Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 3, No. 2, (April 2012), pp. 293- 303
ISSN 2093–9310
http://www.afmi.or.kr

Column generation in the linear programming with trapezoidal fuzzy variables

Behrouz Kheirfam

Received 3 August 2011; Accepted 23 September 2011

Abstract. In this paper, the notions of basic feasible solutions and basic feasible directions with respect to linear ranking functions are defined. We consider the linear programming with trapezoidal fuzzy variables and develop the representation theorem, which provides the basis for the column generation. Then, we illustrate the results by an example.

2010 AMS Classification: 90C70

Keywords: Fuzzy linear programming, Decomposition principle, Resolution theorem, Column generation, Ranking function.

Corresponding Author: Behrouz Kheirfam (b.kheirfam@azaruniv.du)

1. Introduction

Linear programming is one of the most widely used decision making tools for solving real world problems, but it fails to deals with imprecise data. So the many researchers succeed in capturing vague and imprecise information by fuzzy linear programming (FLP) problems [1, 2]. The concept of fuzzy decision making was first proposed by Bellman and Zadeh [1]. Tanaka et al. [10] used this concept for solving mathematical programming problems. The first formulation of fuzzy linear programming (FLP) was given by Zimmermann [14]. Chanas [3] proposed the possibility of the identification of a complete fuzzy decision in FLP problem by use of the parametric programming technique. Fang et al. [4] studied a method for solving linear programming problems with fuzzy coefficients in constraints. Liu [7] introduced a method for solving FLP problem based on the satisfaction degree of the constraints. Maleki [9] studied a method for solving linear programming with vagueness in constraints by using ranking function. Zhang et al. [13] formulated a FLP problem as four objective constrained optimization problem where the cost coefficients are fuzzy and also presented it's solution. Ganesan and Veeramani [5] introduced an approach to solve of FLP problem with symmetric trapezoidal fuzzy numbers without converting it into crisp model. By using of certain linear ranking function for ordering trapezoidal fuzzy numbers, defined the dual of a fuzzy linear programming problem with fuzzy variables, then the duality results and complementary slackness have been given [8]. However, in contrast with the vast literature on modeling and solution procedures for a FLP problem, the studies in the large structured problems are rather scarce. Therefore, from practical point of view, it seems necessary which consider the large structured programming problems, but even if it is theoretically possible to solve these problems, in practice it is not always so. There are some certain barriers which restrict the attempt of the analyst. This suggests the idea of developing methods of solution that should not use simultaneously all the data of the problem. In linear programming (LP) problems, such idea is due to Dantzig and Wolfe [6] as the decomposition principle or the column generation, which the original problem reformulates in terms of extreme points and extreme rays of feasible region.

In this paper, we first attempt to extend one of the most important theorems of the linear programming to fuzzy linear programming with trapezoidal fuzzy variables under ranking functions, which we will refer to as fuzzy representation theorem and then expand the column generation to these problems

This paper is organized as follows: In Section 2, some preliminary summaries on the fuzzy numbers and the fuzzy linear programming is introduced as well as fuzzy representation theorem. In Section 3, we introduced the column generation for FLP problem with fuzzy variables. Finally, in Section 4, a numerical example is proposed to illustrate of method.

2. Preliminaries

In this section, we introduce some basic concepts and results of fuzzy numbers, fuzzy arithmetic and ranking of fuzzy numbers which are needed in the rest of the paper.

2.1. **Fuzzy numbers.** Let R denote the set of all real numbers. In this paper, a fuzzy number will be a fuzzy set $\tilde{a} : R \longrightarrow [0, 1]$ with the following properties:

(1) Its membership function is defined by

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - (a^L - \alpha)}{\alpha}, & a^L - \alpha \le x \le a^L; \\ 1, & a^L \le x \le a^U; \\ \dfrac{(a^U + \beta) - x}{\beta}, & a^U \le x \le a^U + \beta; \\ 0, & \text{otherwise}, \end{cases}$$

(2) The membership function $\mu_{\tilde{a}}(x)$ is a piecewise continuous one,

(3) $\tilde{a}$ is fuzzy convex; that is, $\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \ge \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$, $\forall x, y \in R$ and $\lambda \in [0, 1]$,

which is denoted by $\tilde{a} = (a^L, a^U, \alpha, \beta)_{LR}$. The set of fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$, where $a^L \le a^U, \alpha > 0, \beta > 0$ and $a^L, a^U, \alpha, \beta \in R$ will be denoted by $F(\mathbb{R})$. Let $\tilde{a} = (a^L, a^U, \alpha, \beta)_{LR}$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)_{LR}$ be two fuzzy numbers belonging to

$F(\mathbb{R})$, then the arithmetic operations between fuzzy numbers, or fuzzy numbers and classical numbers, is described as follows:

(1) $x > 0, \quad x \in R; \quad x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta)$,
(2) $x < 0, \quad x \in R; \quad x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha)$,
(3) $\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta)$.

Next, we define a ranking function that represents a fuzzy number by means of a classical number.

2.2. **Ranking function.** One of the ways for solving mathematical programming problems in a fuzzy environment is to compare fuzzy numbers. The comparison between fuzzy numbers is achieved by using a ranking function that fulfils certain conditions, described in [11]. An appropriate approach to ordering the elements of $F(\mathbb{R})$ is to define a ranking function $\mathcal{R} : F(\mathbb{R}) \to R$, which maps each fuzzy number in the real line, where a natural order exists. Some orders on $F(\mathbb{R})$ are defined as follows:

(1) $\tilde{a} \leq^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$;
(2) $\tilde{a} <^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b})$;
(3) $\tilde{a} =^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$,

where $\tilde{a}$ and $\tilde{b}$ belong to $F(\mathbb{R})$, $\mathcal{R}$ is a ranking function, and the symbols " $\leq^f, <^f$ " and " $=^f$ " mean inequalities and equality with respect to the ranking function $\mathcal{R}$.

We will restrict our attention to linear ranking functions; that is, a ranking function $\mathcal{R}$ such that

$$\mathcal{R}(k\tilde{a} + \tilde{b}) = k\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b}), \tag{2.1}$$

for any $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ and any $k \in R$.

Hence, we can choose a linear ranking function which satisfies Equation (2.1) as

$$\mathcal{R}(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta, \tag{2.2}$$

where c_L, c_U, c_α and c_β are arbitrary constants.

Yager [12] proposed a procedure for ordering fuzzy sets in which a ranking $\mathcal{R}(\tilde{a})$ is calculated for the fuzzy number $\tilde{a} = (a^L, a^U, \alpha, \beta)_{LR}$ from its λ-cut, that is, $\tilde{a}_\lambda = [a^L - (1-\lambda)\alpha, a^U + (1-\lambda)\beta]$, according to the following formula:

$$\mathcal{R}(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf \tilde{a}_\lambda + \sup \tilde{a}_\lambda) d\lambda,$$

which leads to

$$\mathcal{R}(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{\beta - \alpha}{4}. \tag{2.3}$$

Therefore, for trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$, we have

$$\tilde{a} \geq^f \tilde{b} \text{ if and only if } a^L + a^U + \frac{1}{2}(\beta - \alpha) \geq b^L + b^U + \frac{1}{2}(\theta - \gamma).$$

Remark 2.1 (Definition 1.3. in [5])**.** For any fuzzy number $\tilde{a}$, let us define $\tilde{a} \geq^f \tilde{0}$ if there exist $\epsilon \geq 0$ and $\xi \geq 0$ such that $\tilde{a} \geq^f (-\epsilon, \epsilon, \xi, \xi)$. We also denote $(-\epsilon, \epsilon, \xi, \xi)$

by $\tilde{0}$. Hence, without loss of generality, we consider $\tilde{0} = (0, 0, 0, 0)$ as a trapezoidal fuzzy zero.

Although all the results are shown on the basis of the ranking function defined (2.2), all these results remain valid for general linear ranking function as defined by (2.1). For the sake of simplicity, we have selected the ranking function (2.2) in order to facilitate the reading of the paper.

2.3. **Fuzzy linear programming.** Consider the primal problem in standard form

$$\begin{array}{ll} \min & \tilde{z} =^f c\tilde{x} \\ \text{s.t.} & A\tilde{x} =^f \tilde{b} \\ & \tilde{x} \geq^f \tilde{0}, \end{array} \qquad (FLP)$$

with dual

$$\begin{array}{ll} \max & \tilde{w} =^f y\tilde{b} \\ \text{s.t.} & yA \leq c, \end{array} \qquad (FLD)$$

where $\tilde{b} \in (\mathcal{F}(\mathbb{R}))^m, A \in \mathbb{R}^{m\times n}, c^T \in \mathbb{R}^n$ are data, $\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n$ and $y^T \in \mathbb{R}^m$ are to be determined, and $\mathcal{R}$ is a linear ranking function as defined by (2.2).

Definition 2.2. A trapezoidal fuzzy vector $\tilde{x} \geq^f \tilde{0}$ is said to be a fuzzy feasible solution for FLP if $\tilde{x}$ satisfies the constraints $A\tilde{x} =^f \tilde{b}$.

Definition 2.3. A fuzzy feasible solution $\tilde{x}^*$ is called a fuzzy optimal solution for FLP if for all fuzzy feasible solutions $\tilde{x}$, we have $c\tilde{x} \geq^f c\tilde{x}^*$.

Definition 2.4. Let A be the coefficient matrix of the problem FLP with full row rank and B be a nonsingular sub-matrix $m\times m$ of A. Let $\{B_1, \dots, B_m\} \subset \{1, \dots, n\}$ denotes the index set of the columns of matrix B. Let $N = \{1, 2, \dots, n\} \setminus B$. In this case, vector $\tilde{x} =^f (\tilde{x}_B^T, \tilde{x}_N^T)^T =^f (B^{-1}\tilde{b}, \tilde{0})$ is called a basic solution. If $\tilde{x}_B \geq^f \tilde{0}$, then the fuzzy basic solution $\tilde{x}$ is called a fuzzy basic feasible solution and simply denoted by $FBFS$.

Definition 2.5. A fuzzy vector $\tilde{d} \in (\mathcal{F}(\mathbb{R}))^n$ is said to be a fuzzy feasible direction (ray) of the FLP problem at $\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n$, if there exists a positive scalar θ for which $\tilde{x} + \theta\tilde{d}$ is a fuzzy feasible solution of FLP. That is,

$$\begin{aligned} A(\tilde{x} + \theta\tilde{d}) &=^f \tilde{b} \\ \tilde{x} + \theta\tilde{d} &\geq^f \tilde{0}. \end{aligned} \tag{2.4}$$

It is easily seen that (2.4) is the same as

$$\begin{aligned} A\tilde{d} &=^f \tilde{0} \\ \tilde{d} &\geq^f \tilde{0}. \end{aligned} \tag{2.5}$$

In this case, a fuzzy vector $\tilde{d} =^f \begin{bmatrix} \tilde{d}_B \\ \tilde{d}_N \end{bmatrix} =^f \begin{bmatrix} -B^{-1}A_{.j}\tilde{1} \\ \tilde{e}_j \end{bmatrix}$, with $B^{-1}A_{.j}\tilde{1} \leq^f \tilde{0}$, where $\tilde{1} = (1, 1, 1, 1)$ and $\tilde{e}_j$ is an unit fuzzy vector, is called a fuzzy basic feasible direction.

Lemma 2.6. *The FLP problem is unbounded if for a fuzzy basic feasible direction* $\tilde{d}$, $c\tilde{d} <^f \tilde{0}$.

Proof. By definition 2.5, $\tilde{x} + \theta\tilde{d}$ is a fuzzy feasible solution of FLP for a positive fixed θ. The objective value is equal to $\tilde{z} =^f c\tilde{x} + \theta c\tilde{d}$, which for large enough θ, $\tilde{z} \longrightarrow -\infty$. This completes the proof. □

In the following, we generalize one of the fundamental theorems of linear programming to linear programming with trapezoidal fuzzy variables. In particular, we show that any element of a set $P = \{\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n : A\tilde{x} \geq^f \tilde{b}\}$ that has at least one fuzzy basic feasible solution can be represented as a convex combination of fuzzy basic feasible solutions plus a nonnegative linear combination of fuzzy basic feasible directions. A precise statement is given by the following theorem, which we refer to as fuzzy representation theorem.

Theorem 2.7. (Fuzzy Representation Theorem) *Let $P = \{\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n : A\tilde{x} \geq^f \tilde{b}\}$ be a nonempty set with at least a fuzzy basic feasible solution. Let $\tilde{x}^{(1)}, \tilde{x}^{(2)}, \ldots, \tilde{x}^{(k)}$ be the fuzzy basic feasible solutions, and let $\tilde{d}^{(1)}, \tilde{d}^{(2)}, \ldots, \tilde{d}^{(t)}$ be the fuzzy basic feasible directions of P. Then for any $\tilde{x} \in P$*

$$\begin{aligned} \tilde{x} &=^f \sum_{i=1}^{k} \lambda_i \tilde{x}^{(i)} + \sum_{j=1}^{t} \mu_j \tilde{d}^{(j)} \\ &\sum_{i=1}^{k} \lambda_i \tilde{1} =^f \tilde{1} \\ &\lambda_i \geq 0, \;\; i = 1, 2, \ldots, k \\ &\mu_j \geq 0, \;\; j = 1, 2, \ldots, t, \end{aligned} \tag{2.6}$$

where $\tilde{1} = (1, 1, 1, 1)$ is a trapezoidal fuzzy number.

Proof. Let $\tilde{x}$ be an element of P that does not as

$$\sum_{i=1}^{k} \lambda_i \tilde{x}^{(i)} + \sum_{j=1}^{t} \mu_j \tilde{d}^{(j)},$$

with $\sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0$ and $\mu_j \geq 0$ under linear ranking function $\mathcal{R}$. That is, the system (2.6) or equivalently

$$\begin{bmatrix} \mathcal{R}(\tilde{x}_1^{(1)}) & \cdots & \mathcal{R}(\tilde{x}_1^{(k)}) & \mathcal{R}(\tilde{d}_1^{(1)}) & \cdots & \mathcal{R}(\tilde{d}_1^{(t)}) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \mathcal{R}(\tilde{x}_n^{(1)}) & \cdots & \mathcal{R}(\tilde{x}_n^{(k)}) & \mathcal{R}(\tilde{d}_n^{(1)}) & \cdots & \mathcal{R}(\tilde{d}_n^{(t)}) \\ \mathcal{R}(\tilde{1}) & \cdots & \mathcal{R}(\tilde{1}) & \mathcal{R}(\tilde{0}) & \cdots & \mathcal{R}(\tilde{0}) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ \mu_1 \\ \vdots \\ \mu_t \end{bmatrix} = \begin{bmatrix} \mathcal{R}(\tilde{x}_1) \\ \vdots \\ \mathcal{R}(\tilde{x}_n) \\ \mathcal{R}(\tilde{1}) \end{bmatrix}$$

$$\lambda_i \geq 0, \quad \mu_j \geq 0, i = 1, \ldots, k, j = 1, \ldots, t,$$

has no a solution, then by the Farkas Lemma,

$$\begin{bmatrix} \mathcal{R}(\tilde{x}_1^{(1)}) & \cdots & \mathcal{R}(\tilde{x}_n^{(1)}) & \mathcal{R}(\tilde{1}) \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{R}(\tilde{x}_1^{(k)}) & \cdots & \mathcal{R}(\tilde{x}_n^{(k)}) & \mathcal{R}(\tilde{1}) \\ \mathcal{R}(\tilde{d}_1^{(1)}) & \cdots & \mathcal{R}(\tilde{d}_n^{(1)}) & \mathcal{R}(\tilde{0}) \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{R}(\tilde{d}_1^{(t)}) & \cdots & \mathcal{R}(\tilde{d}_n^{(t)}) & \mathcal{R}(\tilde{0}) \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} \leq 0$$

$$\begin{bmatrix} \mathcal{R}(\tilde{x}_1) & \cdots & \mathcal{R}(\tilde{x}_n) & \mathcal{R}(\tilde{1}) \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} > 0,$$

or equivalently

$$y^T\tilde{x}^{(i)} + y_{n+1} \leq^f \tilde{0}, \quad i = 1, \ldots, k \tag{2.7}$$

$$y^T\tilde{d}^{(j)} \leq^f \tilde{0}, \quad j = 1, \ldots, t \tag{2.8}$$

$$y^T\tilde{x} + y_{n+1} >^f \tilde{0}, \tag{2.9}$$

where $y = [y_1, \ldots, y_n]^T$, has a solution. Now consider the following problem

$$\begin{array}{c} \max \tilde{z}_0 =^f y^T\tilde{x} \\ \text{s.t.} \ \ A\tilde{x} \geq^f \tilde{b}, \end{array}$$

which is feasible and $y^T\tilde{d}^{(j)} \leq^f \tilde{0},\ j = 1, \ldots, t$ implies that it is bounded. Therefore there exists a fuzzy basic feasible solution $\tilde{x}^{(\alpha)}$, which is an optimal solution. Thus for any fuzzy feasible solution $\tilde{x}$, $y^T\tilde{x} \leq^f y^T\tilde{x}^{(\alpha)}$ which implies a contradiction by the relations (2.7) and (2.9) and completes the proof. □

3. Column generation

Consider the linear programming problem with trapezoidal fuzzy variables

$$\begin{array}{lrl} \min & \tilde{z} =^f c\tilde{x} & \\ \text{s.t.} & A_H\tilde{x} =^f \tilde{b}_H & \qquad (FP) \\ & A_E\tilde{x} =^f \tilde{b}_E & \\ & \tilde{x} \geq^f \tilde{0}, & \end{array}$$

where $A_H \in R^{m_1\times n},\ A_E \in R^{m_2\times n}, \tilde{b}_H \in (\mathcal{F}(\mathbb{R}))^{m_1}, \tilde{b}_E \in (\mathcal{F}(\mathbb{R}))^{m_2}$ and $c^T \in R^n$ are data and $\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n$ is to be determined. Let $X = \{\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n : A_E\tilde{x} =^f \tilde{b}_E, \tilde{x} \geq^f \tilde{0}\}$. Then by Theorem 2.7, any feasible point $\tilde{x} \in X$ can be written as:

$$\begin{array}{c} \tilde{x} =^f \sum_{i=1}^{k} \lambda_i\tilde{x}^{(i)} + \sum_{j=1}^{t} \mu_j\tilde{d}^{(j)} \\ \sum_{i=1}^{k} \lambda_i\tilde{1} =^f \tilde{1} \\ \lambda_i \geq 0,\ i = 1, 2, \ldots, k \\ \mu_j \geq 0,\ j = 1, 2, \ldots, t, \end{array} \tag{3.1}$$

where $\tilde{x}^{(i)}$ and $\tilde{d}^{(j)}$ are the basic feasible solutions and the basic feasible directions of X respectively. The problem (FP) can be rewritten in the terms of basic feasible solutions and basic feasible directions as follows:

$$\tilde{z} =^f c\tilde{x}$$

or

$$\begin{aligned}\mathcal{R}(\tilde{z}) = \mathcal{R}(c\tilde{x}) &= c\mathcal{R}(\tilde{x})\\ &= c\mathcal{R}(\sum_{i=1}^{k}\lambda_i\tilde{x}^{(i)} + \sum_{j=1}^{t}\mu_j\tilde{d}^{(j)})\\ &= \mathcal{R}(\sum_{i=1}^{k}c\lambda_i\tilde{x}^{(i)} + \sum_{j=1}^{t}c\mu_j\tilde{d}^{(j)})\\ &= \mathcal{R}(\sum_{i=1}^{k}(c\tilde{x}^{(i)})\lambda_i + \sum_{j=1}^{t}(c\tilde{d}^{(j)})\mu_j).\end{aligned}$$

Thus,

$$\tilde{z} =^f \sum_{i=1}^{k}(c\tilde{x}^{(i)})\lambda_i + \sum_{j=1}^{t}(c\tilde{d}^{(j)})\mu_j,$$

similarly for $A_H\tilde{x} =^f \tilde{b}_H$

$$\sum_{i=1}^{k}(A_H\tilde{x}^{(i)})\lambda_i + \sum_{j=1}^{t}(A_H\tilde{d}^{(j)})\mu_j =^f \tilde{b}_H.$$

Therefore

$$\begin{aligned}\min \tilde{z} =^f &\sum_{i=1}^{k}(c\tilde{x}^{(i)})\lambda_i + \sum_{j=1}^{t}(c\tilde{d}^{(j)})\mu_j\\ \text{s.t.}\ &\sum_{i=1}^{k}(A_H\tilde{x}^{(i)})\lambda_i + \sum_{j=1}^{t}(A_H\tilde{d}^{(j)})\mu_j =^f \tilde{b}_H\\ &\sum_{i=1}^{k}\lambda_i\tilde{1} =^f \tilde{1}\\ &\lambda_i \geq 0,\ i = 1, 2, \ldots, k\\ &\mu_j \geq 0,\ j = 1, 2, \ldots, t.\end{aligned} \tag{3.2}$$

This is a linear programming problem with fuzzy coefficients that can be solved by simplex method. It is referred to as the *master problem.*

Theorem 3.1. *The* (FP) *and* (3.2) *problems have the same optimal value and every optimal solution of* (FP) *corresponds to an optimal solution of* (3.2) *with the same objective function value and rice-versa.*

Proof. Let (λ^*, μ^*) be an optimal solution of (3.2). In this way

$$\tilde{z} =^f \sum_{i=1}^{k}(c\tilde{x}^{(i)})\lambda_i^* + \sum_{j=1}^{t}(c\tilde{d}^{(j)})\mu_j^* \leq^f \sum_{i=1}^{k}(c\tilde{x}^{(i)})\lambda_i^{'} + \sum_{j=1}^{t}(c\tilde{d}^{(j)})\mu_j^{'}$$

or

$$c(\sum_{i=1}^{k}\tilde{x}^{(i)}\lambda_i^* + \sum_{j=1}^{t}\tilde{d}^{(j)}\mu_j^*) \leq^f c(\sum_{i=1}^{k}\tilde{x}^{(i)}\lambda_i^{'} + \sum_{j=1}^{t}\tilde{d}^{(j)}\mu_j^{'}),$$

for every feasible solution $(\lambda^{'}, \mu^{'})$ of (3.2). Then by (3.1), we have

$$c\tilde{x}^* \leq^f c\tilde{x}^{'}.$$

This shows that $\tilde{x}^* =^f \sum_{i=1}^{k}\tilde{x}^{(i)}\lambda_i^* + \sum_{j=1}^{t}\tilde{d}^{(j)}\mu_j^*$ is an optimal solution of (FP). □

Now suppose that given a basic feasible solution (λ, μ) of (3.2) with corresponding basis matrix $B_{(m_1+1)\times(m_1+1)} = \mathcal{R}(\tilde{B})$, where $\tilde{B}$ is a fuzzy sub-matrix of the constraints matrix of (3.2). Let $(y, y_{m_1+1}) = \hat{c}_B B^{-1}$, where $\hat{c}_B = \mathcal{R}(\tilde{c}_B)$ ($\tilde{c}_B$ is the corresponding vector to the basis $\tilde{B}$). The optimality criterion is carried out by computing the components of

$$c\tilde{x}^{(i)} - (y, y_{m_1+1})\begin{bmatrix} A_H\tilde{x}^{(i)} \\ \tilde{1} \end{bmatrix} \geq^f \tilde{0}, \quad i = 1, 2, \ldots, k \tag{3.3}$$

and

$$c\tilde{d}^{(j)} - (y, y_{m_1+1})\begin{bmatrix} A_H\tilde{d}^{(j)} \\ \tilde{0} \end{bmatrix} \geq^f \tilde{0}, \quad j = 1, 2, \ldots, t. \tag{3.4}$$

Hence, the relations (3.3) and (3.4) can be checked by determining

$$\min_i \tilde{w}_0 =^f (c - yA_H)\tilde{x}^{(i)} - y_{m_1+1}\tilde{1}$$
$$\min_j \tilde{w}_{00} =^f (c - yA_H)\tilde{d}^{(j)},$$

and verifying to see the optimal values are nonnegative. Since $\{\tilde{x}^{(i)}\}$ and $\{\tilde{d}^{(j)}\}$ are the sets of basic feasible solutions and basic feasible directions of X respectively, thus the optimality condition can be verified by

$$\begin{aligned} \min \quad & \tilde{w} =^f (c - yA_H)\tilde{x} - y_{m_1+1}\tilde{1} \\ \text{s.t.} \quad & A_E\tilde{x} =^f \tilde{b}_E \\ & \tilde{x} \geq^f \tilde{0}. \end{aligned} \tag{3.5}$$

Case 1: Suppose that the optimal value of this problem is unbounded. Recall that this is only possible if a basic feasible direction $\tilde{d}^{(\alpha)}$ is found such that $(c - yA_H)\tilde{d}^{(\alpha)} <^f \tilde{0}$. This means that condition (3.4) is violated. In this case, μ_α with the column $B^{-1}\begin{bmatrix} A_H\tilde{d}^{(\alpha)} \\ \tilde{0} \end{bmatrix}$ is entering to basis.

Case 2: Suppose that the optimal value is bounded. Let $\tilde{x}^{(\beta)}$ be an optimal solution of (3.5). If the optimal value is nonnegative, then by optimality $\tilde{x}^{(\beta)}$, for each basic feasible solution $\tilde{x}^{(i)}$, the optimality condition (3.3) holds and stop with an optimal solution of the overall problem. Otherwise, λ_β with the column $B^{-1}\begin{bmatrix} A_H\tilde{x}^{(\beta)} \\ \tilde{1} \end{bmatrix}$ is the entering to basis.

We therefore summarize the solution method for obtaining the fuzzy optimal solution as well as the dual evaluators at the final stage as follows

step 1: From the corresponding master problem (3.2) to (FP) and let (λ, μ) be an optimal solution and y be the dual optimal.

step 2: Form the sub-problem (3.5) by using optimal solutions obtained in step 1.

step 3: Solve the sub-problem of obtained in step 2, which one of the following two cases occurs:

3-1: If the sub-problem is unbounded in direction $\tilde{d}^{\alpha}$, then a variable μ_α with the column $B^{-1}\begin{bmatrix} A_H\tilde{d}^{(\alpha)} \\ \tilde{0} \end{bmatrix}$ enters to basis.

3-2: If the optimal value is nonnegative, stop and the present solution is optimal. Otherwise, a variable λ_β with the column $B^{-1}\begin{bmatrix} A_H\tilde{x}^{(\beta)} \\ \tilde{1} \end{bmatrix}$ is the entering variable to basis.

step 4: Perform a ratio test to determine the leaving a variable in the problem (3.2), then update $\tilde{B}$ and return to step 2.

4. A numerical example

Consider the following fuzzy linear programming problem

$$\begin{aligned} \min \tilde{z} =^f & -\tilde{x}_1 - 2\tilde{x}_2 - \tilde{x}_3 \\ \text{s.t.} \quad & \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \leq^f (7,9,4,20) \\ & -\tilde{x}_1 + \tilde{x}_2 \leq^f (1,3,1,1) \\ & -\tilde{x}_1 + 2\tilde{x}_2 \leq^f (6,7,3,9) \\ & \tilde{x}_3 \leq^f (2,3,1,3) \\ & \tilde{x}_1, \ \tilde{x}_2, \ \tilde{x}_3 \geq^f \tilde{0}. \end{aligned}$$

The first constraint is handled as $A_H\tilde{x} =^f \tilde{b}_H$ and the rest of the constraints are treated by X. Note that $\tilde{x}^{(1)} =^f (\tilde{0},\tilde{0},\tilde{0})^T$ is a basic feasible solution of X. An initial basic feasible solution for the master problem is $(s_1, \lambda_1) = (12, 1)$ which corresponded to basis

$$\tilde{B} =^f \begin{bmatrix} \tilde{1} & A_H\tilde{x}^{(1)} \\ \tilde{0} & \tilde{1} \end{bmatrix} =^f \begin{bmatrix} \tilde{1} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{bmatrix}.$$

For this basis,

$$\tilde{c}_B =^f (\tilde{0}, c\tilde{x}^{(1)}) =^f (\tilde{0},\tilde{0}),$$

and

$$(y_1, y_2) = \hat{c}_B B^{-1} = (0, 0).$$

Therefore, the sub-problem (3.5) becomes

$$\begin{aligned} \min \tilde{z} =^f & -\tilde{x}_1 - 2\tilde{x}_2 - \tilde{x}_3 \\ \text{s.t.} \ & -\tilde{x}_1 + \tilde{x}_2 \leq^f (1,3,1,1) \\ & -\tilde{x}_1 + 2\tilde{x}_2 \leq^f (6,7,3,9) \\ & \tilde{x}_3 \leq^f (2,3,1,3) \\ & \tilde{x}_1, \ \tilde{x}_2, \ \tilde{x}_3 \geq^f \tilde{0}. \end{aligned}$$

This sub-problem is unbounded in direction $\tilde{d}^{(1)} =^f \begin{bmatrix}(2,2,2,2)\\(1,1,1,1)\\(0,0,0,0)\end{bmatrix}$. Thus, μ_1 with column

$$B^{-1}\begin{bmatrix}A_H\tilde{d}^{(1)}\\ \tilde{0}\end{bmatrix} =^f \begin{bmatrix}(3,3,3,3)\\(0,0,0,0)\end{bmatrix}$$

is the entering to the basis and s_1 leaves from the basis.
At the second iteration:

$$\tilde{B} =^f \begin{bmatrix}A_H\tilde{d}^{(1)} & A_H\tilde{x}^{(1)}\\ \tilde{0} & \tilde{1}\end{bmatrix} =^f \begin{bmatrix}(3,3,3,3) & (0,0,0,0)\\(0,0,0,0) & (1,1,1,1)\end{bmatrix},$$

$$\tilde{c}_B =^f (c\tilde{d}^{(1)}, c\tilde{x}^{(1)}) =^f ((-4,-4,4,4),\ \ (0,0,0,0)),$$

$$(y_1,y_2) = (-\frac{4}{3},0),$$

$$(\mu_1,\lambda_1) = (4,1).$$

In this way, the sub-problem (3.5) becomes

$$\begin{aligned}\min \tilde{z} =^f\ & \tfrac{1}{3}\tilde{x}_1 - \tfrac{2}{3}\tilde{x}_2 + \tfrac{1}{3}\tilde{x}_3\\ \text{s.t. } & -\tilde{x}_1 + \tilde{x}_2 \leq^f (1,3,1,1)\\ & -\tilde{x}_1 + 2\tilde{x}_2 \leq^f (6,7,3,9)\\ & \tilde{x}_3 \leq^f (2,3,1,3)\\ & \tilde{x}_1,\ \ \tilde{x}_2,\ \ \tilde{x}_3 \geq^f \tilde{0},\end{aligned}$$

which has an optimal solution $\tilde{x}^{(2)} =^f \begin{bmatrix}(0,5,5,11)\\(1,8,6,12)\\(0,0,0,0)\end{bmatrix}$ with the objective function $\tilde{w} =^f (-\frac{16}{3},1,\frac{29}{3},\frac{23}{3}) <^f \tilde{0}$. Therefore λ_2 with the column

$$B^{-1}\begin{bmatrix}A_H\tilde{x}^{(2)}\\ \tilde{1}\end{bmatrix} =^f \begin{bmatrix}(\frac{1}{3},\frac{13}{3},\frac{11}{3},\frac{23}{3})\\(1,1,1,1)\end{bmatrix},$$

is the entering to basis and λ_1 leaves from the basis.
At the third iteration,

$$\tilde{B} =^f \begin{bmatrix}A_H\tilde{d}^{(1)} & A_H\tilde{x}^{(2)}\\ \tilde{0} & \tilde{1}\end{bmatrix} =^f \begin{bmatrix}(3,3,3,3) & (1,13,11,23)\\(0,0,0,0) & (1,1,1,1)\end{bmatrix},$$

$$\tilde{c}_B =^f (c\tilde{d}^{(1)}, c\tilde{x}^{(2)}) =^f ((-4,-4,4,4),\ \ (-21,-2,35,17)),$$

$$(y_1,y_2) = (-\frac{4}{3},-\frac{8}{3}),$$

$$(\mu_1,\lambda_2) = (\frac{2}{3},1).$$

The optimal solution of the corresponding sub-problem remains the same and the optimal value is equal to zero. Therefore, the optimal $\tilde{x}^*$ is given by

$$\tilde{x}^* =^f \lambda_2\tilde{x}^{(2)} + \mu_1\tilde{d}^{(1)} =^f 1\begin{bmatrix}(0,5,5,11)\\(1,8,6,12)\\(0,0,0,0)\end{bmatrix} + \frac{2}{3}\begin{bmatrix}(2,2,2,2)\\(1,1,1,1)\\(0,0,0,0)\end{bmatrix}$$

Acknowledgements. The author kindly acknowledge the help of the associate editor and anonymous referees in improving the readability of the paper.

References

[1] R. E. Bellman and L. A. Zadeh, Decision making in a fuzzy environment, Management Sci. 17 (1970) 141–164.
[2] L. Campose and J. L. Verdegay, Linear programming problems and ranking of fuzzy numbers, Fuzzy Sets and Systems 32(1) (1989) 1–11.
[3] S. Chanas, The use of parametric programming in fuzzy linear programming, Fuzzy Sets and Systems 11 (1983) 229–241.
[4] S. C. Fang, C. F. Hu, H. F. Wang and S. Y. Wu, Linear programming with fuzzy coefficients in constraints, Comput. Math. Appl. 37 (1999) 63–76.
[5] K. Ganesan and P. Veeramani, Fuzzy linear programs with trapezoidal fuzzy numbers, Ann. Oper. Res. 143 (2006) 305–315.
[6] J. J. Jarvis, H. D. Sherali and M. S. Bazaraa, Linear Programming and Network Flows, John Wiley and Sons, 1997.
[7] X. Liu, Measuring the satisfacation of constraints in fuzzy linear programming, Fuzzy Sets and Systems 122 (2001) 263–275.
[8] N. Mahdavi-Amiri and S. H. Nasseri, Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, Fuzzy Sets and Systems 158 (2007) 1961–1978.
[9] H. R. Maleki, Ranking functions and their applications to fuzzy linear programming, Far East J. Math. Sci. 4 (2002) 21–33.
[10] H. Tanaka, T. Okuda and K. Asia, On fuzzy mathematical programming, Journal of Cybernetics and Systems 3 (1973) 37–46.
[11] X. Wang and E. Kerre, Reasonable properties for the ordering of fuzzy quantities(2 part), Fuzzy Sets and Systems 118 (2001) 375–405.
[12] R. R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Inform. Sci. 24 (1981) 143–161.
[13] G. Zhang, Y. H. Wu, M. Remias and J. Lu, Formulation of fuzzy linear programming problems as four-objective constrained optimization problems, Appl. Math. Comput. 139 (2003) 383–399.
[14] H. J. Zimmermann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1978) 45–55.

Behrouz Kheirfam (b.kheirfam@azaruniv.du)
Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I. R. Iran