**T-locality groups**

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**Abstract.** The aim of this paper is to introduce the concepts of $T$-locality groups, where $T$ stands for any continuous triangular norm. Our construct mainly will deal and relate with both fuzzy $T$-locality spaces and fuzzy $TL$-uniform spaces. We establish some basic results and characterization theorems of $T$-locality groups. We give the necessary and sufficient conditions for a group structure and a fuzzy $T$-locality system to be compatible. Moreover, we show that all initial and final lifts exist uniquely in the concrete category of $T$-locality groups and hence all initial and final $T$-locality groups exist and can be characterized. As consequences the $T$-locality subgroups, $T$-locality product groups and $T$-locality quotient groups are exist.

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1. Introduction

In 1995, N. N. Morsi introduced the fuzzy $T$-locality spaces, for each continuous triangular norm $T$. In 2006, we deduced the fuzzy $TL$-uniform spaces which compatible with fuzzy $T$-locality spaces. In this manuscript, we introduce a new structure of $T$-locality groups. We show that these structure is conforms well with fuzzy $T$-locality spaces. We give some other important results of $T$-locality groups and we give the notions of the left and right translations for a $T$-locality group. Also, we show that every $T$-locality group is $TL$-uniformizable and we characterize the uniformly continuous functions. Moreover, we study the initial and final $T$-locality groups.

We proceed as follows: In Section 2, we present some basic definitions and ideas on the classes of fuzzy sets, $I$-topological spaces, fuzzy $T$-locality spaces and fuzzy $TL$-uniform spaces. In Section 3, we introduce our definition of $T$-locality groups and we prove some of their properties, we show that for the $T$-locality group, the
left and right translations are homeomorphisms. Some other results for these T-locality groups are studied. We also generalize the two important characterization theorems, which give necessary and sufficient conditions for a T-locality system and a group structure to be compatible. Moreover, we study the relations between T-locality groups and fuzzy TL-uniform spaces. Some examples of T-locality groups are given. In Section 4, we show that all initial lifts exist uniquely in the concrete category T-LocGrp of T-locality groups and hence all initial T-locality groups exist and can be characterized, thus the category T-LocGrp is a topological category. Therefore all final lifts and all final T-locality groups also exist. The subgroups and the product groups of T-locality groups in the categorical sense are special initial T-locality groups and hence exist and can be characterized. The quotient groups of T-locality groups are characterized as special final T-locality groups.

2. Preliminaries

A triangular norm T (cf. [18]) is a binary operation on the unit interval I = [0,1] that is associative, symmetric, isotone in each argument and has neutral element 1. The basic three (continuous) triangular norms are their simplest, namely Min (also denoted by \(\land\)), \(\prod\) (product) and \(T_m\) (the Lukasiewicz conjunction), where for all \(\alpha, \beta \in I\), \(\alpha \prod \beta = \alpha \beta\) and \(\alpha T_m \beta = (\alpha + \beta) \land 1\). The binary operation \(\land\) above is the truncated subtraction, defined on non-negative real numbers by \(r \land s = \max\{r - s, 0\}, r, s \geq 0\).

A continuous triangular norm T is uniformly continuous, that is for all \(\epsilon > 0\) there is \(\theta = \theta_{T,\epsilon} > 0\) such that for every \((\alpha, \beta) \in I \times I\), we have
\[
(\alpha T \beta - \epsilon) \leq (\alpha - \theta) T (\beta - \theta) \leq \alpha T \beta \leq (\alpha + \theta) T (\beta + \theta) \leq (\alpha T \beta) + \epsilon.
\]
Obviously, for every real numbers \(r, s \geq 0, \epsilon > 0\) and the above \(\theta = \theta_{T,\epsilon} > 0\), we have
\[
(r T s) \land \epsilon \leq (r \land \theta) T (s \land \theta).
\]
For a continuous triangular norm T the following binary operation on I,
\[
j(\alpha, \gamma) = \sup\{\theta \in I : \alpha T \theta \leq \gamma\}, \alpha, \gamma \in I,
\]
is called the residual implication of T [15]. For this implication, we shall use the following property [17], \(\forall \alpha, \beta, \theta, \gamma \in I:\)
\[
j(\alpha, \alpha) = 1.
\]
\[
j(\alpha T \beta, \theta T \gamma) \geq j(\alpha, \theta) T j(\beta, \gamma).
\]
A fuzzy set \(\lambda\) in a universe set X, introduced by Zadeh in [19], is a function \(\lambda : X \to I = [0,1]\). The collection of all fuzzy sets of X is denoted by \(I^X\). The height of a fuzzy set \(\lambda\) is the following real number:
\[
hgt\lambda = \sup\{\lambda(x) : x \in X\}.
\]
If \(H \subseteq X\), then its characteristic function is denoted by \(1_H\) and the set of all (crisp) subsets of X is denoted by \(2^X\). We also denote the constant fuzzy set of X with value \(\alpha \in I\) by \(\alpha\).
Given two fuzzy sets $\mu, \lambda \in I^X$, we denote by $\mu T \lambda$ the following fuzzy set of $X$:

$$(\mu T \lambda)(x) = \mu(x) T \lambda(x), \; x \in X.$$  

The degree of containment of $\mu$ in $\lambda$ according to $T$ is the real number in $I$ [7], defined by:

$$j < \mu, \lambda >= \inf_{x \in X} j(\mu(x), \lambda(x)).$$

We follow Lowen’s definition of a fuzzy interior operator on a set $X$ [12]. This is an operator $^o : I^X \rightarrow I^X$ that satisfies $\mu^o \leq \mu, (\mu \land \lambda)^o = \mu^o \land \lambda^o$ for all $\mu, \lambda \in I^X$ and $a^o = a$ for all $a \in I$. We may define an an $I$-topology in the usual way, namely assuming a fuzzy set $\mu$ to be open if and only if $\mu^o = \mu$. We denote this $I$-topology by $\tau$. The pair $(X, \alpha)$ is called an $I$-topological space.

A function $f : (X, \alpha) \rightarrow (Y, \beta)$, between two $I$-topological spaces, is said to be continuous [12] if $f^-(\mu) \in \tau$ for all $\mu \in \tau$, where $(f^-(\mu))(x) = \mu(f(x)), \forall x \in X$. It is said to be open if $f(\lambda) \in \tau$ for all $\lambda \in \tau$.

In [13], Lowen introduced the concepts of Initial and final $I$-topological spaces. Consider a family of $I$-topological spaces $(Y_r, \tau_r)_{r \in \Lambda}$ and for each $r \in \Lambda$, a mapping $f_r : X \rightarrow Y_r$. The initial $I$-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$ is meant the $I$-topology $\tau$ on $X$ for which the conditions of an initial lift in the category of $I$-topological spaces are fulfilled, that is,

(i) All mappings $f_r : (X, \tau) \rightarrow (Y_r, \tau_r)$ are continuous,

(ii) For any $I$-topological space $(Z, \sigma)$ and any mapping $f : (Z, \sigma) \rightarrow (X, \tau)$ is continuous if and only if for all $r \in \Lambda$ the mappings $f_r \circ f : (Z, \sigma) \rightarrow (Y_r, \tau_r)$ are continuous. The union $\bigcup_{r \in \Lambda} f_r^-(\tau_r)$ of the family $(f_r^-(\tau_r))_{r \in \Lambda}$ where

$$f_r^-(\tau_r) = \{f_r^-(\mu) : \mu \in \tau_r\},$$

is a subbase for an $I$-topology on $X$, for which the conditions (i) and (ii) of the initial lift in the category of $I$-topological spaces are fulfilled [12, 13], called the initial $I$-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$, and $f_r^-(\tau_r)$ is the initial $I$-topology of $\tau_r$ with respect to $f_r$. Therefore, all initial lifts and all initial $I$-topological spaces exist uniquely in the category of $I$-topological spaces and hence the category of $I$-topological spaces is a topological category. Consequently, all final lifts also exist.

Assume that $f_r : X_r \rightarrow Y$ is a mapping of $X_r$ to $Y$. By the final $I$-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$ we mean the $I$-topology $\tau$ on $Y$ which fulfills the conditions of a final lift in the category of $I$-topological spaces, that is,

(i) All mappings $f_r : (X_r, \tau_r) \rightarrow (Y, \tau)$ are continuous,

(ii) For any $I$-topological space $(Z, \sigma)$ and any mapping $f : (Y, \tau) \rightarrow (Z, \sigma)$ is continuous if and only if for all $r \in \Lambda$ the mappings $f \circ f_r : (X_r, \tau_r) \rightarrow (Z, \sigma)$ are continuous. It is shown in [13] that the infimum $\bigcap_{r \in \Lambda} f_r(\tau_r)$ of the family $(f_r(\tau_r))_{r \in \Lambda}$ with respect to the finer relation on $I$-topologies, where

$$f_r(\tau_r) = \{\lambda \in I^Y : f_r^-(\lambda) \in \tau_r\}$$

is the final $I$-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$.

$I$-filters and $I$-filterbases were introduced by R. Lowen in [13]. An $I$-filter in a universe $X$ is a nonempty collection $\mathfrak{F} \subset I^X$ which satisfies: $\emptyset \not\in \mathfrak{F}$, $\mathfrak{F}$ is closed under finite meets and contains all the fuzzy supersets of its individual members.
An $I$-filterbase in $X$ is a nonempty collection $\mathcal{B} \subset I^X$ which satisfies $\emptyset \not\in \mathcal{B}$ and the meets of two members of $\mathcal{B}$ contain a member of $\mathcal{B}$.

**Definition 2.1** ([16]). The $T$-saturation operator is the operator $\sim^T$ which sends an $I$-filterbase $\mathcal{B}$ in $X$ to the following subsets of $I^X$

$$\mathcal{B}^{\sim^T} = \{ \mu \in I^X : \bigvee_{\gamma \in I^i} (\gamma T \mu_\gamma) \leq \mu, \forall \gamma \in I, \mu \in \mathcal{B} \},$$

said to be the $T$-saturation of $\mathcal{B}$.

The fuzzy $T$-locality spaces ($T$-locality spaces, for short) were introduced by N.N. Morsi, for more definitions and properties, we can refer to [19].

**Definition 2.2** ([16]). A $T$-locality space is an $I$-topological space $(X, \rho)$ whose fuzzy interior operator is induced by some indexed family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, of $I$-filterbases in $X$, in the following manner:

$$(2.6) \quad \mu^o(x) = \sup_{\nu \in \mathcal{B}(x)} \exists < \nu, \mu, \mu \in I^X, x \in X.$$

The family $\mathcal{B}$ is said to be a $T$-locality basis for $(X, \rho)$, and $\mathcal{B}^{\sim^T}$ is called a $T$-locality system of $(X, \rho)$. The $I$-topology of $(X, \rho)$ will be denoted by $\tau(\mathcal{B})$. Also, a $T$-locality base $\mathcal{B}$ and a $T$-locality system $\mathcal{B}^{\sim^T}$ induce the same $T$-locality space, that is $\tau(\mathcal{B}) = \tau(\mathcal{B}^{\sim^T})$.

**Theorem 2.3** ([16]). A family of $I$-filterbases in $X$, $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, will be a $T$-locality base in $X$ if and only if it satisfies the following two conditions, for all $x \in X$

(TLB 1) $\nu(x) = 1$ for all $\nu \in \mathcal{B}(x)$.

(TLB 2) Every $\nu \in \mathcal{B}(x)$ has a $T$-kernel. This consists of two families $(\nu, \gamma \in \mathcal{B}(x))_{\gamma \in I^i}$ and $(\nu, \gamma, \theta \in \mathcal{B}(y))_{y, \gamma, \theta \in X \times I^i \times I^i}$ such that for all $(y, \gamma, \theta) \in X \times I^i \times I^i$, $\left(\nu T \nu_{\gamma, \theta} T \nu_{\gamma, \theta} \right) \leq \nu$.

**Definition 2.4** ([16]). A $T$-locality space $(X, \tau(\mathcal{B}))$ is said to be L-regular, if for every $(H, x, \epsilon) \in I^X \times I^2$, such that there is $\nu \in \mathcal{B}(x)$ with $\text{hgt}(\nu \wedge 1_H) < \epsilon$, then there are an open set $\mu$ and $\rho \in \mathcal{B}(x)$ such that, $1_H \subseteq \mu$ and $\text{hgt}(\rho \mu) < \epsilon$.

**Theorem 2.5** ([16]). Let $(X, \rho)$ and $(Y, \gamma)$ be $T$-locality spaces with $T$-locality basis $\mathcal{B}$ and $\mathcal{E}$, respectively, and $x \in X$. Then a function $f : (X, \rho) \rightarrow (Y, \gamma)$ will be continuous at the point $x \in X$, if and only if for all $\rho \in \mathcal{E}(f(x))$, we have $f^- (\rho) \in (\mathcal{B}(x))^{\sim^T}$ if and only if for all $\rho \in \mathcal{E}(f(x))$ and all $\gamma \in I$ there is $\rho_\gamma \in \mathcal{B}(x)$ such that $\gamma T \rho_\gamma \leq f^- (\rho)$ if and only if for all $\rho \in \mathcal{E}(f(x))$ and all $\gamma \in I$ there is $\rho_\gamma \in \mathcal{B}(x)$ such that $\gamma T f(\rho_\gamma) \leq \rho$. If follows that $f$ will be continuous if it is continuous at all points of its domain.

Now, we deduce the following result on the $T$-locality spaces.

**Proposition 2.6.** Let $(X, \tau(\mathcal{B}_1))$ and $(Y, \tau(\mathcal{B}_2))$ be two $T$-locality spaces with basis $\mathcal{B}_1 = (\mathcal{B}_1(x))_{x \in X}$ and $\mathcal{B}_2 = (\mathcal{B}_2(y))_{y \in Y}$ in $X$ and $Y$, respectively. Then their $T$-product $(X \times Y, \tau(\mathcal{B}_1) \otimes_T \tau(\mathcal{B}_2))$ is a $T$-locality space with a base $\mathcal{B} = \mathcal{B}_1 \otimes_T \mathcal{B}_2$, defined by

$$\mathcal{B}(x, y) = \{ \nu_1 \otimes_T \nu_2 : \nu_1 \in \mathcal{B}_1(x), \nu_2 \in \mathcal{B}_2(y) \},$$

where $(\nu_1 \otimes_T \nu_2)(x, y) = \nu_1(x) T \nu_2(y)$, for every $(x, y) \in X \times Y$. 

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Proof. First, we show that for each \((x, y) \in X \times Y\), \(\mathfrak{B}(x, y)\) is an \(I\)-filterbase. Obviously, \(\mathfrak{B} \neq \emptyset\) and \(\emptyset \notin \mathfrak{B}\). Let \(\lambda_1, \lambda_2 \in \mathfrak{B}(x, y)\). Then there are \(\nu_1, \nu_2 \in \mathfrak{B}_1(x)\) and \(\mu_1, \mu_2 \in \mathfrak{B}_2(y)\) such that \(\lambda_1 = \nu_1 \otimes T \mu_1\) and \(\lambda_2 = \nu_2 \otimes T \mu_2\). So, for every \((x, y) \in X \times Y\), we have

\[
(\lambda_1 \land \lambda_2)(x, y) = \lambda_1(x, y) \land \lambda_2(x, y) = (\nu_1 \otimes T \mu_1)(x, y) \land (\nu_2 \otimes T \mu_2)(x, y)
\]

\[
= [\nu_1(x) T \mu_1(y)] \land [\nu_2(x) T \mu_2(y)]
\]

\[
\geq [\nu_1(x) \land \nu_2(x)] T [\mu_1(y) \land \mu_2(y)], \text{ clear}
\]

\[
= (\nu_1 \land \nu_2)(x) T (\mu_1 \land \mu_2)(y)
\]

\[
\geq \nu(x) T \mu_1(y), \text{ by hypothesis, where } \nu \in \mathfrak{B}_1(x) \text{ and } \mu \in \mathfrak{B}_2(y)
\]

\[
= (\nu \otimes T \mu)(x, y)
\]

\[
= \lambda(x, y), \text{ where } \lambda = (\nu \otimes T \mu) \in \mathfrak{B}(x, y).
\]

Hence, the intersection of any two members of \(\mathfrak{B}(x, y)\) contain a member of \(\mathfrak{B}(x, y)\), which proving that \(\mathfrak{B}(x, y)\) is an \(I\)-filterbase in \(X \times Y\).

Now, we fulfill the conditions of Theorem 2.3:

(TLB 1) For every \(\nu \in \mathfrak{B}_1(x)\) and \((x, x) \in X \times Y\), we have

\[
\lambda(x, x) = (\nu \otimes T \mu)(x, x), \text{ for some } \nu \in \mathfrak{B}_1(x) \text{ and } \mu \in \mathfrak{B}_2(x)
\]

\[
= \nu(x) T \mu(x)
\]

\[
= 1 T 1, \text{ by hypothesis}
\]

\[
= 1.
\]

(TLB 2) Let \(\lambda \in \mathfrak{B}(x, y)\) and \((x, y) \in X \times Y\). Then there are \(\nu \in \mathfrak{B}_1(x)\) and \(\mu \in \mathfrak{B}_2(y)\) such that \(\lambda = \nu \otimes T \mu\). Now, since \(\nu\) has a \(T\)-kernel, that is a two families \((\nu_\gamma \in \mathfrak{B}_1(x))_{\gamma \in I_1}\) and \((\nu_{\gamma \theta} \in \mathfrak{B}_1(z))_{(z, \gamma, \theta) \in X \times I_1 \times I_0}\) such that for all \((z, \gamma, \theta) \in X \times I_1 \times I_0\),

\[
[(\gamma T \nu_\gamma(z)) \otimes T \nu_{\gamma \theta}] T \nu_{\gamma \theta} \leq \nu.
\]

Also, since \(\mu\) has a \(T\)-kernel, that is a two families \((\nu_\gamma \in \mathfrak{B}_2(y))_{\gamma \in I_1}\) and \((\mu_{\gamma \theta} \gamma \in \mathfrak{B}_2(s))_{(s, \gamma, \theta) \in Y \times I_1 \times I_0}\),

\[
[(\gamma T \nu_\gamma(s)) \otimes T \nu_{\gamma \theta}] T \nu_{\gamma \theta} \leq \nu.
\]

Hence, for every \(\alpha \in I_1\) and \(\epsilon \in I_0\), we can get, by continuity of \(T\), \(\gamma \in I_1\) in such a way that \(\alpha = \gamma T \gamma\) and then \(\theta = \theta T \epsilon\), be as in (2.1). For all which by taking \(\lambda_\alpha = \nu_\gamma \otimes T \mu_\gamma \in \mathfrak{B}(x, y)\) and \(\lambda_{\epsilon \mu} = \nu_{\gamma \theta} \otimes T \mu_{\gamma \theta} \in \mathfrak{B}(x, y)\), we have

\[
[(\alpha T \lambda_\alpha(z, s)) \otimes T \lambda_{\epsilon \mu}] T \lambda_{\epsilon \mu} = [(\gamma T \nu_\gamma(z) T \mu_\gamma(s)) \otimes T \nu_{\gamma \theta} \otimes T \mu_{\gamma \theta}]
\]

\[
\leq [(\gamma T \nu_\gamma(z) \otimes T \nu_{\gamma \theta}) T \nu_{\gamma \theta} \otimes T \nu_{\gamma \theta}], \text{ by (2.2)}
\]

\[
= [(\gamma T \nu_\gamma(z) \otimes T \nu_{\gamma \theta}) T \nu_{\gamma \theta}] T \nu_{\gamma \theta}
\]

\[
\leq \nu \otimes T \mu = \lambda.
\]

This proves that \(\lambda\) has a \(T\)-kernel and thus \(\mathfrak{B}\) satisfies (TLB 2). Which winds up the proof. \(\square\)
In [10], Höhle defines for every $\psi, \varphi \in I^X \times X$ and $\lambda \in I^X$:

The $T$-section of $\psi$ over $\lambda$ by

$$(\psi < \lambda > T)(x) = \sup_{z \in X} [\lambda(z)T\psi(z, x)], \ x \in X.$$ 

The $T$-composition of $\psi, \varphi$ by

$$(\psi \circ_T \varphi)(x, y) = \sup_{z \in X} [\varphi(x, z)T\psi(z, y)], \ x, y \in X.$$ 

The symmetric of $\psi$ by $\psi^\#(x, y) = \psi(y, x), \ x, y \in X$.

The fuzzy $TL$-uniform spaces ($TL$-uniform spaces, for short) were introduced by K. A. Hashem and N. N. Morsi, for more definitions and properties, we can refer to [9].

**Definition 2.7 (9).** (i) A $TL$-uniform base on a set $X$ is a subset $\vartheta \subseteq I^X \times X$ which fulfills the following properties

1. ($TLUB$ 1) $\vartheta$ is an $I$-filterbase,
2. ($TLUB$ 2) For all $\varphi \in \vartheta$ and $x \in X$, we have $\varphi(x, x) = 1$,
3. ($TLUB$ 3) For all $\varphi \in \vartheta$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \vartheta$ with $T(\varphi_\gamma, \varphi \gamma) \leq \varphi$,
4. ($TLUB$ 4) For all $\varphi \in \vartheta$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \vartheta$ with $T(\varphi_\gamma, \varphi) \leq \gamma \varphi$.

(ii) A $TL$-uniformity on $X$ is a $T$-saturated $TL$-uniform base on $X$.

(iii) If $\Sigma$ is a $TL$-uniformity on $X$, then we shall say that $\vartheta$ is a basis for $\Sigma$ if $\vartheta$ is an $I$-filterbase and $\vartheta^T = \Sigma$.

It follows that for a $TL$-uniformity $\Sigma$ on a set $X$ and all $\varphi \in \Sigma$, we find that $\varphi^\# \in \Sigma$. The pair $(X, \Sigma)$ consisting of a set $X$ and a $TL$-uniformity $\Sigma$ on $X$ is called $TL$-uniform space.

**Definition 2.8 (9).** Let $(X, \Sigma)$ and $(Y, \omega)$ be $TL$-uniform spaces, with bases $\vartheta$ and $\vartheta^\omega$, respectively, and $f : X \to Y$ be a function. We say that $f$ is uniformly continuous if for every $\varphi \in \vartheta$ and $\gamma \in I_1$, there is $\psi \in \omega$ such that $\gamma T \psi \leq (f \times f)^\omega - (\varphi)$.

**Proposition 2.9 (9).** If $\Sigma$ is a $TL$-uniformity on a set $X$, then the indexed family $(\Sigma(x))_{x \in X}$ given by $\Sigma(x) = \{\psi < 1_x > T: \psi \in \Sigma\}$ is a $T$-locality system on $X$.

3. $T$-Locality groups

The concept of $T$-locality group is introduce in this section and some of their properties and results are deduced, we show that for a given $T$-locality group the left and right translations are homeomorphisms. Also, we study the relations between $T$-locality groups and $TL$-uniform spaces. Precisely, we show that every $T$-locality group is $TL$-uniformizable and induces $TL$-uniformities.

In what follows, we consider $(G, \ast)$ as a group with $e$ as the identity element, and for every $\lambda : G \to I$, we define $\ast \lambda : G \to I$, as $\ast \lambda(x) = \lambda(x^{-1})$, for each $x \in G$, where $x^{-1}$ is the inverse element of $x$.

Now, we define the structure of $T$-locality groups as follows:

Let $(G, \ast)$ be a group and $(G, \tau(\mathfrak{B}))$ a $T$-locality space with base $\mathfrak{B}$ on $G$. Then the triple $(G, \ast, \tau(\mathfrak{B}))$ is called a $T$-locality group if the following mappings $\Gamma :$
For any group \((G, \tau(\mathfrak{B}))\), we have \((G, \tau(\mathfrak{B}))\) is a T-locality group if and only if the mapping \(\Omega : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \rightarrow (G, \tau(\mathfrak{B}))\) defined by
\[
\Omega(x, y) = xy^{-1}
\]
for all \(x, y \in G\).

**Theorem 3.1.** For any group \((G, *)\), we have \((G, *, \tau(\mathfrak{B}))\) is a T-locality group if and only if the mapping \(\Upsilon : (G \times G, \tau(\mathfrak{B}) \otimes T \tau(\mathfrak{B})) \rightarrow (G, \tau(\mathfrak{B}))\) defined by
\[
\Upsilon(x, y) = x \ast y = xy
\]
is continuous.

**Proof.** Let \((G, *, \tau(\mathfrak{B}))\) be a T-locality group and \(h : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \rightarrow (G \times G, \tau(\mathfrak{B}) \otimes T \tau(\mathfrak{B}))\) the mapping defined by \(h(x, y) = (x, y^{-1})\). Then \(h\) is the product of the identity mapping \(J_G\) and the continuous mapping \(\Upsilon\), therefore obviously, \(h\) is continuous. Hence, \(\Omega = \Gamma \circ h\) is the composition of continuous mappings \(\Gamma\) and \(h\), that is, \(\Omega\) is continuous.

On the other hand, let \(\Omega\) be a continuous mapping and \(i : (G, \tau(\mathfrak{B})) \rightarrow (G \times G, \tau(\mathfrak{B}) \otimes T \tau(\mathfrak{B}))\) the canonical injection map defined by \(i(x) = (e, x)\), where \(e\) is the identity element of \(G\). Then \(\Upsilon = \Omega \circ i\) is the composition of continuous mappings and therefore is continuous. Since \(\Gamma = \Omega \circ h\) and since \(h = J_G \times \Upsilon\) is the product of continuous mappings \(J_G\) and \(\Upsilon\), then \(h\) is continuous and therefore \(\Gamma\) also is continuous. Hence \((G, *, \tau(\mathfrak{B}))\) is a T-locality group. \(\Box\)

If \(*\) is a binary operation on \(G\), then we define a binary operation \(\mathcal{O}_T\) on \(G^2\) by,
\[
(\lambda \mathcal{O}_T \nu)(x) = \sup_{y \in x \in G} [\lambda(y)T\nu(z)]
\]

**Lemma 3.2.** If \((G, *)\) is a group and \(\lambda : G \rightarrow I\), then for all \(x, y \in G\), we have
\[
(1_x \mathcal{O}_T \lambda)(y) = \lambda(x^{-1}y)\quad \text{and} \quad (\lambda \mathcal{O}_T 1_x)(y) = \lambda(yx^{-1}).
\]

**Proof.** Let \(\lambda : G \rightarrow I\) and \(x, y \in G\). Then
\[
(1_x \mathcal{O}_T \lambda)(y) = \sup_{y \in x \in G} [1_z(y)T\lambda(s)] = \sup_{z \in x} \lambda(s)
\]
\[
= \sup_{z \in x^{-1}y} \lambda(s) = \lambda(x^{-1}y).
\]

Analogously, we can show that \((\lambda \mathcal{O}_T 1_x)(y) = \lambda(yx^{-1})\). Which winds up the proof. \(\Box\)

For each group \((G, *)\) and \(\alpha \in G\), the left and right translations are the homomorphisms \(\mathcal{L}_\alpha : (G, \ast) \rightarrow (G, \ast)\) defined by \(\mathcal{L}_\alpha(x) = \alpha x\) and \(\mathcal{R}_\alpha : (G, \ast) \rightarrow (G, \ast)\), defined by \(\mathcal{R}_\alpha(x) = x\alpha\), for each \(x \in G\), respectively.

The left and right translation in T-locality groups fulfill the following results.

**Proposition 3.3.** Let \((G, *, \tau(\mathfrak{B}))\) be a T-locality group. Then for each \(\alpha \in G\), we have
(i) \(\mathcal{L}_\alpha\) and \(\mathcal{R}_\alpha\) are homeomorphisms,
(ii) \((1_\alpha \mathcal{O}_T \lambda) = \mathcal{L}_\alpha(\lambda)\) and \((\lambda \mathcal{O}_T 1_\alpha) = \mathcal{R}_\alpha(\lambda)\), for every \(\lambda \in I^G\),
(iii) \(\nu \in (\mathfrak{B}(e))^{-T}\) if and only if \(\mathcal{L}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{-T}\) if and only if \(\mathcal{R}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{-T}\),

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(iv) \( \nu \in (\mathcal{B}(\alpha))^\sim_T \) if and only if \( \mathcal{L}_{\alpha^{-1}}(\nu) \in (\mathcal{B}(e))^\sim_T \) if and only if \( \mathcal{R}_{\alpha^{-1}}(\nu) \in (\mathcal{B}(e))^\sim_T \).

(v) If \( \mathcal{B} \) is \( T \)-saturated, then \( \lambda \in \mathcal{B}(e) \) if and only if \( (1_\alpha \circ_T \lambda) \in \mathcal{B}(\alpha) \).

**Proof.** (i) The left translation \( \mathcal{L}_\alpha \) is the composition of the mapping \( \Gamma \) defined above and the injection map \( i : (G, \tau(\mathcal{B})) \rightarrow (G \times G, \tau(\mathcal{B}) \otimes_T \tau(\mathcal{B})) \) defined by \( i(x) = (\alpha, x) \), that is \( \mathcal{L}_\alpha = \Gamma \circ i \). Hence, \( \mathcal{L}_\alpha \) is continuous and bijective. Since \( (\mathcal{L}_\alpha)^- = \mathcal{L}_{\alpha^{-1}} \), then \( (\mathcal{L}_\alpha)^- \) is also continuous. Therefore, \( \mathcal{L}_\alpha \) is a homeomorphism. Similarly, one can prove that \( \mathcal{R}_\alpha \) is a homeomorphism.

(ii) Let \( \lambda \in I^G \) and \( \alpha \in G \). Then for every \( y \in G \), we have

\[
(\mathcal{L}_\alpha(\lambda))(y) = \sup\{\lambda(z) : z \in (\mathcal{L}_\alpha)^-(y)\} = \sup\{\lambda(z) : z \in \mathcal{L}_{\alpha^{-1}}(\lambda y)\} = \sup\{\lambda(z) : z = \alpha^{-1} y\} = \lambda(\alpha^{-1} y) = (1_\alpha \circ_T \lambda)(y), \text{ by Lemma 3.2.}
\]

That is, \( (1_\alpha \circ_T \lambda) = \mathcal{L}_\alpha(\lambda) \). Similarly, we can prove \( (1_\alpha \circ_T \lambda) = \mathcal{R}_\alpha(\lambda) \).

(iii) Let \( \nu \in (\mathcal{B}(e))^\sim_T \). Then for every \( \gamma \in I_1 \), there is

\[
\nu_\gamma \in \mathcal{B}(e) = \mathcal{B}(\alpha^{-1} \alpha) = \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)) \text{ such that } \gamma T \nu_\gamma \leq \nu.
\]

Since \( \mathcal{L}_{\alpha^{-1}} \) is continuous, then in view of Theorem 2.5, we get for every \( \theta \in I_1 \), there is \( \nu_{\gamma \theta} \in \mathcal{B}(e) \) such that \( \theta T \nu_{\gamma \theta} \leq (\mathcal{L}_{\alpha^{-1}})^-(\nu_{\gamma \theta}) = \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) \). Thus \( \gamma T \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) = \mathcal{L}_{\alpha^{-1}}(\mathcal{L}_{\alpha^{-1}})(\mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta})) \leq \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) \). By putting \( \beta = (\gamma T \theta) \in I_1 \) and \( \nu_{\beta \theta} = \nu_{\gamma \theta} \in \mathcal{B}(e) \), we have \( \beta T \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) \leq \mathcal{L}_{\alpha^{-1}}(\nu_{\beta \theta}) \) which implies \( \mathcal{L}_{\alpha^{-1}}(\nu_{\beta \theta}) \in (\mathcal{B}(e))^\sim_T \).

Conversely, let \( \mathcal{L}_{\alpha^{-1}}(\nu) \in (\mathcal{B}(e))^\sim_T \). Then for every \( \gamma \in I_1 \), there is \( \nu_\gamma \in \mathcal{B}(e) = \mathcal{B}(\alpha e) = \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)) \) such that \( \gamma T \nu_\gamma \leq \mathcal{L}_{\alpha^{-1}}(\nu_\gamma) \). Since \( \mathcal{L}_{\alpha^{-1}} \) is continuous, then again by Theorem 2.5, we get for every \( \theta \in I_1 \), there is \( \nu_{\gamma \theta} \in \mathcal{B}(e) \) such that \( \theta T \nu_{\gamma \theta} \leq (\mathcal{L}_{\alpha^{-1}})^-(\nu_{\gamma \theta}) \). By putting \( \beta = (\gamma T \theta) \in I_1 \) and \( \nu_{\beta \theta} = \nu_{\gamma \theta} \in \mathcal{B}(e) \), we get

\[
\beta T \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) = \mathcal{L}_{\alpha^{-1}}(\mathcal{L}_{\alpha^{-1}})(\mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta})) \leq \mathcal{L}_{\alpha^{-1}}(\nu_{\beta \theta}) \leq \mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta}) = (\mathcal{L}_{\alpha^{-1}})^-(\mathcal{L}_{\alpha^{-1}}(\nu_{\gamma \theta})) = \nu_{\beta \theta} = \nu, \text{ for } \mathcal{L}_{\alpha^{-1}} \text{ is injective.}
\]

This implies that \( \nu \in (\mathcal{B}(e))^\sim_T \). Analogously, we can show that \( \nu \in (\mathcal{B}(e))^\sim_T \) if \( \mathcal{R}_{\alpha^{-1}}(\nu) \in (\mathcal{B}(e))^\sim_T \).

(iv) Follows immediately from (iii).

(v) Let \( \mathcal{B} \) be a \( T \)-saturated \( I \)-filterbase. Then

\[
\lambda \in \mathcal{B}(e)
\]

if \( \lambda \in \mathcal{B}(\alpha^{-1} \alpha) = \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)), \alpha \in G \)

if \( \exists \nu \in \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)) \) such that \( \nu \leq \lambda \)

if \( \exists \nu \in \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)), \) for which \( (\mathcal{L}_{\alpha^{-1}})^-(\nu) \in (\mathcal{B}(e))^\sim_T \) and \( \nu \leq \lambda \), by Theorem 2.5

if \( \exists \nu \in \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)), \) for which \( \mathcal{L}_{\alpha}(\nu) \in \mathcal{B}(\alpha) \) and \( \nu \leq \lambda \), by hypothesis

if \( \exists \nu \in \mathcal{B}(\mathcal{L}_{\alpha^{-1}}(\alpha)), \) for which \( \mathcal{L}_{\alpha}(\nu) \in \mathcal{B}(\alpha) \) and \( \mathcal{L}_{\alpha}(\nu) \leq \mathcal{L}_{\alpha}(\lambda) \)

if \( \mathcal{L}_{\alpha}(\lambda) \in \mathcal{B}(\alpha) \)

if \( (1_\alpha \circ_T \lambda) \in \mathcal{B}(\alpha) \), by (ii).

This completes the proof. \( \square \)
We shall now give some characterization theorems of $T$-locality groups. The first gives necessary and sufficient conditions for a group structure and a $T$-locality system to be compatible and the second gives necessary and sufficient conditions for an $I$-filterbase to be a $T$-locality group.

**Theorem 3.4.** If $(G, *)$ is a group, then the triple $(G, *, \tau(\mathcal{B}))$ is a $T$-locality group if and only if the following are hold:

(i) for every $x \in G$, we have $(\mathcal{B}(x))^{-T} = \{1_x \odot_T \nu : \nu \in (\mathcal{B}(e))^{-T}\}$,

(ii) for all $\nu \in \mathcal{B}(e)$ and $\gamma \in I_1$, there is $\nu_\gamma \in \mathcal{B}(e)$, with $\gamma T(\nu_\gamma \odot_T, \nu_\gamma) \leq \nu$.

**Proof.** Let $(G, *, \tau(\mathcal{B}))$ be a $T$-locality group. Then (i) follows from Proposition 3.3. Now, since the map $\Omega : (G \times G, \tau(\mathcal{B}) \otimes_T \tau(\mathcal{B})) \to (G, \tau(\mathcal{B}))$ is continuous at all $(x, y) \in G \times G$. Then, by view of Proposition 2.6 and Theorem 2.5, we have for every $\nu \in \mathcal{B}(e) = \mathcal{B}(ee^{-1}) = \mathcal{B}(\Omega(e, e))$ and $\gamma \in I_1$, there is $(\nu_\gamma \odot_T \nu_\gamma) \in \mathcal{B}(e,e)$ such that $\gamma T(\nu_\gamma \odot_T, \nu_\gamma) \leq \nu$.

Hence, for every $x \in G$, we get

$$\nu(x) \geq [\gamma T(\nu_\gamma \odot_T, \nu_\gamma)](x) = \gamma T(\nu_\gamma \odot_T, \nu_\gamma)(x) = \gamma T \sup \nu_\gamma(y)T \nu_\gamma(z) : (y, z) \in \Omega^{-1}(x) = \gamma T \sup \nu_\gamma(y)T \nu_\gamma(z) : \Omega(y, z) = x = \gamma T \sup \nu_\gamma(y)T \nu_\gamma(z) : yz^{-1} = x = \gamma T \sup \nu_\gamma(y)T \nu_\gamma(z^{-1}) : yz^{-1} = x.$$ 

that is, $\gamma T(\nu_\gamma \odot_T, \nu_\gamma) \leq \nu$.

Which holds (ii).

Conversely, let the stated conditions be hold. If $\nu \in \mathcal{B}(\Omega(e, e)) = \mathcal{B}(ee^{-1}) = \mathcal{B}(e)$ and $\gamma \in I_1$, then from (ii) we can get $\nu_\gamma \in \mathcal{B}(e)$ such that $\gamma T(\nu_\gamma \odot_T, \nu_\gamma) \leq \nu$.

So, as above, we can reach to $\gamma T(\nu_\gamma \odot_T, \nu_\gamma) \leq \nu$, which meaning, by view of Theorem 2.5, that the mapping $\Omega^{-1} : (G \times G, \tau(\mathcal{B}) \otimes_T \tau(\mathcal{B})) \to (G, \tau(\mathcal{B}))$ is continuous at $(e,e) \in G \times G$.

Also, as it follows from (i) and Proposition 3.3, that the left translation $\mathcal{L}_e$ is continuous at the elements $e^{-1}, e \in G$. Therefore, the continuity of the mapping $\Omega$ follows from the following composition:

$$\Omega = \mathcal{L}_e^{-1} \circ \Omega \circ (\mathcal{L}_e^{-1} \times \mathcal{L}_e^{-1}) : G \times G \to G, \text{ where we have } (e,e) \to (ee^{-1}) \to e \to alpha^{-1}, \text{ for all } (\alpha, \theta) \in G \times G.$$

Completing, by Theorem 3.1, that $(G, *, \tau(\mathcal{B}))$ is a $T$-locality group. This proves our assertion. \[ \square \]

**Theorem 3.5.** Let $(G, *)$ be a group and consider a collection $\mathcal{F} \subset I^G$, which satisfies that:

(i) $\mathcal{F}$ is an $I$-filterbase;

(ii) For all $\lambda \in \mathcal{F}$, we have $\lambda(e) = 1$;

(iii) For all $\lambda \in \mathcal{F}$, we get $\lambda \in \mathcal{F}$;

(iv) For all $\gamma \in I_1$, there is $\lambda_\gamma \in \mathcal{F}$, with $\gamma T(\lambda_\gamma \odot_T, \lambda_\gamma) \leq \lambda$.

Then there exists a unique $T$-locality system compatible with the group structure of $G$. 

\( G \) such that \( \mathcal{F} \) is a \( T \)-locality basis at \( e \in G \). This \( T \)-locality system is given by, for every \( x \in G \)

\[(3.1) \quad (\mathcal{B}(x))^{-T} = \{(1_x \odot_T \lambda) \in I^G : \lambda \in \mathcal{F}\}^{-T}.\]

Moreover \((G, \tau(\mathcal{B}))\) is an I-Regularity \( T \)-locality space.

**Proof.** It follows already from the preceding theorem that if a \( T \)-locality system exists, compatible with the group structure of \( G \), it must be given by (3.1) and so it is unique. Now, we show that \( \mathcal{B} = (\mathcal{B}(x))_{x \in G} \) is a \( T \)-locality base in \( G \) as:


Obviously, \( \mathcal{B}(x) \) is an \( I \)-filterbase.

(TLB1) For every \( \nu \in \mathcal{B}(x) \) and \( x \in G \), we have

\[
\nu(x) = (1_x \odot_T \lambda)(x), \text{ for some } \lambda \in \mathcal{F} = \lambda(x^{-1} x), \text{ by Lemma 3.2} \\
= \lambda(e), \text{ by (ii)} \\
= 1, \text{ by (ii).}
\]

(TLB2) Let \( \nu \in \mathcal{B}(x) \). Then for every \( z \in G \), we have

\[
\nu(z) = (1_x \odot_T \lambda)(z), \text{ for some } \lambda \in \mathcal{F} = \lambda(x^{-1} z), \text{ by Lemma 3.2} \\
\geq [\gamma T(\lambda_1 \odot_T \lambda_2)](x^{-1} z), \text{ by (iv)} \\
= \gamma T \sup_{z \in G} [\lambda_1(x^{-1} y) T \lambda_2(y^{-1} z)] \\
\geq \gamma T \sup_{z \in G} \sup_{x \in G} [\lambda_1(x^{-1} y) T \lambda_2(y^{-1} z)] T \lambda_3(e) \\
\geq \gamma T \lambda T(1_x \odot_T \lambda_1)(y) T(1_y \odot_T \lambda_2)(z) T \lambda_3(1), \text{ by Lemma 3.2 and (ii)}
\]

Since, for all \( \theta \in I_0 \), we can get (by continuity of \( T \)) \( \alpha = \alpha_\theta \in I_1 \), for which \( [\gamma T \lambda_1 \odot_T \lambda_2](y) \geq ([\gamma T \lambda_1 \odot_T \lambda_2](y)) \wedge \theta \), hence by taking \( \nu_\gamma \lambda_3 = (1_x \odot_T \lambda_3) \in \mathcal{B}(x) \) and \( \nu_\gamma \lambda_3 = (1_y \odot_T \lambda_3) \in \mathcal{B}(y) \), we get \( ([\gamma T \nu_\gamma \lambda_3](y)) \wedge \theta \nu_\gamma \lambda_3 \leq \nu \). That is \( \nu \) has a \( T \)-kernel, and therefore \( \mathcal{B} = (\mathcal{B}(x))_{x \in G} \) is a \( T \)-locality base in \( G \). We show that the \( T \)-locality space \((G, \tau(\mathcal{B}))\) is an \( I \)-Regularity as follows: Let \( H \in 2^G \), \( x \in G \) and \( e \in I_0 \) be are given such that there is \( \nu \in \mathcal{B}(x) \) with \( hgt(\nu \wedge 1_H) < e \). Consequently, we can find \( e_0 \) very small such that \( hgt(\nu \wedge 1_H) + e_0 < e \). Since \( \nu \in \mathcal{B}(x) \), then for all \( \gamma \in I_1 \) there are \( \lambda, \gamma, \lambda_1 \in \mathcal{F} \) such that \( \nu = (1_X \odot_T \lambda) \) and \( \lambda \geq 2 \gamma T(\lambda_1 \odot_T \lambda_2) \).
Hence
\[
\varepsilon > \text{hgt}[(1_x \odot_T \lambda) \land 1_H] + \varepsilon_0
= \sup_{z \in H}(1_x \odot_T \lambda)(z) + \varepsilon_0
= \sup_{z \in H} \lambda(x^{-1}z) + \varepsilon_0
\geq \sup_{z \in H}[\gamma T(\lambda_{\gamma} \odot_T \lambda_{\nu})](x^{-1}z) + \varepsilon_0, \text{ by (iv)}
= \sup_{z \in H, y \in G}[\gamma T\lambda_{\gamma}(x^{-1}y)T\lambda_{\gamma}(y^{-1}z)] + \varepsilon_0
\geq (\gamma + \theta)T \sup_{y \in G}[\lambda_{\gamma}(x^{-1}y)T\lambda_{\gamma}(y^{-1}z)], \theta = \theta_{\omega, \varepsilon_0} \text{ as in (2.1)}
= (\gamma + \theta)T \sup_{y \in G}[\lambda_{\gamma}(x^{-1}y)T \sup_{z \in H}(1_x \odot_T \lambda_{\gamma})(y)]
= (\gamma + \theta)T \sup_{y \in G}[\lambda_{\gamma}(x^{-1}y)(1_x \odot_T \lambda_{\gamma})(y)]
\]

Choose $\gamma_0 \in I_1$ for which $(\gamma_0 + \theta) = 1$, and taking
\[
\rho = (1_x \odot_T \lambda_{\gamma_0}), \mu = [\nu \in H(1_x \odot_T \lambda_{\gamma_0})]^{o},
\]
we get $\rho \in \mathcal{B}(x)$ and $\mu$ is an open set which satisfy, $1_H \leq \mu$ and $\text{hgt}(\rho T\mu) < \varepsilon$.
Since, for every $x \in H$, we have
\[
\mu(x) = [\nu \in H(1_x \odot_T \lambda_{\gamma_0})]^{o}(x) \geq (1_x \odot_T \lambda_{\gamma_0})^{o}(x)
= \sup_{\nu \in \mathcal{B}(x)} \nu < 1_x \odot_T \lambda_{\gamma_0}, \text{ by (2.6)}
\geq 1_x \odot_T \lambda_{\gamma_0}, \text{ since } 1_x \odot_T \lambda_{\gamma_0} \in \mathcal{B}(x), \text{ by (iii)}
= 1, \text{ by (2.4)}.
\]
This proves the $L$-Regularity of $(G, \tau(\mathcal{B}))$ and completing the proof. \hfill \square

**Proposition 3.6.** Let $(G, *)$ be a group and for all $\lambda \in I^G$, we define $\lambda_L, \lambda_R : G \times G \to I$, by $\lambda_L(x, y) = \lambda(x^{-1}y), \lambda_R(x, y) = \lambda(yx^{-1}), x, y \in G$. Then for every $\nu \in I^G$, the following hold:

(i) $\lambda_L \land \nu = \nu \odot_T \lambda_L$ and $\lambda_R \land \nu = \nu \odot_T \lambda_R$ ;

(ii) $(\lambda T \nu)_L = \lambda_L T \nu_L$ and $(\lambda T \nu)_R = \lambda_R T \nu_R$ ;

(iii) $(s\lambda)_L = s(\lambda_L)$ and $(s\lambda)_R = s(\lambda_R)$ ;

(iv) $(\lambda \odot_T \nu)_L = \nu_L \odot_T \lambda_L$ and $(\lambda \odot_T \nu)_R = \nu_R \odot_T \lambda_R$.

**Proof.** (i) For every $\nu \in I^G$ and $y \in G$, we have
\[
(\lambda_L < \nu >_T)(y) = \sup_{z \in G}[\nu(z)T\lambda_L(z, y)] = \sup_{z \in G}[\nu(z)T\lambda(z^{-1}y)]
= \sup_{z, s = z^{-1}y}[\nu(z)T\lambda(s)] = \sup_{z, s = y}[\nu(z)T\lambda(s)]
= (\nu \odot_T \lambda)(y),
\]
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and
\[ (\lambda_R < \nu \triangleright_T)(y) = \sup_{z \in G} [\nu(z) T \lambda_R(z, y)] = \sup_{z \in G} [\nu(z) T \lambda(yz^{-1})] \]
\[ = \sup_{z, s = yz^{-1}} [\nu(z) T \lambda(s)] = \sup_{z, s = yz^{-1}} [\lambda(s) T \nu(z)] \]
\[ = \sup_{s = y} [\lambda(s) T \nu(z)] = (\lambda \triangleright_T \nu)(y). \]
This proves the required equalities.

(ii) Obviously hold.

(iii) Let \( \lambda \in I^G \) and \( x, y \in G \). Then
\[ (s, \lambda)_L(x, y) = x \lambda(x^{-1} y) = \lambda((x^{-1} y)^{-1}) = \lambda(y^{-1} x) = \lambda_L(y, x) \]
\[ = s(\lambda_L)(x, y). \]
Thus \( (s, \lambda)_L = s(\lambda_L) \), and similarly we can prove \( (s, \lambda)_R = s(\lambda_R) \).

(iv) For \( \lambda, \nu \in I^G \) and \( x, y \in G \), we have
\[ (\lambda \triangleright_T \nu)_L(x, y) = (\lambda \triangleright_T \nu)(x^{-1} y) = \sup_{r = x^{-1} y} [\lambda(r) T \nu(s)] \]
\[ = \sup_{(x^{-1} y)(z^{-1} y) = x^{-1} y} [\lambda(x^{-1} z) T \nu(z^{-1} y)] \]
\[ = \sup_{z \in G} [\lambda_L(x, z) T \nu_L(z, y)] = (\nu_L \circ_T \lambda_L)(x, y), \]
and
\[ (\lambda \triangleright_T \nu)_R(x, y) = (\lambda \triangleright_T \nu)(yx^{-1}) = \sup_{r = yx^{-1}} [\lambda(r) T \nu(s)] \]
\[ = \sup_{(yx^{-1})(z^{-1} y) = yx^{-1}} [\lambda(xz^{-1}) T \nu(1)] \]
\[ = \sup_{z \in G} [\lambda_R(x, z) T \nu_R(z, y)] = (\nu_R \circ_T \lambda_R)(x, y). \]
Rendering (iv) and winds up the proof. \( \Box \)

In the following, we devote to proving that every \( T \)-locality group is \( TL \)-uniformizable. In doing so, we introduce the concepts of left and right \( TL \)-uniformities. Generally, these two \( TL \)-uniformities are not equal unless the group under consideration is commutative.

**Theorem 3.7.** Let \( (G, *, (\mathcal{B})) \) be a \( T \)-locality group, if we define \( \partial_L = \{ \lambda_L \in I^{G \times G} : \lambda \in \mathcal{B}(e) \} \) and \( \partial_R = \{ \lambda_R \in I^{G \times G} : \lambda \in \mathcal{B}(e) \} \), then \( \partial_L \) and \( \partial_R \) are \( TL \)-uniform bases.

**Proof.** If \( (G, *, (\mathcal{B})) \) be a \( T \)-locality group, then \( (G, \tau(\mathcal{B})) \) is a \( T \)-locality space with \( T \)-locality basis \( \mathcal{B} = (\mathcal{B}(x))_{x \in G} \). We claim that \( \partial_L \) is \( TL \)-uniform base.

(\text{TLUB1}) Obviously \( \partial_L \) is an \( I \)-filterbase.

(\text{TLUB2}) If \( \varphi \in \partial_L \), then there is \( \lambda \in \mathcal{B}(e) \) such that \( \varphi = \lambda_L \), and for all \( x \in G \), we get \( \varphi(x, x) = \lambda_L(x, x) = \lambda(x^{-1} x) = \lambda(e) = 1 \).

(\text{TLUB3}) If \( \varphi \in \partial_L \), then there exists a \( \lambda \in \mathcal{B}(e) \) such that \( \varphi = \lambda_L \). Thus by virtue of Theorem 3.4 (i), for all \( \gamma \in I_1 \), we can find \( \lambda_{\gamma} \in \mathcal{B}(e) \) for which
Let \( (\lambda_\gamma \circ_T \lambda_\gamma) \leq \lambda \). By taking \( \varphi_\gamma = (\lambda_\gamma)_L \in \partial_L \), we can obtain

\[
\gamma T(\varphi_\gamma \circ_T \varphi_\gamma) = \gamma T[(\lambda_\gamma)_L \circ_T (\lambda_\gamma)_L] \\
= \gamma T[\lambda_\gamma \circ_T \lambda_\gamma]_L, \text{ by Proposition 3.6 (iv)} \\
= \frac{\gamma T(\lambda_\gamma \circ_T \lambda_\gamma)_L}{\lambda_L}, \text{ clear} \\
\leq \lambda_L = \varphi.
\]

(TLUB4) If \( \varphi \in \partial_L \), then there is \( \lambda \in \mathcal{B}(e) \) such that \( \varphi = \lambda_L \). Consequently, by Theorem 3.4 (ii), for all \( \gamma \in I_1 \), there exists a \( \lambda_\gamma \in \mathcal{B}(e) \) for which \( \gamma T \lambda_\gamma \leq s_\lambda \).

Therefore, by Proposition 3.6 (iii), we get

\[
\gamma T(\lambda_\gamma)_L = (\gamma T \lambda_\gamma)_L \leq (s \lambda)_L = s(\lambda_L) \text{ implies } \gamma T \varphi_\gamma \leq s \varphi.
\]

This shows in accordance with Definition 2.7, that the collection \( \partial_L \) is a TL-uniform base, which in turn gives rise to a TL-uniformity \( \Sigma_L = \partial_L^{-1} \), and similarly for \( \partial_R \).

This completes the proof. \( \square \)

We will call \( \Sigma_L \) and \( \Sigma_R \), respectively, the left and right TL-uniformity associated with \( \mathcal{B} \).

**Definition 3.8.** An I-topological space \((X, \tau)\) is called TL-uniformizable if there is a TL-uniformity \( \Sigma \) on \( X \) such that \( \tau = \tau(\Sigma) \).

**Theorem 3.9.** Every T-locality group is TL-uniformizable.

*Proof. Let \((G, \ast, \tau(\mathcal{B}))\) be a T-locality group. Then \((G, \tau(\mathcal{B}))\) is a T-locality space with the T-locality system \( \mathcal{B}^{-T} = ((\mathcal{B}(x))^{-T})_{x \in G} \). Now, suppose that \((\Sigma(x))_{x \in G}\) is a T-locality system associated with the left TL-uniformity \( \Sigma \). Then we get

\[
\Sigma(x) = \{T \psi : T : \gamma \in \mathcal{B}(e)\}^{-T}, \text{ by Proposition 2.9} \\
= \{\lambda_L < 1_{\gamma} >_{T} : \lambda \in (\mathcal{B}(e))^{-T}\}, \text{ by Theorem 3.5} \\
= \{1_{\gamma} \circ_T \lambda : \lambda \in (\mathcal{B}(e))^{-T}\}, \text{ clear} \\
= (\mathcal{B}(x))^{-T}, \text{ by Proposition 3.3 (iv)}
\]

Therefore, in view of Definition 2.2, we have \( \tau(\mathcal{B}) = \tau(\mathcal{B}^{-T}) = \tau(\Sigma) \). Which proves that \((G, \ast, \tau(\mathcal{B}))\) is TL-uniformizable. \( \square \)

**Proposition 3.10.** Let \((G, \ast, \tau(\mathcal{B}))\) and \((E, \ast, \tau(\mathcal{B}))\) be T-locality groups. If \( \vartheta^{-T} \) and \( \vartheta^{-T} \) are the associated left TL-uniformities on \( G \) and \( E \), respectively, then \( f : (G, \vartheta^{-T}) \rightarrow (E, \vartheta^{-T}) \) is uniformly continuous if and only if for all \( \rho \in (\mathcal{E}(e))^{-1} \) and \( \gamma \in I_1 \), there is \( \lambda \in \mathcal{B}(e) \) such that \( \gamma T(1_{\gamma} \circ_T \lambda) \leq f^{-1}(f(x) \circ_T \rho) \), for each \( x \in G \), where \( e \) and \( e \) are identity elements of \( G \) and \( E \), respectively.

*Proof. In view of Definition 3.8, we have \( f : (G, \vartheta^{-T}) \rightarrow (E, \vartheta^{-T}) \) is uniformly continuous

\[
\text{if } \forall \varphi \in \mathcal{W}, \gamma \in I_1 \exists \psi \in \partial \text{ such that } \gamma T \psi \leq (f \times f)^{-1}(\varphi) \\
\text{if } \forall \rho \in \mathcal{E}(e), \gamma \in I_1 \exists \lambda \in \mathcal{B}(e) \text{ such that } \gamma T \lambda \leq (f \times f)^{-1}(\rho_L) \\
\text{if } \forall \rho \in \mathcal{E}(e), \gamma \in I_1 \exists \lambda \in \mathcal{B}(e) \text{ such that } \gamma T \lambda (x, y) \leq \rho L(f(x), f(y)), \forall x, y \in G \\
\text{if } \forall \rho \in \mathcal{E}(e), \gamma \in I_1 \exists \lambda \in \mathcal{B}(e) \text{ such that } \gamma T \lambda (x^{-1} y) \leq \rho((f(x))^{-1}, f(y)), \forall x, y \in G
\]
iff $\forall \rho \in \xi(e^\downarrow), \gamma \in I_1 \exists \lambda \in B(e)$ such that $\gamma T(1_x \circ_T \lambda)(y) \leq (1_{f(x)} \circ_T \rho)(f(y)), \forall x, y \in G$, by Lemma 3.2

iff $\forall \rho \in \xi(e^\downarrow), \gamma \in I_1 \exists \lambda \in B(e)$ such that $\gamma T(1_x \circ_T \lambda) \leq f^- (1_{f(x)} \circ_T \rho), \forall x \in G$. This winds up the proof. □

**Proposition 3.11.** Let $(G, \ast, \tau(B))$ and $(E, \sharp, \tau(\xi))$ be $T$-locality groups, with $e$ and $e^\downarrow$ as the identity elements of $G$ and $E$, respectively. Then

(i) If $\partial^T_\ast$ and $W^T$ are the associated right TL-uniformities on $G$ and $E$, respectively, then $f : (G, \partial^T_\ast \rightarrow (E, W^T)$ is uniformly continuous if and only if for all $\rho \in \xi(e^\downarrow)$ and $\gamma \in I_1$, there is $\lambda \in B(e)$ such that $\gamma T(\lambda \circ_T 1_x) \leq f^- (\rho \circ_T 1_{f(x)}),$ for each $x \in G$.

(ii) If $\partial^T_L$ and $W^T_L$ are the associated left and right (resp. right and left) TL-uniformities on $G$ and $E$, respectively, then $f : G \rightarrow E$ is uniformly continuous if and only if for all $\rho \in \xi(e^\downarrow)$ and $\gamma \in I_1$, there is $\lambda \in B(e)$ such that $\gamma T(1_x \circ_T \lambda) \leq f^- (\rho \circ_T 1_{f(x)}),$ (resp. $\gamma T(\lambda \circ_T 1_x) \leq f^- (1_{f(x)} \circ_T \rho)$), for each $x \in G$.

**Proof.** Analogous to that of Proposition 3.10. □

**Proposition 3.12.** Let $(G, \ast, \tau(B))$ and $(E, \sharp, \tau(\xi))$ be $T$-locality groups. If $\partial^T_\ast$ and $W^T$ are the associated right TL-uniformities on $G$ and $E$, respectively, then a continuous homomorphism $f : G \rightarrow E$ is uniformly continuous.

**Proof.** Let $f : G \rightarrow E$ be a continuous homomorphism, $\gamma \in I_1$ and $\rho \in \xi(e^\downarrow) = \xi f(e)$. Then by Theorem 2.5, there is $\lambda \in B(e)$ such that $\gamma T \lambda \leq f^- (\rho)$ and hence, we obtain for every $x, y \in G$ that

$$[\gamma T(1_x \circ_T \lambda)](y) = \gamma T(1_x \circ_T \lambda)(y) = \gamma T\lambda(x^{-1} y),$$

by Lemma 3.2

$\leq (f^- (\rho))(x^{-1} y) = \rho(f(x^{-1} y)),$

$= \rho(\lambda f(x^{-1} y), \text{ for } f \text{ homomorphism})$

$= \rho((f(x))^{-1} y), \text{ clear}$

$= (1_{f(x)} \circ_T \rho)(f(y)), \text{ by Lemma 3.2 again}$

$[f^- (1_{f(x)} \circ_T \rho)](y).$

That is $\gamma T(1_x \circ_T \lambda) \leq f^- (1_{f(x)} \circ_T \rho)$, which proves that $f$ is uniformly continuous. □

**Definition 3.13.** A $T$-locality space $(G, o)$ is called homogeneous space if for all $x, y \in G$, there is a homeomorphism $h : (G, o) \rightarrow (G, o)$ such that $h(x) = y$.

**Proposition 3.14.** Every $T$-locality group is a homogeneous space.

**Proof.** This follows easily from the fact that, if $(G, \ast, \tau(B))$ is a $T$-locality group, then for all $\alpha, \theta \in G$, the function $\mathcal{R}_{\alpha^{-1}} \theta : (G, \ast) \rightarrow (G, \ast)$ for which, $\mathcal{R}_{\alpha^{-1}} \theta(\alpha) = \alpha \theta = \theta$, is a homomorphism. □

**Example 3.15.** (i) For every topological group $(G, \ast, T)$, we have the topologically generated group $(G, \ast, \mathcal{M}(T))$ is a $T$-locality group, since all topologically generated spaces are $T$-locality spaces (cf. [16]). Where, $\mathcal{M}(T) = \{ \Sigma \in I^G : \Sigma \text{ is lower semicontinuous from } (G, T) \rightarrow [0, 1] \}$. 206
(ii) Let \((G, \ast)\) be a group and for every \(x \in G\), take \(\mathfrak{B}(x) = \{(1_x \ast \alpha \ast) \in I^G : \alpha \geq 1/2\}\). Then obviously \(\mathfrak{B} = (\mathfrak{B}(x))_{x \in G}\) is a \(T\)-locality base, because for every \(x \in G\), \(\mathfrak{B}(x)\) is an \(I\)-filterbase. Moreover, for every \(\partial \in \mathfrak{B}(x)\), we have
\[
\partial(x) = (1_x \ast \alpha)(x), \text{ for some } \alpha \geq 1/2
\]
Which holds (TLB1). Also, for every \(\partial \in \mathfrak{B}(x)\) and all \((y, \gamma, \theta) \in X \times I_1 \times I_0\), we can take \(\partial_\gamma = (1_y \ast \gamma \ast) \in \mathfrak{B}(x)\) and \(\partial_{y\gamma\theta} = (1_y \ast \gamma \ast \theta) \in \mathfrak{B}(y)\), which satisfy \([\gamma \partial \gamma(y) \leq \partial] \gamma \partial \gamma(y) \leq \partial\). Holds (TLB2). That is \((G, \ast, \tau(\mathfrak{B}))\) is a \(T\)-locality group.

Furthermore the left \(TL\)-uniform base induced by \((G, \ast, \tau(\mathfrak{B}))\) is \(\partial = \{\lambda_L \in I^G \times G : \lambda \in \mathfrak{B}(x)\} = \{(1_e \ast \alpha)_L \in I^G \times G : \alpha \geq 1/2\}\).

(iii) The set \(R^+\) of all positive real numbers equipped with the usual multiplication is a group. Now, if we take
\[
\mathcal{F} = \{(1_1 \ast 1_{(H_U, H)} \ast 1_x \ast 1_{(H_U, H)} = 1) \in I^{R^+} : H \subseteq R^+\} \subseteq I^{R^+},
\]
then, we have the collection \(\mathcal{F}\) satisfies the conditions in Theorem 3.5, thus there is a \(T\)-locality base \(\mathfrak{B} = (\mathfrak{B}(x))_{x \in R^+}\) in \(R^+\), given by for all \(x \in R^+\),
\[
\mathfrak{B}(x) = \{1_x \circ_T (1_1 \ast 1_{(H_U, H)} \ast 1_x \ast 1_{(H_U, H)} = 1) \in I^{R^+} : H \subseteq R^+\}.
\]
Moreover, \((R^+, \tau(\mathfrak{B}))\) is an \(L\)-Regularity \(T\)-locality space.

4. Initial and final \(T\)-locality groups

This section shows that the category \(T\text{-LocGrp}\) of \(T\)-locality groups is a topological category [1] and hence all initial and final \(T\)-locality groups exist and can be characterized.

For any class \(\Lambda\), let \(\{(H_r, \tau_r)\}_{r \in \Lambda}\) be a family of \(T\)-locality groups and \((f_r)_{r \in \Lambda}\) a family of homomorphisms of \(G\) into groups \(H_r\). For any \(T\)-locality group \((G, \tau)\), the family \((f_r : (G, \tau) \rightarrow (H_r, \tau_r))_{r \in \Lambda}\) is called an initial lift of \((f_r : G \rightarrow (H_r, \tau_r))_{r \in \Lambda}\) in the category \(T\text{-LocGrp}\) provided that \((G, \tau)\) is the \(T\)-locality group for which the following conditions are fulfilled:

(i) All mappings \(f_r : (G, \tau) \rightarrow (H_r, \tau_r)\) are continuous homomorphisms;

(ii) For any \(T\)-locality group \((H, \sigma)\) and any mapping \(f : (H, \sigma) \rightarrow (G, \tau)\) is continuous homomorphism if and only if for all \(r \in \Lambda\) the mappings \(f_r \circ f : (H, \sigma) \rightarrow (H_r, \tau_r)\) are continuous homomorphisms.

By an initial \(T\)-locality group \((G, \sigma)\) we mean the \(T\)-locality group which provides an initial lift in the category \(T\text{-LocGrp}\). To prove that all initial lifts and all initial \(T\)-locality groups exist in \(T\text{-LocGrp}\) we have to prove first that in the case \(f_r : G \rightarrow H_r\) is an injective homomorphism for each \(r \in \Lambda\), and \(\tau\) is the initial \(T\)-locality of \((\tau_r)_{r \in \Lambda}\) with respect to \((f_r)_{r \in \Lambda}\) we get that \((G, \tau)\) also is a \(T\)-locality group.

First, we shall consider the case of \(\Lambda\) being a singleton:

**Proposition 4.1.** Let \((H, \sigma)\) be a \(T\)-locality group and let \(f : G \rightarrow H\) be an injective homomorphism of a group \(G\) into \(H\). Then the initial \(T\)-locality space \((G, f^{-1}(\sigma))\) of \((H, \sigma)\) with respect to \(f\) also is a \(T\)-locality group.

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Proof. Let $\Omega_G : (G \times G, f^- (\sigma) \times f^- (\sigma)) \to (G, f^- (\sigma))$ and $\Omega_H : (H \times H, \sigma \times \sigma) \to (H, \sigma)$ be defined as in Theorem 3.1 and let $\lambda \in f^- (\sigma)$. Then, there is $\Sigma \in \sigma$ such that $\lambda = f^- (\Sigma)$. Since $(H, \sigma)$ is a $T$-locality group, it follows that $\Omega_H$ is continuous and hence $\Omega_H (\Sigma) \in \sigma \times \sigma$. Now, since $f$ is a homomorphism, then for every $x, y \in G$, we have

$$\Omega_G^- (\lambda) (x, y) = \lambda (\Omega_G (x, y)) = (f^- (\Sigma))(xy^{-1}) = \Sigma (f(xy^{-1}))$$

$$= \Sigma (f(x)f(y^{-1})) = \Sigma (f(x)f(y)^{-1})$$

$$= \Sigma (\Omega_H(f(x), f(y))) = (\Omega_H^- (\Sigma))(f(x), f(y))$$

$$= [(f \times f)^- (\Omega_H^- (\Sigma))](x, y),$$

that is, $\Omega_G^- (\lambda) = (f \times f)^- (\Omega_H^- (\Sigma))$. Since $f^- (\sigma)$ is the initial $I$-topology of $\sigma$ with respect to $f$, then $f : (G, f^- (\sigma)) \to (H, \sigma)$ is continuous and hence $f \times f : G \times G \to H \times H$ is obviously continuous. Therefore,

$$\Omega_G^- (\lambda) = (f \times f)^- (\Omega_H^- (\Sigma)) \in (f \times f)^- (\sigma \times \sigma) = f^- (\sigma) \times f^- (\sigma),$$

which means that $\Omega_G$ is continuous and hence, by Theorem 3.1, $(G, f^- (\sigma))$ is $T$-locality group. \hfill \Box

Now, consider the case of any class $\Lambda$ :

**Proposition 4.2.** Let $((H_r, \sigma_r))_{r \in \Lambda}$ be a family of $T$-locality groups and for all $r \in \Lambda$, let $f_r : G \to H_r$ be an injective homomorphism of a group $G$ into $H_r$. Let $\tau$ be the initial $I$-topology of $(\sigma_r)_{r \in \Lambda}$ with respect to $\{f_r\}_{r \in \Lambda}$. Then $(G, \tau)$ also is $T$-locality group.

**Proof.** Let $\Omega_G : (G \times G, \tau \times \tau) \to (G, \tau)$ and $\Omega_H : (H_r \times H_r, \sigma_r \times \sigma_r) \to (H_r, \sigma_r)$ be defined as in Theorem 3.1. Since $f_r \circ \Omega_G = \Omega_H \circ (f_r \times f_r)$ and $\Omega_H, f_r$ are continuous, then $f_r \circ \Omega_G$ is continuous. From condition (ii) of the initial lift in the category of $I$-topological spaces, it follows that $\Omega_G$ is continuous and thus $(G, \tau)$ is $T$-locality group. \hfill \Box

The following theorem shows that the $T$-locality group mentioned in Propositions 4.1 and 4.2 fulfills the conditions of an initial lift in the concrete category $T$-LocGrp.

**Theorem 4.3.** Let $((H_r, \sigma_r))_{r \in \Lambda}$ be a family of $T$-locality groups and for all $r \in \Lambda$, let $f_r : G \to H_r$ be an injective homomorphism of a group $G$ into $H_r$ and let $\tau$ be the initial $I$-topology of $(\sigma_r)_{r \in \Lambda}$ with respect to $\{f_r\}_{r \in \Lambda}$. Then $(f_r : (G, \tau) \to (H_r, \sigma_r))_{r \in \Lambda}$ is an initial lift of $(f_r : G \to (H_r, \sigma_r))_{r \in \Lambda}$ in the category $T$-LocGrp.

**Proof.** First, Propositions 4.1 and 4.2 show that $(G, \tau)$ is a $T$-locality group. From condition (i) of an initial lift in the category of $I$-topological spaces, we get that condition (i) of an initial lift in $T$-LocGrp holds, that is, all $f_r : (G, \tau) \to (H_r, \sigma_r)$ are continuous homomorphisms. Here, let $(H, \sigma)$ be a $T$-locality group and $f$ be a mapping from $H$ into $G$. Then from condition (ii) of an initial lift in the category of $I$-topological spaces, we get $f : (H, \sigma) \to (G, \tau)$ is continuous if and only if all $f_r \circ f : (H, \sigma) \to (H_r, \sigma_r)$ are continuous. Now, if $f$ is a homomorphism and we have all $f_r$ are homomorphisms, then all $f_r \circ f$ are homomorphisms. Conversely, let
all \( f_r \circ f \) be homomorphisms. Since all \( f_r \) are homomorphisms we have for every \( x, y \in H \), that

\[
f_r(f(x, y)) = (f_r \circ f)(xy) = [(f_r \circ f)(x)][(f_r \circ f)(y)] = [f_r(f(x))][f_r(f(y))] = f_r(f(x)f(y)).
\]

Moreover, since \( f_r \) is injective, we get \( f(xy) = f(x)f(y) \), that is, \( f \) is a homomorphism. Hence, \( f : (H, \sigma) \to (G, \tau) \) is continuous homomorphism if and only if all \( f_r \circ f : (H, \sigma) \to (H_r, \sigma_r) \) are continuous homomorphisms, that is, condition (ii) of an initial lift in \( T\text{-LocGrp} \) is fulfilled. □

Theorem 4.3 states that all initial lifts exist uniquely in the concrete category \( T\text{-LocGrp} \) and this means that the category \( T\text{-LocGrp} \) is a topological category [1]. Hence, all initial \( T \)-locality groups exist.

By means of Theorem 4.3, the \( T \)-locality groups introduced in Propositions 4.1 and 4.2 coincide with the initial \( T \)-locality groups, that is, if \( ((H_r, \sigma_r))_{r \in \Lambda} \) is a family of \( T \)-locality groups, and for each \( r \in \Lambda \), \( f_r \) is an injective homomorphism of a group \( G \) into \( H_r \) and \( \tau \) is the initial \( I \)-topology of \( (\sigma_r)_{r \in \Lambda} \) with respect to \( (f_r)_{r \in \Lambda} \), then \( (G, \tau) \) is the initial \( T \)-locality group.

**T-locality subgroups and T-locality product groups** are special initial \( T \)-locality groups and hence the above implies the following result.

**Corollary 4.4.** (i) If \( (G, \tau) \) is a \( T \)-locality groups and \( S \) a subgroup of \( G \), then the \( I \)-topological subspace \( (G, \tau_S) \) also is \( T \)-locality group, called a \( T \)-locality subgroup.

(ii) If \( ((G_r, \tau_r))_{r \in \Lambda} \) is a family of \( T \)-locality groups and \( G \) is the product \( \Pi_{r \in \Lambda} G_r \) of the family \( (G_r)_{r \in \Lambda} \) of a groups and \( \tau = \Pi_{r \in \Lambda} \tau_r \) is the product of the family \( (\tau_r)_{r \in \Lambda} \) of \( I \)-topologies, then \( (G, \tau) \) also is \( T \)-locality group, called a \( T \)-locality product group.

Now, since the concrete category \( T\text{-LocGrp} \) is topological category, then all final lifts also uniquely exist [1]. This even means that also all final \( T \)-locality groups exist.

If \( ((G_r, \tau_r))_{r \in \Lambda} \) is a family of \( T \)-locality groups and \( (f_r)_{r \in \Lambda} \) a family of homomorphisms of \( G_r \) into a group \( H \), indexed by any class \( \Lambda \). For any \( T \)-locality group \( (H, \sigma) \), the family \( (f_r : (G_r, \tau_r) \to (H, \sigma))_{r \in \Lambda} \) is called a final lift of \( (f_r : (G_r, \tau_r) \to H)_{r \in \Lambda} \) in the category \( T\text{-LocGrp} \) provided that \( (H, \sigma) \) is the \( T \)-locality group for which fulfills the following conditions:

(i) All mappings \( f_r : (G_r, \tau_r) \to (H, \sigma) \) are continuous homomorphisms;

(ii) For any \( T \)-locality group \( (G, \tau) \) and any mapping \( f : (H, \sigma) \to (G, \tau) \) is continuous homomorphism if and only if for all \( r \in \Lambda \) the mappings \( f \circ f_r : (G_r, \tau_r) \to (G, \tau) \) are continuous homomorphisms.

By a **final \( T \)-locality group** we mean a \( T \)-locality group which provides a final lift in the category \( T\text{-LocGrp} \).

The following propositions show that if for each \( r \in \Lambda \), \( f_r : (G_r, \tau_r) \to (H, \sigma) \) is a surjective homomorphism and \( \sigma \) is the final \( T \)-locality of \( (\tau_r)_{r \in \Lambda} \) with respect to \( (f_r)_{r \in \Lambda} \), then \( (H, \sigma) \) also is a \( T \)-locality group.

To prove these results we need the following proposition which can be proved easily by means of the properties of the \( T \)-locality group.
Proposition 4.5. If \( f : (G, \tau) \to (H, f(\tau)) \) is a surjective homomorphism from a \( T \)-locality group \((G, \tau)\) to a group \( H \) equipped with the final \( I \)-topology \( f(\tau) \) of \( \tau \) with respect to \( f \), then \( f \) is an open function.

Consider the case of \( \Lambda \) being a singleton:

Proposition 4.6. Let \((G, \tau)\) be a \( T \)-locality group and let \( f : (G, \tau) \to H \) be a homomorphism of a group \( G \) onto \( H \). Then the final \( T \)-locality space \((H, f(\tau))\) of \((G, \tau)\) with respect to \( f \) also is a \( T \)-locality group.

Proof. Let \( \Omega_G : (G \times G, \tau \times \tau) \to (G, \tau) \) and \( \Omega_H : (H \times H, f(\tau) \times f(\tau)) \to (H, f(\tau)) \) be defined as in Theorem 3.1. Now, since \( f \) is a surjective homomorphism, then for every \( \Sigma \in I^H \) and \( x, y \in H \), we have

\[
(\Omega_H^r(\Sigma))(x, y) = \Sigma(\Omega_H(x, y))
\]

that is, \( \Omega_H^r(\Sigma) = (f \times f)(\Omega_G^r(f^\prime(\Sigma))) \). If \( \Sigma \in f(\tau) \), then \( f^\prime(\Sigma) \in \tau \), and from continuity of \( \Omega_G \) we get \( \Omega_G^r(f^\prime(\Sigma)) \in \tau \times \tau \). But from Proposition 4.5, we have \( f \) is an open, hence \( f \times f : G \times G \to H \times H \) is obviously an open. Therefore,

\[
\Omega_H^r(\Sigma) = (f \times f)(\Omega_G^r(f^\prime(\Sigma))) \in f(\tau) \times f(\tau).
\]

Which proves the continuity of \( \Omega_H \) and this implies that \((H, f(\tau))\) is \( T \)-locality group.

For any class \( \Lambda \) we have the following:

Proposition 4.7. Let \(((G_r, \tau_r))_{r \in \Lambda}\) be a family of \( T \)-locality groups and for all \( r \in \Lambda \), let \( f_r : G_r \to H \) be a homomorphism of a group \( G_r \) onto a group \( H \). Let \( \sigma \) be the final \( I \)-topology of \((\tau_r)_{r \in \Lambda}\) with respect to \((f_r)_{r \in \Lambda}\). Then \((H, \sigma)\) is a \( T \)-locality group.

Proof. Let \( \mu \in \sigma \). Since \( f_r : (G_r, \tau_r) \to (H, f(\tau)) \) is continuous, then \( f_r^\prime(\mu) \in \tau_r \) for all \( r \in \Lambda \). But from continuity of \( \Omega_{G_r} : (G_r \times G_r, \tau_r \times \tau_r) \to (G_r, \tau_r) \), we get \( \Omega_{G_r}^r(f_r^\prime(\mu)) \in \tau_r \times \tau_r \). Now, by a similar way to the proof of Proposition 4.6, we have \( \Omega_H^r(\mu) = (f_r \times f_r)(\Omega_G^r(f_r^\prime(\mu))) \), where \( \Omega_H : (H \times H, \sigma \times \sigma) \to (H, \sigma) \), moreover all \( f_r \times f_r \) are open, hence \( \Omega_H^r(\mu) \in \sigma \times \sigma \). This proves that \( \Omega_H \) is continuous and thus \((H, \sigma)\) is a \( T \)-locality group.

Now we are going to show that the \( T \)-locality group given in Propositions 4.6 and 4.7 fulfills the conditions of a final lift in the concrete category \( T \)-LocGrp.
Theorem 4.8. Let \((G_r, \tau_r)_{r \in \Lambda}\) be a family of \(T\)-locality groups and for all \(r \in \Lambda\), let \(f_r : G_r \to H\) be a surjective homomorphism of a group \(G_r\) into \(H\) and let \(\sigma\) be the final I-topology of \((\tau_r)_{r \in \Lambda}\) with respect to \((f_r)_{r \in \Lambda}\). Then \((f_r : (G_r, \tau_r) \to (H, \sigma))_{r \in \Lambda}\) is a final lift of \((f_r : G_r \to H)_{r \in \Lambda}\) in the category \(T\text{-LocGrp}\). The proof goes similarly, using Propositions 4.6 and 4.7 and the properties of final lift in the category \(T\text{-LocGrp}\), as in case of Theorem 4.3.

From Theorem 4.8 we get that the \(T\)-locality groups introduced in Propositions 4.6 and 4.7 can be considered as the final \(T\)-locality groups. The final \(T\)-locality quotient group is special final \(T\)-locality group and hence the above implies the following result.

Corollary 4.9. If \(N\) is a normal subgroup of a \(T\)-locality group \((G, \tau)\) and \(G/N\) is the corresponding quotient group and if \(h : G \to G/N\) is the canonical homomorphism defined by \(h(x) = xN\) for all \(x \in G\), then the I-topological quotient space \((G/N, h(\tau))\) also is \(T\)-locality group, called a \(T\)-locality quotient group.

References

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