Prime fuzzy bi-ideals of Γ-semigroups

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Received 23 March 2012; Accepted 2 May 2012

ABSTRACT. In this paper we define prime, strongly prime and semiprime fuzzy bi-ideals of Γ-semigroups. We also define irreducible and strongly irreducible fuzzy bi-ideals of Γ-semigroups. We characterize regular and intra-regular Γ-semigroups by the properties of these fuzzy bi-ideals.

2010 AMS Classification: 06D72, 20N10.

Keywords: Fuzzy bi-ideal, Prime fuzzy bi-ideal, Semiprime fuzzy bi-ideal

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1. INTRODUCTION

The concept of Γ-semigroup was introduced by Sen in [10]. The Γ-semigroup is a generalization of semigroup as well as of ternary semigroup. Many researchers worked on Γ-semigroups and its sub structures. Many classical notions of semigroups have been extended to Γ-semigroup by Saha and Sen in [8], [9] and [11]. The notion of bi-ideal in Γ-semigroups was introduced by Chinram and Jirojkul in [1]. Dutta and Adhikari introduced the notion of prime ideal in Γ-semigroup in [3]. On the other hand the concepts of prime bi-ideal, strongly prime bi-ideal, semiprime bi-ideal, strongly irreducible and irreducible bi-ideals of Γ-semigroup are studied in [12]. In this paper we define prime fuzzy bi-ideal, strongly prime fuzzy bi-ideal, semiprime fuzzy bi-ideal, strongly irreducible fuzzy bi-ideal and irreducible fuzzy bi-ideal of a Γ-semigroup and characterize regular and intra-regular Γ-semigroups by the properties of these fuzzy bi-ideals.

2. PRELIMINARIES

Let $S = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. If there exist a mapping $: S \times \Gamma \times S \to S$, written $(x, \alpha, y)$ by $x\alpha y$, then $S$ is called a Γ-semigroup if it satisfies

$(x\alpha y)\beta z = x\alpha (y\beta z)$, for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. 
2.1. Example. [2] Let $S = \{5n + 4 : n$ is a positive integer$\}$ and $\Gamma = \{5n + 1 : n$ is a positive integer$\}$. Then $S$ is a $\Gamma$-semigroup, where $a \circ b = a + \alpha + b$. ($a, b \in S$ and $\alpha, \beta \in \Gamma$, $+$ is the usual addition of integers)

2.2. Example. [7] Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty sets. Then $M$ is a $\Gamma$-semigroup with respect to the operation defined below:

\[
\begin{array}{cccc}
\alpha & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & 0 & 0 & b \\
c & 0 & 0 & 0 & c \\
\end{array}
\quad \begin{array}{cccc}
\beta & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & 0 & b & 0 \\
c & 0 & 0 & 0 & c \\
\end{array}
\quad \begin{array}{cccc}
\gamma & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & a \\
b & 0 & 0 & b & c \\
c & 0 & 0 & 0 & b \\
\end{array}
\]

2.3. Example. [10] Let $S = \{a, b, c, d, e\}$ be the semigroup with the following Cayley table and $\Gamma$ be any non-empty set. Define $S \times \Gamma \times S \rightarrow S$ by $x_\gamma y = x \cdot y$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semigroup.

\[
\begin{array}{cccc}
\cdot & a & b & c & d & e \\
a & a & a & a & a & a \\
b & a & b & c & d & e \\
c & a & b & c & d & e \\
d & a & d & e & b & c \\
e & a & c & d & e & b \\
\end{array}
\]

2.4. Example. [10] Let $S = \{0, a, b\}$ and $\Gamma = \{\gamma\}$. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $x_\gamma y = x$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semigroup.

2.5. Example. [10] Let $S$ and $\Gamma$ be any two non-empty sets and $0 \in S$. Define $S \times \Gamma \times S \rightarrow S$ by $x_\gamma y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a $\Gamma$-semigroup.

An element $0$ of a $\Gamma$-semigroup $S$, with at least two elements, is called a zero element of $S$ if $a \gamma 0 = 0 \gamma a = 0$, for all $a \in S$ and $\gamma \in \Gamma$. A $\Gamma$-semigroup $S$ is called a $\Gamma$-semigroup with zero if it contains a zero element. Let $S$ be a $\Gamma$-semigroup. A non-empty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a sub $\Gamma$-semigroup of $S$ if $A \Gamma \subseteq A$, that is $a \gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. An element $x \in S$ is called regular if there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x_\alpha s \beta x$. If every element of $S$ is regular, then $S$ is said to be regular. An element $x \in S$ is called intra-regular if there exist $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = a \alpha x \beta x \gamma b$. If every element of $S$ is intra-regular, then $S$ is said to be intra-regular.

A left (right) ideal of a $\Gamma$-semigroup $S$ is a non-empty subset $A$ of $S$ such that $A \Gamma \subseteq A$ ($A \Gamma S \subseteq A$). If a non-empty subset of $S$ is a left and a right ideal of $S$, then it is called a two-sided ideal or an ideal of $S$. An ideal $A$ of a $\Gamma$-semigroup $S$ is said to be idempotent if $A \Gamma A = A$. A sub $\Gamma$-semigroup $B$ of a $\Gamma$-semigroup $S$ is called a bi-ideal of $S$ if $B \Gamma S \Gamma B \subseteq B$. 

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A bi-ideal \( B \) of a \( \Gamma \)-semigroup \( S \) is called a prime bi-ideal of \( S \) if \( B_1 \Gamma B_2 \subseteq B \) implies \( B_1 \subseteq B \) or \( B_2 \subseteq B \) for any bi-ideals \( B_1, B_2 \) of \( S \). A bi-ideal \( B \) of a \( \Gamma \)-semigroup \( S \) is called a strongly prime bi-ideal of \( S \) if \( B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq B \) implies \( B_1 \subseteq B \) or \( B_2 \subseteq B \) for any bi-ideals \( B_1, B_2 \) of \( S \). A bi-ideal \( B \) of a \( \Gamma \)-semigroup \( S \) is called a semiprime bi-ideal of \( S \) if \( B_1 \Gamma B_1 \subseteq B \) implies \( B_1 \subseteq B \) for any bi-ideal \( B_1 \) of \( S \).

A bi-ideal \( B \) of a \( \Gamma \)-semigroup \( S \) is said to be irreducible bi-ideal of \( S \) if \( B_1 \cap B_2 = B \) implies either \( B_1 = B \) or \( B_2 = B \) for any bi-ideals \( B_1, B_2 \) of \( S \). A bi-ideal \( B \) of a \( \Gamma \)-semigroup \( S \) is said to be strongly irreducible bi-ideal of \( S \) if \( B_1 \cap B_2 \subseteq B \) implies either \( B_1 \subseteq B \) or \( B_2 \subseteq B \) for any bi-ideals \( B_1, B_2 \) of \( S \).

2.6. Example. [10] Let \( S = \{0, 1, 2, 3, 4, 5\} \) be a semigroup as given in the following Cayley table. Let \( \Gamma = \{\gamma\} \). Define

\[
S \times \Gamma \times S \rightarrow S \text{ by } x\gamma y = x \cdot y \text{ for all } x, y \in S \text{ and } \gamma \in \Gamma.
\]

Then \( S \) is a \( \Gamma \)-semigroup.

Bi-ideals of \( S \) are

\[
\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\},
\{0, 1, 2, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 3\}, \{0, 1, 4, 5\} \text{ and } S.
\]

Irreducible bi-ideals of \( S \) are

\[
\{0\}, \{0, 1, 2, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 3\}, \{0, 1, 4, 5\} \text{ and } S.
\]

Strongly irreducible bi-ideals are \( \{0\} \) and \( S \). Other irreducible bi-ideals are not strongly irreducible.

A function \( f \) from a non-empty set \( S \) to the unit closed interval \([0, 1]\) of real numbers is called a fuzzy subset of \( S \), that is \( f : S \rightarrow [0, 1] \). A fuzzy subset \( f : S \rightarrow [0, 1] \) is non-empty if \( f \) is not the constant map which assumes the values 0. For fuzzy subsets \( f \) and \( g \) of \( S \), \( f \leq g \) means that for all \( a \in S \), \( f(a) \leq g(a) \). Let \( f, g \) be two fuzzy subsets of \( S \). Then \( f \cup g : S \rightarrow [0, 1] \) is defined by \((f \cup g)(x) = f(x) \vee g(x)\) and \( f \cap g : S \rightarrow [0, 1] \) by \((f \cap g)(x) = f(x) \land g(x)\) for all \( x \in S \). Let \( \{f_i\}_{i \in I} \) be a family of fuzzy subsets of \( S \). Define the fuzzy subsets \( \bigcap_{i \in I} f_i \) and \( \bigcup_{i \in I} f_i \) of \( S \) as follows \((\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} f_i(x)\) and \((\bigcup_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)\) for all \( x \in S \). If \( f \) and \( g \) are fuzzy subsets of a \( \Gamma \)-semigroup \( S \) and \( x \in S \), then their product \( f \Gamma g \) is a fuzzy subset of \( S \) defined as

\[
(f \Gamma g)(x) = \begin{cases} [f(a) \land g(a)] & \text{if } x \text{ is expressible as } x = aob \\ 0 & \text{otherwise} \end{cases}
\]

Let \( f \) be a fuzzy subset of \( X \) and \( t \in [0, 1] \). Define \( f_t = \{x \in X : f(x) \geq t\} \). We call \( f_t \) a \( t \)-cut or a level set. A fuzzy subset \( f \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy sub \( \Gamma \)-semigroup of \( S \) if \( f(\gamma y) \geq f(x) \land f(y) \) for all \( x, y \in S \) and \( \gamma \in \Gamma \). A fuzzy
subset \( f \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy left (right) ideal of \( S \) if \( f(x\gamma y) \geq f(y) \) \((f(x\gamma y) \geq f(x))\) for all \( x, y \in S \) and \( \gamma \in \Gamma \). A fuzzy subset \( f \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy ideal of \( S \) if it is both a fuzzy left ideal and a fuzzy right ideal of \( S \).

2.7. **Lemma.** For any non-empty subsets \( X \) and \( Y \) of a \( \Gamma \)-semigroup \( S \), we have

1. \( f_X \land f_Y = f_{X \cap Y} \)
2. \( f_X \lor f_Y = f_{X \cup Y} \)
3. \( f_X \Gamma f_Y = f_{XY} \)

**Proof.** (1) Straightforward.

(2) Straightforward.

(3) Let \( a \in S \). If \( a \in X \Gamma Y \), then \( a = x\gamma y \) for some \( x \in X \), \( y \in Y \) and \( \gamma \in \Gamma \).

Now \( f_X \Gamma f_Y(a) = \bigvee \{ f_X(u) \land f_Y(v) \} \geq f_X(x) \land f_Y(y) = 1 \land 1 = 1 = f_{XY}(a) \).

This implies \( f_X \Gamma f_Y(a) = f_{XY}(a) \). If \( a \notin X \Gamma Y \), then \( a \neq x\gamma y \) for all \( x \in X \), \( y \in Y \) and \( \gamma \in \Gamma \). If \( a = u\alpha v \) for some \( u, v \in S \), \( \alpha \in \Gamma \), then we have \( f_X \Gamma f_Y(a) = \bigvee \{ f_X(u) \land f_Y(v) \} \neq 0 = f_{XY}(a) \). If \( a \neq u\alpha v \) for all \( u, v \in S \), \( \alpha \in \Gamma \), then we have \( f_X \Gamma f_Y(a) = 0 = f_{XY}(a) \). In any case, we have \( f_X \Gamma f_Y = f_{XY} \). \( \square \)

### 3. Fuzzy Ideals and Fuzzy Bi-ideals of \( \Gamma \)-semigroups

3.1. **Definition.** A fuzzy subset \( f \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy bi-ideal of \( S \) if

1. \( f(x\gamma y) \geq f(x) \land f(y) \) for all \( x, y \in S \) and \( \gamma \in \Gamma \)
2. \( f(x\alpha y\beta z) \geq f(x) \land f(z) \) for all \( x, y, z \in S \) and \( \alpha, \beta \in \Gamma \).

3.2. **Lemma.** Let \( A \) be a non-empty subset of a \( \Gamma \)-semigroup \( S \). Then \( A \) is a sub \( \Gamma \)-semigroup of \( S \) if and only if \( f_A \) is a fuzzy sub \( \Gamma \)-semigroup of \( S \).

**Proof.** Suppose \( A \) is a sub \( \Gamma \)-semigroup of \( S \). Then \( A \Gamma A \subseteq A \). Let \( x, y \in S \) and \( \gamma \in \Gamma \).

Case 1: If \( x \notin A \) or \( y \notin A \), then \( f_A(x) = 0 \) or \( f_A(y) = 0 \). Hence \( f_A(x) \land f_A(y) = 0 \leq f_A(x\gamma y) \).

Case 2: If \( x \in A \) and \( y \in A \), then \( x\gamma y \in A \Gamma A \subseteq A \). Thus \( f_A(x) = 1 \), \( f_A(y) = 1 \) and \( f_A(x\gamma y) = 1 \). Hence \( f_A(x\gamma y) = f_A(x) \land f_A(y) \). Thus in any case \( f_A(x\gamma y) \geq f_A(x) \land f_A(y) \).

Conversely, suppose that \( f_A \) is a fuzzy sub \( \Gamma \)-semigroup of \( S \). Let \( x, y \in A \) and \( \gamma \in \Gamma \). As \( f_A(x\gamma y) \geq f_A(x) \land f_A(y) = 1 \). Thus \( f_A(x\gamma y) = 1 \). This implies \( x\gamma y \in A \). Hence \( A \Gamma A \subseteq A \), that is \( A \) is a sub \( \Gamma \)-semigroup of \( S \). \( \square \)

3.3. **Lemma.** Let \( A \) be a non-empty subset of a \( \Gamma \)-semigroup \( S \). Then \( A \) is a left(right, two sided, bi) ideal of \( S \) if and only if \( f_A \) is a fuzzy left (right, two sided, bi) ideal of \( S \).

**Proof.** The proof is similar to the proof of Lemma 3.2. \( \square \)
3.4. **Lemma.** The product of two fuzzy bi-ideals of a $\Gamma$-semigroup $S$ is a fuzzy bi-ideal of $S$.

*Proof.* Let $f$ and $g$ be two fuzzy bi-ideals of a $\Gamma$-semigroup $S$. Let $x, y \in S$ and $\alpha \in \Gamma$.

Then

\[
(f \Gamma g)(x) \land (f \Gamma g)(y) = \left( \bigvee_{x=\alpha \beta b} \{f(a) \land g(b)\} \right) \land \left( \bigvee_{y=\gamma \delta d} \{f(c) \land g(d)\} \right)
\]

\[
= \bigvee_{x=\alpha \beta b \ y=\gamma \delta d} \{f(a) \land g(b) \land f(c) \land g(d)\}
\]

\[
\leq \bigvee_{x=\alpha \beta b \ y=\gamma \delta d} \{f(a) \land g(b) \land g(d)\}
\]

\[
\leq \bigvee_{x=\alpha \beta b \ y=\gamma \delta d} \{f(a) \land g(b \alpha \gamma \delta d)\}
\]

\[
\leq \bigvee_{x=y=\alpha \beta b \ y=\gamma \delta d} \{f(x) \land g(z)\}
\]

\[
= (f \Gamma g)(x \alpha y).
\]

This shows that $f \Gamma g$ is a fuzzy sub $\Gamma$-semigroup of $S$. Now, let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then

\[
(f \Gamma g)(x) \land (f \Gamma g)(z) = \left( \bigvee_{x=\alpha \gamma b} \{f(a) \land g(b)\} \right) \land \left( \bigvee_{z=\delta \beta t} \{f(s) \land g(t)\} \right)
\]

\[
= \bigvee_{x=\alpha \gamma b \ z=\delta \beta t} \{f(a) \land g(b) \land f(s) \land g(t)\}
\]

\[
\leq \bigvee_{x=\alpha \gamma b \ z=\delta \beta t} \{f(a) \land g(t)\}
\]

\[
\leq \bigvee_{x=y=\alpha \beta b \ z=\delta \beta t} \{(f(e) \land g(i))\}
\]

\[
= (f \Gamma g)(x \alpha y \beta z).
\]

Thus $f \Gamma g$ is a fuzzy bi-ideal of $S$. \qed

3.5. **Lemma.** A fuzzy subset $f$ of a $\Gamma$-semigroup $S$ is a fuzzy bi-ideal of $S$ if and only if $f \Gamma f \leq f$ and $f \Gamma f \leq f$.

*Proof.* Assume that $f$ is a fuzzy bi-ideal of a $\Gamma$-semigroup $S$. If $f \Gamma f(a) = 0$, then $f \Gamma f(a) \leq f(a)$. Otherwise there exist elements $x, y \in S$ and $\alpha \in \Gamma$ such that
\(a = x\alpha y\). Since \(f\) is a fuzzy bi-ideal of \(S\), we have \(f(x\alpha y) \geq f(x) \land f(y)\). Therefore

\[
f\Gamma f(a) = \bigvee_{a = x\alpha y} \{f(x) \land f(y)\}
\leq \bigvee_{a = x\alpha y} \{f(x\alpha y)\}
= f(a).
\]

This implies that \(f \Gamma f \leq f\). Similarly, if \(f \Gamma f S \Gamma f(a) = 0\), then \(f \Gamma f S \Gamma f(a) \leq f(a)\).

Conversely, assume that \(f \Gamma f \leq f\) and \(f \Gamma f S \Gamma f \leq f\). Let \(a, b\) be any elements of \(S\) and \(\alpha \in \Gamma\). Then we have

\[
f(a\alpha b) \geq (f \Gamma f)(a\alpha b)
= \bigvee_{a\alpha b = a'\alpha b'} \{(f(a') \land f(b'))\}
\geq f(a) \land f(b).
\]
Let \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \). Then we have

\[
\begin{align*}
f(a\alpha b\beta c) & \geq (f \Gamma S \Gamma f)(a\alpha b\beta c) \\
& = \bigvee_{a\alpha b\beta c = a'\alpha' b'} \{ f \Gamma S(a') \wedge f(b') \} \\
& \geq \{ f \Gamma S(a\alpha) \wedge f(c) \} \\
& = \left( \bigvee_{a\alpha b = x_1 \alpha_1 y_1} \{ (f(x_1) \wedge f_S(y_1)) \} \right) \wedge f(c) \\
& \geq \{ (f(a) \wedge f_S(b)) \} \wedge f(c) \\
& = (f(a) \wedge 1) \wedge f(c) \\
& = f(a) \wedge f(c).
\end{align*}
\]

This implies that \( f(a\alpha b\beta c) \geq f(a) \wedge f(c) \). Thus \( f \) is a fuzzy bi-ideal of \( S \). \( \square \)

3.6. **Lemma.** The intersection of any collection of fuzzy bi-ideals of a \( \Gamma \)-semigroup \( S \) is a fuzzy bi-ideal of \( S \).

**Proof.** Let \( \{ f_i \} \) be a family of fuzzy bi-ideals of \( S \). Let \( x, y, z \in S \) and \( \alpha, \beta, \gamma \in \Gamma \). Then

\[
\begin{align*}
(\bigcap f_i)(x\gamma y) &= \bigwedge_i (f_i(x\gamma y)) \\
& \geq \bigwedge_i (f_i(x) \wedge f_i(y)) \\
& = \left( \bigwedge_i f_i(x) \right) \wedge \left( \bigwedge_i f_i(y) \right) \\
& = \left( \bigwedge_i f_i \right)(x) \wedge \left( \bigwedge_i f_i \right)(y).
\end{align*}
\]

This implies \( (\bigcap f_i)(x\gamma y) \geq (\bigcap f_i)(x) \wedge (\bigcap f_i)(y) \). Now

\[
\begin{align*}
(\bigcap f_i)(x\alpha y\beta z) &= \bigwedge_i (f_i(x\alpha y\beta z)) \\
& \geq \bigwedge_i (f_i(x) \wedge f_i(z)) \\
& = \left( \bigwedge_i f_i(x) \right) \wedge \left( \bigwedge_i f_i(z) \right) \\
& = \left( \bigwedge_i f_i \right)(x) \wedge \left( \bigwedge_i f_i \right)(z).
\end{align*}
\]

Thus \( \left( \bigwedge f_i \right)(x\alpha y\beta z) \geq \left( \bigwedge f_i \right)(x) \wedge \left( \bigwedge f_i \right)(z) \). This implies that \( \bigwedge f_i \) is a fuzzy bi-ideal of \( S \). \( \square \)
4. Prime, Strongly Prime and Semiprime Fuzzy Bi-ideals of Γ-semigroups

4.1. Definition. A fuzzy bi-ideal $f$ of a Γ-semigroup $S$ is called a prime fuzzy bi-ideal of $S$ if $f_1 \Gamma f_2 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ for any fuzzy bi-ideals $f_1, f_2$ of $S$.

4.2. Definition. A fuzzy bi-ideal $f$ of a Γ-semigroup $S$ is called a strongly prime fuzzy bi-ideal of $S$ if $f_1 \Gamma f_2 \land f_2 \Gamma f_1 \leq f$ implies $f_1 \leq f$ or $f_2 \leq f$ for any fuzzy bi-ideals $f_1, f_2$ of $S$.

4.3. Definition. A fuzzy bi-ideal $f$ of a Γ-semigroup $S$ is called a semiprime fuzzy bi-ideal of $S$ if $f_1 \Gamma f_1 \leq f$ implies $f_1 \leq f$ for any fuzzy bi-ideal $f_1$ of $S$.

4.4. Proposition. Every strongly prime fuzzy bi-ideal of a Γ-semigroup $S$ is a prime fuzzy bi-ideal of $S$.

Proof. Straightforward.

4.5. Proposition. Every prime fuzzy bi-ideal of a Γ-semigroup $S$ is a semiprime fuzzy bi-ideal of $S$.

Proof. Straightforward.

4.6. Lemma. The intersection of any family of prime fuzzy bi-ideals of a Γ-semigroup $S$ is a semiprime fuzzy bi-ideal of $S$.

Proof. Straightforward.

5. Irreducible and Strongly Irreducible Fuzzy Bi-ideals of Γ-semigroups

5.1. Definition. A fuzzy bi-ideal $f$ of a Γ-semigroup $S$ is called an irreducible fuzzy bi-ideal of $S$ if $f_1 \land f_2 = f$ implies either $f_1 = f$ or $f_2 = f$, for any fuzzy bi-ideals $f_1, f_2$ of $S$.

5.2. Definition. A fuzzy bi-ideal $f$ of a Γ-semigroup $S$ is called a strongly irreducible fuzzy bi-ideal of $S$ if $f_1 \land f_2 \leq f$ implies either $f_1 \leq f$ or $f_2 \leq f$, for any fuzzy bi-ideals $f_1, f_2$ of $S$.

5.3. Proposition. Every strongly irreducible semiprime fuzzy bi-ideal of a Γ-semigroup $S$ is a strongly prime fuzzy bi-ideal of $S$.

Proof. Let $f$ be a strongly irreducible semiprime fuzzy bi-ideal of $S$. Let $f_1, f_2$ be any two fuzzy bi-ideals of $S$ such that $f_1 \Gamma f_2 \land f_2 \Gamma f_1 \leq f$. Since $f_1 \land f_2 \leq f_1$ and $f_1 \land f_2 \leq f_2$, we have $(f_1 \land f_2) \Gamma (f_1 \land f_2) \leq f_1 \Gamma f_2$ and $(f_1 \land f_2) \Gamma (f_1 \land f_2) \leq f_2 \Gamma f_1$. Thus $(f_1 \land f_2) \Gamma (f_1 \land f_2) \leq f_1 \Gamma f_2 \land f_2 \Gamma f_1 \leq f$. This implies $f_1 \land f_2 \leq f$, because $f$ is a semiprime fuzzy bi-ideal of $S$. Since $f$ is a strongly irreducible fuzzy bi-ideal of $S$, we have $f_1 \leq f$ or $f_2 \leq f$. Hence $f$ is a strongly prime fuzzy bi-ideal of $S$. □
Let \( g = g \) with respect to the property \( S \). Hence by Zorn’s Lemma, there exists a fuzzy bi-ideal \( S \) such that \( f \leq g \) and \( g(a) = \alpha \).

**Proof.** Let \( X = \{ h : h \) is a fuzzy bi-ideal of \( S, h(a) = \alpha \) and \( f \leq h \} \). Then \( X \neq \emptyset \), because \( f \in X \). The collection \( X \) is a partially ordered set under inclusion. If \( Y = \{ h_i : h_i \) is a fuzzy bi-ideal of \( S, h_i(a) = \alpha \) and \( f \leq h_i \) for all \( i \in I \} \) is any totally ordered subset of \( X \), then \( \bigvee_{i \in I} h_i \) is a fuzzy bi-ideal of \( S \) such that \( f \leq \bigvee_{i \in I} h_i \). Indeed, if \( a, b, c \in S \) and \( \alpha, \beta, \gamma \in \Gamma \), then

\[
\left( \bigvee_{i \in I} h_i \right) (a) \leq \left( \bigvee_{i \in I} h_i \right) (b) \leq \left( \bigvee_{i \in I} h_i \right) (c). 
\]

This implies \( \left( \bigvee_{i \in I} h_i \right) (a) \leq \left( \bigvee_{i \in I} h_i \right) (b) \). Now

\[
\left( \bigvee_{i \in I} h_i \right) (a \circ b) \leq \left( \bigvee_{i \in I} h_i \right) (a \circ b \circ c). 
\]

Hence \( \bigvee_{i \in I} h_i \) is a fuzzy bi-ideal of \( S \). As \( f \leq h_i \) for all \( i \in I \), we have \( f \leq \bigvee_{i \in I} h_i \). Also

\[
\left( \bigvee_{i \in I} h_i \right) (a) = \bigvee_{i \in I} h_i (a) = \alpha. 
\]

Hence by Zorn’s Lemma, there exists a fuzzy bi-ideal \( g \) of \( S \) which is maximal with respect to the property \( f \leq g \) and \( g(a) = \alpha \). Now, let \( g_1, g_2 \) be any fuzzy bi-ideals of \( S \) such that \( g_1 \wedge g_2 = g \). This implies \( g \leq g_1 \) and \( g \leq g_2 \). We claim that \( g = g_1 \) or \( g = g_2 \). On the contrary, suppose that \( g \neq g_1 \) and \( g \neq g_2 \). This implies \( g < g_1 \) and \( g < g_2 \). So \( g_1 \neq \alpha \) and \( g_2 \neq \alpha \). Hence \( g_1 \wedge g_2 = g \neq \alpha \). Which is a contradiction to the fact that \( g_1 \wedge g_2 = g \). Hence either \( g = g_1 \) or \( g = g_2 \). Thus \( g \) is an irreducible fuzzy bi-ideal of \( S \).

The following Theorems are taken from [12].
5.5. **Theorem.** For a \( \Gamma \)-semigroup \( S \) the following conditions are equivalent:

1. \( S \) is intra-regular.
2. \( L \cap R \subseteq \Lambda \Gamma R \), for every left ideal \( L \) and every right ideal \( R \) of \( S \).

5.6. **Theorem.** For a \( \Gamma \)-semigroup \( S \), the following assertions are equivalent:

1. \( S \) is both regular and intra-regular.
2. \( B \cap \Gamma B = B \) for every bi-ideal \( B \) of \( S \).
3. \( B_1 \cap B_2 = B_1 \cap B_2 \cap B_2 \cap \Gamma B_1 \) for all bi-ideals \( B_1 \) and \( B_2 \) of \( S \).
4. Each bi-ideal of \( S \) is semiprime.
5. Each proper bi-ideal of \( S \) is the intersection of irreducible semiprime bi-ideals of \( S \) which contains it.

5.7. **Theorem.** For a \( \Gamma \)-semigroup \( S \) the following conditions are equivalent:

1. \( S \) is intra-regular.
2. \( f \wedge g \leq g \Gamma f \), for every fuzzy right ideal \( f \) and every fuzzy left ideal \( g \) of \( S \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose \( S \) is an intra-regular \( \Gamma \)-semigroup and \( x \in S \). Then there exist \( a, b \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( x = a \alpha x \beta a \). Let \( f \) be a fuzzy right ideal and \( g \) be a fuzzy left ideal of \( S \). Then we have

\[
(g \Gamma f)(x) = \bigvee_{x = u \gamma v} \{g(u) \wedge f(v)\} \\
\geq \{g(a \alpha x) \wedge f(x \gamma b)\} \\
\geq g(x) \wedge f(x).
\]

This implies \( f \wedge g \leq g \Gamma f \).

(2) \( \Rightarrow \) (1) Let \( R \) be a right ideal and \( L \) be a left ideal of \( S \). Then by Lemma 3.3, the characteristic function \( f_R \) and \( f_L \) are the fuzzy right and left ideals of \( S \), respectively. Thus by hypothesis \( f_R \wedge f_L \leq f_L \Gamma f_R \). Then by Lemma 2.7, \( f_R \cap L \leq f_L \Gamma R \). Hence by Theorem 5.5, \( S \) is intra-regular.

5.8. **Theorem.** For a \( \Gamma \)-semigroup \( S \), the following assertions are equivalent:

1. \( S \) is both regular and intra-regular.
2. \( f \Gamma f = f \) for every fuzzy bi-ideal \( f \) of \( S \).
3. \( g \wedge h = g \Gamma h \wedge h \Gamma g \) for all fuzzy bi-ideals \( g \) and \( h \) of \( S \).
4. Each fuzzy bi-ideal of \( S \) is semiprime.
5. Each proper fuzzy bi-ideal of \( S \) is the intersection of irreducible semiprime fuzzy bi-ideals of \( S \) which contains it.

**Proof.** (1) \( \Rightarrow \) (2) Let \( f \) be a fuzzy bi-ideal of \( S \). Then by Lemma 3.3, \( f \Gamma f \leq f \). Now, let \( a \in S \). As \( S \) is both regular and intra-regular, there exist elements \( x, y, z \in S \) and \( \alpha, \beta, \delta, \theta, \xi \in \Gamma \) such that \( a = a \alpha x \beta \) and \( a = y \delta \theta a \xi z \). Then \( a = a \alpha x \beta a = a \alpha x \beta (a \alpha x \beta a) = a \alpha x \beta (y \delta \theta a \xi z) \alpha x \beta a = (a \alpha x \beta y \delta \theta a) \theta (a \xi z a \alpha x \beta a) \). Thus we have

\[
(f \Gamma f)(a) = \bigvee_{a = x \gamma y} [(f(x) \wedge f(y))] \\
\geq \{f(a \alpha x \beta y \delta \theta a) \wedge f(a \xi z a \alpha x \beta a)\} \\
\geq f(a) \wedge f(a) \\
= f(a)
\]

for all \( a \in S \).
This implies \( f \Gamma f \geq f \). Hence \( f \Gamma f = f \).

(2) \( \Rightarrow \) (1) Let \( B \) be any bi-ideal of \( S \). Then \( f_B \), the characteristic function of \( B \), is a fuzzy bi-ideal of \( S \). Thus by hypothesis,

\[
f_B \Gamma f_B = f_B
\]

\[
\Rightarrow f_B \Gamma B = f_B
\]

\[
\Rightarrow B \Gamma B = B.
\]

Then by Theorem 5.6, \( S \) is both regular and intra-regular.

(2) \( \Rightarrow \) (3) Let \( g \) and \( h \) be two fuzzy bi-ideal of \( S \). Then by Lemma 3.6 \( g \wedge h \) is also a fuzzy bi-ideal of \( S \). Thus by hypothesis, we have \( g \wedge h = (g \wedge h) \Gamma (g \wedge h) \leq g \Gamma h \). Similarly \( g \wedge h \leq h \Gamma g \). This implies 
\[
g \wedge h \leq g \Gamma h \wedge h \Gamma g
\]

(1)

Now \( g \Gamma h \) and \( h \Gamma g \), being the product of two fuzzy bi-ideals of \( S \), are fuzzy bi-ideals of \( S \) and so \( g \Gamma h \wedge h \Gamma g \) is a fuzzy bi-ideal of \( S \). Thus by hypothesis, we have

\[
g \Gamma h \wedge h \Gamma g = (g \Gamma h \wedge h \Gamma g) \Gamma (g \Gamma h \wedge h \Gamma g)
\]

\[
\leq (g \Gamma h) \Gamma (h \Gamma g)
\]

\[
= g \Gamma h \Gamma g
\]

as \( h \Gamma h = h \) (by hypothesis)

\[
\leq g \Gamma f_S \Gamma g
\]

\[
\leq g
\]

as \( g \) is a fuzzy bi-ideal of \( S \).

Similarly \( g \Gamma h \wedge h \Gamma g \leq h \). Thus

\[
g \Gamma h \wedge h \Gamma g \leq g \wedge h
\]

(II)

From (I) and (II) \( g \Gamma h \wedge h \Gamma g = g \wedge h \).

(3) \( \Rightarrow \) (4) Let \( g \) be a fuzzy bi-ideal of \( S \) such that \( f \Gamma f \leq g \) for any fuzzy bi-ideal \( f \) of \( S \). Then by hypothesis, \( f = f \wedge f = f \Gamma f \wedge f \Gamma f = f \Gamma f \leq g \). This implies \( f \leq g \). Thus \( g \) is a semiprime fuzzy bi-ideal of \( S \). Hence every fuzzy bi-ideal of \( S \) is semiprime.

(4) \( \Rightarrow \) (5) Let \( f \) be a proper fuzzy bi-ideal of \( S \) and \( \{ f_i : i \in I \} \) be the collection of all irreducible fuzzy bi-ideals of \( S \) such that \( f \leq f_i \) for all \( i \in I \). This implies \( f \leq \bigwedge_{i \in I} f_i \). Let \( a \in S \) and \( \alpha \in [0,1] \) be such that \( f(a) = \alpha \). Then by Theorem 5.4

there exists an irreducible fuzzy bi-ideal \( f_\alpha \) of \( S \) such that \( f \leq f_\alpha \) and \( f(a) = f_\alpha (a) \). This implies \( f_\alpha \in \{ f_i : i \in I \} \). Thus \( \bigwedge_{i \in I} f_i \leq f_\alpha \), so \( \bigwedge_{i \in I} f(a) \leq f_\alpha (a) = f(a) \) for all \( a \in S \). This implies \( \bigwedge_{i \in I} f_i \leq f \). Hence \( \bigwedge_{i \in I} f_i = f \). By hypothesis, each fuzzy bi-ideal of \( S \) is semiprime, thus each fuzzy bi-ideal of \( S \) is the intersection of all irreducible semiprime fuzzy bi-ideals of \( S \) which contain it.

(5) \( \Rightarrow \) (2) Let \( f \) be a fuzzy bi-ideal of \( S \). Then by Lemma 3.5, we have \( f \Gamma f \leq f \).

Also \( f \Gamma f \) being the product of two fuzzy bi-ideals of \( S \), is a fuzzy bi-ideal of \( S \). Thus by hypothesis \( f \Gamma f = \bigwedge_{i \in I} f_i \), where each \( f_i \) is an irreducible semiprime fuzzy bi-ideal of \( S \) such that \( f \Gamma f \leq f_i \) for all \( i \in I \). This implies \( f \leq f_i \) for all \( i \in I \), because each \( f_i \) is a semiprime fuzzy bi-ideal of \( S \). Thus \( f \leq \bigwedge_{i \in I} f_i = f \Gamma f \). Hence \( f \Gamma f = f \).

\[ \square \]
5.9. Theorem. Let $S$ be a regular and intra-regular $\Gamma$-semigroup. Then the following assertions for a fuzzy bi-ideal $f$ of $S$ are equivalent:

(1) $f$ is strongly irreducible.

(2) $f$ is strongly prime.

Proof. (1) $\Rightarrow$ (2) Let $S$ be a regular and intra-regular $\Gamma$-semigroup and $f$ be a strongly irreducible fuzzy bi-ideal of $S$. Suppose that $g$ and $h$ are two fuzzy bi-ideals of $S$ such that $g\Gamma h \leq h\Gamma f$. As $S$ is both regular and intra-regular $\Gamma$-semigroup, we have by Theorem 5.8 $g\wedge h = g\Gamma h \wedge h\Gamma f \leq f$. This implies either $g \leq f$ or $h \leq f$, because $f$ is strongly irreducible. Thus $f$ is strongly prime fuzzy bi-ideal of $S$.

(2) $\Rightarrow$ (1) Suppose $f$ is strongly prime fuzzy bi-ideal of $S$. Let $g, h$ be any two fuzzy bi-ideals of $S$ such that $g \wedge h \leq f$. Since $S$ is both regular and intra-regular, we have by Theorem 5.8 $g\Gamma h \wedge h\Gamma g = g \wedge h \leq f$. This implies either $g \leq f$ or $h \leq f$, because $f$ is strongly prime fuzzy bi-ideal of $S$. Thus $f$ is strongly irreducible. □

5.10. Theorem. Each fuzzy bi-ideal of a $\Gamma$-semigroup $S$ is strongly prime if and only if $S$ is regular and intra-regular and the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion.

Proof. Suppose that each fuzzy bi-ideal of a $\Gamma$-semigroup $S$ is strongly prime. Then each fuzzy bi-ideal of $S$ is semiprime. Thus by Theorem 5.8 $S$ is both regular and intra-regular. Now, let $g, h$ be any two fuzzy bi-ideals of $S$. Then by Theorem 5.8 $g\Gamma h \leq h\Gamma g = g \wedge h$. As each fuzzy bi-ideal of $S$ is strongly prime, so is $g \wedge h$. Thus either $g \leq g \wedge h$ or $h \leq g \wedge h$. Now, if $g \leq g \wedge h$, then $g \leq h$ and if $h \leq g \wedge h$, then $h \leq g$.

Conversely, assume that $S$ is regular and intra-regular and the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion. Let $f$ be an arbitrary fuzzy bi-ideal of $S$ and $g, h$ be any two fuzzy bi-ideals of $S$ such that $g\Gamma h \wedge h\Gamma g \leq f$. As $S$ is both regular and intra-regular, we have by Theorem 5.8

$$g \wedge h = g\Gamma h \wedge h\Gamma g \leq f \quad \text{(i)}.$$ 

Since the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion, we have either $g \leq h$ or $h \leq g$. This implies either $g \wedge h = g$ or $g \wedge h = h$. Then (i) implies either $g \leq f$ or $h \leq f$. Hence $f$ is strongly prime. □

5.11. Theorem. If the set of fuzzy bi-ideals of a $\Gamma$-semigroup $S$ is totally ordered by inclusion, then $S$ is both regular and intra-regular if and only if each fuzzy bi-ideal of $S$ is prime.

Proof. Suppose $S$ is both regular and intra-regular. Let $f$ be an arbitrary fuzzy bi-ideal of $S$ and $g, h$ be any two fuzzy bi-ideals of $S$ such that $g\Gamma h \leq f$. Since the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion, so either $g \leq h$ or $h \leq g$. If $g \leq h$, then $g\Gamma g \leq g\Gamma h \leq f$. This implies $g \leq f$, because $f$ is semiprime by Theorem 5.8. If $h \leq g$, then $h\Gamma h \leq g\Gamma h \leq f$. This implies $h \leq f$, because $f$ is semiprime by Theorem 5.8.

Conversely, suppose that every fuzzy bi-ideal of $S$ is prime. Since every prime fuzzy bi-ideal of $S$ is semiprime, so by Theorem 5.8, $S$ is both regular and intra-regular. □
5.12. Proposition. If the set of fuzzy bi-ideals of a $\Gamma$-semigroup $S$ is totally ordered, then the concept of primeness and strongly primeness coincides.

Proof. Let $f$ be a prime fuzzy bi-ideal of $S$. Let $f_1, f_2$ be any two fuzzy bi-ideals of $S$ such that $f_1 \Gamma f_2 \wedge f_2 \Gamma f_1 \leq f$. As the set of fuzzy bi-ideals of a $\Gamma$-semigroup $S$ is totally ordered, we have either $f_1 \leq f_2$ or $f_2 \leq f_1$. If $f_1 \leq f_2$, then $f_1 \Gamma f_1 = f_1 \Gamma f_1 \wedge f_1 \Gamma f_1 \leq f_1 \Gamma f_2 \wedge f_2 \Gamma f_1 \leq f$. This implies $f_1 \leq f$, as $f$ is a prime fuzzy bi-ideal of $S$. Similarly, if $f_2 \leq f_1$, then $f_2 \Gamma f_2 = f_2 \Gamma f_2 \wedge f_2 \Gamma f_2 \leq f_1 \Gamma f_2 \wedge f_2 \Gamma f_1 \leq f$. This implies $f_2 \leq f$, as $f$ is a prime fuzzy bi-ideal of $S$. This shows that $f$ is a strongly prime fuzzy bi-ideal of $S$. Thus every prime fuzzy bi-ideal of $S$ is a strongly prime fuzzy bi-ideal of $S$. Also by Proposition 4.3 every strongly prime fuzzy bi-ideal of $S$ is a prime fuzzy bi-ideal of $S$. \hfill $\Box$

5.13. Theorem. For a $\Gamma$-semigroup $S$ the following assertions are equivalent:
1. The set of all fuzzy bi-ideals of $S$ is totally ordered under inclusion.
2. Each fuzzy bi-ideal of $S$ is strongly irreducible.
3. Each fuzzy bi-ideal of $S$ is irreducible.

Proof. (1) $\Rightarrow$ (2) Let the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion. Assume that $f$ is an arbitrary fuzzy bi-ideal of $S$ and $f_1, f_2$ be any two fuzzy bi-ideals of $S$ such that $f_1 \wedge f_2 \leq f$. Since the set of fuzzy bi-ideals of $S$ is totally ordered by inclusion, we have either $f_1 \leq f_2$ or $f_2 \leq f_1$. This implies that either $f_1 \wedge f_2 = f_1$ or $f_1 \wedge f_2 = f_2$. This implies that either $f_1 \leq f$ or $f_2 \leq f$. Thus $f$ is strongly irreducible. Hence each fuzzy bi-ideal of $S$ is strongly irreducible.

(2) $\Rightarrow$ (3) Suppose each fuzzy bi-ideal of $S$ is strongly irreducible and let $f$ be an arbitrary fuzzy bi-ideal of $S$ and $f_1, f_2$ be any two fuzzy bi-ideals of $S$ such that $f_1 \wedge f_2 = f$. This implies that $f \leq f_1$ and $f \leq f_2$. By hypothesis either $f_1 \leq f$ or $f_2 \leq f$. Thus either $f_1 = f$ or $f_2 = f$. This implies $f$ is an irreducible fuzzy bi-ideal of $S$. Hence each fuzzy bi-ideal of $S$ is irreducible.

(3) $\Rightarrow$ (1) Suppose each fuzzy bi-ideal of $S$ is irreducible. Let $f_1, f_2$ be any two fuzzy bi-ideals of $S$. Then by Lemma 4.6 $f_1 \wedge f_2$ is also a fuzzy bi-ideal of $S$ and so is irreducible fuzzy bi-ideal of $S$. Since $f_1 \wedge f_2 = f_1 \wedge f_2$, thus by hypothesis $f_1 = f_1 \wedge f_2$ or $f_2 = f_1 \wedge f_2$. That is $f_1 \leq f_2$ or $f_2 \leq f_1$. Hence the set of all fuzzy bi-ideals of $S$ is totally ordered by inclusion. \hfill $\Box$

Acknowledgements. The Authors are very thankful to the learned referees/editor in Chief Young Bae Jun for their suggestions to improve the present paper.

References

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