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Prime fuzzy bi-ideals of Γ-semigroups

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Abstract. In this paper we define prime, strongly prime and semiprime fuzzy bi-ideals of Γ-semigroups. We also define irreducible and strongly irreducible fuzzy bi-ideals of Γ-semigroups. We characterize regular and intra-regular Γ-semigroups by the properties of these fuzzybi-ideals.

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1. Introduction

The concept of Γ-semigroup was introduced by Sen in [10]. The Γ-semigroup is a generalization of semigroup as well as of ternary semigroup. Many researchers worked on Γ-semigroups and its sub structures. Many classical notions of semigroups have been extended to Γ-semigroup by Saha and Sen in [8], [9] and [11]. The notion of bi-ideal in Γ-semigroups was introduced by Chinram and Jirojkul in [1]. Dutta and Adhikari introduced the notion of prime ideal in Γ-semigroup in [3]. On the other hand the concepts of prime bi-ideal, strongly prime bi-ideal, semiprime bi-ideal, strongly irreducible and irreducible bi-ideals of Γ-semigroup are studied in [12]. In this paper we define prime fuzzy bi-ideal, strongly prime fuzzy bi-ideal, semiprime fuzzy bi-ideal, strongly irreducible fuzzy bi-ideal and irreducible fuzzy bi-ideal of a Γ-semigroup and characterize regular and intra-regular Γ-semigroups by the properties of these fuzzy bi-ideals.

2. Preliminaries

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. If there exist a mapping : $S \times \Gamma \times S \to S$, written (x, α, y) by $x\alpha y$, then S is called a Γ-semigroup if it satisfies

$(x\alpha y)\,\beta z = x\alpha\,(y\beta z)$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

2.1. **Example.** [2] Let $S = \{5n+4 : n \text{ is a positive integer}\}$ and $\Gamma = \{5n+1 : n \text{ is a positive integer}\}$. Then S is a Γ-semigroup, where $a\alpha b = a + \alpha + b$. ($a, b \in S$ and $\alpha, \beta \in \Gamma$, $+$ is the usual addition of integers)

2.2. **Example.** [7] Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty sets. Then M is a Γ-semigroup with respect to the operation defined below:

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

2.3. **Example.** [10] Let $S = \{a, b, c, d, e\}$ be the semigroup with the following Cayley table and Γ be any non-empty set. Define

$S \times \Gamma \times S \to S$ by $x\gamma y = x \cdot y$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Then S is a Γ-semigroup.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

2.4. **Example.** [10] Let $S = \{0, a, b\}$ and $\Gamma = \{\gamma\}$. Define a mapping $S \times \Gamma \times S \to S$ by $x\gamma y = x$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup.

2.5. **Example.** [10] Let S and Γ be any two non-empty sets and $0 \in S$. Define $S \times \Gamma \times S \to S$ by

$$x\gamma y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup.

An element 0 of a Γ-semigroup S, with at least two elements, is called a zero element of S if $a\gamma 0 = 0\gamma a = 0$, for all $a \in S$ and $\gamma \in \Gamma$. A Γ-semigroup S is called a Γ-semigroup with zero if it contains a zero element. Let S be a Γ-semigroup. A non-empty subset A of a Γ-semigroup S is said to be a sub Γ-semigroup of S if $A\Gamma A \subseteq A$, that is $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. An element $x \in S$ is called regular if there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha s\beta x$. If every element of S is regular, then S is said to be regular. An element $x \in S$ is called intra-regular if there exist $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = a\alpha x\beta x\gamma b$. If every element of S is inrta-regular, then S is said to be intra-regular.

A left (right) ideal of a Γ-semigroup S is a non-empty subset A of S such that $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). If a non-empty subset of S is a left and a right ideal of S, then it is called a two-sided ideal or an ideal of S. An ideal A of a Γ-semigroup S is said to be idempotent if $A\Gamma A = A$. A sub Γ-semigroup B of a Γ-semigroup S is called a bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$.

A bi-ideal B of a Γ-semigroup S is called a prime bi-ideal of S if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of S. A bi-ideal B of a Γ-semigroup S is called a strongly prime bi-ideal of S if $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of S. A bi-ideal B of a Γ-semigroup S is called a semiprime bi-ideal of S if $B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S.

A bi-ideal B of a Γ-semigroup S is said to be irreducible bi-ideal of S if $B_1 \cap B_2 = B$ implies either $B_1 = B$ or $B_2 = B$ for any bi-ideals B_1, B_2 of S. A bi-ideal B of a Γ-semigroup S is said to be strongly irreducible bi-ideal of S if $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of S.

2.6. **Example.** [10] Let $S = \{0, 1, 2, 3, 4, 5\}$ be a semigroup as given in the following Cayley table. Let $\Gamma = \{\gamma\}$. Define

$S \times \Gamma \times S \to S$ by $x\gamma y = x \cdot y$ for all $x, y \in S$ and $\gamma \in \Gamma$.

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then S is a Γ-semigroup.

Bi-ideals of S are

$$\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\},$$
$$\{0, 1, 2, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 3\}, \{0, 1, 4, 5\} \text{ and } S.$$

Irreducible bi-ideals of S are

$$\{0\}, \{0, 1, 2, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 3\}, \{0, 1, 4, 5\} \text{ and } S.$$

Strongly irreducible bi-ideals are $\{0\}$ and S. Other irreducible bi-ideals are not strongly irreducible.

A function f from a non-empty set S to the unit closed interval $[0, 1]$ of real numbers is called a fuzzy subset of S, that is $f : S \to [0, 1]$. A fuzzy subset $f : S \to [0, 1]$ is non-empty if f is not the constant map which assumes the values 0. For fuzzy subsets f and g of S, $f \leq g$ means that for all $a \in S$, $f(a) \leq g(a)$. Let f, g be two fuzzy subsets of S. Then $f \cup g : S \to [0, 1]$ is defined by $(f \cup g)(x) = f(x) \vee g(x)$ and $f \cap g : S \to [0, 1]$ by $(f \cap g)(x) = f(x) \wedge g(x)$ for all $x \in S$. Let $\{f_i\}_{i\in I}$ be a family of fuzzy subsets of S. Define the fuzzy subsets $\cap_{i\in I} f_i$ and $\cup_{i\in I} f_i$ of S as follows $(\cap_{i\in I} f_i)(x) = \bigwedge_{i\in I} f_i(x)$ and $(\cup_{i\in I} f_i)(x) = \bigvee_{i\in I} f_i(x)$ for all $x \in S$. If f and g are fuzzy susets of a Γ-semigroup S and $x \in S$, then their product $f\Gamma g$ is a fuzzy subset of S defined as

$$(f\Gamma g)(x) = \begin{cases} \bigvee_{x=a\alpha b} [f(a) \wedge g(a)] & \text{if } x \text{ is expressible as } x = a\alpha b \\ 0 & \text{otherwise} \end{cases}$$

Let f be a fuzzy subset of X and $t \in [0, 1]$. Define $f_t = \{x \in X : f(x) \geq t\}$. We call f_t a t-cut or a level set. A fuzzy subset f of a Γ-semigroup S is called a fuzzy sub Γ-semigroup of S if $f(x\gamma y) \geq f(x) \wedge f(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. A fuzzy

subset f of a Γ-semigroup S is called a fuzzy left (right) ideal of S if $f(x\gamma y) \geq f(y)$ ($f(x\gamma y) \geq f(x)$) for all $x, y \in S$ and $\gamma \in \Gamma$. A fuzzy subset f of a Γ-semigroup S is called a fuzzy ideal of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.

2.7. **Lemma.** For any non-empty subsets X and Y of a Γ-semigroup S, we have
(1) $f_X \wedge f_Y = f_{X\cap Y}$
(2) $f_X \vee f_Y = f_{X\cup Y}$
(3) $f_X \Gamma f_Y = f_{X\Gamma Y}$

Proof. (1) Straightforward.
(2) Straightforward.
(3) Let $a \in S$. If $a \in X\Gamma Y$, then $a = x\gamma y$ for some $x \in X$, $y \in Y$ and $\gamma \in \Gamma$. Now $f_X \Gamma f_Y(a) = \bigvee_{a=u\alpha v} \{f_X(u) \wedge f_Y(v)\} \geq f_X(x) \wedge f_Y(y) = 1 \wedge 1 = 1 = f_{X\Gamma Y}(a)$. This implies $f_X \Gamma f_Y(a) = f_{X\Gamma Y}(a)$. If $a \notin X\Gamma Y$, then $a \neq x\gamma y$ for all $x \in X$, $y \in Y$ and $\gamma \in \Gamma$. If $a = u\alpha v$ for some $u, v \in S$, $\alpha \in \Gamma$, then we have $f_X \Gamma f_Y(a) = \bigvee_{a=u\alpha v} \{f_X(u) \wedge f_Y(v)\} = 0 = f_{X\Gamma Y}(a)$. If $a \neq u\alpha v$ for all $u, v \in S$, $\alpha \in \Gamma$, then we have $f_X \Gamma f_Y(a) = 0 = f_{X\Gamma Y}(a)$. In any case, we have $f_X \Gamma f_Y = f_{X\Gamma Y}$. □

3. Fuzzy Ideals and Fuzzy Bi-ideals of Γ-semigroups

3.1. **Definition.** A fuzzy subset f of a Γ-semigroup S is called a fuzzy bi-ideal of S if
(1) $f(x\gamma y) \geq f(x) \wedge f(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$
(2) $f(x\alpha y\beta z) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

3.2. **Lemma.** Let A be a non-empty subset of a Γ-semigroup S. Then A is a sub Γ-semigroup of S if and only if f_A is a fuzzy sub Γ-semigroup of S.

Proof. Suppose A is a sub Γ-semigroup of S. Then $A\Gamma A \subseteq A$. Let $x, y \in S$ and $\gamma \in \Gamma$.

Case 1: If $x \notin A$ or $y \notin A$, then $f_A(x) = 0$ or $f_A(y) = 0$. Hence $f_A(x) \wedge f_A(y) = 0 \leq f_A(x\gamma y)$.

Case 2: If $x \in A$ and $y \in A$, then $x\gamma y \in A\Gamma A \subseteq A$. Thus $f_A(x) = 1$, $f_A(y) = 1$ and $f_A(x\gamma y) = 1$. Hence $f_A(x\gamma y) = f_A(x) \wedge f_A(y)$. Thus in any case $f_A(x\gamma y) \geq f_A(x) \wedge f_A(y)$.

Conversely, suppose that f_A is a fuzzy sub Γ-semigroup of S. Let $x, y \in A$ and $\gamma \in \Gamma$. As $f_A(x\gamma y) \geq f_A(x) \wedge f_A(y) = 1$. Thus $f_A(x\gamma y) = 1$. This implies $x\gamma y \in A$. Hence $A\Gamma A \subseteq A$, that is A is a sub Γ-semigroup of S. □

3.3. **Lemma.** Let A be a non-empty subset of a Γ-semigroup S. Then A is a left(right, two sided, bi) ideal of S if and only if f_A is a fuzzy left (right, two sided, bi) ideal of S.

Proof. The proof is similar to the proof of Lemma 3.2. □

3.4. **Lemma.** The product of two fuzzy bi-ideals of a Γ-semigroup S is a fuzzy bi-ideal of S.

Proof. Let f and g be two fuzzy bi-ideals of a Γ-semigroup S. Let $x, y \in S$ and $\alpha \in \Gamma$.

Then

$$\begin{aligned}
(f\Gamma g)(x) \wedge (f\Gamma g)(y) &= \left(\bigvee_{x=a\beta b} \{f(a) \wedge g(b)\}\right) \wedge \left(\bigvee_{y=c\gamma d} \{f(c) \wedge g(d)\}\right) \\
&= \bigvee_{x=a\beta b} \bigvee_{y=c\gamma d} \{f(a) \wedge g(b) \wedge f(c) \wedge g(d)\} \\
&\leq \bigvee_{x=a\beta b} \bigvee_{y=c\gamma d} \{f(a) \wedge g(b) \wedge g(d)\} \\
&\leq \bigvee_{x=a\beta b} \bigvee_{y=c\gamma d} \{f(a) \wedge g(b\alpha c\gamma d)\} \\
&\leq \bigvee_{x\alpha y=e\delta z} \{f(e) \wedge g(z)\} \\
&= (f\Gamma g)(x\alpha y).
\end{aligned}$$

This shows that $f\Gamma g$ is a fuzzy sub Γ-semigroup of S. Now, let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
(f\Gamma g)(x) \wedge (f\Gamma g)(z) &= \left(\bigvee_{x=a\gamma b} \{f(a) \wedge g(b)\}\right) \wedge \left(\bigvee_{z=s\delta t} \{f(s) \wedge g(t)\}\right) \\
&= \bigvee_{x=a\gamma b} \bigvee_{z=s\delta t} \{f(a) \wedge g(b) \wedge f(s) \wedge g(t)\} \\
&\leq \bigvee_{x=a\gamma b} \bigvee_{z=s\delta t} \{f(a) \wedge g(t)\} \\
&\leq \bigvee_{x\alpha y\beta z=e\theta i} \{(f(e) \wedge g(i)\} \\
&= (f\Gamma g)(x\alpha y\beta z).
\end{aligned}$$

Thus $f\Gamma g$ is a fuzzy bi-ideal of S. □

3.5. **Lemma.** A fuzzy subset f of a Γ-semigroup S is a fuzzy bi-ideal of S if and only if $f\Gamma f \leq f$ and $f\Gamma f_S\Gamma f \leq f$.

Proof. Assume that f is a fuzzy bi-ideal of a Γ-semigroup S. If $f\Gamma f(a) = 0$, then $f\Gamma f(a) \leq f(a)$. Otherwise there exist elements $x, y \in S$ and $\alpha \in \Gamma$ such that

$a = x\alpha y$. Since f is a fuzzy bi-ideal of S, we have $f(x\alpha y) \geq f(x) \wedge f(y)$. Therefore

$$\begin{aligned} f\Gamma f(a) &= \bigvee_{a=x\alpha y} \{f(x) \wedge f(y)\} \\ &\leq \bigvee_{a=x\alpha y} \{f(x\alpha y)\} \\ &= f(a). \end{aligned}$$

This implies that $f\Gamma f \leq f$. Similarly, if $f\Gamma f_S\Gamma f(a) = 0$, then $f\Gamma f_S\Gamma f(a) \leq f(a)$. Otherwise there exist elements $x, y, p, q \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha y$, $x = p\beta q$. Since f is a fuzzy bi-ideal of S, we have $f(p\beta q\alpha y) \geq f(p) \wedge f(y)$. Therefore

$$\begin{aligned} f\Gamma f_S\Gamma f(a) &= \bigvee_{a=x\alpha y} \{f\Gamma f_S(x) \wedge f(y)\} \\ &= \bigvee_{a=x\alpha y} \left\{ \bigvee_{x=p\beta q} (f(p) \wedge f_S(q)) \right\} \wedge f(y) \\ &= \bigvee_{a=x\alpha y} \left\{ \bigvee_{x=p\beta q} (f(p) \wedge 1) \right\} \wedge f(y) \\ &= \bigvee_{a=x\alpha y} \left\{ \bigvee_{x=p\beta q} (f(p) \right\} \wedge f(y) \\ &= \bigvee_{a=x\alpha y} \bigvee_{x=p\beta q} \{f(p) \wedge f(y)\} \\ &\leq \bigvee_{a=x\alpha y} \bigvee_{x=p\beta q} \{f(p\beta q\alpha y)\} \\ &= f(a). \end{aligned}$$

Conversely, assume that $f\Gamma f \leq f$ and $f\Gamma f_S\Gamma f \leq f$. Let a, b be any elements of S and $\alpha \in \Gamma$. Then we have

$$\begin{aligned} f(a\alpha b) &\geq (f\Gamma f)(a\alpha b) \\ &= \bigvee_{a\alpha b=a'\alpha b'} \{f(a') \wedge f(b')\} \\ &\geq f(a) \wedge f(b). \end{aligned}$$

Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned} f(a\alpha b\beta c) &\geq (f\Gamma f_S\Gamma f)(a\alpha b\beta c) \\ &= \bigvee_{a\alpha b\beta c = a'\alpha' b'} \{f\Gamma f_S(a') \wedge f(b')\} \\ &\geq \{f\Gamma f_S(a\alpha b) \wedge f(c)\} \\ &= \left(\bigvee_{a\alpha b = x_1\alpha_1 y_1} \{(f(x_1) \wedge f_S(y_1))\right) \wedge f(c)\} \\ &\geq \{(f(a) \wedge f_S(b))\} \wedge f(c) \\ &= (f(a) \wedge 1) \wedge f(c) \\ &= f(a) \wedge f(c). \end{aligned}$$

This implies that $f(a\alpha b\beta c) \geq f(a) \wedge f(c)$. Thus f is a fuzzy bi-ideal of S. □

3.6. **Lemma.** The intersection of any collection of fuzzy bi-ideals of a Γ-semigroup S is a fuzzy bi-ideal of S.

Proof. Let $\{f_i\}$ be a family of fuzzy bi-ideals of S. Let $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} (\bigcap_i f_i)(x\gamma y) &= \bigwedge_i (f_i(x\gamma y)) \\ &\geq \bigwedge_i (f_i(x) \wedge f_i(y))) \\ &= \left(\bigwedge_i f_i(x)\right) \wedge \left(\bigwedge_i f_i(y)\right) \\ &= \left(\bigwedge_i f_i\right)(x) \wedge \left(\bigwedge_i f_i\right)(y). \end{aligned}$$

This implies $(\bigwedge_i f_i)(x\gamma y) \geq (\bigwedge_i f_i)(x) \wedge (\bigwedge_i f_i)(y)$. Now

$$\begin{aligned} (\bigwedge_i f_i)(x\alpha y\beta z) &= \bigwedge_i (f_i(x\alpha y\beta z)) \\ &\geq \bigwedge_i (f_i(x) \wedge f_i(z)) \\ &= \left(\bigwedge_i f_i(x)\right) \wedge \left(\bigwedge_i f_i(z)\right) \\ &= \left(\bigwedge_i f_i\right)(x) \wedge \left(\bigwedge_i f_i\right)(z). \end{aligned}$$

Thus $\left(\bigwedge_i f_i\right)(x\alpha y\beta z) \geq \left(\bigwedge_i f_i\right)(x) \wedge \left(\bigwedge_i f_i\right)(z)$. This implies that $\bigwedge_i f_i$ is a fuzzy bi-ideal of S. □

4. Prime, Strongly Prime and Semiprime Fuzzy Bi-ideals of Γ-semigroups

4.1. **Definition.** A fuzzy bi-ideal f of a Γ-semigroup S is called a prime fuzzy bi-ideal of S if $f_1\Gamma f_2 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ for any fuzzy bi-ideals f_1, f_2 of S.

4.2. **Definition.** A fuzzy bi-ideal f of a Γ-semigroup S is called a strongly prime fuzzy bi-ideal of S if $f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ for any fuzzy bi-ideals f_1, f_2 of S.

4.3. **Definition.** A fuzzy bi-ideal f of a Γ-semigroup S is called a semiprime fuzzy bi-ideal of S if $f_1\Gamma f_1 \leq f$ implies $f_1 \leq f$ for any fuzzy bi-ideal f_1 of S.

4.4. **Proposition.** Every strongly prime fuzzy bi-ideal of a Γ-semigroup S is a prime fuzzy bi-ideal of S.

Proof. Straightforward. □

4.5. **Proposition.** Every prime fuzzy bi-ideal of a Γ-semigroup S is a semiprime fuzzy bi-ideal of S.

Proof. Straightforward. □

4.6. **Lemma.** The intersection of any family of prime fuzzy bi-ideals of a Γ-semigroup S is a semiprime fuzzy bi-ideal of S.

Proof. Straightforward. □

5. Irreducible and Strongly Irreducible Fuzzy Bi-ideals of Γ-semigroups

5.1. **Definition.** A fuzzy bi-ideal f of a Γ-semigroup S is called an irreducible fuzzy bi-ideal of S if $f_1 \wedge f_2 = f$ implies either $f_1 = f$ or $f_2 = f$, for any fuzzy bi-ideals f_1, f_2 of S.

5.2. **Definition.** A fuzzy bi-ideal f of a Γ-semigroup S is called a strongly irreducible fuzzy bi-ideal of S if $f_1 \wedge f_2 \leq f$ implies either $f_1 \leq f$ or $f_2 \leq f$, for any fuzzy bi-ideals f_1, f_2 of S.

5.3. **Proposition.** Every strongly irreducible semiprime fuzzy bi-ideal of a Γ-semigroup S is a strongly prime fuzzy bi-ideal of S.

Proof. Let f be a strongly irreducible semiprime fuzzy bi-ideal of S. Let f_1, f_2 be any two fuzzy bi-ideals of S such that $f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$. Since $f_1 \wedge f_2 \leq f_1$ and $f_1 \wedge f_2 \leq f_2$, so we have $(f_1 \wedge f_2)\Gamma(f_1 \wedge f_2) \leq f_1\Gamma f_2$ and $(f_1 \wedge f_2)\Gamma(f_1 \wedge f_2) \leq f_2\Gamma f_1$. Thus $(f_1 \wedge f_2)\Gamma(f_1 \wedge f_2) \leq f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$. This implies $f_1 \wedge f_2 \leq f$, because f is a semiprime fuzzy bi-ideal of S. Since f is a strongly irreducible fuzzy bi-ideal of S, we have $f_1 \leq f$ or $f_2 \leq f$. Hence f is a strongly prime fuzzy bi-ideal of S. □

5.4. **Theorem.** Let f be a fuzzy bi-ideal of a Γ-semigroup S with $f(a) = \alpha$, where $a \in S$ and $\alpha \in (0, 1]$. Then there exists an irreducible fuzzy bi-ideal g of S such that $f \leq g$ and $g(a) = \alpha$.

Proof. Let $X = \{h : h$ is a fuzzy bi-ideal of S, $h(a) = \alpha$ and $f \leq h\}$. Then $X \neq \varnothing$, because $f \in X$. The collection X is a partially ordered set under inclusion. If $Y = \{h_i : h_i$ is a fuzzy bi-ideal of S, $h_i(a) = \alpha$ and $f \leq h_i$ for all $i \in I$ $\}$ is any totally ordered subset of X, then $\bigvee_{i\in I} h_i$ is a fuzzy bi-ideal of S such that $f \leq \bigvee_{i\in I} h_i$. Indeed, if $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$, then

$$\begin{aligned}
\left(\bigvee_{i\in I} h_i\right)(a\alpha b) &= \bigvee_{i\in I}(h_i(a\alpha b)) \\
&\geq \bigvee_{i\in I}(h_i(a) \wedge h_i(b)) \quad \text{as each } h_i \text{ is a fuzzy bi-ideal of } S \\
&= \left(\bigvee_{i\in I} h_i(a)\right) \wedge \left(\bigvee_{i\in I} h_i(b)\right) \\
&= \left(\bigvee_{i\in I} h_i\right)(a) \wedge \left(\bigvee_{i\in I} h_i\right)(b).
\end{aligned}$$

This implies $\left(\bigvee_{i\in I} h_i\right)(a\alpha b) \geq \left(\bigvee_{i\in I} h_i\right)(a) \wedge \left(\bigvee_{i\in I} h_i\right)(b)$. Now

$$\begin{aligned}
\left(\bigvee_{i\in I} h_i\right)(a\alpha b\beta c) &= \bigvee_{i\in I}(h_i(a\alpha b\beta c)) \\
&\geq \bigvee_{i\in I}(h_i(a) \wedge h_i(c)) \quad \text{as each } h_i \text{ is a fuzzy bi-ideal of } S \\
&= \left(\bigvee_{i\in I} h_i(a)\right) \wedge \left(\bigvee_{i\in I} h_i(c)\right) \\
&= \left(\bigvee_{i\in I} h_i\right)(a) \wedge \left(\bigvee_{i\in I} h_i\right)(c).
\end{aligned}$$

Hence $\bigvee_{i\in I} h_i$ is a fuzzy bi-ideal of S. As $f \leq h_i$ for all $i \in I$, we have $f \leq \bigvee_{i\in I} h_i$. Also $\left(\bigvee_{i\in I} h_i\right)(a) = \bigvee_{i\in I}(h_i(a)) = \alpha$. Thus $\bigvee_{i\in I} h_i \in X$ and $\bigvee_{i\in I} h_i$ is an upper bound of Y. Hence by Zorn's Lemma, there exists a fuzzy bi-ideal g of S which is maximal with respect to the property $f \leq g$ and $g(a) = \alpha$. Now, let g_1, g_2 be any fuzzy bi-ideals of S such that $g_1 \wedge g_2 = g$. This implies $g \leq g_1$ and $g \leq g_2$. We claim that $g = g_1$ or $g = g_2$. On the contrary, suppose that $g \neq g_1$ and $g \neq g_2$. This implies $g < g_1$ and $g < g_2$. So $g_1(a) \neq \alpha$ and $g_2(a) \neq \alpha$. Hence $(g_1 \wedge g_2)(a) = g_1(a) \wedge g_2(a) \neq \alpha$. Which is a contradiction to the fact that $g_1(a) \wedge g_2(a) = g(a) = \alpha$. Hence either $g = g_1$ or $g = g_2$. Thus g is an irreducible fuzzy bi-ideal of S. □

The following Theorems are taken from [12].

5.5. **Theorem.** For a Γ-semigroup S the following conditions are equivalent:
(1) S is intra-regular.
(2) $L \cap R \subseteq L\Gamma R$, for every left ideal L and every right ideal R of S.

5.6. **Theorem.** For a Γ-semigroup S, the following assertions are equivalent:
(1) S is both regular and intra-regular.
(2) $B\Gamma B = B$ for every bi-ideal B of S.
(3) $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$ for all bi-ideals B_1 and B_2 of S.
(4) Each bi-ideal of S is semiprime.
(5) Each proper bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contains it.

5.7. **Theorem.** For a Γ-semigroup S the following conditions are equivalent:
(1) S is intra-regular.
(2) $f \wedge g \leq g\Gamma f$, for every fuzzy right ideal f and every fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (2) Suppose S is an intra-regular Γ-semigroup and $x \in S$. Then there exist $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = a\alpha x\beta x\gamma b$. Let f be a fuzzy right ideal and g be a fuzzy left ideal of S. Then we have

$$\begin{aligned}(g\Gamma f)(x) &= \bigvee_{x=u\gamma v} \{g(u) \wedge f(v)\} \\ &\geq \{g(a\alpha x) \wedge f(x\gamma b)\} \\ &\geq g(x) \wedge f(x).\end{aligned}$$

This implies $f \wedge g \leq g\Gamma f$.

(2) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then by Lemma 3.3, the characteristic function f_R and f_L are the fuzzy right and left ideals of S, respectively. Thus by hypothesis $f_R \wedge f_L \leq f_L\Gamma f_R$. Then by Lemma 2.7, $f_{R\cap L} \leq f_{L\Gamma R}$. This implies $R \cap L \leq L\Gamma R$. Hence by Theorem 5.5, S is intra-regular. □

5.8. **Theorem.** For a Γ-semigroup S, the following assertions are equivalent:
(1) S is both regular and intra-regular.
(2) $f\Gamma f = f$ for every fuzzy bi-ideal f of S.
(3) $g \wedge h = g\Gamma h \wedge h\Gamma g$ for all fuzzy bi-ideals g and h of S.
(4) Each fuzzy bi-ideal of S is semiprime.
(5) Each proper fuzzy bi-ideal of S is the intersection of irreducible semiprime fuzzy bi-ideals of S which contains it.

Proof. (1) $\Rightarrow$ (2) Let f be a fuzzy bi-ideal of S. Then by Lemma 3.5, $f\Gamma f \leq f$. Now, let $a \in S$. As S is both regular and intra-regular, so there exist elements $x, y, z \in S$ and $\alpha, \beta, \delta, \theta, \xi \in \Gamma$ such that $a = a\alpha x\beta a$ and $a = y\delta a\theta a\xi z$. Then $a = a\alpha x\beta a = a\alpha x\beta(a\alpha x\beta a) = a\alpha x\beta(y\delta a\theta a\xi z)\alpha x\beta a = (a\alpha x\beta y\delta a)\theta(a\xi z\alpha x\beta a)$. Thus we have

$$\begin{aligned}(f\Gamma f)(a) &= \bigvee_{a=x\gamma y} [\{f(x) \wedge f(y)\}] \\ &\geq \{f(a\alpha x\beta y\delta a) \wedge f(a\xi z\alpha x\beta a)\} \\ &\geq f(a) \wedge f(a) \\ &= f(a) \qquad \text{for all } a \in S.\end{aligned}$$

This implies $f\Gamma f \geq f$. Hence $f\Gamma f = f$.

(2) $\Rightarrow$ (1) Let B be any bi-ideal of S. Then f_B, the charcateristic function of B, is a fuzzy bi-ideal of S. Thus by hypothesis,

$$\begin{aligned} f_B\Gamma f_B &= f_B \\ &\Rightarrow f_{B\Gamma B} = f_B \\ &\Rightarrow B\Gamma B = B. \end{aligned}$$

Then by Theorem 5.6, S is both regular and intra-regular.

(2) $\Rightarrow$ (3) Let g and h be two fuzzy bi-ideal of S. Then by Lemma 3.6, $g \wedge h$ is also a fuzzy bi-ideal of S. Thus by hypothesis, we have $g \wedge h = (g \wedge h)\Gamma(g \wedge h) \leq g\Gamma h$. Similarly $g \wedge h \leq h\Gamma g$. This implies

$g \wedge h \leq g\Gamma h \wedge h\Gamma g$ **(i)**

Now $g\Gamma h$ and $h\Gamma g$, being the product of two fuzzy bi-ideals of S, are fuzzy bi-ideals of S and so $g\Gamma h \wedge h\Gamma g$ is a fuzzy bi-ideal of S. Thus by hypothesis, we have

$$\begin{aligned} g\Gamma h \wedge h\Gamma g &= (g\Gamma h \wedge h\Gamma g)\Gamma(g\Gamma h \wedge h\Gamma g) \\ &\leq (g\Gamma h)\Gamma(h\Gamma g) \\ &= g\Gamma h\Gamma g \qquad \text{as } h\Gamma h = h \text{ (by hypothesis)} \\ &\leq g\Gamma f_S\Gamma g \qquad \text{as } h \leq f_S \\ &\leq g \qquad \text{as } g \text{ is a fuzzy bi-ideal of } S. \end{aligned}$$

Similarly $g\Gamma h \wedge h\Gamma g \leq h$. Thus

$g\Gamma h \wedge h\Gamma g \leq g \wedge h$ **(ii)**

From **(i)** and **(ii)** $g\Gamma h \wedge h\Gamma g = g \wedge h$.

(3) $\Rightarrow$ (4) Let g be a fuzzy bi-ideal of S such that $f\Gamma f \leq g$ for any fuzzy bi-ideal f of S. Then by hypothesis, $f = f \wedge f = f\Gamma f \wedge f\Gamma f = f\Gamma f \leq g$. This implies $f \leq g$. Thus g is a semiprime fuzzy bi-ideal of S. Hence every fuzzy bi-ideal of S is semiprime.

(4) $\Rightarrow$ (5) Let f be a proper fuzzy bi-ideal of S and $\{f_i : i \in I\}$ be the collection of all irreducible fuzzy bi-ideals of S such that $f \leq f_i$ for all $i \in I$. This implies $f \leq \bigwedge_{i\in I} f_i$. Let $a \in S$ and $\alpha \in (0, 1]$ be such that $f(a) = \alpha$. Then by Theorem 5.4, there exists an irreducible fuzzy bi-ideal f_α of S such that $f \leq f_\alpha$ and $f(a) = f_\alpha(a)$. This implies $f_\alpha \in \{f_i : i \in I\}$. Thus $\bigwedge_{i\in I} f_i \leq f_\alpha$, so $\bigwedge_{i\in I} f_i(a) \leq f_\alpha(a) = f(a)$ for all $a \in S$. This implies $\bigwedge_{i\in I} f_i \leq f$. Hence $\bigwedge_{i\in I} f_i = f$. By hypothesis, each fuzzy bi-ideal of S is semiprime, thus each fuzzy bi-ideal of S is the intersection of all irreducible semiprime fuzzy bi-ideals of S which contain it.

(5) $\Rightarrow$ (2) Let f be a fuzzy bi-ideal of S. Then by Lemma 3.5, we have $f\Gamma f \leq f$. Also $f\Gamma f$ being the product of two fuzzy bi-ideals of S, is a fuzzy bi-ideal of S. Thus by hypothesis $f\Gamma f = \bigwedge_{i\in I} f_i$, where each f_i is an irreducible semiprime fuzzy bi-ideal of S such that $f\Gamma f \leq f_i$ for all $i \in I$. This implies $f \leq f_i$ for all $i \in I$, because each f_i is a semiprime fuzzy bi-ideal of S. Thus $f \leq \bigwedge_{i\in I} f_i = f\Gamma f$. Hence $f\Gamma f = f$.

□

5.9. **Theorem.** Let S be a regular and intra-regular Γ-semigroup. Then the following assertions for a fuzzy bi-ideal f of S are equivalent:

(1) f is strongly irreducible.
(2) f is strongly prime.

Proof. (1) $\Rightarrow$ (2) Let S be a regular and intra-regular Γ-semigroup and f be a strongly irreducible fuzzy bi-ideal of S. Suppose that g and h are two fuzzy bi-ideals of S such that $g\Gamma h \wedge h\Gamma g \leq f$. As S is both regular and intra-regular Γ-semigroup, we have by Theorem 5.8, $g \wedge h = g\Gamma h \wedge h\Gamma g \leq f$. This implies either $g \leq f$ or $h \leq f$, because f is strongly irreducible. Thus f is strongly prime fuzzy bi-ideal of S.

(2) $\Rightarrow$ (1) Suppose f is strongly prime fuzzy bi-ideal of S. Let g, h be any two fuzzy bi-ideals of S such that $g \wedge h \leq f$. Since S is both regular and intra-regular, we have by Theorem 5.8, $g\Gamma h \wedge h\Gamma g = g \wedge h \leq f$. This implies either $g \leq f$ or $h \leq f$, because f is strongly prime fuzzy bi-ideal of S. Thus f is strongly irreducible. □

5.10. **Theorem.** Each fuzzy bi-ideal of a Γ-semigroup S is strongly prime if and only if S is regular and intra-regular and the set of fuzzy bi-ideals of S is totally ordered by inclusion.

Proof. Suppose that each fuzzy bi-ideal of a Γ-semigroup S is strongly prime. Then each fuzzy bi-ideal of S is semiprime. Thus by Theorem 5.8, S is both regular and intra-regular. Now, let g, h be any two fuzzy bi-ideals of S. Then by Theorem 5.8, $g\Gamma h \wedge h\Gamma g = g \wedge h$. As each fuzzy bi-ideal of S is strongly prime, so is $g \wedge h$. Thus either $g \leq g \wedge h$ or $h \leq g \wedge h$. Now, if $g \leq g \wedge h$, then $g \leq h$ and if $h \leq g \wedge h$, then $h \leq g$.

Conversely, assume that S is regular and intra-regular and the set of fuzzy bi-ideals of S is totally ordered by inclusion. Let f be an arbitrary fuzzy bi-ideal of S and g, h be any two fuzzy bi-ideals of S such that $g\Gamma h \wedge h\Gamma g \leq f$. As S is both regular and intra-regular, we have by Theorem 5.8,

$$g \wedge h = g\Gamma h \wedge h\Gamma g \leq f \ ...(\mathbf{i}).$$

Since the set of fuzzy bi-ideals of S is totally ordered by inclusion, we have either $g \leq h$ or $h \leq g$. This implies either $g \wedge h = g$ or $g \wedge h = h$. Then **(i)** implies either $g \leq f$ or $h \leq f$. Hence f is strongly prime. □

5.11. **Theorem.** If the set of fuzzy bi-ideals of a Γ-semigroup S is totally ordered by inclusion, then S is both regular and intra-regular if and only if each fuzzy bi-ideal of S is prime.

Proof. Suppose S is both regular and intra-regular. Let f be an arbitrary fuzzy bi-ideal of S and g, h be any two fuzzy bi-ideals of S such that $g\Gamma h \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered by inclusion, so either $g \leq h$ or $h \leq g$. If $g \leq h$, then $g\Gamma g \leq g\Gamma h \leq f$. This implies $g \leq f$, because f is semiprime by Theorem 5.8. If $h \leq g$, then $h\Gamma h \leq g\Gamma h \leq f$. This implies $h \leq f$, because f is semiprime by Theorem 5.8.

Conversely, suppose that every fuzzy bi-ideal of S is prime. Since every prime fuzzy bi-ideal of S is semiprime, so by Theorem 5.8, S is both regular and intra-regular. □

5.12. **Proposition.** If the set of fuzzy bi-ideals of a Γ-semigroup S is totally ordered, then the concept of primeness and strongly primeness coincides.

Proof. Let f be a prime fuzzy bi-ideal of S. Let f_1, f_2 be any two fuzzy bi-ideals of S such that $f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$. As the set of fuzzy bi-ideals of a Γ-semigroup S is totally ordered, we have either $f_1 \leq f_2$ or $f_2 \leq f_1$. If $f_1 \leq f_2$, then $f_1\Gamma f_1 = f_1\Gamma f_1 \wedge f_1\Gamma f_1 \leq f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$. This implies $f_1 \leq f$, as f is a prime fuzzy bi-ideal of S. Similarly, if $f_2 \leq f_1$, then $f_2\Gamma f_2 = f_2\Gamma f_2 \wedge f_2\Gamma f_2 \leq f_1\Gamma f_2 \wedge f_2\Gamma f_1 \leq f$. This implies $f_2 \leq f$, as f is a prime fuzzy bi-ideal of S. This shows that f is a strongly prime fuzzy bi-ideal of S. Thus every prime fuzzy bi-ideal of S is a strongly prime fuzzy bi-ideal of S. Also by Proposition 4.4, every strongly prime fuzzy bi-ideal of S is a prime fuzzy bi-ideal of S. □

5.13. **Theorem.** For a Γ-semigroup S the following assertions are equivalent:
(1) The set of all fuzzy bi-ideals of S is totally ordered under inclusion.
(2) Each fuzzy bi-ideal of S is strongly irreducible.
(3) Each fuzzy bi-ideal of S is irreducible.

Proof. (1) $\Rightarrow$ (2) Let the set of fuzzy bi-ideals of S is totally ordered by inclusion. Assume that f is an arbitrary fuzzy bi-ideal of S and f_1, f_2 be any two fuzzy bi-ideals of S such that $f_1 \wedge f_2 \leq f$. Since the set of fuzzy bi-ideals of S is totally ordered by inclusion, we have either $f_1 \leq f_2$ or $f_2 \leq f_1$. This implies that either $f_1 \wedge f_2 = f_1$ or $f_1 \wedge f_2 = f_2$. This implies that either $f_1 \leq f$ or $f_2 \leq f$. Thus f is strongly irreducible. Hence each fuzzy bi-ideal of S is strongly irreducible.

(2) $\Rightarrow$ (3) Suppose each fuzzy bi-ideal of S is strongly irreducible and let f be an arbitrary fuzzy bi-ideal of S and f_1, f_2 be any two fuzzy bi-ideals of S such that $f_1 \wedge f_2 = f$. This implies that $f \leq f_1$ and $f \leq f_2$. By hypothesis either $f_1 \leq f$ or $f_2 \leq f$. Thus either $f_1 = f$ or $f_2 = f$. This implies f is an irreducible fuzzy bi-ideal of S. Hence each fuzzy bi-ideal of S is irreducible.

(3) $\Rightarrow$ (1) Suppose each fuzzy bi-ideal of S is irreducible. Let f_1, f_2 be any two fuzzy bi-ideals of S. Then by Lemma 3.6, $f_1 \wedge f_2$ is also a fuzzy bi-ideal of S and so is irreducible fuzzy bi-ideal of S. Since $f_1 \wedge f_2 = f_1 \wedge f_2$, thus by hypothesis $f_1 = f_1 \wedge f_2$ or $f_2 = f_1 \wedge f_2$. That is $f_1 \leq f_2$ or $f_2 \leq f_1$. Hence the set of all fuzzy bi-ideals of S is totally ordered by inclusion. □

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