only for solving special type of FLP problems in which some parameters are taken as symmetric fuzzy numbers and a special type of product is used:

$$\begin{aligned} & \text{Maximize (or Minimize)} \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \\ & \text{subject to} \\ & \sum_{j=1}^n a_{ij} \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ & \tilde{x}_j \succeq 0 \end{aligned} \tag{P_4}$$

where, a_{ij} is a real number, $\tilde{x}_j = (x_j, y_j, \alpha_j'', \beta_j'')_{LR}$, $\tilde{b}_i = (b_i, g_i, \lambda_i'', \rho_i'')_{LR}$, $\tilde{c}_j = (t_j, u_j, \alpha_j''', \beta_j''')_{LR}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$

Example 6.3 ([7]). Maximize $((13, 15, 2, 2)_{LR} \otimes \tilde{x}_1 \oplus (12, 14, 3, 3)_{LR} \otimes \tilde{x}_2) \oplus (15, 17, 2, 2)_{LR} \otimes \tilde{x}_3$ subject to

$$\begin{aligned} 12\tilde{x}_1 \oplus 13\tilde{x}_2 \oplus 12\tilde{x}_3 & \preceq (475, 505, 6, 6) \\ 14\tilde{x}_1 \oplus 13\tilde{x}_3 & \preceq (460, 480, 8, 8) \\ 12\tilde{x}_1 \oplus 15\tilde{x}_2 & \preceq (465, 495, 5, 5) \\ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 & \succeq 0 \end{aligned}$$

where, $\tilde{x}_1 = (x_1, y_1, \alpha_1'', \beta_1'')_{LR}$, $\tilde{x}_2 = (x_2, y_2, \alpha_2'', \beta_2'')_{LR}$, $\tilde{x}_3 = (x_3, y_3, \alpha_3'', \beta_3'')_{LR}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$.

(iv) The existing method [16] may be used for only for solving following type of FLP problems:

$$\begin{aligned} & \text{Maximize (or Minimize)} \sum_{j=1}^n (\tilde{c}_j x_j) \\ & \text{subject to} \\ & \sum_{j=1}^n \tilde{a}_{ij} x_j \preceq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ & x_j \geq 0 \end{aligned} \tag{P_5}$$

where x_j is a real number, $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{b}_i = (b_i, g_i, \lambda_i'', \rho_i'')_{LR}$, $\tilde{c}_j = (t_j, u_j, \alpha_j''', \beta_j''')_{LR}$ and $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$.

Example 6.4 ([16]). Maximize $((3, 4, 2, 3)_{LR} x_1 \oplus (3, 5, 1, 4)_{LR} x_2)$ subject to

$$\begin{aligned} ((3, 6, 1, 2)_{LR} x_1 \oplus ((2, 4, 1, 2)_{LR} x_2) & \preceq (10, 13, 2, 4)_{LR} \\ ((5, 6, 2, 4)_{LR} x_1 \oplus ((7, 10, 3, 4)_{LR} x_2) & \preceq (18, 24, 3, 6)_{LR} \\ x_1, x_2 & \geq 0 \end{aligned}$$

where, $L(x) = R(x) = \text{maximum}\{0, 1 - x\}$.

(v) In the Section 5, it is pointed out that the existing method [1] can be used for only for solving following type of FLP problems:

$$\begin{aligned} & \text{Maximize (or Minimize)} \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \\ & \text{subject to} \\ & \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \end{aligned} \tag{P_6}$$

\tilde{x}_j is non-negative fuzzy number
 where, $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\tilde{x}_j = (x_j, \alpha'_j, \beta''_j)_{LR}$, $\tilde{b}_i = (b_i, \lambda''_i, \rho''_i)_{LR}$ and
 $\tilde{c}_j = (t_j, \alpha'''_j, \beta'''_j)_{LR}$.

Example 6.5. Maximize $((1, 1, 1)_{LR} \otimes \tilde{x}_1 \oplus (2, 1, 2)_{LR} \otimes \tilde{x}_2)$
 subject to

$$\begin{aligned} (4, 1, 0)_{LR} \otimes \tilde{x}_1 \oplus (-3, 2, 1)_{LR} \otimes \tilde{x}_2 &\preceq (2, 1, 2)_{LR} \\ (-3, 1, 2)_{LR} \otimes \tilde{x}_1 \oplus (2, 1, 1)_{LR} \otimes \tilde{x}_2 &\preceq (1, 0, 1)_{LR} \end{aligned}$$

where, $\tilde{x}_1 = (x_1, y_1, \alpha''_1, \beta''_1)_{LR}$, $\tilde{x}_2 = (x_2, y_2, \alpha''_2, \beta''_2)_{LR}$ are non-negative L - R flat fuzzy numbers.

Remark 6.6. Since in the all the fuzzy linear programming problems P_1 to P_5 there is need to find the product of a fuzzy number with a scalar quantities only so either the restriction $\tilde{x}_j \succeq 0$ or \tilde{x}_j is non-negative fuzzy number is used there will be no change in rank of the obtained fuzzy optimal solution and fuzzy optimal value i.e. by replacing $\tilde{x}_j \succeq 0$ by \tilde{x}_j is non-negative fuzzy number an alternative fuzzy optimal solution and an alternative fuzzy optimal value will be obtained.

7. LIMITATION OF EXISTING METHODS

The existing methods [1, 7, 13, 16, 17] cannot be used for solving following type of FLP problems:

$$(i) \quad \text{Maximize (or Minimize) } \sum_{j=1}^n (\tilde{c}_j x_j) \quad (P_7)$$

subject to

$$\begin{aligned} \sum_{j=1}^n a_{ij} x_j &\leq, =, \geq b_i, \quad i = 1, 2, \dots, m \\ x_j &\geq 0 \end{aligned}$$

where, a_{ij}, b_i, x_j are real numbers and $\tilde{c}_j = (t_j, u_j, \alpha'''_j, \beta'''_j)_{L_j R_j}$.

Example 7.1. Maximize $((3, 4, 3, 1)_{L_1 R_1})x_1 \oplus ((2, 2, 1, 2)_{L_2 R_2})x_2)$
 subject to

$$\begin{aligned} 3x_1 + 2x_2 &\leq 5 \\ 5x_1 + 4x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

where, $L_1(x) = \text{maximum}(0, 1 - x^4)$, $R_1(x) = \text{maximum}(0, 1 - x^2)$, $L_2(x) = \text{maximum}(0, 1 - x^2)$ and $R_2(x) = e^{-|x|}$.

$$(ii) \quad \text{Maximize (or Minimize) } \sum_{j=1}^n (c_j \tilde{x}_j) \quad (P_8)$$

subject to

$$\begin{aligned} \sum_{j=1}^n a_{ij} \tilde{x}_j &\preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ \tilde{x}_j &\succeq 0 \end{aligned}$$

where, c_j, a_{ij} are real numbers and $\tilde{x}_j = (m_j, n_j, \alpha'_j, \beta'_j)_{L_j R_j}$, $\tilde{b}_i = (p_i, q_i, \alpha''_i, \beta''_i)_{L'_i R'_i}$.

Example 7.2. Maximize $(3\tilde{x}_1 \oplus 1\tilde{x}_2)$
 subject to

$$\begin{aligned} 1\tilde{x}_1 \oplus 5\tilde{x}_2 &\preceq (6, 8, 3, 1)_{L'_1 R'_1} \\ 3\tilde{x}_1 \oplus 3\tilde{x}_2 &\preceq (5, 10, 4, 3)_{L'_2 R'_2} \\ \tilde{x}_1, \tilde{x}_2 &\succeq 0 \end{aligned}$$

where, $\tilde{x}_1 = (m_1, n_1, \alpha'_1, \beta'_1)_{L_1 R_1}$, $\tilde{x}_2 = (m_2, n_2, \alpha'_2, \beta'_2)_{L_2 R_2}$ and $L_1(x) = L'_1(x) = \text{maximum}(0, 1 - x^4)$, $R_1(x) = R'_1(x) = \text{maximum}(0, 1 - x^2)$, $L_2(x) = L'_2(x) = \text{maximum}(0, 1 - x^2)$, $R_2(x) = R'_2(x) = e^{-|x|}$

$$\begin{aligned} \text{(iii) Maximize (or Minimize)} \quad & \sum_{j=1}^n (\tilde{c}_j \otimes \tilde{x}_j) \\ \text{subject to} \quad & \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \preceq, \approx, \succeq \tilde{b}_i, \quad i = 1, 2, \dots, m \\ & \tilde{x}_j \text{ ia a non-negative } L\text{-}R \text{ flat fuzzy number} \end{aligned} \tag{P_9}$$

where, $\tilde{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})_{L_j R_j}$, $\tilde{x}_j = (m_j, n_j, \alpha'_j, \beta'_j)_{L_j R_j}$, $\tilde{b}_i = (p_i, q_i, \alpha''_i, \beta''_i)_{L'_i R'_i}$, $\tilde{c}_j = (t_j, u_j, \alpha'''_j, \beta'''_j)_{L_j R_j}$.

Example 7.3. Maximize $((3, 4, 3, 1)_{L_1 R_1} \otimes \tilde{x}_1 \oplus (2, 2, 1, 2)_{L_2 R_2} \otimes \tilde{x}_2)$
subject to

$$\begin{aligned} (1, 2, 1, 3)_{L_1 R_1} \otimes \tilde{x}_1 \oplus (2, 5, 1, 2)_{L_2 R_2} \otimes \tilde{x}_2 &\preceq (6, 8, 3, 1)_{L'_1 R'_1} \\ (4, 5, 1, 3)_{L_1 R_1} \otimes \tilde{x}_1 \oplus (2, 2, 2, 1)_{L_2 R_2} \otimes \tilde{x}_2 &\preceq (5, 10, 4, 3)_{L'_2 R'_2} \end{aligned}$$

where, $\tilde{x}_1 = (m_1, n_1, \alpha, \beta)_{L_1 R_1}$, $\tilde{x}_2 = (m_2, n_2, \alpha_2, \beta_2)_{L_2 R_2}$ and $L_1(x) = L'_1(x) = \text{maximum}(0, 1 - x^4)$, $R_1(x) = R'_1(x) = \text{maximum}(0, 1 - x^2)$, $L_2(x) = L'_2(x) = \text{maximum}(0, 1 - x^2)$ and $R_2(x) = R'_2(x) = e^{-|x|}$.

8. PROPOSED METHOD TO FIND THE FUZZY OPTIMAL SOLUTION OF FLP PROBLEMS

In this section, to overcome the shortcomings and limitations of the existing methods, a new method is proposed to find the fuzzy optimal solution of FLP problems.

The fuzzy optimal solution of FLP problems P_7 to P_9 may be obtained by using following steps:

Step 1 Assuming all the parameters $\tilde{c}_j, \tilde{x}_j, \tilde{a}_{ij}$ and \tilde{b}_i as L - R flat fuzzy numbers $(p_j, q_j, \alpha'_j, \beta'_j)_{L_j R_j}$, $(x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j}$, $(a_{ij}, b_{ij}, \lambda_{ij}, \rho_{ij})_{L_j R_j}$, and $(b_i, g_i, \lambda''_i, \rho''_i)_{L'_i R'_i}$ respectively then the FLP problem P_9 may be written as:

$$\begin{aligned} \text{Maximize (or Minimize)} \quad & \sum_{j=1}^n (p_j, q_j, \alpha'_j, \beta'_j)_{L_j R_j} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j} \\ \text{subject to} \quad & \end{aligned}$$

$$\sum_{j=1}^n (a_{ij}, b_{ij}, \lambda_{ij}, \rho_{ij})_{L_j R_j} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j} \preceq, \approx, \succeq (b_i, g_i, \lambda''_i, \rho''_i)_{L'_i R'_i} \quad \forall i = 1, 2, \dots, m.$$

Step 2 Assuming $\tilde{x}_j = (x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j}$ is a non-negative L - R flat fuzzy number,

$$(a_{ij}, b_{ij}, \lambda_{ij}, \rho_{ij})_{L_j R_j} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{L_j R_j}$$

and $(p_j, q_j, \alpha'_j, \beta'_j)_{L_j R_j} \otimes (x_j, y_j, \alpha''_j, \beta''_j)_{L_j R_j} = (t_j, u_j, \alpha'''_j, \beta'''_j)_{L_j R_j}$ the FLP problem, obtained in Step 1, may be written as:

$$\begin{aligned} & \text{Maximize (or Minimize)} \quad \left(\sum_{j=1}^n (t_j, u_j, \alpha_j'', \beta_j'')_{L_j R_j} \right) \\ & \text{subject to} \\ & \sum_{j=1}^n (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{L_j R_j} \preceq, \approx, \succeq (b_i, g_i, \lambda_i'', \rho_i'')_{L'_i R'_i} \quad \forall i = 1, 2, \dots, m \\ & \tilde{x}_j \text{ is a non-negative } L\text{-}R \text{ flat fuzzy number.} \end{aligned}$$

Step 3 Convert FLP problem, obtained in Step 2, into the following crisp linear programming problems.

$$\begin{aligned} & \text{Maximize (or Minimize)} \quad \mathfrak{R} \left(\sum_{j=1}^n (t_j, u_j, \alpha_j'', \beta_j'')_{L_j R_j} \right) \\ & \text{subject to} \\ & \mathfrak{R} \left(\sum_{j=1}^n (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{L_j R_j} \right) \leq, = \geq \mathfrak{R}((b_i, g_i, \lambda_i'', \rho_i'')_{L'_i R'_i}) \end{aligned}$$

for all $i = 1, 2, \dots, m$ where, $x_j - \alpha_j'' \geq 0, y_j - x_j \geq 0, x_j \geq 0, y_j \geq 0, \alpha_j'' \geq 0, \beta_j'' \geq 0$.

Step 4 Solve the crisp linear programming problem, obtained in Step 3, to find the optimal solution $\{x_j, y_j, \alpha_j'', \beta_j''\}$ and the optimal value representing the Yager’s ranking index corresponding to maximum objective function value.

Step 5 Find the fuzzy optimal solution by putting the values of x_j, y_j, α_j'' and β_j'' in $\tilde{x}_j = (x_j, y_j, \alpha_j'', \beta_j'')_{L_i R_j}$.

9. ADVANTAGE OF THE PROPOSED METHOD

The main advantage of proposed method over existing methods [1, 7, 13, 15, 16] is that the existing methods can be used only to find the solution of such FLP problems in which all the uncertain parameters are represented by same type of L - R fuzzy numbers and it can’t be used for solving such FLP problems in which the different uncertain parameters are represented by different L - R fuzzy numbers while the proposed method can be used for solving both types of FLP problems i.e. FLP problems in which either the different uncertain parameters are represented by same type of L - R fuzzy numbers or by the different types of L - R fuzzy numbers.

9.1. Fuzzy optimal solution of chosen fully FLP problem. To show the advantage of the proposed method over existing method the FLP problems, chosen in section which can’t be solved by using the existing methods, is solved by using the proposed method.

(i) The fuzzy optimal solution of FLP problem, chosen in Example 7.1, may be obtained by using the following steps:

Step 1 The FLP problem, chosen in Example 7.1, is:

$$\begin{aligned} & \text{Maximize } ((3, 4, 3, 1)_{L_1 R_1})x_1 \oplus ((2, 2, 1, 2)_{L_2 R_2})x_2 \\ & \text{subject to} \\ & 3x_1 + 2x_2 \leq 5 \\ & 5x_1 + 4x_2 \leq 8 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Step 2 Now using the arithmetic operations, defined in Section 2, the FLP problem, chosen in Step 1, may be written as:

$$\text{Maximize } ((3x_1, 4x_1, 3x_1, x_1)_{L_1 R_1} \oplus (2x_2, 2x_2, 1x_2, 2x_2)_{L_2 R_2})$$

$$\begin{aligned} &\text{subject to} \\ &3x_1 + 2x_2 \leq 5 \\ &5x_1 + 4x_2 \leq 8 \\ &x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Step 3 The FLP problem, obtained in Step 2, may be converted into the following crisp linear programming (CLP) problem

$$\begin{aligned} &\text{Maximize } (\mathfrak{R}((3x_1, 4x_1, 3x_1, 1x_1)_{L_1R_1} \oplus (2x_2, 2x_2, 1x_2, 2x_2)_{L_2R_2})) \\ &\text{subject to} \\ &3x_1 + 2x_2 \leq 5 \\ &5x_1 + 4x_2 \leq 8 \\ &x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Step 4 The (CLP) problem, obtained in Step 3, may be written as:

$$\begin{aligned} &\text{Maximize } \left(\frac{103}{30}x_1 + \frac{8}{3}x_2 \right) \\ &\text{subject to} \\ &3x_1 + 2x_2 \leq 5 \\ &5x_1 + 4x_2 \leq 8 \\ &x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Step 5 The optimal solution of the CLP problem, obtained in Step 4, is $x_1 = 1.60, x_2 = 0$ the optimal value, representing the Yager’s ranking index, corresponding to maximum value of objective function, $\frac{27}{5}$.

(ii) The fuzzy optimal solution of FLP problem, chosen in Example 7.2, may be obtained by using the following steps:

Step 1 Let $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ then given FLP problem may be written as:

$$\begin{aligned} &\text{Maximize } (3(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}) \\ &\text{subject to} \end{aligned}$$

$$1(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus 5(x_2, y_2, \alpha_2, \beta_2)_{L_2R_2} \preceq (6, 8, 3, 1)_{L'_1R'_1}$$

$$3(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus 3 \otimes (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2} \preceq (5, 10, 4, 3)_{L'_2R'_2}$$

$(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $(x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ are non-negative L - R flat fuzzy numbers.

Step 2 Now using the arithmetic operations, defined in Section 2, the fully FLP problem, obtained in Step 1, may be written as:

$$\begin{aligned} &\text{Maximize } ((3x_1, 3y_1, 3\alpha_1, 3\beta_1)_{L_1R_1} \oplus (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}) \\ &\text{subject to} \end{aligned}$$

$$(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (5x_2, 5y_2, 5\alpha_2, 5\beta_2)_{L_2R_2} \preceq (6, 8, 3, 1)_{L'_1R'_1}$$

$$(3x_1, 3y_1, 3\alpha_1, 3\beta_1)_{L_1R_1} \oplus (3x_2, 3y_2, 3\alpha_2, 3\beta_2)_{L_2R_2} \preceq (5, 10, 4, 3)_{L'_2R'_2}$$

$(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}, (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ are non-negative L - R flat fuzzy numbers.

Step 3 The FLP problem, obtained in Step 2, may be converted into the following crisp linear programming (CLP) problem

$$\begin{aligned} &\text{Maximize } (\mathfrak{R}((3x_1, 3y_1, 3\alpha_1, 3\beta_1)_{L_1R_1} \oplus (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2})) \\ &\text{subject to} \end{aligned}$$

$$\mathfrak{R}((x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (5x_2, 5y_2, 5\alpha_2, 5\beta_2)_{L_2R_2}) \leq \mathfrak{R}((6, 8, 3, 1)_{L'_1R'_1})$$

$$\mathfrak{R}((3x_1, 3y_1, 3\alpha_1, 3\beta_1)_{L_1R_1} \oplus (3x_2, 3y_2, 3\alpha_2, 3\beta_2)_{L_2R_2}) \leq \mathfrak{R}((5, 10, 4, 3)_{L'_2R'_2})$$

$$x_1 - \alpha_1 \geq 0, y_1 - x_1 \geq 0, x_1 \geq 0, y_1 \geq 0, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 - \alpha_2 \geq 0, y_2 - x_2 \geq 0, x_2 \geq 0, y_2 \geq 0, \alpha_2 \geq 0, \beta_2 \geq 0$$

Step 4 The (CLP) problem, obtained in Step 3, may be written as:

$$\text{Maximize } \left(\frac{3}{2}x_1 + \frac{3}{2}y_1 - \frac{6}{5}\alpha_1 + \beta_1 + \frac{1}{2}x_2 + \frac{1}{2}y_2 - \frac{1}{3}\alpha_2 + \frac{1}{2}\beta_2 \right)$$

subject to

$$\frac{1}{2}x_1 + \frac{1}{2}y_1 - \frac{2}{5}\alpha_1 + \frac{1}{3}\beta_1 + \frac{5}{2}x_2 + \frac{5}{2}y_2 - \frac{5}{3}\alpha_2 + \frac{5}{2}\beta_2 \leq \frac{92}{15}$$

$$\frac{3}{2}x_1 + \frac{3}{2}y_1 - \frac{6}{5}\alpha_1 + \beta_1 + \frac{3}{2}x_2 + \frac{3}{2}y_2 - \alpha_2 + \frac{3}{2}\beta_2 \leq \frac{23}{3}$$

$$x_1 - \alpha_1 \geq 0, x_2 - \alpha_2 \geq 0, y_1 - x_1 \geq 0, y_2 - x_2 \geq 0$$

$$x_1, x_2, y_1, y_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$$

Step 5 The optimal solution of the CLP problem, obtained in Step 4, is $x_1 = \frac{5}{2}, y_1 = \frac{5}{2}, \alpha_1 = 0, \beta_1 = 0, x_2 = 0, y_2 = 0, \alpha_2 = 0, \beta_2 = 0$ the optimal value, representing the Yager's ranking index, corresponding to maximum value of objective function, $\frac{38}{5}$.

Step 6 Putting the values of $x_1, x_2, y_1, y_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ the fuzzy optimal solution of the chosen FLP problem is $\tilde{x}_1 = (\frac{5}{2}, \frac{5}{2}, 0, 0)_{L_1R_1}, \tilde{x}_2 = (0, 0, 0, 0)_{L_2R_2}$.

(iii) The fuzzy optimal solution of the FLP problem, chosen in Example 7.3, may be obtained by using the following steps:

Step 1 Let $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ then given FLP problem may be written as:

$$\text{Maximize } ((3, 4, 3, 1)_{L_1R_1} \otimes (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (2, 2, 1, 2)_{L_2R_2} \otimes (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2})$$

subject to

$$(1, 2, 1, 3)_{L_1R_1} \otimes (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (2, 5, 1, 2)_{L_2R_2} \otimes (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2} \preceq (6, 8, 3, 1)_{L'_1R'_1}$$

$$(4, 5, 2, 1)_{L_2R_2} \otimes (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1} \oplus (2, 3, 2, 1)_{L_2R_2} \otimes (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2} \preceq (5, 10, 4, 3)_{L'_2R'_2}$$

$(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $(x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ are non-negative L - R flat fuzzy numbers.

Step 2 Now using the arithmetic operations, defined in Section 2, the fully FLP problem, obtained in Step 1, may be written as:

$$\text{Maximize } ((3x_1, 4y_1, 3\alpha_1 + 3x_1, 4\beta_1 + y_1)_{L_1R_1} \oplus (2x_2, 2y_2, 2\alpha_2 + x_2, 2\beta_2 + 2y_2)_{L_2R_2})$$

subject to

$$(x_1, 2y_1, \alpha_1 + x_1, 2\beta_1 + 3y_1)_{L_1R_1} \oplus (2x_2, 5y_2, 2\alpha_2 + x_2, 5\beta_2 + 2y_2)_{L_2R_2} \preceq (6, 8, 3, 1)_{L'_1R'_1}$$

$$(4x_1, 5y_1, 4\alpha_1 + 2x_1, 5\beta_1 + y_1)_{L_1R_1} \oplus (2x_2, 3y_2, 2\alpha_2 + 2x_2, 3\beta_2 + y_2)_{L_2R_2} \preceq (5, 10, 4, 3)_{L'_2R'_2}$$

$(x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}, (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ are non-negative L - R flat fuzzy numbers.

Step 3 The FLP problem, obtained in Step 2, may be converted into the following crisp linear programming (CLP) problem

$$\text{Maximize } \mathfrak{R}((3x_1, 4y_1, 3\alpha_1 + 3x_1, 4\beta_1 + y_1)_{L_1R_1} \oplus (2x_2, 2y_2, 2\alpha_2 + x_2, 2\beta_2 + 2y_2)_{L_2R_2})$$

subject to

$$\mathfrak{R}((x_1, 2y_1, \alpha_1 + x_1, 2\beta_1 + 3y_1)_{L_1R_1} \oplus (2x_2, 5y_2, 2\alpha_2 + x_2, 5\beta_2 + 2y_2)_{L_2R_2}) \leq \mathfrak{R}((6, 8, 3, 1)_{L'_1R'_1})$$

$$\mathfrak{R}((4x_1, 5y_1, 4\alpha_1 + 2x_1, 5\beta_1 + y_1)_{L_1R_1} \oplus (2x_2, 3y_2, 2\alpha_2 + 2x_2, 3\beta_2 + y_2)_{L_2R_2}) \leq \mathfrak{R}((5, 10, 4, 3)_{L'_2R'_2})$$

$$x_1 - \alpha_1 \geq 0, y_1 - x_1 \geq 0, x_1 \geq 0, y_1 \geq 0, \alpha_1 \geq 0, \beta_1 \geq 0$$

$$x_2 - \alpha_2 \geq 0, y_2 - x_2 \geq 0, x_2 \geq 0, y_2 \geq 0, \alpha_2 \geq 0, \beta_2 \geq 0.$$

Step 4 The (CLP) problem, obtained in Step 3, may be written as:

$$\text{Maximize } \left(\frac{3}{10}x_1 + \frac{7}{3}y_1 - \frac{6}{5}\alpha_1 + \frac{4}{3}\beta_1 + \frac{2}{3}x_2 + 2y_2 - \frac{2}{3}\alpha_2 + \beta_2 \right)$$

subject to

$$\frac{1}{10}x_1 + 2y_1 - \frac{2}{5}\alpha_1 + \frac{2}{3}\beta_1 + \frac{2}{3}x_2 + \frac{7}{3}y_2 - \frac{2}{3}\alpha_2 + \frac{5}{2}\beta_2 \leq \frac{92}{15}$$

$$\frac{6}{5}x_1 + \frac{17}{6}y_1 - \frac{8}{5}\alpha_1 + \frac{5}{3}\beta_1 + \frac{1}{3}x_2 + 2y_2 - \frac{2}{3}\alpha_2 + \frac{3}{2}\beta_2 \leq \frac{23}{3}$$

$$x_1 - \alpha_1 \geq 0, x_2 - \alpha_2 \geq 0, y_1 - x_1 \geq 0, y_2 - x_2 \geq 0$$

$$x_1, x_2, y_1, y_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$$

Step 5 The optimal solution of the CLP problem, obtained in Step 4, is $x_1 = 0, y_1 = 0, \alpha_1 = 0, \beta_1 = \frac{16}{5}, x_2 = \frac{47}{50}, y_2 = \frac{47}{50}, \alpha_2 = 0, \beta_2 = 0$ the optimal value, representing the Yager’s ranking index, corresponding to maximum value of objective function, $\frac{34}{5}$.

Step 6 Putting the values of $x_1, x_2, y_1, y_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ in $\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1)_{L_1R_1}$ and $\tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2)_{L_2R_2}$ the fuzzy optimal solution of the chosen FLP problem is $\tilde{x}_1 = (0, 0, 0, \frac{16}{5})_{L_1R_1}, \tilde{x}_2 = (\frac{47}{50}, \frac{47}{50}, 0, 0)_{L_2R_2}$.

10. RESULTS AND DISCUSSION

To show the advantages of the proposed method over existing methods the result of the FLP problems, chosen in Example 6.1 to Example 6.5 and Example 7.1 to Example 7.3, obtained by using the existing method and proposed method are shown in Table 1

Table 1

Examples	Existing method [1]	Proposed method
Example 6.1	$\frac{267}{7}$	$\frac{267}{7}$
Example 6.2	$\frac{191}{10}$	$\frac{191}{10}$
Example 6.3	$\frac{536}{5}$	$\frac{536}{5}$
Example 6.4	$\frac{237}{20}$	$\frac{237}{20}$
Example 6.5	$\frac{31}{10}$	$\frac{31}{10}$
Example 7.1	Not applicable	$\frac{27}{5}$
Example 7.2	Not applicable	$\frac{38}{5}$
Example 7.3	Not applicable	$\frac{34}{5}$

11. CONCLUSION

In this paper the shortcomings of existing methods [1, 7, 13, 15, 16] for solving FLP problems are pointed out. A new method is proposed for finding the optimal

solution of FLP problems also the advantages of proposed method over existing method are discussed.

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