Fuzzy $h$-ideals with operators in $\Gamma$-hemirings

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Abstract. In this paper, we introduce the concept of fuzzy left $h$-ideals with operators in $\Gamma$-hemirings and establish a new fuzzy left $h$-ideal with operators. In particular, we consider the characterizations of $M$-Noetherian $M\Gamma$-hemirings. Finally, we investigate cartesian products of $M$-fuzzy left $h$-ideals in $M\Gamma$-hemirings.

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1. Introduction

Semirings play an important role in studying matrices and determinants. Many researchers studied the theory of matrices and determinants over semirings [11,14]. A special semiring with a zero and endowed with the commutative addition is said to be a hemiring. Although ideals in semirings are useful for ways, they do not in general coincide with the usual ring ideals if $S$ is a ring. Indeed, many results in rings apparently have no analogues in semirings using only ideals. We note that the ideals of semirings play a crucial role in the structure theory. Henriksen [9] defined a more restricted class of ideals in semirings, which is called $k$-ideals, a still more restricted of ideals in hemirings has been given by Iizuka [11]. In 2004, Jun [15] considered the fuzzy $h$-ideals of hemirings. By using the fuzzy $h$-ideals, Zhan et al. described the $h$-hemiregular hemirings [22]. Furthermore, many researchers gave some basic definitions and results related with fuzzy $h$-ideals of hemirings [3,5,20].

The concept of $\Gamma$-ring was first introduced in 1966 by Barnes [1], a concept more general than a ring. After the paper of Barnes, many researchers are engaged in studying some particular $\Gamma$-ring. In 1992, applying the concept of fuzzy sets to the theory of $\Gamma$-ring, Y. B. Jun and C. Y. Lee [13] gave the notion of fuzzy ideals in $\Gamma$-ring and some properties of fuzzy ideals of $\Gamma$-ring. After that, Hong and Jun...
Let $A$ be a fuzzy set and $0 +$ is a $\Gamma$-hemirings, where $\Gamma = S$. Example 2.3 defined the normalized fuzzy ideals and fuzzy maximal ideals in a $\Gamma$-ring and Jun further characterized the fuzzy prime ideals of a $\Gamma$-ring. In particular, Dutta-Chanda studied the fuzzy ideals of a $\Gamma$-ring and characterized the $\Gamma$-fields and Noetherian $\Gamma$-rings by considering the fuzzy ideals via operator rings of $\Gamma$-rings. The concept of $\Gamma$-semiring was introduced by M. K. Rao, these concepts are extended by Dutta and Sardar. And some properties of such $\Gamma$-semiring have been studied, for example, see [8, 17, 19, 23].

Dudek discussed quasigroups and BCC-algebras with operators, respectively. In 2007, Zhan et al. investigated fuzzy $h$-ideals with operators in hemirings. Now, in this paper, we consider this theory to $\Gamma$-hemirings, we introduce the concept of fuzzy left $h$-ideals with operators in $M-\Gamma$-hemirings $S$ and establish a new fuzzy left $h$-ideal with operators. Using the left $M$-$h$-ideals, we establish $M$-fuzzy left $h$-ideals of $S$. Moreover, we introduce the concept of $M$-Noetherian $\Gamma$-$h$-hemirings and cartesian product of $M$-fuzzy left $h$-ideals, we prove that if $\mu$ and $\nu$ are $M$-fuzzy left $h$-ideals of $S$, then $\mu \times \nu$ is an $M$-fuzzy left $h$-ideal of $S \times S$.

2. Preliminaries

Definition 2.1 ([23]). Let $S$ and $\Gamma$ be two additive semigroups. Then $S$ is said to be a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (images to be denoted by $a\alpha b$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$) satisfies the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;
(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;
(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$;
(iv) $\alpha(a\beta c) = (a\alpha b)\beta c$.

By a zero of a $\Gamma$-semiring $S$, we mean an element $0 \in S$ such that $0a = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$ and $\alpha \in \Gamma$. A $\Gamma$-semiring with zero is said to be a $\Gamma$-hemiring.

A left ideal of a $\Gamma$-hemirings $S$ is a subset $A$ of $S$ which is closed under the addition such that $S TA \subseteq A$, where $S TA = \{x\alpha y | x, y \in S, \alpha \in \Gamma\}$.

A left ideal $A$ of $\Gamma$-hemirings $S$ is called a left $h$-ideal of $S$, respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + x$ implies $x \in A$.

Right $h$-ideals are defined similarly.

Definition 2.2 ([23]). A fuzzy set $\mu$ of $\Gamma$-hemirings $S$ is said to be a fuzzy left $h$-ideal of $S$ if it satisfies the following conditions:

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$,
(ii) $\mu(x\alpha y) \geq \mu(y)$ for all $x, y \in S$ and $\alpha \in \Gamma$,
(iii) $x + a + z = b + z$ implies $\mu(x) \geq \min\{\mu(a), \mu(b)\}$ for all $a, b, x, z \in S$.

Fuzzy right $h$-ideals of $S$ are defined similarly.

Example 2.3 ([23]). Let $S$ be a hemiring with the multiplicative identity $1$. Then $S$ is a $\Gamma$-hemirings, where $\Gamma = S$ and $a\alpha b$ denotes the product of elements $a, \alpha, b$ in $S$. Now any fuzzy $h$-ideal of the hemiring $S$ is a fuzzy $h$-ideal of the $\Gamma$-hemiring $S$. 

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3. M-Fuzzy left h-ideals

**Definition 3.1.** A $\Gamma$-hemiring $S$ with operators is an algebraic system consisting of a $\Gamma$-hemiring $S$, a set $M$ and a function defined in the product set $M \times \Gamma \times S$ and having values in $S$ such that if the product $\max$ denotes the elements in $S$ determined by the element $m$ of $M$, $x$ of $S$ and the element $\alpha, \beta$ of $\Gamma$, then

$$\max(x + y) = \max + \max y$$

and

$$\max(x\beta y) = (\max)\beta(\max y)$$

hold for any $x, y \in S$, $m \in M$ and $\alpha, \beta \in \Gamma$. We usually use the phrase “$S$ is an $M$-$\Gamma$-hemiring” instead of a “$\Gamma$-hemiring with operators”.

**Example 3.2.** Let $(S, +)$ be a semigroup, where $S$ is the sets of all non-negative integers and the operation a is the usual additive operation. Let $(\Gamma, +) = \{1\}$. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $aab = a\cdot \alpha \cdot b$ for all $a, b \in S$ and $\alpha \in \Gamma$, where “$\cdot$” is usual multiplication. Then, it can be easily verified that $S$, under the above multiplication and the structure $\Gamma$-mapping, is a $\Gamma$-hemiring. We consider $M = \{0, 1\}$, then $S$ is an $M$-$\Gamma$-hemiring.

**Definition 3.3.** A left $h$-ideal $I$ of an $M$-$\Gamma$-hemiring $S$ is called a left $M$-$h$-ideal of $S$ if $\max \in I$ for all $m \in M$, $x \in I$ and $\alpha \in \Gamma$.

**Definition 3.4.** Let $S$ be an $M$-$\Gamma$-hemiring and $\mu$ a fuzzy $h$-ideal of $S$. If the inequality $\mu(\max) \geq \mu(x)$ holds for any $x \in S$, $m \in M$ and $\alpha \in \Gamma$, then $\mu$ is said to be a fuzzy left $h$-ideal with operators of $S$. We use the phrases “an $M$-fuzzy left $h$-ideal of $S$” instead of “an fuzzy $h$-ideal with operators of $S$”.

**Proposition 3.5.** Let $A$ be a non-empty subset of an $M$-$\Gamma$-hemiring $S$, and $\mu$ a fuzzy set in $S$ defined by

$$\mu(x) = \begin{cases} s, & \text{if } x \in A; \\ t, & \text{otherwise}, \end{cases}$$

for all $x \in S$, where $s > t$ in $[0, 1]$. Then $\mu$ is an $M$-fuzzy left $h$-ideal of $S$ if and only if $A$ is a left $M$-$h$-ideal of $S$.

**Proof.** Suppose that $A$ is a left $M$-$h$-ideal of $S$. we know that $\mu$ is a fuzzy left $h$-ideal of $S$. Let $x \in S$, $m \in M$ and $\alpha \in \Gamma$. If $x \in A$, then $\max \in A$ as $A$ is a left $M$-$h$-ideal of $S$, and so $\mu(\max) = s = \mu(x)$. If $x \notin A$, then $\mu(x) = t \leq \mu(\max)$. Thus $\mu$ is an $M$-fuzzy left $h$-ideal of $S$.

Conversely, if $\mu$ is an $M$-fuzzy left $h$-ideal of $S$, then it is easy to show that $A$ is a left $h$-ideal of $S$. Then, for any $x \in A$, $m \in M$ and $\alpha \in \Gamma$, $\mu(\max) \geq \mu(x) = s$ and so, $\mu(\max) = s$. This shows that $\max \in A$. Consequently, $A$ is a left $M$-$h$-ideal of $S$. \(\square\)

For any subset $A$ of a $\Gamma$-hemiring $S$, $\chi_A$ will denote the characteristic function of $A$.

**Corollary 3.6.** Let $A$ be a non-empty subset of an $M$-$\Gamma$-hemirings $S$. Then $A$ is a left $M$-$h$-ideal of $S$ if and only if $\chi_A$ is an $M$-fuzzy left $h$-ideal of $S$. 185
Proposition 3.7. Let $\mu$ be an $M$-fuzzy left $h$-ideal of an $M$-$\Gamma$-hemiring $S$. For any $m \in M$, $\alpha \in \Gamma$, define a fuzzy set $\mu[\alpha m]$ in $S$ by $\mu[\alpha m](x) = \mu(\alpha mx)$, $\forall x \in S$. Then $\mu[\alpha m]$ is a fuzzy left $h$-ideal of $S$.

Proof. It is obvious.

For any $t \in [0,1]$, the set
$$U(\mu; t) = \{x \in S | \mu(x) \geq t\}$$
is called a level subset of $\mu$.

The following is a simple consequence of the transfer principle for fuzzy sets in [10].

Lemma 3.8 ([23]). A fuzzy set $\mu$ in a $\Gamma$-hemiring $S$ is a fuzzy left $h$-ideal of $S$ if and only if the each non-empty level subset $U(\mu; t), t \in (0,1)$, of $\mu$ is a left $h$-ideal of $S$.

Theorem 3.9. A fuzzy set $\mu$ in an $M$-$\Gamma$-hemiring $S$ is an $M$-fuzzy left $h$-ideal of $S$ if and only if the each non-empty level subset $U(\mu; t)$ of $\mu$ is a left $M$-h-ideal of $S$.

Proof. Let $\mu$ be an $M$-fuzzy left $h$-ideal of $S$, and assume that $U(\mu; t) \neq \emptyset$ for $t \in [0,1]$. Then by Lemma 3.8, $U(\mu; t)$ is a left $h$-ideal of $S$. For every $x \in U(\mu; t)$, $\alpha \in \Gamma$, and $m \in M$, we have
$$\mu(\alpha max) \geq \mu(x) \geq t,$$
and so $\alpha max \in U(\mu; t)$. Hence $U(\mu; t)$ is a left $M$-$h$-ideal of $S$.

Conversely, suppose that $U(\mu; t) \neq \emptyset$ is a left $M$-$h$-ideal of $S$. Then $\mu$ is a fuzzy left $h$-ideal of $S$ by Lemma 3.8. Now assume that there exist $y \in S$, $\gamma \in \Gamma$ and $k \in M$ such that
$$\mu(\gamma y) < \mu(ky).$$

Taking
$$t_0 = \frac{1}{2}(\mu(\gamma y) + \mu(y)),$$
we get $t_0 \in [0,1]$ and
$$\mu(\gamma y) < t_0 < \mu(y)$$
This implies that $\gamma y \notin U(\mu; t_0)$ and $y \in U(\mu; t_0)$, this leads a contradiction. And therefore
$$\mu(\gamma y) \geq \mu(y),$$
for all $y \in S$, $\gamma \in \Gamma$ and $k \in M$. This completes the proof.

Proposition 3.10. Let $\mu$ and $\nu$ be two fuzzy sets in an $M$-$\Gamma$-hemiring $S$. If they are $M$ fuzzy left $h$-ideals of $S$, then so is $\mu \cap \nu$, where $\mu \cap \nu$ is defined by
$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad x \in S.$$

Proof. For $a, b \in S$,
$$(\mu \cap \nu)(a + b) = \min\{\mu(a + b), \nu(a + b)\}$$
$$\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\}$$
$$= \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\}$$
$$= \min\{\min\{\mu \cap \nu\}(a), (\mu \cap \nu)(b)\}.$$
Let $\alpha, \beta \in \Gamma$, since $\mu(ab) \geq \mu(b)$, and $\nu(ab) \geq \mu(b)$, it follows that

\[
(\mu \cap \nu)(ab) = \min\{\mu(ab), \nu(ab)\} \\
\geq \min\{\mu(b), \nu(b)\} = (\mu \cap \nu)(b).
\]

Now, $\mu \cap \nu$ is a fuzzy left ideal of $S$. Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

\[
(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \\
\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\} = \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\} = \min\{\min\{(\mu \cap \nu)(a), (\mu \cap \nu)(b)\}\}.
\]

Therefore $\mu \cap \nu$ is a fuzzy left $h$-ideal of $S$. Let $m \in M$, $\alpha \in \Gamma$ we have

\[
(\mu \cap \nu)(\max) = \min\{\min\{\mu(\max), \nu(\max)\}\} \\
\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x).
\]

Consequently, $\mu \cap \nu$ is an $M$-fuzzy left $h$-ideal of $S$. \hfill $\Box$

**Theorem 3.11** ([23]). Let $\{A_t\}_{t \in \Lambda} \subseteq [0, 1]$ be a collection of $M$-$h$-ideals of an $M$-$\Gamma$-hemiring $S$ such that

(i) $S = \bigcup_{t \in \Lambda} A_t$,

(ii) $t < s$ if and only if $A_t \subset A_s$ for all $t, s \in \Lambda$.

Define a fuzzy set $\mu$ in $S$ by

\[
\mu(x) = \sup\{t \in \Lambda | x \in A_t\}, \quad \forall x \in S.
\]

Then $\mu$ is an $M$-fuzzy left $h$-ideal of $S$.

**Definition 3.12.** An $M$-$\Gamma$-hemiring $S$ is said to satisfy the ascending (descending) chain condition (briefly, $ACC(\text{DCC})$) if for every ascending (descending) sequence $A_1 \subseteq A_2 \subseteq \cdots (A_1 \supseteq A_2 \supseteq \cdots)$ of left $M$-$h$-ideals of $S$ there exists a nature number $n$ such that

\[
A_i = A_n, \quad \forall i \geq n.
\]

**Definition 3.13.** An $M$-$\Gamma$-hemiring $S$ is said to $M$-Noetherian if every left $M$-$h$-ideals of $S$ satisfies $ACC$ for left $M$-$h$-ideals.

**Theorem 3.14.** Let $\{A_n\}_{n \in N}$ be a family of left $M$-$h$-ideals of an $M$-$\Gamma$-hemiring $S$ which is nested, that is, $S = A_1 \supseteq A_2 \supseteq \cdots$. Let $\mu$ be a fuzzy set in $S$ defined by

\[
\mu(x) = \begin{cases} 
\frac{n}{n+1}, & \text{for } x \in A_n/A_{n+1}, \quad n = 1, 2, 3, \cdots; \\
1, & \text{for } x \in \bigcap_{n=1}^{\infty} A_n,
\end{cases}
\]

for all $x \in S$. Then $\mu$ is an $M$-fuzzy left $h$-ideal of $S$.

**Proof.** Suppose that $x \in A_k/A_{k+1}$ and $y \in A_r/A_{r+1}$ for $k = 1, 2, \cdots; r = 1, 2, \cdots$. Without loss of generality, we may assume that $k \leq r$. Then clearly $y \in A_k$, so $x + y \in A_k$. Hence

\[
\mu(x + y) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}.
\]

If $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x + y \in \bigcap_{n=1}^{\infty} A_n$, and clearly that

\[
\mu(x + y) = 1 = \min\{\mu(x), \mu(y)\}.
\]
If \( x \in \bigcap_{n=1}^{\infty} A_n \), and \( y \notin \bigcap_{n=1}^{\infty} A_n \), then there exists \( l \in N \) such that \( y \in A_l/A_{l+1} \), it follows that \( x + y \in A_l \), so that
\[
\mu(x + y) \geq \frac{l}{l+1} = \min\{\mu(x), \mu(y)\}.
\]
Similarly, we know that
\[
\mu(x + y) \geq \frac{l}{l+1} = \min\{\mu(x), \mu(y)\}.
\]
whenever \( x \notin \bigcap_{n=1}^{\infty} A_n \), and \( y \in \bigcap_{n=1}^{\infty} A_n \).
Now if \( y \in A_r/A_{r+1} \) for some \( r = 1, 2, \cdots \), then \( xoy \in A_r \) for all \( x \in S \) and \( \alpha \in \Gamma \). Hence
\[
\mu(xoy) \geq \frac{r}{r+1} = \mu(y).
\]
If \( y \in \bigcap_{n=1}^{\infty} A_n \), then \( xoy \in \bigcap_{n=1}^{\infty} A_n \) for all \( x \in S \) and \( \alpha \in \Gamma \). So
\[
\mu(xoy) = 1 = \mu(y).
\]
Let \( a, b, x, z \in S \) be such that \( x + a + z = b + z \). If \( a, b \in A_r/A_{r+1} \) for some \( r = 1, 2, \cdots \), then \( x \in A_r \) as \( A_r \) is a left \( M\)-h-ideal of \( S \). Thus
\[
\mu(x) \geq \frac{r}{r+1} = \min\{\mu(a), \mu(b)\}.
\]
If \( a, b \in \bigcap_{n=1}^{\infty} A_n \), then \( x \in \bigcap_{n=1}^{\infty} A_n \), and so
\[
\mu(x) = 1 = \min\{\mu(a), \mu(b)\}.
\]
Assume that \( a \in A_r/A_{r+1} \) for some \( r = 1, 2, \cdots \), and \( b \in \bigcap_{n=1}^{\infty} A_n \) (or, \( a \in \bigcap_{n=1}^{\infty} A_n \) for some and \( b \in A_r/A_{r+1} \) for some \( r = 1, 2, \cdots \)). Then \( x \in A_r \), and so
\[
\mu(x) = \frac{r}{r+1} = \min\{\mu(a), \mu(b)\}.
\]
Consequently, \( \mu \) is a fuzzy left \( h \)-ideal of \( S \).

The last, let \( x \in \bigcap_{n=1}^{\infty} A_n \), \( m \in M \) and \( \alpha \in \Gamma \). Then \( \mu(x) = 1 \) and \( \max \in \bigcap_{n=1}^{\infty} A_n \), so
\[
\mu(\max) = 1 = \mu(x).
\]
If \( x \in A_r/A_{r+1} \), \( m \in M \) and \( \alpha \in \Gamma \), then \( \max \in A_r \), we have
\[
\mu(\max) \geq \frac{r}{r+1} = \mu(x).
\]
So, \( \mu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \). \( \square \)

4. **Cartesian Product of \( M \)-fuzzy \( h \)-ideals**

A fuzzy relation on any set \( S \) is a fuzzy set \( \mu : S \times S \rightarrow [0, 1] \).

If \( \mu \) is a fuzzy relation on a set \( S \) and \( \nu \) is a fuzzy set in \( S \), then \( \mu \) is a fuzzy relation on \( \nu \) if \( \mu(x, y) \leq \min\{\nu(x), \nu(y)\} \), \( \forall x, y \in S \).

**Definition 4.1** \((\text{2})\). Let \( \mu \) and \( \nu \) be fuzzy sets in a set \( S \). Then the Cartesian product of \( \mu \) is defined by \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} \), \( \forall x, y \in S \).

**Lemma 4.2** \((\text{2})\). Let \( \mu \) and \( \nu \) be fuzzy sets in a set \( S \). Then
(i) \( \mu \times \nu \) is a fuzzy relation on \( S \),
(ii) \( U(\mu \times \nu; t) = U(\mu; t) \times U(\nu; t) \) for all \( t \in [0, 1] \).
Definition 4.3. If $\nu$ is a fuzzy set in a set $S$, then the strongest fuzzy relation on $S$ that is a fuzzy relation on $\nu$ is $\mu_\nu$, which is given by $\mu_\nu(x,y) = \min\{\nu(x), \nu(y)\}$ $\forall x, y \in S$.

Lemma 4.4. For a given fuzzy set $\nu$ on a set $S$, let $\mu_\nu$ be strongest fuzzy relation on $S$. Then for $t \in [0,1]$, we have that $U(\mu_\nu; t) = U(\nu; t) \times U(\nu; t)$.

The following proposition is an immediate consequence of Lemma 4.4, and we omit the proof.

Proposition 4.5. If $\nu$ is a fuzzy left $h$-ideal of an M-$\Gamma$-hemiring $S$. Then the level left $h$-ideals of $\mu_\nu$ are given by $U(\mu_\nu; t) = U(\nu; t) \times U(\nu; t)$ $\forall t \in [0,1]$.

Let $S_1$ and $S_2$ be two M-$\Gamma$-hemirings. Now we can easy to check that $S_1 \times S_2$ is an M-$\Gamma$-hemiring by the operations which we define as follows:

(i) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$;
(ii) $(x_1, x_2)\alpha(y_1, y_2) = (x_1\alpha y_1, x_2\alpha y_2)$;
(iii) $m\alpha(x_1, x_2) = (\max x_1, \max x_2)$,

for all $x_1, x_2, y_1, y_2 \in S$, $\alpha \in \Gamma$ and $m \in M$.

Theorem 4.6. Let $\mu$ and $\nu$ be M-fuzzy left $h$-ideals of an M-$\Gamma$-hemiring $S$. Then $\mu \times \nu$ is an M-fuzzy left $h$-ideal of $S \times S$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\alpha \in \Gamma$. Then we have

$$(\mu \times \nu)((x_1, x_2) + (y_1, y_2)) = (\mu \times \nu)(x_1 + y_1, x_2 + y_2)$$
$$= \min\{\mu(x_1 + y_1), \nu(x_2 + y_2)\}$$
$$\geq \min\{\min\{\mu(x_1), \mu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\}$$
$$= \min\{\min\{\mu(x_1), \nu(x_2)\}, \min\{\mu(y_1), \nu(y_2)\}\}$$
$$= \min\{(\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)\},$$

and then

$$(\mu \times \nu)((x_1, x_2)\alpha(y_1, y_2)) = (\mu \times \nu)(x_1\alpha y_1, x_2\alpha y_2)$$
$$= \min\{\mu(x_1\alpha y_1), \nu(x_2\alpha y_2)\}$$
$$\geq \min\{\mu(y_1), \nu(y_2)\}$$
$$= (\mu \times \nu)(y_1, y_2).$$

Now let $(a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S \times S$ be such that

$$(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2),$$

i.e. $(x_1 + a_1 + z_1, x_2 + a_2 + z_2) = (b_1 + z_1, b_2 + z_2)$, it follows that

$$x_1 + a_1 + z_1 = b_1 + z_1, x_2 + a_2 + z_2 = b_2 + z_2,$$

so that

$$(\mu \times \nu)(x_1, x_2) = \min\{\mu(x_1), \nu(x_2)\}$$
$$\geq \min\{\min\{\mu(a_1), \mu(b_1)\}, \min\{\nu(a_2), \nu(b_2)\}\}$$
$$= \min\{\min\{\mu(a_1), \nu(a_2)\}, \min\{\mu(b_1), \nu(b_2)\}\}$$
$$= \min\{(\mu \times \nu)(a_1, a_2), (\mu \times \nu)(b_1, b_2)\}. $$
Therefor \( \mu \times \nu \) is a fuzzy left \( h \)-ideal of \( S \times S \). Now let \( x = (x_1, x_2) \in S \times S, m \in M \), then
\[
(\mu \times \nu)(max) = (\mu \times \nu)(max(x_1, x_2)) = (\mu \times \nu)(max_1, max_2) = \min\{\mu(max_1), \nu(max_2)\} \geq \min\{\mu(x_1), \nu(x_2)\} = (\mu \times \nu)(x_1, x_2) = (\mu \times \nu)(x).
\]
Hence, \( \mu \times \nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \times S \). \( \square \)

**Theorem 4.7.** Let \( \nu \) be a fuzzy set in an \( M\Gamma \)-hemiring \( S \) and let \( \mu_\nu \) be the strongest fuzzy relation on \( S \). Then \( \nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \) if and only if \( \mu_\nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \times S \).

**Proof.** Assume that \( \nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \). Let \( (x_1, x_2), (y_1, y_2) \in S \times S \), and \( \alpha \in \Gamma \). Then
\[
\mu_\nu((x_1, x_2) + (y_1, y_2)) = \mu_\nu(x_1 + y_1, x_2 + y_2) = \min\{\nu(x_1 + y_1), \nu(x_2 + y_2)\} \geq \min\{\min\{\nu(x_1), \nu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} = \min\{\min\{\nu(x_1), \nu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} = \min\{\mu_\nu(x_1, x_2), \mu_\nu(y_1, y_2)\},
\]
and
\[
\mu_\nu((x_1, x_2)\alpha(y_1, y_2)) = \mu_\nu(x_1\alpha y_1, x_2\alpha y_2) = \min\{\nu(x_1\alpha y_1), \nu(x_2\alpha y_2)\} \geq \min\{\min\{\nu(x_1), \nu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} = \mu_\nu(y_1, y_2).
\]

Now let \( (a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S \times S \) be such that
\[
(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2).
\]
So
\[
x_1 + a_1 + z_1 = b_1 + z_1, x_2 + a_2 + z_2 = b_2 + z_2.
\]
Thus
\[
\mu_\nu(x_1, x_2) = \min\{\nu(x_1), \nu(x_2)\} \geq \min\{\min\{\nu(a_1), \nu(b_1)\}, \min\{\nu(a_2), \nu(b_2)\}\} = \min\{\min\{\nu(a_1), \nu(a_2)\}, \min\{\nu(b_1), \nu(b_2)\}\} = \min\{\mu_\nu(a_1, a_2), \mu_\nu(b_1, b_2)\}.
\]
Therefor \( \mu_\nu \) is a fuzzy left \( h \)-ideal of \( S \times S \). Now, for any \( (x_1, x_2) \in S \times S, m \in M \), we have
\[
\mu_\nu(max(x_1, x_2)) = \mu_\nu(max_1, max_2) = \min\{\nu(max_1), \nu(max_2)\} \geq \min\{\nu(x_1), \nu(x_2)\} = \mu_\nu(x_1, x_2).
\]
Thus \( \mu_\nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \times S \).
Conversely, suppose that \( \mu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \times S \). By Theorem 4.12 in [15], we know that \( \nu \) is a fuzzy left \( h \)-ideal of \( S \). Now, for any \( x_1, x_2, y_1, y_2 \in S \), and \( \alpha \in \Gamma \), by using Proposition 4.7 in [15], we have
\[
\min \{ \nu(x_1 \alpha y_1), \nu(x_2 \alpha y_2) \} = \mu \nu(x_1 \alpha y_1, x_2 \alpha y_2) = \mu \nu((x_1, x_2) \alpha (y_1, y_2)) \geq \mu \nu(y_1, y_2) = \min \{ \nu(y_1), \nu(y_2) \},
\]
and so \( \nu(x_1 \alpha y_1) \geq \min \{ \nu(y_1), \nu(y_2) \} \). Taking \( x_1 = x, y_1 = y \) and \( y_2 = 0 \), we get \( \nu(x \alpha y) \geq \min \{ \nu(y), \nu(0) \} = \nu(y) \).

Then let \( m \in M \), we have
\[
\min \{ \nu(m \alpha x_1), \nu(m \alpha x_2) \} = \mu \nu(m \alpha x_1, m \alpha x_2) = \mu \nu(m \alpha (x_1, x_2)) \geq \mu \nu(x_1, x_2) = \min \{ \nu(x_1), \nu(x_2) \}.
\]
Taking \( x_1 = x_2 = x \), we have \( \nu(max) \geq \nu(x) \). Consequently, \( \nu \) is an \( M \)-fuzzy left \( h \)-ideal of \( S \). \qed

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