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Some study of (α, β)-fuzzy ideals in ordered semigroups

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Abstract. Algebraic structures especially an ordered semigroups play a prominent role in mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages. A theory of fuzzy sets in terms of fuzzy points on ordered semigroups can be developed. In this paper, we generalize the concept of (α, β)-fuzzy left (right) ideal of an ordered semigroup S and introduce a new sort of fuzzy left (right) ideals called $(\in, \in \vee q_k)$-fuzzy left (right) ideals, where $k \in [0, 1)$. In particular, we describe the relationships among ordinary fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy ideals of an ordered semigroup S. Finally, we characterize regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals.

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1. Introduction

After Zadeh's pioneering paper [27] on fuzzy sets in 1965, which provides a natural framework for generalizing the basic notions of algebra e.g. set theory, group theory, groupoids, real analysis etc, several researchers get sufficient motivation to review various concepts and results from the realm of algebra in broader framework of fuzzy setting. Mordeson et al. in [21] introduced an up to date account of fuzzy subsemigroups and fuzzy ideals of a semigroup. Although semigroups concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science and information science. Kehayopulu and Tsingelis applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals in ordered semigroups see [11, 12, 13]. Shabir and Khan characterized ordered semigroups by the properties of their fuzzy ideals and fuzzy generalized bi-ideals see [14, 24]. In [1, 2] the idea of a quasi-coincidence of a fuzzy point with a fuzzy set was mentioned, which played a significant role to generate different types of fuzzy subgroups. Bhakat and Das see [1] gave the concept of (α, β)-fuzzy subgroups and introduced $(\in, \in \vee q)$-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. Specifically, an $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup [23]. These new concepts constituted a platform for further development of other algebraic structures. With this objective in view, Ma et al. [19] introduced the interval valued $(\in, \in \vee q)$-fuzzy filters of R_0-algebras and gave important results of R_0-algebras see [20]. The concept of $(\in, \in \vee q)$-fuzzy subalgebras and (ideals) of a Lie algebra were introduced by Davvaz and Mozafar in [7]. Davvaz et al. used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems see [3, 4, 5, 6, 7, 29, 30]. Jun et al. [9] gave the concept of a generalized fuzzy bi-ideals in ordered semigroups and characterized regular ordered semigroups in terms of this notions also see [10]. Several other researchers used this idea in other branches of algebra to characterize different results see [8, 22, 28]. Khan and Shabir in [16] introduced the concept of (α, β)-fuzzy interior ideals in ordered semigroups and discussed several important generalizations of ordered semigroups using these notions. Shabir et al. [26] initiated more general forms of $(\in, \in \vee q)$-fuzzy ideals in semigroups and defined $(\in, \in \vee q_k)$-fuzzy ideals in semigroups and gave some useful results in terms of these notions also see [17, 25].

Recently, Khan et al. [14] characterized ordered semigroups in terms of (α, β)-fuzzy left (resp. right) ideals. In this paper, we give more general forms of (α, β)-fuzzy left (resp. right) ideals and introduce a new sort of fuzzy left (resp. right) ideals called $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals in ordered semigroup S where $k \in [0, 1)$. In particular, we describe the relationships among ordinary fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy ideals of an ordered semigroup S. Finally, we characterize regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and some essential results are obtained.

2. Preliminaries

An *ordered semigroup* (or *po-semigroup*) is a system $(S, \cdot, \leq)$ satisfying the following properties:
$(OS1)$ $(S, \cdot)$ is a semigroup;
$(OS2)$ $(S, \leq)$ is a poset;
$(OS3)$ $a \leq b \longrightarrow ax \leq bx$ and $xa \leq xb$ for all $a, b, x \in S$.

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h \text{ for some } h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote, $AB := \{ab | a \in A, b \in B\}$. If $A \subseteq B$, then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$ and $((A]] = (A]$, (see [15]).

In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes.

In the following S will denote an ordered semigroup, unless otherwise stated. A non-empty subset A of S is called a right (resp. left) ideal [15] of S if: (i) $AS \subseteq A$

(resp. $SA \subseteq A$) and (ii) $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right ideal of S then A is called a two-sided ideal or simply an ideal of S.

Now we recall some basic concepts of fuzzy logic. By a fuzzy subset F of an ordered semigroup S we mean a function $F : X \longrightarrow [0, 1]$.

A fuzzy subset F of S is called a *fuzzy left* (resp. *right*) *ideal* of S if
(1) $(\forall x, y \in S)(x \leq y \longrightarrow F(x) \geq F(y))$ and
(2) $(\forall x, y \in S)(F(xy) \geq F(y)$ (resp. $F(xy) \geq F(x)$).

A fuzzy subset F of S is called a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S.

For a non empty subset X of S, define

$$X_a = \{(y, z) \in S \times S | a \leq yz\} \text{ (see [15])}.$$

From the Transfer Principle in fuzzy set theory [18] it follows that a fuzzy subset F defined on a non-empty set X can be characterized by level subset, i.e. by sets of the form

$$U(F; t) := \{x \in S | F(x) \geq t\},$$

where $t \in [0, 1]$. As shown in [18], for any algebraic system $\mathcal{U} = (X; \mathcal{F})$ where $\mathcal{F}$ is a family of operations defined on X, the transfer principle can be formulated in the following way:

Lemma 2.1 ([18]). *A fuzzy subset F defined on $\mathcal{U}$ has the property $\mathcal{P}$ if and only if all non-empty level subsets $U(F; t)$ have the property $\mathcal{P}$.*

As a consequence of the above property, we have the following theorem:

Theorem 2.2. *A fuzzy subset F of S is a fuzzy left (resp. right) ideal of S if and only if each non-empty level subset $U(F; t)$ for all $0 < t \leq 1$, is a left (resp. right) ideal of S.*

We denote by $F(S)$ the set of all fuzzy subsets of S and define an order relation "$\subseteq$" on the fuzzy subsets of $F(S)$ as follows:

$$F \subseteq G \text{ if and only if } F(x) \leq G(x) \text{ for all } x \in S,$$

and for F, $G \in F(S)$, we define the following:

$$(\forall x \in S)(F \cap G)(x) = F(x) \wedge G(x);$$

and

$$(\forall x \in S)(F \cup G)(x) = F(x) \vee G(x).$$

The product $F \circ G$ of the fuzzy subsets F and G is defined as follows:

$$(F \circ G)(a) = \begin{cases} \bigvee_{(y,z) \in X_a} (F(y) \wedge G(z)) & \text{if } X_a \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

An ordered semigroup $(S, \cdot, \leq)$ is called regular [3] if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalent definitions:
(i) $(\forall a \in S)(a \in (aSa])$ and
(ii) $(\forall A \subseteq S)(A \subseteq (ASA])$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\varnothing \neq T \subseteq S$. Then the *characteristic function* χ_T of T is defined by:

$$\chi_T(x) := \begin{cases} 1 \text{ if } x \in T, \\ 0 \text{ if } x \notin T. \end{cases}$$

Theorem 2.3 ([11]). *A nonempty subset T of an ordered semigroup S is a left (resp. right) ideal of S if and only if the characteristic function χ_T of T is a fuzzy left (resp. right) ideal of S.*

3. $(\in, \in\vee q_k)$-fuzzy ideals

In this section, we define a more generalized form of $(\in, \in \vee q)$-fuzzy ideals of an ordered semigroup S given in [14] and introduce $(\in, \in \vee q_k)$-fuzzy ideals of S, where k is an arbitrary element of $[0, 1)$ unless otherwise stated. A fuzzy subset F of S of the form

$$F : S \longrightarrow [0, 1], \quad y \longmapsto \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value $t \in (0, 1]$ and is denoted by $[x; t]$. A fuzzy point $[x; t]$ is said to belong to (resp. quasi-coincident with) a fuzzy set F, written as $[x; t] \in F$ (resp. $[x; t]qF$) if $F(x) \geq t$ (resp. $F(x) + t > 1$). If $[x; t] \in F$ or $[x; t]qF$, then we write $[x; t] \in \vee qF$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. Generalizing the concept of $[x; t]qF$ in semigroups, Shabir et al. [26] defined $[x; t]q_kF$ as $F(x) + t + k > 1$, where $k \in [0, 1)$.

Definition 3.1. A fuzzy subset F of S is called an *$(\in, \in\vee q_k)$-fuzzy left (resp. right) ideal* of S if it satisfies the conditions:
$(C1)$ $(\forall x, y \in S)(\forall t \in (0, 1])(x \leq y \longrightarrow [y; t] \in F \longrightarrow [x; t] \in \vee q_kF)$,
$(C2)$ $(\forall x, y \in S)(\forall t \in (0, 1])([y; t] \in F \longrightarrow [xy; t] \in \vee q_kF$ (resp. $[yx; t] \in \vee q_kF$)).

Example 3.2. Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the multiplication table and order relation as follows:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (c, c), (c, e), (d, d), (e, e)\}.$$

Let $F : S \longrightarrow [0, 1]$ be a fuzzy subset of S defined as follows:

$$F(x) = \begin{cases} 0.8 \text{ if } x = a, \\ 0.2 \text{ if } x = b, \\ 0.7 \text{ if } x = c, \\ 0.5 \text{ if } x = d, \\ 0.6 \text{ if } x = e. \end{cases}$$

Then F is an $(\in, \in \vee q_k)$-fuzzy ideal of S for all $t \in (0, \frac{1-k}{2}]$ with $k = 0.4$.

Theorem 3.3. *Let I be a left (resp. right) ideal of S and F a fuzzy subset in S defined by:*

$$F(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *F is a $(q, \in\vee q_k)$-fuzzy ideal of S.*
(2) *F is an $(\in, \in\vee q_k)$-fuzzy ideal of S.*

Proof. (1) We discuss only the case of left ideals since the case for right ideals can be proved similarly. Let $x, y \in S$, $x \leq y$ and $t \in (0, 1]$ be such that $[y; t]\text{q}F$. Then $F(y) + t > 1$ and so $y \in I$. Since I is a left ideal of S and $x \leq y \in I$, we have $x \in I$. Thus $F(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x; t] \in F$. If $t > \frac{1-k}{2}$, then $F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x; t]\text{q}_k F$. Therefore $[x; t] \in \vee\text{q}_k F$.

Let $x, y \in S$ and $t \in (0, 1]$ be such that $[y; t]\text{q}F$. Then $F(y) + t > 1$ so $y \in I$. Since I is a left ideal of S, we have $xy \in I$. Thus $F(xy) \geq \frac{1-k}{2}$. Now, if $t > \frac{1-k}{2}$, then $F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t]\text{q}_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq t$ and so $[xy; t] \in F$. Therefore, $[xy; t] \in \vee\text{q}_k F$.

(2) Let $x, y \in S$, $x \leq y$ and $t \in (0, 1]$ be such that $[y; t] \in F$. Then $F(y) \geq t$ and $y \in I$. Since I is a left ideal of S and $x \leq y \in I$, we have $x \in I$. Thus $F(x) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x; t] \in F$. If $t > \frac{1-k}{2}$, then $F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[x; t]\text{q}_k F$. Therefore $[x; t] \in \vee\text{q}_k F$.

If $x, y \in S$ and $t \in (0, 1]$ be such that $[y; t] \in F$, then $x, y \in I$ and so $xy \in I$. Thus $F(xy) \geq \frac{1-k}{2}$. If $t > \frac{1-k}{2}$, then $F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t]\text{q}_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq t$ and we have $[xy; t] \in F$. Therefore $[xy; t] \in \vee\text{q}_k F$. Consequently, F is an $(\in, \in\vee\text{q}_k)$-fuzzy left ideal of S. □

If we take $k = 0$ in Theorem 3.3, then we get the following corollary:

Corollary 3.4 ([14])**.** *Let I be a left (resp. right) ideal of S and F a fuzzy subset in S defined by:*

$$F(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *F is a $(q, \in \vee q)$-fuzzy ideal of S.*
(2) *F is an $(\in, \in \vee q)$-fuzzy ideal of S.*

Theorem 3.5. *A fuzzy subset F of S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if and only if*

($C3$) *$(\forall x, y \in S)(x \leq y \rightarrow F(x) \geq F(y) \wedge \frac{1-k}{2})$,*
($C4$) *$(\forall x, y \in S)(F(xy) \geq F(y) \wedge \frac{1-k}{2}$ (resp. $F(xy) \geq F(x) \wedge \frac{1-k}{2}$)).*

Proof. Suppose that F is an $(\in, \in \vee q_k)$-fuzzy ideal of S. On the contrary, assume that there exist $x, y \in S$, $x \leq y$ such that $F(x) < F(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $F(x) < t \leq F(y) \wedge \frac{1-k}{2}$. Then $[y; t] \in F$, but $F(x) < t$ and $F(x) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[x; t]\overline{\in \vee q_k}F$, a contradiction. Hence $F(x) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ with $x \leq y$. Suppose there exist $x, y \in S$ such that $F(xy) < F(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $F(xy) < t \leq F(y) \wedge \frac{1-k}{2}$. Then $[y; t] \in F$ but $F(xy) < t$

and $F(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $[xy; t]\overline{q}_k F$. Thus, $[xy; t]\overline{\in \vee q_k} F$, a contradiction. Therefore, $F(xy) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$.

Conversely, let $[y; t] \in F$ for some $t \in (0, 1]$. Then $F(y) \geq t$. Now, $F(x) \geq F(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2}$. If $t > \frac{1-k}{2}$, then $F(x) \geq \frac{1-k}{2}$ and $F(x)+t+k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, it follows that $[x; t]\mathrm{q}_k F$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq t$ and so $[x; t] \in F$. Thus, $[x; t] \in \vee q_k F$.

Let $[y; t] \in F$, then $F(y) \geq t$ and so $F(xy) \geq F(y) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2}$. If $t > \frac{1-k}{2}$, then $F(xy) \geq \frac{1-k}{2}$ and $F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $[xy; t]\mathrm{q}_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq t$ implies that $[xy; t] \in F$. Thus $[xy; t] \in \vee \mathrm{q}_k F$ and consequently, F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Similarly we can prove that F is an $(\in, \in \vee q_k)$-fuzzy right ideal of S. □

If we take $k = 0$ in Theorem 3.5, then we have the following corollary:

Corollary 3.6 ([14])**.** *A fuzzy subset F of S is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if and only if*
(1) $(\forall x, y \in S)(x \leq y \longrightarrow F(x) \geq F(y) \wedge 0.5)$,
(2) $(\forall x, y \in S)(F(xy) \geq F(y) \wedge 0.5\,(resp\ F(xy) \geq F(x) \wedge 0.5))$.

Using Theorem 3.5, we have the following characterization of fuzzy left (resp. right) ideals of ordered semigroups.

Proposition 3.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\varnothing \neq I \subseteq S$. Then I is a left (resp. right) ideal of S if and only if the characteristic function χ_I of I is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S.*

Corollary 3.8. *A fuzzy subset F of an ordered semigroup S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if and only if it satisfies conditions $(C3)$ and $(C4)$ of Theorem 3.5.*

Remark 3.9. Every fuzzy left (resp. right) ideal of an ordered semigroup is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S. But the converse is not true. As shown in the following example.

Example 3.10. Consider the ordered semigroup as given in Example 3.2, and define a fuzzy subset $F : S \to [0, 1]$ as follows:

$$F(x) = \begin{cases} 0.8 \text{ if } x = a, \\ 0.2 \text{ if } x = b, \\ 0.7 \text{ if } x = c, \\ 0.5 \text{ if } x = d, \\ 0.6 \text{ if } x = e. \end{cases}$$

Then F is an $(\in, \in \vee q_k)$-fuzzy ideal but $U(F; t) = \{a, c\}$ for all $t \in (0.6, 0.7]$ is not an ideal of S. Hence by Theorem 2.2, F is not a fuzzy ideal of S for all $t \in (0.6, 0.7]$.

Theorem 3.11. *Every $(\in, \in)$-fuzzy left (resp. right) ideal is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S.*

Proof. The proof is straightforward and omitted here. □

The converse of the above theorem is not true in general as shown in Example 3.12.

Example 3.12. Consider the ordered semigroup as given in Example 3.2, and define a fuzzy subset $F : S \rightarrow [0,1]$ as follows:

$$F(x) = \begin{cases} 0.8 \text{ if } x = a, \\ 0.2 \text{ if } x = b, \\ 0.7 \text{ if } x = c, \\ 0.5 \text{ if } x = d, \\ 0.6 \text{ if } x = e. \end{cases}$$

Then F is an $(\in, \in \vee q_k)$-fuzzy ideal but not an $(\in, \in)$-fuzzy ideal of S. This is because $[c; 0.65] \in F$ but $[ce; 0.65] = [e; 0.65] \overline{\in} F$.

Theorem 3.13. *Every* $(\in \vee q_k, \in \vee q_k)$*-fuzzy left (resp. right) ideal is an* $(\in, \in \vee q_k)$*-fuzzy left (resp. right) ideal of* S.

Proof. Suppose that F is an $(\in \vee q_k, \in \vee q_k)$-fuzzy left ideal of S. Let $x, y \in S$, and $t \in (0,1]$ be such that $x \leq y$, $[y,t] \in F$. Then $[y,t] \in \vee q_k F$. Since $x \leq y$ by hypothesis, we have $[x,t] \in \vee q_k F$. Let $x, y \in S$, and $t \in (0,1]$ be such that $[y,t] \in F$. Then $[y,t] \in \vee q_k F$. Hence $[xy,t] \in \vee q_k F$. Similarly we can prove that $[yx,t] \in \vee q_k F$. □

Theorem 3.14. *A fuzzy subset* F *of* S *is an* $(\in, \in \vee q_k)$*-fuzzy left (resp. right) ideal of* S *if and only if* $U(F;t)\,(\neq \varnothing)$ *is left (resp. right) ideal of* S *for all* $t \in (0, \frac{1-k}{2}]$.

Proof. Assume that F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Let $x, y \in S$, $x \leq y$ be such that $y \in U(F;t)$ for some $t \in (0, \frac{1-k}{2}]$. Then $F(y) \geq t$ and by hypothesis

$$\begin{aligned} F(x) &\geq F(y) \wedge \frac{1-k}{2} \\ &\geq t \wedge \frac{1-k}{2} = t. \end{aligned}$$

Hence $x \in U(F;t)$. Let $x, y \in S$ be such that $y \in U(F;t)$ for some $t \in (0, \frac{1-k}{2}]$. Then $F(y) \geq t$ and we have

$$\begin{aligned} F(xy) &\geq F(y) \wedge \frac{1-k}{2} \\ &\geq t \wedge \frac{1-k}{2} = t. \end{aligned}$$

Thus $xy \in U(F;t)$.

Conversely, suppose that $U(F;t)(\neq \varnothing)$ is a left ideal of S for all $t \in (0, \frac{1-k}{2}]$. Let $x, y \in S$, $x \leq y$ be such that $F(x) < F(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, \frac{1-k}{2}]$ such that $F(x) < t \leq F(y) \wedge \frac{1-k}{2}$ then $F(y) \geq t$ implies that $y \in U(F;t)$ but $x \notin U(F;t)$, a contradiction. Hence $F(x) \geq F(y) \wedge \frac{1-k}{2}$. Suppose $x, y \in S$ such that $F(xy) < F(y) \wedge \frac{1-k}{2}$. Then choose $t \in (0, \frac{1-k}{2}]$, and $F(xy) < t \leq F(y) \wedge \frac{1-k}{2}$. Thus $y \in U(F;t)$ but $xy \notin U(F;t)$, a contradiction. Hence $F(xy) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ and $k \in [0,1)$. Therefore F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By a similar way, we can prove that F is an $(\in, \in \vee q_k)$-fuzzy right ideal of S. □

Proposition 3.15. *Let* F *be a nonzero* $(\in, \in \vee q_k)$*-fuzzy left (resp. right) ideal of* S*. Then the set* $F_0 = \{x \in S | F(x) > 0\}$ *is a left (resp. right) ideal of* S.

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy left ideal of S. If $x, y \in S$, $x \leq y$ and $y \in F_0$, then, $F(y) > 0$. Since F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, we have

$$F(x) \geq F(y) \wedge \frac{1-k}{2} > 0, \text{ because } F(y) > 0.$$

Thus $F(x) > 0$ and so $x \in F_0$. Let $y \in F_0$ then, $F(y) > 0$ and by hypothesis,

$$\begin{aligned} F(xy) &\geq F(y) \wedge \frac{1-k}{2} \\ &> 0. \end{aligned}$$

Thus $xy \in F_0$. Consequently, F_0 is a left ideal of S. Similarly we can prove that F_0 is a right ideal of S. □

In the following lemma, we establish a relationship between $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and $(\in, \in)$-fuzzy left (resp. right) ideals of S.

Lemma 3.16. *Suppose that F is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S such that $F(x) < \frac{1-k}{2}$ for all $x \in S$. Then F is an $(\in, \in)$-fuzzy left (resp. right) ideal of S.*

Proof. Let $x, y \in S$ and $t \in (0, 1]$ be such that $x \leq y$, $[y; t] \in F$, then $F(y) \geq t$. Since F is an $(\in, \in \vee q_k)$-fuzzy left ideal and $F(x) < \frac{1-k}{2}$. we have,

$$\begin{aligned} F(x) &\geq F(y) \wedge \frac{1-k}{2} \\ &\geq t \wedge \frac{1-k}{2} = t. \end{aligned}$$

Hence $[x; t] \in F$. If $x, y \in S$ and $t \in (0, 1]$ be such that $[y; t] \in F$, then $F(y) \geq t$ and we have

$$\begin{aligned} F(xy) &\geq F(y) \wedge \frac{1-k}{2} \\ &\geq t \wedge \frac{1-k}{2} = t. \end{aligned}$$

Thus $[xy; t] \in F$. By a similar way, we can prove that $[yx; t] \in F$. □

For any fuzzy subset F of S and $t \in (0, 1]$ we denote $Q^k(F; t)$ as follows:

$$Q^k(F; t) := \{x \in S \mid [x; t] q_k F\} \text{ and } [F]_t^k := \{x \in S \mid [x; t] \in \vee q_k F\}.$$

Clearly $[F]_t^k = U(F; t) \cup Q^k(F; t)$.

We call $[F]_t^k$ an $(\in \vee q_k)$-fuzzy level left (resp. right) ideal I and $Q^k(F; t)$ a q-level left (resp. right) ideal of S.

We give another characterization of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals by using $[F]_t^k$.

Theorem 3.17. *A fuzzy subset F of S is an $(\in, \in \vee q_k)$-fuzzy left ideal of S if and only if $[F]_t^k$ is a left (resp. right) ideal of S for all $t \in (0, 1]$.*

Proof. Assume F is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S. Let $x, y \in S, x \leq y$ and $t \in (0, 1]$ be such that $y \in [F]_t^k$. Then $[y; t] \in \vee q_k F$, that is $F(y) \geq t$ or $F(y) + t + k > 1$. Since F is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy left ideal of S and $x \leq y$, we have $F(x) \geq F(y) \wedge \frac{1-k}{2}$. We discuss the following cases:

Case 1 $F(y) \geq t$. If $t > \frac{1-k}{2}$, then $F(x) \geq F(y) \wedge \frac{1-k}{2} = \frac{1-k}{2}$ and so

$$F(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

so $[x;t]q_k F$. If $t \leq \frac{1-k}{2}$, then $F(x) \geq F(y) \wedge \frac{1-k}{2} \geq t$. Hence $[x;t] \in F$.

Case 2 $F(y) + t + k > 1$. If $t > \frac{1-k}{2}$, then

$$F(x) \geq F(y) \wedge \frac{1-k}{2} > 1 - t - k \wedge \frac{1-k}{2} = 1 - t - k,$$

that is, $F(x) + t + k > 1$. Thus $[x;t]q_k F$. If $t \leq \frac{1-k}{2}$, then

$$F(x) \geq F(y) \wedge \frac{1-k}{2} > 1 - t - k \wedge \frac{1-k}{2} = \frac{1-k}{2} \geq t.$$

Therefore $[x;t] \in F$. Hence $[x;t] \in \vee q_k F$. Let $x \in S$ and $y \in [F]_t^k$ for $t \in (0,1]$. Then $[y;t] \in \vee q_k F$, that is $F(y) \geq t$ or $F(y) + t + k > 1$. Since F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, we have $F(xy) \geq F(y) \wedge \frac{1-k}{2}$.

Case 1 $F(y) \geq t$. If $t > \frac{1-k}{2}$, then $F(xy) \geq F(y) \wedge \frac{1-k}{2} = \frac{1-k}{2}$ and so

$$F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

so $[xy;t]q_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq F(y) \wedge \frac{1-k}{2} \geq t$. Hence $[xy;t] \in F$.

Case 2 $F(y) + t + k > 1$. If $t > \frac{1-k}{2}$, then

$$F(xy) \geq F(y) \wedge \frac{1-k}{2} > 1 - t - k \wedge \frac{1-k}{2} = 1 - t - k,$$

that is, $F(xy) + t + k > 1$. Thus $[xy;t]q_k F$. If $t \leq \frac{1-k}{2}$, then

$$F(xy) \geq F(y) \wedge \frac{1-k}{2} > 1 - t - k \wedge \frac{1-k}{2} = \frac{1-k}{2} \geq t.$$

Therefore $[xy;t] \in F$. Hence $[xy;t] \in \vee q_k F$. Thus $[F]_t^k$ is a left ideal. Similarly we can prove that $[yx;t] \in \vee q_k F$. Consequently $[F]_t^k$ is a left (resp. right) ideal of S.

Conversely, assume that F is a fuzzy subset of S and $t \in (0,1]$ be such that $[F]_t^k$ is a left ideal of S. If there exist $x, y \in S, x \leq y$ and $t \in (0, \frac{1-k}{2}]$ such that

$$F(x) < t \leq F(y) \wedge \frac{1-k}{2}.$$

Then $y \in U(F;t) \subseteq [F]_t^k$. Hence $y \in [F]_t^k$, since $[F]_t^k$ is left ideal and $x \leq y$. Thus $x \in [F]_t^k$ and we have $F(x) \geq t$ or $F(x) + t + k > 1$, a contradiction. Hence $F(x) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$ with $x \leq y$. Let $x, y \in S$ be such that

$$F(xy) < t \leq F(y) \wedge \frac{1-k}{2}$$

for some $t \in (0, \frac{1-k}{2}]$. Then $y \in U(F;t) \subseteq [F]_t^k$. Hence $y \in [F]_t^k$, since $[F]_t^k$ is left ideal. Thus $xy \in [F]_t^k$ and we have $F(xy) \geq t$ or $F(xy) + t + k > 1$, a contradiction. Hence $F(xy) \geq F(y) \wedge \frac{1-k}{2}$ for all $x, y \in S$. Thus F is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Similarly we can prove that F is an $(\in, \in \vee q_k)$-fuzzy right ideal of S. □

From Theorem 2.2 and 3.17, $U(F;t)$ and $[F]_t^k$ are left (resp. right) ideals of S for all $t \in (0, \frac{1-k}{2}]$, but $Q^k(F;t)$ is not a left ideal of S for $t \in (0, \frac{1-k}{2}]$ in general, as shown in the next example.

Example 3.18. Consider an ordered semigroup is given in example 3.2, and define a fuzzy subset $F : S \to [0,1]$ as follows:

$$F(a) = 0.8, \quad F(b) = 0.3, \quad F(c) = 0.6, \quad F(d) = 0.4, \quad F(e) = 0.1.$$

Then $Q(F;t) = \{a,b,c,d\}$ for $t \in (0.1, \frac{1-k}{2}]$ and $k = 0.4$. Since $[a, 0.2]q_k F$ but $[ae; 0.2] = [d; 0.2]\overline{q_k}F$.

4. Regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (right) ideals

Definition 4.1. A fuzzy subset F of S is called an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S if it satisfies the conditions:
$(C5)$ $(\forall x, y \in S)(x \leq y \longrightarrow F(x) \geq F(y) \wedge \frac{1-k}{2})$,
$(C6)$ $(\forall x \in S)(F(x) \geq ((F \circ 1) \cap (1 \circ F))(x) \wedge \frac{1-k}{2})$.

Theorem 4.2. *Let F be a non-zero $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S, then the set $F_0 = \{x \in S \mid F(x) > 0\}$ is a quasi-ideal of S.*

Proof. Let $x, y \in S$ be such that $x \leq y$. If $y \in F_0$, then $F(y) > 0$. Since F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S, thus

$$F(x) \geq F(y) \wedge \frac{1-k}{2} > 0$$

Thus $F(x) > 0$ and so $x \in F_0$. Let $a \in (F_0 S] \cap (S F_0]$, then $a \in (F_0 S]$ and $a \in (S F_0]$. Hence there exist $r, s \in S$ and $x, y \in F_0$ such that $a \leq xs$, and $a \leq ry$. Then $(x,s), (r,y) \in X_a$. Since $X_a \neq \phi$, we have

$$\begin{aligned}
F(x) &\geq ((F \circ 1) \cap (1 \circ F))(x) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{(p,q) \in X_a} (F(p) \wedge 1(q) \right\} \wedge \left\{ \bigvee_{(q,p) \in X_a} (1(q) \wedge F(p) \right\} \wedge \frac{1-k}{2} \\
&\geq \{F(x) \wedge 1(r)\} \wedge \{1(r) \wedge F(y)\} \wedge \frac{1-k}{2} \\
&= F(x) \wedge F(y) \wedge \frac{1-k}{2} > 0, \text{ because } x, y \in F_0.
\end{aligned}$$

Hence $a \in F_0$ implies that $(F_0 S] \cap (S F_0] \subseteq F_0$. □

The proof of the following proposition is straightforward and omitted.

Proposition 4.3. *Every $(\in, \in \vee q_k)$-fuzzy left (resp. right) or two sided ideal of S is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.*

The converse of Proposition 4.3 is not true in general. This is shown in the following example.

Example 4.4. Consider an ordered semigroup $S = \{0, 1, 2, 3\}$ with multiplication table and order relation:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	0	0	0
3	0	3	0	0

$$\leq := \{(0,0),(1,1),(2,2),(3,3),(0,1),(0,2),(0,3)\}.$$

Then $\{0, 1\}$ is a quasi-ideal of S, but not a left (resp. right) ideal of S. Define a fuzzy subset $F : S \to [0, 1]$ as follows:

$$F(0) = F(1) = 0.25, \quad F(2) = F(3) = 0.$$

Then F is fuzzy quasi-ideal of S but not an $(\in, \in \vee q_k)$-fuzzy left ideal of S for $k = 0.4$ since $F(12) = F(2) = 0 < F(1) \wedge \frac{1-k}{2}$.

Definition 4.5. Let $(S, \cdot, \leq)$ be an ordered semigroup and F, G fuzzy subsets of S. Then the $\frac{1-k}{2}$-product of F and G is defined as:

$$(F \circ_{\frac{1-k}{2}} G)(a) = \begin{cases} \bigvee\limits_{(y,z)\in X_a} (F(y) \wedge G(z) \wedge \frac{1-k}{2}) & \text{if } X_a \neq \varnothing, \\ 0 & \text{if } X_a = \varnothing. \end{cases}$$

Proposition 4.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup and F_1, F_2, G_1, G_2 fuzzy subsets of S such that $F_1 \subseteq F$ and $G_1 \subseteq G_2$. Then $F_1 \circ_{\frac{1-k}{2}} G_1 \subseteq F_2 \circ_{\frac{1-k}{2}} G_2$.*

Lemma 4.7. *If F and G are $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals of an ordered semigroup S, then $F \cap_{\frac{1-k}{2}} G$ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S, where $(F \cap_{\frac{1-k}{2}} G)(x) := F(x) \wedge G(x) \wedge \frac{1-k}{2}$ for all $x \in S$.*

Proof. Let $x, y \in S$ such that $x \leq y$. Since F and G are $(\in, \in \vee \mathrm{q}_k)$-fuzzy left ideals of S, then $F(x) \geq F(y) \wedge \frac{1-k}{2}$ and $G(x) \geq G(y) \wedge \frac{1-k}{2}$. Therefore

$$\begin{aligned} (F \cap_{\frac{1-k}{2}} G)(x) &= F(x) \wedge G(x) \wedge \frac{1-k}{2} \\ &\geq \left\{F(y) \wedge \frac{1-k}{2}\right\} \wedge \left\{G(y) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2} \\ &= \left(F(y) \wedge G(y) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2} \\ &= (F \cap_{\frac{1-k}{2}} G)(y) \wedge \frac{1-k}{2}. \end{aligned}$$

Hence $(F \cap_{\frac{1-k}{2}} G)(x) \geq (F \cap_{\frac{1-k}{2}} G)(y) \wedge \frac{1-k}{2}$.

Let $x, y \in S$. Since F and G are $(\in, \in \vee q_k)$-fuzzy left ideals of S, then $F(xy) \geq F(y) \wedge \frac{1-k}{2}$ and $G(xy) \geq G(y) \wedge \frac{1-k}{2}$. Thus

$$\begin{aligned}(F \cap_{\frac{1-k}{2}} G)(xy) &= F(xy) \wedge G(xy) \wedge \frac{1-k}{2} \\ &\geq \{F(y) \wedge \frac{1-k}{2}\} \wedge \{G(y) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\ &= (F(y) \wedge G(y) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\ &= (F \cap_{\frac{1-k}{2}} G)(y) \wedge \frac{1-k}{2}.\end{aligned}$$

Hence $F \cap_{\frac{1-k}{2}} G$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Similarly we can prove that $F \cap_{\frac{1-k}{2}} G$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy right ideal of S. □

Let F and G be fuzzy subsets of S, we define $F \cup G$ as follows:

$$(F \cup_{\frac{1-k}{2}} G)(x) = (F(x) \vee G(x)) \wedge \frac{1-k}{2}$$

for all $x \in S$.

Lemma 4.8. *If F is an $(\in, \in \vee q_k)$-fuzzy right ideal of S and G is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, respectively, then $F \circ_{\frac{1-k}{2}} G \subseteq F \cap_{\frac{1-k}{2}} G$.*

Proof. Let $a \in S$. If $X_a = \phi$, then $(F \circ_{\frac{1-k}{2}} G)(a) = 0 \leq \left(F \cap_{\frac{1-k}{2}} G\right)(a)$. Assume that $X_a \neq \phi$, then

$$(F \circ_{\frac{1-k}{2}} G)(a) := \bigvee_{(y,z) \in X_a} (F(y) \wedge G(z) \wedge \frac{1-k}{2}).$$

Since $a \leq yz$, and F is an $(\in, \in \vee q_k)$-fuzzy right ideal and G is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, thus

$$F(a) \geq F(yz) \wedge \frac{1-k}{2} \geq (F(y) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} = F(y) \wedge \frac{1-k}{2}$$

and

$$G(a) \geq G(yz) \wedge \frac{1-k}{2} \geq (F(y) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} = G(y) \wedge \frac{1-k}{2}.$$

Hence, $F(y) \wedge G(y) \wedge \frac{1-k}{2} \leq F(a) \wedge G(a) \wedge \frac{1-k}{2} = (F \cap_{\frac{1-k}{2}} G)(a)$. □

Regular ordered semigroups are characterized by right and left ideals in [12]. In the following we give a characterization of regular ordered semigroups in terms of $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee \mathrm{q}_k)$-fuzzy left ideals.

Lemma 4.9 ([12])**.** *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular.*
(2) *$R \cap L = (RL]$ for every right ideal R and every left ideal L of S.*

Lemma 4.10 ([17])**.** *Let S be an ordered semigroup and $\phi \neq A, B \subseteq S$. Then, we have*

(i) $(\chi_A \circ_{\frac{1-k}{2}} \chi_B)(a) = \chi_{(AB]}(a) \wedge \frac{1-k}{2}$ *for all* $a \in S$.
(ii) $\chi_A \cap_{\frac{1-k}{2}} \chi_B = \chi_{A\cap B} \wedge \frac{1-k}{2}$.
(iii) $\chi_A \cup_{\frac{1-k}{2}} \chi_B = \chi_{A\cup B} \wedge \frac{1-k}{2}$.

Theorem 4.11. *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy right ideal F and every $(\in, \in \vee q_k)$-fuzzy left ideal G of S, we have $F \circ_{\frac{1-k}{2}} G = F \cap_{\frac{1-k}{2}} G$.*

Proof. Let S be a regular ordered semigroup and $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. Hence $(ax, a) \in X_a$ and we have

$$\begin{aligned}(F \circ_{\frac{1-k}{2}} G)(a) &= \bigvee_{(y,z)\in X_a} (F(y) \wedge G(z) \wedge \frac{1-k}{2}) \\ &\geq F(ax) \wedge G(a) \wedge \frac{1-k}{2}.\end{aligned}$$

Since F is an $(\in, \in \vee q_k)$-fuzzy right ideal of S, therefore $F(ax) \geq F(a) \wedge \frac{1-k}{2}$. Thus

$$\begin{aligned}F(ax) \wedge G(a) \wedge \frac{1-k}{2} &\geq (F(a) \wedge \frac{1-k}{2}) \wedge G(a) \wedge \frac{1-k}{2} \\ &= F(a) \wedge G(a) \wedge \frac{1-k}{2} \\ &= \left(F \cap_{\frac{1-k}{2}} G\right)(a).\end{aligned}$$

Hence $\left(F \cap_{\frac{1-k}{2}} G\right)(a) \leq (F \circ_{\frac{1-k}{2}} G)(a)$. On the other hand, by Lemma 4.8, we have $(F \circ_{\frac{1-k}{2}} G)(a) \leq (F \cap_{\frac{1-k}{2}} G)(a)$. Therefore $(F \circ_{\frac{1-k}{2}} G)(a) = (F \cap_{\frac{1-k}{2}} G)(a)$.

Conversely, suppose for every $(\in, \in \vee q_k)$-fuzzy right ideal F and every $(\in, \in \vee q_k)$-fuzzy left ideal G of S we have $F \circ_{\frac{1-k}{2}} G = F \cap_{\frac{1-k}{2}} G$. By Lemma 4.9, we need to show that $R \cap L = (RL]$ for every right ideal R and left ideal L of S. Let $y \in R \cap L$, then $y \in R$ and $y \in L$ since R is right ideal and L is left ideal of S. Thus by Lemma 4.10, χ_R is a fuzzy right and χ_L is a fuzzy left ideal of S. Therefore by Proposition 3.7, χ_R is an $(\in, \in \vee q_k)$-fuzzy right ideal and χ_L is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By hypothesis, $(\chi_R \circ_{\frac{1-k}{2}} \chi_L)(y) = (\chi_R \cap_{\frac{1-k}{2}} \chi_L)(y)$. Since $y \in R$ and $y \in L$ so $\chi_R(y) = 1$ and $\chi_L(y) = 1$, then $(\chi_R \cap_{\frac{1-k}{2}} \chi_L)(y) = \chi_R(y) \wedge \chi_L(y) \wedge \frac{1-k}{2} = 1 \wedge 1 \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $(\chi_R \circ_{\frac{1-k}{2}} \chi_L)(y) = \frac{1-k}{2}$. By Lemma 4.10, $(\chi_R \circ_{\frac{1-k}{2}} \chi_L)(y) = \chi_{(RL]}(y) \wedge \frac{1-k}{2}$. Thus $\chi_{(RL]}(y) = \frac{1-k}{2}$. It follows that $y \in (RL]$. Hence $R \cap L \subseteq (RL]$ but $(RL] \subseteq R \cap L$ always holds. Therefore $R \cap L = (RL]$. Consequently S is regular. □

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