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On fuzzy fully invariant congruence relations

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Abstract. In this article, we consider the variety $Alg(\tau)$ of all algebras of type τ and fuzzy fully invariant congruence relations on the algebra $\mathcal{F}_\tau(X)$ of all terms algebra of type τ, where co-domain of them is a complete lattice L with the least element 0 and the greatest element 1 then we show that the lattice of all equational theories can be embedded into the lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$ and the lattice of all varieties of type τ is dually isomorphic to a sublattice of the lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. Using a Galois connection, we obtain the Birkhoff-type characterization, namely, every variety can be defined by a set of fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. In a special case, if L is the unit interval $[0,1]$ of real numbers, we show that the set of all fuzzy subvarieties of $Alg(\tau)$, where the value of an algebra $\mathcal{A} \in Alg(\tau)$ is 1 if and only if $|\mathcal{A}| = 1$ and the set of all fuzzy fully invariant congruence equality relations on $\mathcal{F}_\tau(X)$ are coincide, whenever 0 is an image of elements in the both sets.

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1. Introduction

A variety of type τ is a class of algebras of type τ closed under taking of subalgebras, homomorphic images and direct products. It is known that every class of algebras defined by a set of equations, set of pairs of terms, is a variety so-called an equational class. In the opposite side, we call every subset of equations which equals to the set of all identities of a class of algebras an equational theory. In [4], the lattice of all equational classes is dually isomorphic to the lattice of all equational theories as complete lattices. The notion of fuzzy set was introduced first by Zadeh [14] as a function from set X to the unit interval $[0,1]$. The first inspiration application to many algebraic structures was the concept of fuzzy group introduced by Rosenfeld [10]. In universal algebra, Murali [7] introduced and investigated fuzzy subalgebras. Fuzzification was applied to classes of algebras by Mordeson [6]. The notion of a fuzzy subvariety was introduced by Šešelja [11] as a poset valued variety. Then Pibaljomme [9] specified co-domain of fuzzy subvarieties to the unit interval $[0, 1]$ and showed that the lattice of all subvarieties of a given varieties can be embedded into the lattice of all fuzzy subvarieties of the variety. The concept of fuzzy equality was introduced first by Höhle [5] who has been used by many others see e.g. [6, 12, 1]. In unversal algebra, the concept of fuzzy equality was introduced and investigated by Bělohávek (see [1, 13]) to many algebraic structures such as subalgebras, products and varieties of **L**-algebras, where **L** is a residuated lattice. In [12], Šešelja and Tepavčević investigated compatible fuzzy equality so-called fuzzy identities in universal algebra. In [1], Bělohávek introduced the concept of fuzzy fully invariant congruence relations on the **L**-algebra of terms and generalized well-known Birkhoff's result [2] in Pavelka style approach [8].

Our investigation is to find an interconnection between the lattice of all equational theories, the lattice of all fuzzy fully invariant congruence relations on the algebra of terms, the lattice of all fuzzy subvarieties of the variety of all algebras of the same type and the lattice of all subvarieties of the same type. Finally, we use a Galois connection to show the Birkhoff-type theorem, namely, every variety can be defined by a set of fuzzy fully invariant congruence relations.

2. Preliminaries

Let $\tau = (n_i)_{i\in I}$ be a type of *algebras* with operation symbols $(f_i)_{i\in I}$, where f_i is an n_i-ary operation. An *algebra of type* τ is an ordered pair $\mathcal{A} := (A; (f_i^A)_{i\in I})$, where A is a non-empty set and $(f_i^A)_{i\in I}$ is a sequence of operations on A indexed by a non-empty index set I such that to each n_i- ary operation symbol f_i there is a corresponding n_i-ary operation f_i^A on A. The set A is called the *universe* of $\mathcal{A}$ and the sequence $(f_i^A)_{i\in I}$ is called the *sequence of fundamental operations* of $\mathcal{A}$. We often write $\mathcal{A}$ instead of $\mathcal{A} := (A; (f_i^A)_{i\in I})$. We denote by $Alg(\tau)$ the class of all algebras of type τ. A class V of algebras of type τ is called a *variety* if it is closed under taking of homomorphic images (**H**), subalgebras (**S**) and direct products (**P**).

Let $X_n := \{x_1, \ldots, x_n\}$ be a finite set of variables, $W_\tau(X_n)$ be the set of all n-ary terms of type τ, and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n), X := \{x_1, \ldots, x_n, \ldots\}$ be the set of all terms of type τ. Then we denote by $\mathcal{F}_\tau(X)$ the absolutely free algebra; $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i\in I})$ with $\bar{f}_i : (t_1, \ldots, t_{n_i}) \mapsto f_i(t_1, \ldots, t_{n_i})$.

An *equation* of type τ is a pair $(s, t) \in W_\tau(X)^2$; such pairs are commonly written as $s \approx t$. An equation $s \approx t$ is an *identity* of an algebra $\mathcal{A}$, denoted by $\mathcal{A} \models s \approx t$ if $s^{\mathcal{A}} = t^{\mathcal{A}}$, where $s^{\mathcal{A}}$ and $t^{\mathcal{A}}$ are the term operations induced by terms s and t on $\mathcal{A}$. A class K of algebras of type τ satisfies an equation $s \approx t$, denoted by $K \models s \approx t$, if $\mathcal{A} \models s \approx t$, for every $\mathcal{A} \in K$. Let Σ be a set of equations of type τ. A class K of algebras of type τ is said to satisfy Σ , denoted by $K \models \Sigma$, if $K \models s \approx t$ for every $s \approx t \in \Sigma$. The notation $K \not\models \Sigma$ means that K dose not satisfy Σ. We observe that $\models = \{(\mathcal{A}, s \approx t) \in Alg(\tau) \times W_\tau(X)^2 \mid \mathcal{A} \models s \approx t\}$ is a relation between $Alg(\tau)$ and $W_\tau(X)^2$. Therefore, the relation $\models$ gives a Galois connection (Mod, Id) between the class $Alg(\tau)$ and the set $W_\tau(X)^2$ defined by letting for every $\Sigma \subseteq W_\tau(X)^2$ and $K \subseteq Alg(\tau)$,

$$\begin{aligned} Mod\Sigma &:= \{\mathcal{A} \in Alg(\tau) \mid \mathcal{A} \models \Sigma\} \text{ and} \\ IdK &:= \{s \approx t \in W_\tau(X)^2 \mid K \models s \approx t\}. \end{aligned}$$

A class K of algebras of type τ is called an *equational class* if there is a set $\Sigma \subseteq W_\tau(X)^2$ such that $K = Mod\Sigma$. A class Σ of equations of type τ is called an *equational theory* if there is a class K of algebras of type τ such that $\Sigma = IdK$. We note that V is a variety if and only if $ModIdV = V$.

In [4], we know that the set of all equational classes $\mathcal{L}(\tau)$ forms a complete lattice, where the meet $\wedge$ and the join $\vee$ operations are defined as following

$$\begin{aligned} K \wedge V &= K \cap V \text{ and} \\ K \vee V &= \cap\{T \in \mathcal{L}(\tau) \mid T \supseteq K \cup V\} \end{aligned}$$

for every $K, V \subseteq Alg(\tau)$ and the set of all equational theories $\mathcal{E}(\tau)$ forms a complete lattice, where the meet $\wedge$ and the join $\vee$ operations are defined as following

$$\begin{aligned} \Sigma \wedge \Omega &= \Sigma \cap \Omega \text{ and} \\ \Sigma \vee \Omega &= \cap\{\Psi \in \mathcal{E}(\tau) \mid \Psi \supseteq \Sigma \cup \Omega\} \end{aligned}$$

for every $\Sigma, \Omega \subseteq W_\tau(X)^2$. Moreover, the lattice of all equational classes is dually isomorphic to the lattice of all equational theories.

A congruence relation θ on $\mathcal{F}_\tau(X)$ is said to be *fully invariant* if whenever $(x, y) \in \theta$, we also have $(\varphi(x), \varphi(y)) \in \theta$, for every endomorphism φ of $\mathcal{F}_\tau(X)$; that is, if θ compatible with all endomorphisms φ of $\mathcal{F}_\tau(X)$. It is clear that $W_\tau(X)^2$ and $\Delta_{W_\tau(X)} := \{(t, t) \mid t \in W_\tau(X)\}$ are fully invariant congruence relation on $\mathcal{F}_\tau(X)$. We denote by $\mathrm{End}(\mathcal{F}_\tau(X))$ the set of all endomorphisms of $\mathcal{F}_\tau(X)$.

Theorem 2.1 ([4]). *Let $\Sigma \subseteq W_\tau(X)^2$ be a set of equations of type τ. Then Σ is an equational theory if and only if it is a fully invariant congruence relation on the term algebra $\mathcal{F}_\tau(X)$.*

Theorem 2.2 ([4]). *A class K of algebras of type τ is an equational class if and only if it is a variety.*

For more information about terms, identities, and varieties see, e.g., [3, 4].

Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice with the least element 0 and the greatest element 1 in L. A mapping $\mu : Alg(\tau) \to L$ is called a *fuzzy subvariety* of $Alg(\tau)$ if the following properties are satisfied:

(1) $\forall \mathcal{B} \in Alg(\tau) \forall \mathcal{A} \in \mathbf{H}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B})$,
(2) $\forall \mathcal{B} \in Alg(\tau) \forall \mathcal{A} \in \mathbf{S}(\mathcal{B}), \mu(\mathcal{A}) \geq \mu(\mathcal{B})$, and
(3) $\mu(\prod_{j \in J} \mathcal{A}_j) \geq \inf\{\mu(\mathcal{A}_j) \mid j \in J\}$ for all $A_j \in Alg(\tau)$.

The set of all fuzzy subvarieties of $Alg(\tau)$ is denoted by $FS(Alg(\tau))$ and the set of all fuzzy subvarieties of $Alg(\tau)$, where $\mu(\mathcal{A}) = 1$ if and only if $|A| = 1$ is denoted by $FSE(Alg(\tau))$. In ([9]), we know that the set $FS(Alg(\tau))$ with compositions the meet $\wedge$ and the join $\vee$ is a complete lattice, where

$$\bigwedge_{j\in J}\mu_j := \bigcap_{j\in J}\mu_j \text{ and}$$
$$\bigvee_{j\in J}\mu_j := \bigcap\{\mu\in FS(Alg(\tau)) \mid \bigcup_{j\in J}\mu_j\subseteq\mu\}$$

for every family $\{\mu_j \mid j\in J\}\subseteq FS(Alg(\tau))$.

A mapping $E : W_\tau(X)^2 \to L$ is called a *compatible fuzzy equivalence relation* on $W_\tau(X)$ if

(1) $E(t,t) = 1, \quad \forall t\in W_\tau(X)$ (reflexivity),
(2) $E(s,t) = E(t,s), \quad \forall s,t\in W_\tau(X)$ (symmetry),
(3) $E(s,t)\wedge E(t,p)\le E(s,p), \quad \forall s,t,p\in W_\tau(X)$ (transitivity) and
(4) $\bigwedge_{j=1}^{n_i} E(s_j,t_j) \le E(\bar f_i(s_1,\ldots,s_{n_i}),\bar f_i(t_1,\ldots,t_{n_i}))$, for all n_i-ary operation $\bar f_i$ on $W_\tau(X)$ and $s_1,\ldots,s_{n_i},t_1,\ldots,t_{n_i}\in W_\tau(X)$ (compatibility).

A compatible fuzzy equivalence relation E on $W_\tau(X)$, where $E(s,t)=1$ implies $s=t$ will be called a *compatible fuzzy equality relation* on $W_\tau(X)$.

Form now on and in the rest of the paper, we assume that $(L,\wedge,\vee,0,1)$ denotes an arbitrary complete lattice with the least element 0 and the greatest element 1 unless specified otherwise.

3. Fuzzy fully invariant congruence relations

In this section, we mention the concept of fuzzy fully invariant congruence relation introductioned by Bělohávek ([1]) and prove that the lattice of all equational theories is embedded into the lattice of all fuzzy fully invariant congruence relations.

Definition 3.1 ([1])**.** A compatible fuzzy equivalence relation $E : W_\tau(X)^2\to L$ is called a *fuzzy fully invariant congruence relation* on $\mathcal{F}_\tau(X)$ if $E(\varphi(s),\varphi(t))\ge E(s,t)$ for all $s,t\in W_\tau(X)$ and $\varphi\in \mathrm{End}(\mathcal{F}_\tau(X))$.

A fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$, where $E(s,t)=1$ implies $s=t$ will be called a *fuzzy fully invariant congruence equality relation*.

We denote by $FF(\mathcal{F}_\tau(X))$ the set of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$ and by $FFE(\mathcal{F}_\tau(X))$ the set of all fuzzy fully invariant congruence equality relations on $\mathcal{F}_\tau(X)$.

Let E be a fuzzy set of $W_\tau(X)^2$ and $\alpha\in L$. We note that the set

$$E_\alpha := \{(s,t)\in W_\tau(X)^2 \mid E(s,t)\ge\alpha\}$$

is called $\alpha-$ *cut set* of E.

The next lemma is equivalent to the definition of a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$.

Lemma 3.2. *Let $E : W_\tau(X)^2\to L$ be a fuzzy set. Then $E\in FF(\mathcal{F}_\tau(X))$ if and only if E_α is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$, for all $\alpha\in L$.*

Proof. $(\Rightarrow)$: Let $E\in FF(\mathcal{F}_\tau(X))$ and $\alpha\in L$. By Theorem 1 in [12], E_α is a congruence relation on $\mathcal{F}_\tau(X)$. Let $(s,t)\in E_\alpha$ and $\varphi\in \mathrm{End}(\mathcal{F}_\tau(X))$. Then $\alpha\le$

$E(s,t) \leq E(\varphi(s), \varphi(t))$. Hence, $(\varphi(s), \varphi(t)) \in E_\alpha$. Thus, E_α is a fully invariant on $\mathcal{F}_\tau(X)$.

$(\Leftarrow)$: Suppose that for all $\alpha \in L, E_\alpha$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$. Since $1 \in L, E_1$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$. We have $(t,t) \in E_1$ for all $t \in W_\tau(X)$, i.e., $E(t,t) = 1$ for all $t \in W_\tau(X)$.

Let $s, t \in W_\tau(X)$. Since $(s,t) \in E_{E(s,t)}$ and $E_{E(s,t)}$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$, we have $(t,s) \in E_{E(s,t)}$. Hence, $E(t,s) \geq E(s,t)$. Similarly, $E(s,t) \geq E(t,s)$. Thus, $E(s,t) = E(t,s)$.

Let $s, t, p \in W_\tau(X)$ and $E(s,t) \wedge E(t,p) = \alpha$. Then $E(s,t) \geq \alpha$ and $E(t,p) \geq \alpha$. Hence, $(s,t), (t,p) \in E_\alpha$. By assumption and $(s,t), (t,p) \in E_\alpha$, we have $(s,p) \in E_\alpha$. Thus, $E(s,p) \geq \alpha = E(s,t) \wedge E(t,p)$.

Next, we show that E is compatible with all n_i-ary operation $\bar{f}_i \in (\bar{f}_i)_{i \in I}$. Let $s_1, \ldots, s_{n_i}, t_1, \ldots, t_{n_i} \in W_\tau(X)$ and $\bar{f}_i \in (\bar{f}_i)_{i \in I}$ be an n_i-ary operation on $W_\tau(X)$. Let $\bigwedge_{j=1}^{n_i} E(s_j, t_j) = \alpha$. Then $(s_j, t_j) \in E_\alpha$, for all $j = 1, \ldots, n_i$. Since E_α is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$, $(\bar{f}(s_1, s_2, \ldots, s_{n_i}), \bar{f}(t_1, t_2, \ldots, t_{n_i})) \in E_\alpha$. Hence, $E(\bar{f}(s_1, s_2, \ldots, s_{n_i}),\ \bar{f}(t_1, t_2, \ldots, t_{n_i})) \geq \alpha = \bigwedge_{j=1}^{n_i} E(s_j, t_j)$.

Finally, we show that $E(\varphi(s), \varphi(t)) \geq E(s,t)$ for all $\varphi \in \mathrm{End}(\mathcal{F}_\tau(X))$ and $s, t \in W_\tau(X)$. Let $s, t \in W_\tau(X)$ and $\varphi \in \mathrm{End}(\mathcal{F}_\tau(X))$. Since $E_{E(s,t)}$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$, $(\varphi(s), \varphi(t)) \in E_{E(s,t)}$. Hence, $E(\varphi(s), \varphi(t)) \geq E(s,t)$. Therefore, $E \in FF(\mathcal{F}_\tau(X))$. □

By Theorem 2.1 and Lemma 3.2, we obtain the following corollary.

Corollary 3.3. *Let $E : W_\tau(X)^2 \to L$ be a fuzzy set. Then $E \in FF(\mathcal{F}_\tau(X))$ if and only if E_α is an equational theory, for all $\alpha \in L$.*

Let $\Sigma \subseteq W_\tau(X)^2$. A function $E_\Sigma : W_\tau(X)^2 \to L$ define by

$$E_\Sigma(s,t) = \begin{cases} 1, & \text{if } (s,t) \in \Sigma; \\ 0, & \text{otherwise,} \end{cases}$$

for all $(s,t) \in W_\tau(X)^2$ is called a *characteristic function*.

Proposition 3.4. *Let $\Sigma \subseteq W_\tau(X)^2$. Then the characteristic function E_Σ is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$ if and only if Σ is an equational theory.*

Proof. $(\Rightarrow)$: Assume that E_Σ is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$. By Corollary 3.3, E_1 is an equational theory. It follows that Σ is an equational theory.

$(\Leftarrow)$: Assume that Σ is an equational theory. We have

$$(E_\Sigma)_\alpha = \begin{cases} \Sigma, & \text{if } \alpha > 0; \\ W_\tau(X)^2, & \text{if } \alpha = 0. \end{cases}$$

By Corollary 3.3, this implies that E_Σ is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$. □

Proposition 3.5. *Let Σ be an equational theory and $\alpha \in L \setminus \{0\}$. Then there exists a fuzzy fully invariant congruence relation E on $\mathcal{F}_\tau(X)$ such that $\Sigma = E_\alpha$.*

Proof. Let Σ be an equational theory and $\alpha \in L \setminus \{0\}$. Let $E : W_\tau(X)^2 \to L$ be a fuzzy set defined by letting for every $(s,t) \in W_\tau(X)^2$,

$$E(s,t) = \begin{cases} 1, & \text{if } s = t; \\ \alpha, & \text{if } (s,t) \in \Sigma \setminus \Delta_{W_\tau(X)}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, it can be easily verified that E is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$ such that $\Sigma = E_\alpha$. □

Proposition 3.6. *Let $\mu : Alg(\tau) \to L$ be a fuzzy subvariety and $\alpha \in L \setminus \{0\}$ such that $\mu_\alpha \neq \varnothing$. Then there exists a fuzzy fully invariant congruence relation E on $\mathcal{F}_\tau(X)$ such that $\mu_\alpha = ModE_\alpha$.*

Proof. Let $\mu : Alg(\tau) \to L$ be a fuzzy subvariety and $\alpha \in L \setminus \{0\}$ such that $\mu_\alpha \neq \varnothing$. Let $E : W_\tau(X)^2 \to L$ be a fuzzy set defined by letting for every $(s,t) \in W_\tau(X)^2$,

$$E(s,t) = \begin{cases} 1, & \text{if } s = t; \\ \alpha, & \text{if } (s,t) \in Id\mu_\alpha \setminus \Delta_{W_\tau(X)}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is easy to verify that E is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$. Since $E_\alpha = Id\mu_\alpha$, and by Proposition 3.3 in [9] and μ_α is a variety, we have $ModE_\alpha = ModId\mu_\alpha = \mu_\alpha$. □

Proposition 3.7. *Let $E \in FF(\mathcal{F}_\tau(X))$ and $\{\alpha_i \mid i \in I\} \subseteq L$. Then*

$$\bigwedge_{i\in I} E_{\alpha_i} = E_{\bigvee\limits_{i\in I} \alpha_i}.$$

Proof. Let $E \in FF(\mathcal{F}_\tau(X))$ and $\{\alpha_i \mid i \in I\} \subseteq L$. Since $E_{\bigvee\limits_{i\in I} \alpha_i} \subseteq E_{\alpha_i}$ for all $i \in I$, we have $E_{\bigvee\limits_{i\in I} \alpha_i} \subseteq \bigwedge\limits_{i\in I} E_{\alpha_i}$. Let $(s,t) \in \bigwedge\limits_{i\in I} E_{\alpha_i}$. Then $E(s,t) \geq \alpha_i$ for all $i \in I$. Hence, $E(s,t) \geq \bigvee\limits_{i\in I} \alpha_i$. Thus, $(s,t) \in E_{\bigvee\limits_{i\in I} \alpha_i}$. Therefore, $\bigwedge\limits_{i\in I} E_{\alpha_i} = E_{\bigvee\limits_{i\in I} \alpha_i}$. □

At the end of this section, we present another example of a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$.

Example 3.8. Let type $\tau = (2)$, $B := Mod\{x(yz) \approx (xy)z,\ \ x^2 \approx x\}$ be the variety of bands and $RB := Mod\{x(yz) \approx (xy)z,\ \ x^2 \approx x,\ \ xyz \approx xz\}$ be the variety of rectangular bands. We define the mapping $E : W_\tau(X)^2 \to [0,1]$ by letting for every $(s,t) \in W_\tau(X)^2$,

$$E(s,t) = \begin{cases} 1, & \text{if } s = t; \\ 0.8, & \text{if } (s,t) \in IdB \setminus \Delta_{W_\tau(X)}; \\ 0.4, & \text{if } (s,t) \in Id(RB) \setminus IdB; \\ 0, & \text{otherwise.} \end{cases}$$

Then E is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$, since every level subset of E is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$.

4. Lattice of fuzzy fully invariant congruence relations

Let E and $F \in FF(\mathcal{F}_\tau(X))$. We define the order on $FF(\mathcal{F}_\tau(X))$ by $E \leq F$ (sometimes written by $E \subseteq F$) which $E \leq F$ means $E(s,t) \leq F(s,t)$ for all $(s,t) \in W_\tau(X)^2$. We repeat the meaning of unions and intersections of fuzzy subsets ([8]) of $\mathcal{F}_\tau(X)$. Let $E, F \in FF(\mathcal{F}_\tau(X))$ and $(s,t) \in W_\tau(X)^2$. We define

$$\begin{aligned}(E \cap F)(s,t) &:= \min\{E(s,t), F(s,t)\} \text{ and}\\ (E \cup F)(s,t) &:= \max\{E(s,t), F(s,t)\}.\end{aligned}$$

For arbitrary intersections and unions, we define

$$\begin{aligned}\left(\bigcap_{i\in I} E_i\right)(s,t) &:= \inf\{E_i(s,t) \mid i \in I\} \text{ and}\\ \left(\bigcup_{i\in I} E_i\right)(s,t) &:= \sup\{E_i(s,t) \mid i \in I\},\end{aligned}$$

for a family $\{E_i \mid i \in I\}$ of fuzzy subsets of $\mathcal{F}_\tau(X)$ and $(s,t) \in W_\tau(X)^2$. Then the following lemma is easy to verify.

Lemma 4.1. *Let $\{E_i \mid i \in I\}$ be a family of fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. Then $\bigcap_{i\in I} E_i$ is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$.*

In general, the union of fuzzy fully invariant congruence relations need not to be a fuzzy fully invariant congruence relation as the following example shows.

Example 4.2. Let L be a lattice given in Figure 1 and type $\tau = (2)$.

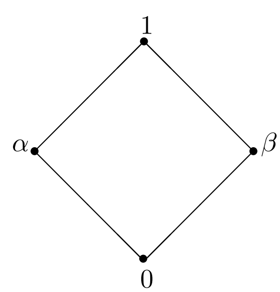

Figure 1.

Let $\Sigma_1 = IdMod\{x(yz) \approx (xy)z,\ \ x^2 \approx x\}$ and $\Sigma_2 = IdMod\{x(yz) \approx (xy)z,\ \ xy \approx yx\}$. We define a fuzzy fully invariant congruence relation $E : W_\tau(X)^2 \to L$ by

$$E(s,t) := \begin{cases} 1, & \text{if } \ (s,t) \in \Sigma_1 \cap \Sigma_2;\\ \alpha, & \text{if } \ (s,t) \in \Sigma_1 \setminus \Sigma_1 \cap \Sigma_2;\\ \beta, & \text{if } \ (s,t) \in \Sigma_2 \setminus \Sigma_1 \cap \Sigma_2;\\ 0, & \text{otherwise,}\end{cases}$$

for every $(s,t) \in W_\tau(X)^2$ and define a fuzzy fully invariant congruence relation $F : W_\tau(X)^2 \to L$ by letting for every $(s,t) \in W_\tau(X)^2$,

$$F(s,t) := \begin{cases} 1, & \text{if} \quad (s,t) \in \Sigma_1 \cap \Sigma_2; \\ \alpha, & \text{if} \quad (s,t) \in \Sigma_2 \setminus \Sigma_1 \cap \Sigma_2; \\ \beta, & \text{if} \quad (s,t) \in \Sigma_1 \setminus \Sigma_1 \cap \Sigma_2; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have for every $(s,t) \in W_\tau(X)^2$,

$$(E \cup F)(s,t) := \begin{cases} 1, & \text{if} \quad (s,t) \in \Sigma_1 \cup \Sigma_2; \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $(E \cup F)_1 = \Sigma_1 \cup \Sigma_2$ which is not an equational theory. By Corollary 3.3, we obtain that $E \cup F$ is not a fuzzy fully invariant congruence relation of $\mathcal{F}_\tau(X)$.

Next, we consider the lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. Let $\{E_j \mid j \in J\}$ be a family of fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. We define the meet $\wedge$ and the join $\vee$ on $FF(\mathcal{F}_\tau(X))$ as follow:

$$\begin{aligned} \bigwedge_{j \in J} E_j &:= \bigcap_{j \in J} E_j \text{ and} \\ \bigvee_{j \in J} E_j &:= \bigcap \{E \in FF(\mathcal{F}_\tau(X)) \mid \bigcup_{j \in J} E_j \subseteq E\}. \end{aligned}$$

Then we have the following theorem which is easy to verify.

Theorem 4.3. *The lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$ denoted by $FF(\mathcal{F}_\tau(X)) := (FF(\mathcal{F}_\tau(X)), \wedge, \vee)$ forms a complete lattice which has the least and the greatest elements, say $\mathbf{0}, \mathbf{1}$ respectively, where $\mathbf{1}(s,t) = 1$ and*

$$\mathbf{0}(s,t) = \begin{cases} 1, & \textit{if} \quad s = t, \\ 0, & \text{otherwise,} \end{cases}$$

for all $(s,t) \in W_\tau(X)^2$.

In the following example, we show that $FFE(\mathcal{F}_\tau(X))$ is not a sublattice of $FF(\mathcal{F}_\tau(X))$.

Example 4.4. Let $\tau = (2)$, L be a lattice given in Figure 1 and $\Sigma = IdMod\{x(yz) \approx (xy)z, x^2 \approx x, xy \approx yx\}$. We define a fuzzy fully invariant congruence equality relation $E : W_\tau(X)^2 \to L$ by letting for every $(s,t) \in W_\tau(X)^2$,

$$E(s,t) := \begin{cases} 1, & \text{if} \quad s = t; \\ \alpha, & \text{if} \quad (s,t) \in \Sigma \setminus \Delta_{W_\tau(X)}; \\ 0, & \text{otherwise,} \end{cases}$$

and define a fuzzy fully invariant congruence equality relation $F : W_\tau(X)^2 \to L$ by letting for every $(s,t) \in W_\tau(X)^2$,

$$F(s,t) := \begin{cases} 1, & \text{if} \quad s = t; \\ \beta, & \text{if} \quad (s,t) \in \Sigma \setminus \Delta_{W_\tau(X)}; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$(E \vee F)(s,t) := \begin{cases} 1, & \text{if } \ (s,t) \in \Sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we obtain $(E \vee F)_1 = \Sigma$. Thus, $E \vee F \notin FFE(\mathcal{F}_\tau(X))$.

Using Proposition 3.4, we prove that the lattice $\mathcal{E}(\tau)$ of all equational theories of type τ can be embedded into the lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$.

Theorem 4.5. *The lattice $\mathcal{E}(\tau)$ of all equational theories of type τ can be embedded into the lattice $FF(\mathcal{F}_\tau(X))$ of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$.*

Proof. We define a mapping $\varphi : \mathcal{E}(\tau) \to FF(\mathcal{F}_\tau(X))$ by $\Sigma \mapsto E_\Sigma$ for every equational theory Σ in $\mathcal{E}(\tau)$. By Proposition 3.4, φ is an injective mapping. Let Σ, Σ' be equational theories in $\mathcal{E}(\tau)$. It is clear that $\varphi(\Sigma \wedge \Sigma') = E_\Sigma \wedge E_{\Sigma'}$. Next, we show that $\varphi(\Sigma \vee \Sigma') = E_\Sigma \vee E_{\Sigma'}$, i.e., $E_{\Sigma \vee \Sigma'} = E_\Sigma \vee E_{\Sigma'}$. Now, we have $E_{\Sigma \vee \Sigma'} \supseteq E_\Sigma \cup E_{\Sigma'}$. Let $(s,t) \in W_\tau(X)^2$ and $F \in FF(\mathcal{F}_\tau(X))$ be such that $F \supseteq E_\Sigma \cup E_{\Sigma'}$.

Case 1. Let $(s,t) \in \Sigma \vee \Sigma'$. Since $F \supseteq E_\Sigma \cup E_{\Sigma'}, F_1 \supseteq (E_\Sigma \cup E_{\Sigma'})_1 = \Sigma \cup \Sigma'$. It follows that $F_1 \supseteq \Sigma \vee \Sigma'$. Hence, $F(s,t) = 1 = E_{\Sigma \vee \Sigma'}(s,t)$.

Case 2. Let $(s,t) \notin \Sigma \vee \Sigma'$. Then we have $E_{\Sigma \vee \Sigma'}(s,t) = 0 \leq F(s,t)$. Thus, $\varphi(\Sigma \vee \Sigma') = E_\Sigma \vee E_{\Sigma'}$. This completes the proof. □

Since the lattice of all equational theories of type τ is dually isomorphic to the lattice of all varieties of type τ and by Theorem 4.5, we have the following theorem.

Theorem 4.6. *The lattice $\mathcal{L}(\tau)$ of all varieties of type τ is dually isomorphic to a sublattice of the lattice $FF(\mathcal{F}_\tau(X))$.*

In the case $L = [0,1]$, we define the fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$ generated by a fuzzy subset of $W_\tau(X)^2$ as follows. Let E be a fuzzy subset of $W_\tau(X)^2$. The fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$ generated by E is defined by

$$< E >_{FF} := \bigcap \{F \in FF(\mathcal{F}_\tau(X)) \mid E \subseteq F\}.$$

Now, we want to describe the construction of the fuzzy invariant congruence relation on $\mathcal{F}_\tau(X)$ generated by a fuzzy subset E of $W_\tau(X)^2$.

Theorem 4.7. *Let $E : W_\tau(X)^2 \to [0,1]$ be a fuzzy subset of $W_\tau(X)^2$. Define a mapping $F : W_\tau(X)^2 \to [0,1]$ by*

$$F(s,t) = \sup\{\alpha \in [0,1] \mid (s,t) \in IdModE_\alpha\}$$

for every $(s,t) \in W_\tau(X)^2$. Then $F = < E >_{FF}$.

Proof. First, we prove that F is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$. It is sufficient to prove that for all $\alpha \in Im(F), F_\alpha$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$. Let $\alpha \in Im(F)$. Since $(t,t) \in IdModE_1$ for all $t \in W_\tau(X), F(t,t) = 1$. Hence, $(t,t) \in F_\alpha$ for all $t \in W_\tau(X)$.

Let $(s,t) \in F_\alpha$. We have $(s,t) \in IdModE_\alpha$ if and only if $(t,s) \in IdModE_\alpha$. Hence, we obtain $\{\alpha \in [0,1] \mid (s,t) \in IdModE_\alpha\} = \{\alpha \in [0,1] \mid (t,s) \in IdModE_\alpha\}$. Then $F(s,t) = F(t,s)$. This implies that $(t,s) \in F_\alpha$.

Let $(s,t),(t,p) \in F_\alpha$. Then $F(s,t) \geq \alpha$ and $F(t,p) \geq \alpha$. Without loss of generality, we assume that $F(s,t) \leq F(t,p)$. If $F(s,t) = 0$ then $\alpha = 0$. Hence, $(s,p) \in F_\alpha$. Suppose that $F(s,t) > 0$. Let $\gamma \in [0,1]$ be such that $\gamma < F(s,t)$. Hence,$(s,t),(t,p) \in IdModE_\gamma$. Since $IdModE_\gamma$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$ and $(s,t),(t,p) \in IdModE_\gamma$, we have $(s,p) \in IdModE_\gamma$. This implies that $F(s,p) \geq F(s,t) \geq \alpha$. Therefore, $(s,p) \in F_\alpha$.

Let $(s_1,t_1),\ldots,(s_{n_i},t_{n_i}) \in F_\alpha$ and $\bar{f}_i \in (\bar{f}_i)_{i\in I}$ be an n_i-ary operation. Then $F(s_j,t_j) \geq \alpha$ for all $j = 1,\ldots,n_i$. Without loss of generality, we may assume that $F(s_1,t_1) \leq \ldots \leq F(s_{n_i},t_{n_i})$. If $F(s_1,t_1) = 0$ then $\alpha = 0$. Hence, $(\bar{f}_i(s_1,\ldots,s_{n_i}),$ $\bar{f}_i(t_1,\ldots,t_{n_i})) \in F_\alpha$. Suppose that $F(s_1,t_1) > 0$. Let $\gamma \in [0,1]$ be such that $F(s_1,t_1) > \gamma$. Then $(s_j,t_j) \in IdModE_\gamma$ for all $j = 1,\ldots,n_i$. Since $IdModE_\gamma$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$ and $(s_j,t_j) \in IdModE_\gamma$, for every $j = 1,\ldots,n_i$, we have $(\bar{f}_i(s_1,\ldots,s_{n_i}),\bar{f}_i(t_1,\ldots,t_{n_i})) \in IdModE_\gamma$. It follows that $F(\bar{f}_i(s_1,\ldots,s_{n_i}),\ \bar{f}_i(t_1,\ldots,t_{n_i})) \geq F(s_1,t_1) \geq \alpha$. Thus,

$$(\bar{f}_i(s_1,\ldots,s_{n_i}),\bar{f}_i(t_1,\ldots,t_{n_i})) \in F_\alpha.$$

Let $(s,t) \in F_\alpha$ and $\varphi \in \mathrm{End}(\mathcal{F}_\tau(X))$. Then $F(s,t) \geq \alpha$. If $F(s,t) = 0$ then $\alpha = 0$. Hence, $(\varphi(s),\varphi(t)) \in F_\alpha$. Suppose that $F(s,t) > 0$. Let $\gamma \in [0,1]$ be such that $F(s,t) > \gamma$. Then $(s,t) \in IdModE_\gamma$. Since $IdModE_\gamma$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$ and $(s,t) \in IdModE_\gamma, (\varphi(s),\varphi(t)) \in IdModE_\gamma$. This implies that $F(\varphi(s),\varphi(t)) \geq F(s,t) \geq \alpha$. Hence, $(\varphi(s),\varphi(t)) \in F_\alpha$. Thus, F is a fuzzy fully invariant congruence relation on $\mathcal{F}_\tau(X)$.

Next step is to show that $E \subseteq F$. Let $(s,t) \in W_\tau(X)^2$. Since $(s,t) \in IdModE_{E(s,t)}$, we have $F(s,t) \geq E(s,t)$.

Finally, we want to prove that for any fuzzy fully invariant congruence relation G on $\mathcal{F}_\tau(X)$ containing E we have $F \subseteq G$. It is clear that for all $\alpha \in [0,1]$, we have $IdModE_\alpha \subseteq G_\alpha$, since $(s,t) \in E_\alpha$ implies $\alpha \leq E(s,t) \leq G(s,t)$, i.e., $(s,t) \in G_\alpha$. Let $(s,t) \in W_\tau(X)^2$ and $\beta \in \{\alpha \in [0,1] \mid (s,t) \in IdModE_\alpha\}$. Then $(s,t) \in IdModE_\beta \subseteq G_\beta$ implies $G(s,t) \geq \beta$. Therefore, $F(s,t) = \sup\{\alpha \in [0,1] \mid (s,t) \in IdModE_\alpha\} \leq G(s,t)$. This completes the proof. □

Proposition 4.8. *$FFE(\mathcal{F}_\tau(X))$ is a sublattice of $FF(\mathcal{F}_\tau(X))$ but not a complete sublattice.*

Proof. First,we show that if $E \in FFE(\mathcal{F}_\tau(X))$ then there exists $\alpha \in [0,1)$ such that $E_\alpha = E_1$. Let $E \in FFE(\mathcal{F}_\tau(X))$. Suppose that $E_\alpha \neq E_1$ for all $\alpha \in [0,1)$. Let $\alpha_i = 1 - \frac{1}{i}$ for all $i \in \mathbb{N}$. Then we have $E_0 = E_{\alpha_1} \supseteq E_{\alpha_2} \supseteq \ldots \supseteq E_{\alpha_i} \supseteq \ldots \supset E_1$. Hence, there exist $s,t \in W_\tau(X)$ with $s \neq t$ such that $(s,t) \in E_{\alpha_i}$ for all $i \in \mathbb{N}$. By Theorem 4.7, $E(s,t) = \sup\{\alpha \in [0,1] \mid (s,t) \in E_\alpha\} = 1$. Then $E \notin FFE(\mathcal{F}_\tau(X))$. It is a contradiction. Thus, there exists $\alpha \in [0,1)$ such that $E_\alpha = E_1$.

Next, we show that if $E,F \in FFE(\mathcal{F}_\tau(X))$ then $E \wedge F$ and $E \vee F \in FFE(\mathcal{F}_\tau(X))$. Let $E,F \in FFE(\mathcal{F}_\tau(X))$. It is clear that $E \wedge F \in FFE(\mathcal{F}_\tau(X))$. Now, we show that $E \vee F \in FFE(\mathcal{F}_\tau(X))$. It is sufficient to show that $(E \vee F)(s,t) \neq 1$ for all $s,t \in W_\tau(X)$ with $s \neq t$. Let $s,t \in W_\tau(X)$ be such that $s \neq t$. Since $E,F \in FFE(\mathcal{F}_\tau(X))$, there exist $\beta_1, \beta_2 \in [0,1)$ such that $E_{\beta_1} = E_1 = F_1 = F_{\beta_2}$. Let $\gamma = \max\{\beta_1, \beta_2\}$. So, $\gamma < 1$ and $E_\gamma = E_1 = F_1 = F_\gamma$. Then $(s,t) \notin E_\gamma \cup F_\gamma = E_\gamma \vee F_\gamma = E_1 = F_1$. By Theorem 4.7, $(E \vee F)(s,t) = \sup\{\alpha \in [0,1] \mid (s,t) \in IdMod(E \cup F)_\alpha\} \leq \gamma < 1$. Hence, $E \vee F \in FFE(\mathcal{F}_\tau(X))$. Thus, $FFE(\mathcal{F}_\tau(X))$ is a sublattice of $FF(\mathcal{F}_\tau(X))$.

Finally, we show that $FFE(\mathcal{F}_\tau(X))$ is not a complete sublattice of $FF(\mathcal{F}_\tau(X))$. Let

$$E_n(s,t) := \begin{cases} 1, & \text{if } \ s = t; \\ 1 - \frac{1}{n}, & \text{otherwise,} \end{cases}$$

for all $(s,t) \in W_\tau(X)^2$ and $n \in \mathbb{N}$. Then we have $\{E_n \mid n \in \mathbb{N}\} \subseteq FFE(\mathcal{F}_\tau(X))$. But $\bigvee_{n\in\mathbb{N}} E_n(s,t) = 1$ for all $(s,t) \in W_\tau(X)^2$. Hence, $\bigvee_{n\in\mathbb{N}} E_n \notin FFE(\mathcal{F}_\tau(X))$. Therefore, $FFE(\mathcal{F}_\tau(X))$ is not a complete sublattice of $FF(\mathcal{F}_\tau(X)$. □

Now, we want to show that the set $\{E \in FFE(\mathcal{F}_\tau(X)) \mid 0 \in Im(E)\}$ is a one to one correspondence to the set $\{\mu \in FSE(Alg(\tau)) \mid 0 \in Im(\mu)\}$.

Lemma 4.9. *If $\mu \in FSE(Alg(\tau))$ with $0 \in Im(\mu)$ then there exists $E \in FFE(\mathcal{F}_\tau(X))$ such that $Id\mu_\alpha = E_{1-\alpha}$ for all $\alpha \in Im(\mu)$.*

Proof. Suppose that $\mu \in FSE(Alg(\tau))$ with $0 \in Im(\mu)$. Let $E_\mu : W_\tau(X)^2 \to [0,1]$ be a fuzzy set defined by letting for every $(s,t) \in W_\tau(X)^2$,

$$E_\mu(s,t) = \begin{cases} 1, & \text{if } s = t, \\ 1 - \alpha, & \text{if } \ (s,t) \in Id\mu_\alpha \setminus Id\mu_\beta, \beta < \alpha \text{ and } \alpha, \beta \in Im(\mu). \end{cases}$$

It is sufficient to prove that for all $\alpha \in Im(\mu), Id\mu_\alpha = (E_\mu)_{1-\alpha}$. Let $\alpha \in Im(\mu)$. It is clear that $Id\mu_0 = (E_\mu)_1$. Suppose that $\alpha > 0$. Let $(s,t) \in (E_\mu)_{1-\alpha}$. Then $E_\mu(s,t) = 1 - \gamma \geq 1 - \alpha$, for some $\gamma \in [0,1]$. Then $(s,t) \in Id\mu_\gamma - Id\mu_\beta$, for all $\beta < \gamma$. Since $\gamma \leq \alpha$, we have $Id\mu_\gamma \subseteq Id\mu_\alpha$. Thus $(s,t) \in Id\mu_\alpha$. Let $(s,t) \in Id\mu_\alpha$. Then $E_\mu(s,t) \geq 1 - \alpha$. It follows that $(s,t) \in (E_\mu)_{1-\alpha}$, i.e., $Id\mu_\alpha \subseteq (E_\mu)_{1-\alpha}$. Thus, $Id\mu_\alpha = (E_\mu)_{1-\alpha}$. □

Remark 4.10. Let $\mu \in FSE(Alg(\tau))$ with $0 \in Im(\mu)$.

(1) $\alpha \in Im(\mu)$ if and only if $1 - \alpha \in Im(E_\mu)$.
(2) If $F \in FFE(\mathcal{F}_\tau(X))$ such that $Id\mu_\alpha = F_{1-\alpha}$, for all $\alpha \in Im(\mu)$, then $E_\mu \leq F$.
(3) If $\nu \in FSE(Alg(\tau))$ with $0 \in Im(\nu)$ and $E_\mu = E_\nu$, then $\mu = \nu$.

Proof. (1) $(\Rightarrow)$: Suppose that $\alpha \in Im(\mu)$. If $\alpha = 0$, then we have $1 - \alpha = 1 \in Im(E_\mu)$. If $\alpha > 0$, then $Id\mu_\alpha \supset Id\mu_\beta$ for all $\beta \in Im(\mu)$ such that $\beta < \alpha$. It follows that $1 - \alpha \in Im(E_\mu)$.

$(\Leftarrow)$: Clear.

(2) Assume that $F \in FFE(\mathcal{F}_\tau(X))$ such that for all $\alpha \in Im(\mu)$, $\ Id\mu_\alpha = F_{1-\alpha}$. Let $(s,t) \in W_\tau(X)^2$. We have $(s,t) \in (E_\mu)_{E_\mu(s,t)} = Id\mu_{(1-E_\mu(s,t))} = F_{E_\mu(s,t)}$. Hence, $F(s,t) \geq E_\mu(s,t)$.

(3) Suppose that $\nu \in FSE(Alg(\tau))$ with $0 \in Im(\nu)$ and $E_\mu = E_\nu$. Let $\mathcal{A} \in Alg(\tau)$. Then we have $Id\mu_{\mu(\mathcal{A})} = (E_\mu)_{1-\mu(\mathcal{A})} = (E_\nu)_{1-\mu(\mathcal{A})} = Id\nu_{\mu(\mathcal{A})}$. This implies that $ModId\mu_{\mu(\mathcal{A})} = ModId\nu_{\mu(\mathcal{A})}$. Hence, $\mu_{\mu(\mathcal{A})} = \nu_{\mu(\mathcal{A})}$. Thus, $\mu(\mathcal{A}) \leq \nu(\mathcal{A})$. Similarly, we can show that $\nu(\mathcal{A}) \leq \mu(\mathcal{A})$. Then $\nu(\mathcal{A}) = \mu(\mathcal{A})$. Therefore, $\nu = \mu$. □

Lemma 4.11. *If $E \in FFE(\mathcal{F}_\tau(X))$ with $0 \in Im(E)$, then there exists $\mu \in FSE(Alg(\tau))$ such that $\mu_{1-\alpha} = ModE_\alpha$ for all $\alpha \in Im(E)$.*

Proof. Let $E \in FFE(\mathcal{F}_\tau(X))$ with $0 \in Im(E)$. We define a mapping $\mu_E : Alg(\tau) \to [0,1]$ by letting for every $\mathcal{A} \in Alg(\tau)$,

$$\mu_E(\mathcal{A}) = \begin{cases} 1, & \text{if } |\mathcal{A}| = 1; \\ 1-\alpha, & \text{if } \mathcal{A} \models E_\alpha \text{ and } \mathcal{A} \not\models E_\beta, \forall \alpha, \beta \in Im(E) \text{ with } \beta < \alpha. \end{cases}$$

It is sufficient to show that $ModE_\alpha = (\mu_E)_{1-\alpha}$ for all $\alpha \in Im(E)$. Let $\alpha \in Im(E)$. It is clear that $(\mu_E)_1 = ModE_0$. Suppose that $\alpha > 0$. Let $\mathcal{A} \in (\mu_E)_{1-\alpha}$. Then $\mu_E(\mathcal{A}) = 1-\gamma \geq 1-\alpha$, for some $\gamma \in [0,1]$. It follows that $\mathcal{A} \models E_\gamma$ and $\gamma \leq \alpha$. Hence, $\mathcal{A} \models E_\alpha$, i.e., $\mathcal{A} \in ModE_\alpha$. Thus, $(\mu_E)_{1-\alpha} \subseteq ModE_\alpha$. Let $\mathcal{A} \in ModE_\alpha$. Then $\mathcal{A} \models E_\alpha$. It turns out $\mu_E(\mathcal{A}) \geq 1-\alpha$, i.e., $\mathcal{A} \in (\mu_E)_{1-\alpha}$. Hence, $ModE_\alpha \subseteq (\mu_E)_{1-\alpha}$. Thus, $ModE_\alpha = (\mu_E)_{1-\alpha}$. □

Remark 4.12. Let $E \in FFE(\mathcal{F}_\tau(X))$ with $0 \in Im(E)$.

(1) $\alpha \in Im(E)$ if and only if $1-\alpha \in Im(\mu_E)$.
(2) If $\nu \in FSE(Alg(\tau))$ such that $ModE_\alpha = \nu_{1-\alpha}$, for all $\alpha \in Im(E)$, then $\mu_E \leq \nu$.
(3) If $F \in FFE(\mathcal{F}_\tau(X))$ with $0 \in Im(F)$ and $\mu_E = \mu_F$, then $E = F$.

Theorem 4.13. *The set $\{E \in FFE(\mathcal{F}_\tau(X)) \mid 0 \in Im(E)\}$ is a one to one correspondence to the set $\{\mu \in FSE(Alg(\tau)) \mid 0 \in Im(\mu)\}$.*

Proof. We define a mapping $\varphi : FSE(Alg(\tau)) \to FFE(\mathcal{F}_\tau(X))$ by letting $\mu \in FSE(Alg(\tau)$,

$$\varphi(\mu) = E_\mu$$

By Lemma 4.9, Remark 4.10 and Lemma 4.11, we have that φ is a bijection. □

5. A Galois connection between $FF(\mathcal{F}_\tau(X))$ and $Alg(\tau)$

In the rest of this note we back to assume that L is a complete lattice with the least element 0 and the greatest element 1. Then we define a Galois connection between the set $FF(\mathcal{F}_\tau(X))$ and the class $Alg(\tau)$. Using Galois connection properties, we obtain the Birkhoff-type theorem, namely, every variety can be defined by a set of fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$.

Theorem 5.1 ([4]). *Let $\sigma : P(FF(\mathcal{F}_\tau(X))) \to P(Alg(\tau))$ and $\iota : P(Alg(\tau)) \to P(FF(\mathcal{F}_\tau(X)))$. For any subset $\Gamma \subseteq FF(\mathcal{F}_\tau(X))$ and any subclass $K \subseteq Alg(\tau)$, we define*

$$\begin{aligned} \sigma(\Gamma) &= \{\mathcal{A} \in Alg(\tau) \mid \forall E \in \Gamma \exists \alpha \in L, \mathcal{A} \models E_\alpha\} \text{ and} \\ \iota(K) &= \{E \in FF(\mathcal{F}_\tau(X)) \mid \forall \mathcal{A} \in K \exists \alpha \in L, \mathcal{A} \models E_\alpha\}. \end{aligned}$$

Then (σ, ι) is a Galois connection between $FF(\mathcal{F}_\tau(X))$ and $Alg(\tau)$.

By the Galois connection (σ, ι) in Theorem 5.1, we obtain the following proposition.

Proposition 5.2. *Let $K \subseteq Alg(\tau)$. Then $\iota(K)$ is a bounded meet-semilattice.*

Proof. Let $K \subseteq Alg(\tau)$. It is clear that $\mathbf{0} \in \iota(K)$. Let $\{E_j \mid j \in J\} \subseteq \iota(K)$. Then $(\bigwedge_{j\in J} E_j)_\alpha \subseteq (E_j)_\alpha$ for all $\alpha \in L$ and $j \in J$. This implies that $\bigwedge_{j\in J} E_j \in \iota(K)$. □

Lemma 5.3. *Let $\Gamma \subseteq FF(\mathcal{F}_\tau(X))$. Then $\sigma(\Gamma)$ is a variety.*

Proof. Let $\Gamma \subseteq FF(\mathcal{F}_\tau(X))$, $\mathcal{B} \in \sigma(\Gamma)$ and $\mathcal{A} \in \mathbf{H}(\mathcal{B})$. Since $\mathcal{B} \in \sigma(\Gamma)$, for all $E \in \Gamma$ there exists $\alpha \in L$ such that $\mathcal{B} \models E_\alpha$. Since $\mathcal{A} \in \mathbf{H}(\mathcal{B})$ and $\mathcal{B} \models E_\alpha$ then $\mathcal{A} \models E_\alpha$. Hence, $\mathcal{A} \in \sigma(\Gamma)$. Similarly, $\sigma(\Gamma)$ is closed under taking the operation $\mathbf{S}$. Let $\mathcal{A}_j \in \sigma(\Gamma)$ for all $j \in J$ and $E \in \Gamma$. Then there exists $\alpha_j \in L$ such that $\mathcal{A}_j \models E_{\alpha_j}$, for all $j \in J$. Let $\alpha = \sup\{\alpha_j \mid j \in J, \mathcal{A}_j \models E_{\alpha_j}\}$. Then $\mathcal{A}_j \models E_\alpha$, for all $j \in J$. Hence, $\prod_{j\in J} \mathcal{A}_j \models E_\alpha$. Thus, $\prod_{j\in J} \mathcal{A}_j \in \sigma(\Gamma)$. Therefore, $\sigma(\Gamma)$ is a variety. □

Lemma 5.4. *Let $K \subseteq Alg(\tau)$ and $E \in FF(\mathcal{F}_\tau(X))$. Then for all $\mathcal{A} \in K$ there exists $\alpha \in L$ such that $\mathcal{A} \models E_\alpha$ if and only if there exists $\beta \in L$ such that for all $\mathcal{A} \in K, \mathcal{A} \models E_\beta$.*

Proof. $(\Leftarrow)$: Obvious.

$(\Rightarrow)$: Suppose that for all $\mathcal{A} \in K$ there exists $\alpha_{\mathcal{A}} \in L, \mathcal{A} \models E_{\alpha_{\mathcal{A}}}$. Let $\beta = \sup\{\alpha_{\mathcal{A}} \in L \mid \mathcal{A} \in K, \mathcal{A} \models E_{\alpha_{\mathcal{A}}}\}$. Then $E_\beta \subseteq E_{\alpha_{\mathcal{A}}}$, for all $\mathcal{A} \in K$. Hence, $\mathcal{A} \models E_\beta$, for all $\mathcal{A} \in K$. □

Lemma 5.5. *Let $K, K' \subseteq Alg(\tau)$. Then $IdK \subseteq IdK'$ if and only if $E_{IdK} \in \iota(K')$.*

Proof. $(\Rightarrow)$: Assume that $IdK \subseteq IdK'$. Let $\mathcal{A} \in K'$. We have $\mathcal{A} \models IdK'$. By assumption, we have $\mathcal{A} \models IdK$. Since $IdK = (E_{IdK})_1, E_{IdK} \in \iota(K')$.

$(\Leftarrow)$: Assume that $E_{IdK} \in \iota(K')$. By Lemma 5.4, there exists $\beta \in L$ such that $\mathcal{A} \models (E_{IdK})_\beta$, for every $\mathcal{A} \in K'$, i.e., $K' \models (E_{IdK})_\beta$. Since $(E_{IdK})_1 \subseteq (E_{IdK})_\beta$, we have $IdK = (E_{IdK})_1 \subseteq (E_{IdK})_\beta \subseteq IdK'$. □

Proposition 5.6. *Let $K \subseteq Alg(\tau)$. Then $\sigma\iota(K) = ModIdK$.*

Proof. By Lemma 5.3, we have $ModIdK \subseteq \sigma\iota(K)$. Let $\mathcal{A} \in \sigma\iota(K)$. Since $E_{IdK} \in \iota(K), \mathcal{A} \models IdK$. It implies that $\mathcal{A} \in ModIdK$. Hence, $\sigma\iota(K) \subseteq ModIdK$. Thus, $\sigma\iota(K) = ModIdK$. □

Corollary 5.7. *K is a variety if and only if $\sigma\iota(K) = K$.*

Now we present the Birkhoff-type theorem, namely every variety can be defined by a set of fuzzy fully invariant congruence relations.

Theorem 5.8. *K is a variety if and only if there exists $\Gamma \subseteq FF(\mathcal{F}_\tau(X))$ such that $K = \sigma(\Gamma)$.*

Proof. $(\Rightarrow)$: Let K be a variety. Let $\Gamma = \{E_{IdK}\}$. We claim that $\sigma(\{E_{IdK}\}) = K$. Clearly, $K \subseteq \sigma(\{E_{IdK}\})$. Let $\mathcal{A} \in \sigma(\{E_{IdK}\})$. We have $\mathcal{A} \models IdK$, since $(E_{IdK})_\alpha = W_\tau(X)^2$ or $(E_{IdK})_\alpha = IdK$ for every $\alpha \in L$. Then $\mathcal{A} \in K$. Therefore, $\sigma(\{E_{IdK}\}) = K$.

$(\Leftarrow)$: For the converse we obtain by Lemma 5.3. □

Finally, we show that the lattice $\mathcal{H}_{\iota\sigma}$ of all closed sets under the closure operator $\iota\sigma$ is isomorphic to the lattice $\mathcal{E}(\tau)$ of all equational theories.

Theorem 5.9. *The lattice of all closed sets under $\iota\sigma$ is isomorphic to the lattice of all equational theories.*

Proof. Define $\varphi : \mathcal{H}_{\iota\sigma} \to \mathcal{E}(\tau)$ defined by letting for every $K \subseteq Alg(\tau)$,

$$\varphi(\iota(K)) = IdK.$$

By Lemma 5.5, φ is well-defined and one to one. It is clear that φ is onto. So, φ is a bijection. Next, we show that φ preserves the meet and join operations. Let $K, K' \subseteq Alg(\tau)$. Now we show $\varphi(\iota(K) \wedge \iota(K')) = \varphi(\iota(K)) \wedge \varphi(\iota(K'))$. Consider

$$\begin{aligned} \varphi(\iota(K) \wedge \iota(K')) &= \varphi(\iota(K) \cap \iota(K')) \\ &= \varphi(\iota(K \cup K')) \\ &= Id(K \cup K') \\ &= IdK \wedge IdK' \\ &= \varphi(\iota(K)) \wedge \varphi(\iota(K')). \end{aligned}$$

Finally, we show that $\varphi(\iota(K) \vee \iota(K')) = \varphi(\iota(K)) \vee \varphi(\iota(K'))$.
We recall :

$$\begin{aligned} \iota(K) \vee \iota(K') &= \cap\{\iota(T) \subseteq FF(\mathcal{F}_\tau(X)) \mid \iota(T) \supseteq \iota(K) \cup \iota(K')\} \text{ and} \\ IdK \vee IdK' &= \cap\{IdT \subseteq W_\tau(X)^2 \mid IdT \supseteq IdK \cup IdK'\}. \end{aligned}$$

Let

$$\begin{aligned} A &= \{T \subseteq Alg(\tau) \mid \iota(T) \supseteq \iota(K) \cup \iota(K')\} \text{ and} \\ B &= \{T \subseteq Alg(\tau) \mid IdT \supseteq IdK \cup IdK'\}. \end{aligned}$$

Now, we show that $A = B$. Let $T \in A$. Then $\iota(T) \supseteq \iota(K) \cup \iota(K')$. It follows that $E_{IdK}, E_{IdK'} \in \iota(T)$. By Lemma 5.5, we have $IdT \supseteq IdK \cup IdK'$. Hence, $T \in B$. Thus, $A \subseteq B$. Let $T \in B$. Then $IdT \supseteq IdK \cup IdK'$. Let $E \in \iota(K) \cup \iota(K')$. If $E \in \iota(K)$ then there exists $\alpha \in L$ such that $K \models E_\alpha$. Since $IdK \subseteq IdT, E_\alpha \subseteq IdT$. Then $T \models E_\alpha$. Hence, $E \in \iota(T)$. Similarly, if $E \in \iota(K')$ then $E \in \iota(T)$. Then $\iota(T) \supseteq \iota(K) \cup \iota(K')$. Hence, $T \in A$. Thus, $B \subseteq A$. Therefore $A = B$.

We assume that $\iota(K) \vee \iota(K') = \iota(T)$. Then $T \in A$. Since $A = B, T \in B$. Then $IdT \supseteq IdK \cup IdK'$. Hence, $IdT \supseteq IdK \vee IdK'$. Let $M \subseteq Alg(\tau)$ be such that $IdM \supseteq IdK \cup IdK'$. Then $M \in B$. Since $A = B, M \in A$. So that $\iota(T) \subseteq \iota(M)$. Since $E_{IdT} \in \iota(T), E_{IdT} \in \iota(M)$. Then we have $IdT = (E_{IdT})_1 \subseteq IdM$. Thus, $IdT = IdK \vee IdK'$. It follows that $\varphi(\iota(K) \vee \iota(K')) = \varphi(\iota(K)) \vee \varphi(\iota(K'))$. Therefore, φ is an isomorphism. This completes the proof. □

6. Conclusions

We have presented that the lattice of all equational theories can be embedded into the lattice of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$. In addition, we have defined a Galois connection between the class of all algebras of type τ and the set of all fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$ and showed that every variety can be defined by a set of fuzzy fully invariant congruence relations on $\mathcal{F}_\tau(X)$.

Our future work is to introduce the concept of fuzzy totally fully invariant congruence relation on $\mathcal{F}_\tau(X)$ and investigate the connection between solid varieties and fuzzy totally fully invariant congruence relations on $\mathcal{F}_\tau(X)$.

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