On the topological structure of intuitionistic fuzzy soft sets

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ABSTRACT. The aim of this paper is to construct topologies on intuitionistic fuzzy soft sets. The concept of intuitionistic fuzzy soft topologies is introduced and their basic properties are given.

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1. INTRODUCTION

To solve complicated problems in economics, engineering, environmental science and social science, methods in classical mathematics are not always successful because of various types of uncertainties present in these problems. There are several theories: probability theory, fuzzy set theory [12], rough set theory [9] and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. For example, probability theory can deal only with stochastically stable phenomena (see [8]). To overcome these kinds of difficulties, Molodtsov [8] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

In soft set theory, we observe that in most cases the parameters are vague words or sentences involve vague words. Considering this point, Maji et al. [7] proposed the concept of intuitionistic fuzzy soft sets by combining soft sets with intuitionistic fuzzy sets. Gunduz et al. [6] introduced the concept of intuitionistic fuzzy soft module and simplified the sigh of intuitionistic fuzzy soft sets.

In this paper, we define intuitionistic fuzzy soft topologies and give their basic properties.

2. Preliminaries

Throughout this paper, $U$ denotes initial universe, $E$ denotes the set of all possible parameters and $I$ denotes $[0, 1]$. We only consider the case where $U$ and $E$ are both nonempty finite sets. “Intuitionistic fuzzy” is briefly “IF”.

2.1. On lattices.

Definition 2.1. Let $(L, \leq)$ be a poset.

1. $L$ is called a lattice, if $a \lor b \in L$, $a \land b \in L$ for any $a, b \in L$.
2. $L$ is called a complete lattice, if $\forall S \subseteq L$ for any $S \subseteq L$.
3. $L$ is called distributive, if $a \lor (b \land c) = (a \lor b) \land (a \lor c)$, $a \land (b \lor c) = (a \land b) \lor (a \land c)$ for any $a, b, c \in L$.
4. $L$ is called a complete distributive lattice (resp. a distributive lattice), if $L$ is a complete lattice (resp. a lattice) and distributive.

Remark 2.5. (1) $(J, \leq)$ is a poset with $0_J = (0, 1)$ and $1_J = (1, 0)$.

2.2. Let $(a, b), (c, d) \in I \times I$. We define

1. $(a, b) = (c, d) \iff a = c, b = d$.
2. $(a, b) \cup (c, d) = (a \lor c, b \land d), (a, b) \cap (c, d) = (a \land c, b \lor d)$.
3. $(a, b)' = (b, a)$.

Moreover, for $\{(a_\alpha, b_\alpha) : \alpha \in \Gamma\} \subseteq I \times I$,

\[
\bigcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigvee_{\alpha \in \Gamma} a_\alpha, \bigwedge_{\alpha \in \Gamma} b_\alpha) \quad \text{and} \quad \bigcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigwedge_{\alpha \in \Gamma} a_\alpha, \bigvee_{\alpha \in \Gamma} b_\alpha).
\]

Definition 2.4. Let $(a, b), (c, d) \in J$ and let $S \subseteq J \times J$. $(a, b)(c, d)$, if $a \leq c$ and $b \geq d$. We denote $S$ by $\leq$.

Remark 2.5. (2) $(J, \leq)$ is a poset with $0_J = (0, 1)$ and $1_J = (1, 0)$.

(1) $(a, b)' = (b, a)$.

(3) \[(a, b) \cup (c, d) \cup (e, f) = (a, b) \cup ((c, d) \cup (e, f)),\]
\[\quad ((a, b) \cap (c, d)) \cap (e, f) = (a, b) \cap ((c, d) \cap (e, f)).\]
(4) \((a, b) \cup (c, d) = (c, d) \cup (a, b)\), \((a, b) \cap (c, d) = (c, d) \cap (a, b)\).

(5) \(\{(a, b) \cup (c, d)\} \cap (e, f) = \{(a, b) \cap (e, f)\} \cup ((c, d) \cap (e, f))\).

(6) \((\bigcup_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha}))' = \bigcap_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha})', (\bigcap_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha}))' = \bigcup_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha})'\).

**Proposition 2.6.** \((J, \leq, \cap, \cup, J)\) is a complete distributive lattice.

**Proof.** By Remark 2.5 \((J, \leq)\) is a poset.

For any \(\{(a_{\alpha}, b_{\alpha}) : \alpha \in \Gamma\} \subseteq J\), \(\bigcup_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha}) = (\bigvee_{\alpha \in \Gamma} a_{\alpha}, \bigwedge_{\alpha \in \Gamma} b_{\alpha})\). Put

\[
\bigwedge_{\alpha \in \Gamma} a_{\alpha} = a, \bigvee_{\alpha \in \Gamma} b_{\alpha} = b.
\]

For any \(\varepsilon > 0\), there exists \(\alpha_{\varepsilon} \in \Gamma\) such that \(a_{\alpha_{\varepsilon}} > a - \varepsilon\). Then \(a + b - \varepsilon < a_{\alpha_{\varepsilon}} + b \leq a_{\alpha_{\varepsilon}} + b \leq 1\). Thus \(a + b \leq 1\). This implies \(\bigcup_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha}) \in J\).

Similarly, we can prove \(\bigcap_{\alpha \in \Gamma} (a_{\alpha}, b_{\alpha}) \in J\). Hence \(J\) is a complete lattice. By Remark 2.5 \(J\) is a distributive. Therefore, \(J\) is a complete distributive lattice. \(\Box\)

2.2. **IF sets and IF topologies.** In this paper, \(\mathcal{F}(U)\) denotes the family of all fuzzy sets in \(U\).

**Definition 2.7 (1][2].** An IF set \(A \in U\) is an object having the from

\[
A = \{x, \mu_A(x), \nu_A(x) > : x \in U\},
\]

where \(\mu_A, \nu_A \in \mathcal{F}(U)\) satisfying \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) for each \(x \in U\), and \(\mu_A(x), \nu_A(x)\) are used to define the degree of membership and the degree of non-membership of the element \(x\) to \(A\), respectively.

For the sake of simplicity, we redefine Definition 2.7 and give Definition 2.8.

**Definition 2.8.** \(A\) is called an IF set in \(U\), if \(A = (\mu_A, \nu_A)\) and for each \(x \in U\), \(A(x) = (\mu_A(x), \nu_A(x)) \in J\), where \(\mu_A, \nu_A \in \mathcal{F}(U)\) are used to define the degree of membership and the degree of non-membership of the element \(x\) to \(A\), respectively.

In this paper, \(\mathcal{IF}(U)\) denotes the family of all IF sets in \(U\). Let \(A, B \in \mathcal{IF}(U)\) and let \((J, \leq, \cap, \cup, J)\) be a complete distributive lattice. Then some IF relations and IF operations are defined as follows:

(1) \(A = B \iff A(x) = B(x)\) for each \(x \in U\).

(2) \(A \subseteq B \iff A(x) \leq B(x)\) for each \(x \in U\).

(3) \((A \cup B)(x) = A(x) \cup B(x)\) for each \(x \in U\).

(4) \((A \cap B)(x) = A(x) \cap B(x)\) for each \(x \in U\).

(5) \(A'(x) = A(x)'\) for each \(x \in U\).

Moreover,

\[
(\bigcup_{\alpha \in \Gamma} A_{\alpha})(x) = \bigcup_{\alpha \in \Gamma} A_{\alpha}(x) \text{ for each } x \in U
\]

and

\[
(\bigcap_{\alpha \in \Gamma} A_{\alpha})(x) = \bigcap_{\alpha \in \Gamma} A_{\alpha}(x) \text{ for each } x \in U,
\]

where \(\{A_{\alpha} : \alpha \in \Gamma\} \subseteq \mathcal{IF}(U)\).

Obviously, \(A = B \iff A \subseteq B\) and \(B \subseteq A\).
Example 2.9. Let \( U = \{x_1, x_2, x_3, x_4, x_5\} \). We defined
\[
A(x_1) = (0.2, 0.8), A(x_2) = (0.6, 0.3), A(x_3) = (0, 1), A(x_4) = (1, 0), A(x_5) = (0.4, 0.4).
\]
Then \( A \in \mathcal{IF}(U) \). We denote it by
\[
A = \left(\frac{0.2, 0.8}{x_1} + \frac{0.6, 0.3}{x_2} + \frac{0.1}{x_3} + \frac{1, 0}{x_4} + \frac{0.4, 0.4}{x_5}\right).\]
We have
\[
A' = \left(\frac{0.8, 0.2}{x_1} + \frac{0.3, 0.6}{x_2} + \frac{0.1}{x_3} + \frac{1, 0}{x_4} + \frac{0.4, 0.4}{x_5}\right).
\]
Let
\[
B = \left(\frac{0.2, 0.5}{x_1} + \frac{0.2, 0.8}{x_2} + \frac{0.8, 0.1}{x_3} + \frac{0.3, 0.6}{x_4} + \frac{0.1}{x_5}\right).
\]
Then
\[
A \cap B = \left(\frac{0.2, 0.8}{x_1} + \frac{0.2, 0.8}{x_2} + \frac{0.1}{x_3} + \frac{0.3, 0.6}{x_4} + \frac{0.1}{x_5}\right),
\]
\[
A \cup B = \left(\frac{0.2, 0.5}{x_1} + \frac{0.6, 0.3}{x_2} + \frac{0.8, 0.1}{x_3} + \frac{1, 0}{x_4} + \frac{0.4, 0.4}{x_5}\right).
\]

Definition 2.10 ([13]). Let \( A \in \mathcal{IF}(U) \).

(1) \( A \) is called an IF empty set, if \( A(x) = (0, 1) \) for each \( x \in U \). We denote it by \( \tilde{0} \).

(2) \( A \) is called an IF universe set, if \( A(x) = (1, 0) \) for each \( x \in U \). We denote it by \( 1 \).

Definition 2.11 ([3]). Let \( \tau \subseteq \mathcal{IF}(U) \). Then \( \tau \) is called an IF topology on \( U \), if
(i) \( \tilde{0}, 1 \in \tau \),
(ii) \( A, B \in \tau \) implies \( A \cap B \in \tau \),
(iii) \( \{A_\alpha : \alpha \in \Gamma\} \subseteq \tau \) implies \( \bigcup \{A_\alpha : \alpha \in \Gamma\} \in \tau \).

The pair \( (U, \tau) \) is called an IF topological space. Every member of \( \tau \) is called an IF open set in \( U \). \( A \) is called an IF closed set in \( U \) if \( A' \in \tau \).

Example 2.12. Let \( A \) and \( B \) are the IF in Example 2.9 Then
\[
\tau = \{\tilde{0}, 1, A, B, A \cap B, A \cup B\}
\]
is an IF topology on \( U \).

2.3. IF soft sets.

Definition 2.13 ([5]). Let \( A \subseteq E \). A pair \( (f, A) \) is called a fuzzy soft set over \( U \), if \( f \) is a mapping given by \( f : A \rightarrow \mathcal{F}(U) \). We denote \( (f, A) \) by \( f_A \).

In other words, a fuzzy soft set \( f_A \) over \( U \) is a parameterized family of fuzzy sets in the universe \( U \).

Definition 2.14 ([6, 7]). Let \( A \subseteq E \). A pair \( (f, A) \) is called an IF soft set over \( U \), where \( f \) is a mapping given by \( f : A \rightarrow \mathcal{IF}(U) \). We denote \( (f, A) \) (resp. \( \mu_{f(e)}, \mu_{f(e)} \)) by \( f_A \) (resp. \( f, f^0 \)).
In other words, an IF soft set $f_A$ over $U$ is a parameterized family of IF sets in the universe $U$, and $\mu_{f(e)} = f_e \in \mathcal{F}(U)$, $\nu_{f(e)} = f_e^c \in \mathcal{F}(U)$, $f(e) = (f_e, f_e^c) \in IFS(U)$ and $f(e)(x) = (f_e(x), f_e^c(x)) \in J$ for any $e \in A$ and $x \in U$.

Let $A \subseteq E$. Denote
$$IFS(U)_A = \{f_A : f_A \text{ is an IF soft set over } U\},$$
$$IFS(U) = \{f_A : f_A \text{ is an IF soft set over } U \text{ and } A \subseteq E\}.$$

Obviously,
$$IFS(U) = \bigcup_{A \subseteq E} IFS(U)_A.$$

**Example 2.15.** Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a universe consisting of five houses as possible alternatives, and let $A = \{e_1, e_2, e_3, e_4\} \subseteq E$ be a set of parameters considered by the decision makers, where $e_1$, $e_2$, $e_3$ and $e_4$ represent the parameters “beautiful”, “modern”, “cheap” and “in the green surroundings”, respectively.

Now, we consider a soft set $f_A$, which describes the “attractiveness of the houses” that Mr. X is going to buy. In this case, to define the soft set $f_A$ means to point out beautiful houses, modern houses and so on. Consider the mapping $f$ given by “houses(.)”, where (.) is to be filled in by one of the parameters $e_i \in A$. For instance, $f(e_1)$ means “houses(beautiful)”, and its functional value is the set consisting of all the beautiful houses in $U$. Let $f_A$ be an IF soft set over $U$, defined as follows

$$f(e_1) = \frac{(0.2, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2} + \frac{(0.2, 0.6)}{x_3} + \frac{(0.4, 0.5)}{x_4} + \frac{(0.4, 0.4)}{x_5},$$
$$f(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.9, 0.1)}{x_2} + \frac{(0.1, 0.8)}{x_3} + \frac{(0.3, 0.6)}{x_4} + \frac{(0.5, 0.3)}{x_5},$$
$$f(e_3) = \frac{(0.1)}{x_1} + \frac{(0.1)}{x_2} + \frac{(0.5, 0.4)}{x_3} + \frac{(0.1)}{x_4} + \frac{(0.1)}{x_5},$$
$$f(e_4) = \frac{(0.1)}{x_1} + \frac{(0.5, 0.5)}{x_2} + \frac{(0.1)}{x_3} + \frac{(0.5, 0.4)}{x_4} + \frac{(0.1)}{x_5}.$$

Then the IF soft set $f_A$ is described by the following Table 1.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.2, 0.8)</td>
<td>(0.2, 0.5)</td>
<td>(0.1)</td>
<td>(0.1)</td>
</tr>
<tr>
<td>(0.6, 0.3)</td>
<td>(0.9, 0.1)</td>
<td>(0.1)</td>
<td>(0.5, 0.5)</td>
</tr>
<tr>
<td>(0.2, 0.6)</td>
<td>(0.1, 0.8)</td>
<td>(0.1)</td>
<td>(0.1)</td>
</tr>
<tr>
<td>(0.4, 0.5)</td>
<td>(0.3, 0.6)</td>
<td>(0.5, 0.4)</td>
<td>(0.5, 0.4)</td>
</tr>
<tr>
<td>(0.4, 0.4)</td>
<td>(0.5, 0.3)</td>
<td>(0.1)</td>
<td>(0.1)</td>
</tr>
</tbody>
</table>

**Definition 2.16** ([7]). Let $A, B \in IFS(U)$.

1. $f_A$ is called IF soft subset of $g_B$, if $A \subseteq B$ and $f(e) \subseteq g(e)$ for any $e \in A$. We write $f_A \subseteq g_B$. 
2. $f_A$ and $g_B$ are called IF soft equal, if $f_A \subseteq g_B$ and $g_A \subseteq f_B$. We write $f_A = g_B$. 

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Obviously, \( f_A = g_B \) if and only if \( A = B \) and \( f(e) = g(e) \) for any \( e \in A \).

**Definition 2.17** ([7]). Let \( A, B \in \mathcal{IFS}(U) \).

1. The intersection of \( f_A \) and \( g_B \) is the IF soft set \( h_C \), where \( C = A \cap B \), and \( h(e) = f(e) \cap g(e) \) for any \( e \in C \). We write \( f_A \cap g_B = h_C \).
2. The union \( f_A \) and \( g_B \) is the IF soft set \( h_C \), where \( C = A \cup B \), and for any \( e \in C \),

\[
h(e) = \begin{cases} f(e), & e \in A - B \\ g(e), & e \in B - A \\ f(e) \cup g(e), & e \in A \cap B \end{cases}
\]

We write \( f_A \cup g_B = h_C \).

### 3. IF SOFT TOPOLOGIES

In this section, we introduce the concept of IF soft topologies.

**Definition 3.1.** The relative complement of an IF soft set \( f_E \) is denoted by \( f_E' \) and is defined by

\[
(f_E)' = f_E',
\]

where

\[
f' : E \to \mathcal{IFS}(U)
\]

is a mapping given by \( f'(e) = (f(e))' \) for each \( e \in E \).

**Proposition 3.2.** Let \( f_E, g_E \in \mathcal{IFS}(U)_E \). Then

\[
(f_E \cap g_E)' = f_E' \cup g_E', \quad (f_E \cup g_E)' = f_E' \cap g_E'.
\]

**Proof.** Let \( f_E \cap g_E = h_E \), where

\[
h'(e) = (h(e))' = (f(e) \cap g(e))' = (f(e))' \cup (g(e))' = f'(e) \cup g'(e)
\]

for any \( e \in E \). Then \( (f_E \cap g_E)' = h'_E \) and \( f_E' \cap g'_E = h'_E \). Thus

\[
(f_E \cap g_E)' = f_E' \cup g_E'.
\]

Similarly, we can prove \( (f_E \cup g_E)' = f_E' \cap g_E' \). \( \square \)

**Definition 3.3.** Let \( f_E \in \mathcal{IFS}(U)_E \).

1. \( f_E \) is called absolute IF soft over \( U \), if \( f(e) = 1 \) for any \( e \in E \). We denoted it by \( U_E \).
2. \( f_E \) is called relative null IF soft over \( U \), if \( f(e) = 0 \) for any \( e \in E \). We denoted it by \( \varnothing_E \).

Obviously, \( \varnothing_E = U'_E \) and \( U_E = \varnothing'_E \).

**Definition 3.4.** Let \( \tau \subseteq \mathcal{IFS}(U)_E \) and \( \tau' = \{ f_E : f'_E \in \tau \} \). Then \( \tau \) is called an IF soft topology on \( U \) if the following conditions are satisfied:

1. \( U_E, \varnothing_E \in \tau \).
2. \( f_E, g_E \in \tau \) implies \( f_E \cap g_E \in \tau \).
3. \( \{ (f_\alpha)_E : \alpha \in \Gamma \} \subseteq \tau \) implies \( \bigcup \{ (f_\alpha)_E : \alpha \in \Gamma \} \in \tau \).

The pair \((U, \tau, E)\) is called an IF soft topological space over \( U \). Every member of \( \tau \) is called an IF soft open set in \( U \). \( f_E \) is called an IF soft closed set in \( U \) if \( f_E \in \tau' \).
Example 3.5. Let \( U = \{x_1, x_2\} \) and \( E = \{e_1, e_2\} \). Let \( f_E, g_E, h_E, l_E \in \mathcal{IFS}(U)_E \) where
\[
\begin{align*}
f(e_1) &= \frac{(0.2, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2}, \quad f(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.9, 0.1)}{x_2}; \\
g(e_1) &= \frac{(0.1, 0.8)}{x_1} + \frac{(0.6, 0.1)}{x_2}, \quad g(e_2) = \frac{(0.2, 0.8)}{x_1} + \frac{(0.8, 0.1)}{x_2}; \\
h(e_1) &= \frac{(0.2, 0.8)}{x_1} + \frac{(0.6, 0.1)}{x_2}, \quad h(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.9, 0.1)}{x_2}; \\
l(e_1) &= \frac{(0.1, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2}, \quad l(e_2) = \frac{(0.2, 0.8)}{x_1} + \frac{(0.8, 0.1)}{x_2}. 
\end{align*}
\]
We have
\[
\begin{align*}
f'(e_1) &= \frac{(0.8, 0.2)}{x_1} + \frac{(0.3, 0.6)}{x_2}, \quad f'(e_2) = \frac{(0.5, 0.2)}{x_1} + \frac{(0.1, 0.9)}{x_2}.
\end{align*}
\]
Then \( h_E = f_E \cup g_E \) and \( l_E = f_E \cap g_E \). Thus \( \tau = \{f_E, g_E, h_E, l_E, \emptyset, E, U_E\} \) is an IF soft topology on \( U \).

Proposition 3.6. Let \((U, \tau_1, E)\) and \((U, \tau_2, E)\) be two IF soft topologies over \( U \). Denote \( \tau_1 \cap \tau_2 = \{f_E : f_E \in \tau_1 \text{ and } f_E \in \tau_2\} \). Then \( \tau_1 \cap \tau_2 \) is an IF soft topology on \( U \).

Proof. Obviously \( \emptyset, U_E \in \tau_1 \cap \tau_2 \). Let \( f_E, g_E \in \tau_1 \cap \tau_2 \). Then \( f_E, g_E \in \tau_1 \) and \( f_E, g_E \in \tau_2 \). Note that \( \tau_1 \) and \( \tau_2 \) are two IF soft topologies on \( U \). Then \( f_E \cap g_E \in \tau_1 \) and \( f_E \cap g_E \in \tau_2 \). Hence \( f_E \cap g_E \in \tau_1 \cap \tau_2 \). Let \( \{(f_\alpha)_E : \alpha \in \Gamma\} \subseteq \tau_1 \cap \tau_2 \). Then \( (f_\alpha)_E \in \tau_1 \) and \( (f_\alpha)_E \in \tau_2 \) for any \( \alpha \in \Gamma \). Since \( \tau_1 \) and \( \tau_2 \) are two IF soft topologies on \( U \), \( \bigcup \{(f_\alpha)_E : \alpha \in \Gamma\} \in \tau_1 \cap \tau_2 \). Thus \( \bigcup \{(f_\alpha)_E : \alpha \in \Gamma\} \in \tau_1 \cap \tau_2 \). \( \square \)

Let \( \tau_1 \) and \( \tau_2 \) be two IF soft topologies on \( U \). Denote \( \tau_1 \cap \tau_2 = \{f_E \cup g_E : f_E \in \tau_1 \text{ and } g_E \in \tau_2\} \), \( \tau_1 \cup \tau_2 = \{f_E \cap g_E : f_E \in \tau_1 \text{ and } g_E \in \tau_2\} \).

Example 3.7. Let \( f_E \) and \( g_E \) be two IF soft sets in Example 3.5. Then \( \tau_1 = \{\emptyset, U_E, f_E\} \), \( \tau_2 = \{\emptyset, U_E, g_E\} \) and \( \tau_1 \cap \tau_2 = \{\emptyset, U_E\} \) are three IF soft topologies on \( U \). But \( \tau_1 \cup \tau_2 = \{\emptyset, U_E, f_E, g_E\} \), \( \tau_1 \cap \tau_2 = \{\emptyset, U_E, f_E, g_E, f_E \cup g_E\} \) and \( \tau_1 \cap \tau_2 = \{\emptyset, U_E, f_E, g_E, f_E \cap g_E\} \) are not IF soft topologies on \( U \).

Theorem 3.8. Let \((U, \tau, E)\) be an IF soft topologies over \( U \). for any \( e \in E \),
\[
\tau(e) = \{f(e) : f_E \in \tau\}
\]
is an IF topology on \( U \).
Proof. Let \( e \in E \). (1) By \( \emptyset, U_E \in \tau, \emptyset = \emptyset(e) \) and \( \tilde{1} = U(e) \), we have \( \emptyset, \tilde{1} \in \tau(e) \).

(2) Let \( V, W \in \tau(e) \). Then there exist \( f_E, g_E \in \tau \) such that \( V = f(e) \) and \( W = g(e) \). By \( \tau \) be an IF soft topologies on \( U \), \( f_E \cap g_E \in \tau \). Put \( h_E = f_E \cap g_E \). Then \( h_E \in \tau \). Note that \( V \cap W = f(e) \cap g(e) = h(e) \) and \( \tau(e) = \{ f(e) : f_E \in \tau \} \). Then \( V \cap W \in \tau(e) \).

(3) Let \( \{ V_\alpha : \alpha \in \Gamma \} \subseteq \tau(e) \). Then for every \( \alpha \in \Gamma \), there exist \( (f_\alpha)_E \in \tau \) such that \( V_\alpha = f_\alpha(e) \). By \( \tau \) be an IF soft topologies on \( U \), \( \bigcup \{(f_\alpha)_E : \alpha \in \Gamma \} \in \tau \). Put \( f_E = \bigcup \{(f_\alpha)_E : \alpha \in \Gamma \} \). Then \( f_E \in \tau \). Note that \( \bigcup_{\alpha \in \Gamma} V_\alpha = \bigcup \{ f_\alpha(e) : \alpha \in \Gamma \} = f(e) \) and \( \tau(e) = \{ f(e) : f_E \in \tau \} \). Then \( \bigcup_{\alpha \in \Gamma} V_\alpha \in \tau(e) \).

Therefore \( \tau(e) = \{ f(e) : f_E \in \tau \} \) is an IF topology on \( U \).

\( \Box \)

Definition 3.9. Let \( (U, \tau, E) \) be an IF soft topological space and \( \mathcal{B} \subseteq \tau \). \( \mathcal{B} \) is a basis for \( \tau \), if for each \( g_E \in \tau \), there exists \( \mathcal{B}' \subseteq \mathcal{B} \) such that \( g_E = \bigcup \mathcal{B}' \).

Example 3.10. Let \( \tau \) be the IF soft topology in Example 3.5. Then \( \mathcal{B} = \{ \emptyset, f_E, g_E, \{ E \}, U_E \} \) is a basis for \( \tau \).

Theorem 3.11. Let \( \mathcal{B} \) is a basis for IF soft topology \( \tau \). Denote \( \mathcal{B}_c = \{ f(e) : f_E \in \mathcal{B} \} \) and \( \tau(e) = \{ f(e) : f_E \in \tau \} \) for any \( e \in E \). Then \( \mathcal{B}_c \) is a basis for IF topology \( \tau(e) \).

Proof. Let \( e \in E \). For any \( V \in \tau(e) \), \( V = g(e) \) for some \( g_E \in \tau \). Note that \( \mathcal{B} \) is a basis for \( \tau \). Then there exists \( \mathcal{B}' \subseteq \mathcal{B} \) such that \( g_E = \bigcup \mathcal{B}' \). So \( V = \bigcup \mathcal{B}'_{e} \), where \( \mathcal{B}'_{e} = \{ f(e) : f_E \in \mathcal{B}' \} \subseteq \mathcal{B}_c \). Thus \( \mathcal{B}_c \) is basis for IF topology \( \tau(e) \) for any \( e \in E \).

\( \Box \)

4. Some properties of IF soft topologies

In this section, we give some properties of IF soft topologies.

Definition 4.1. Let \( (U, \tau, E) \) be an IF soft topological space and let \( f_E \in \mathcal{IFS}(U)_E \). Then interior and closure of \( f_E \) denoted respectively by \( \text{int}(f_E) \) and \( \text{cl}(f_E) \), are defined as follows:

\[
\text{int}(f_E) = \bigcup \{ g_E \in \tau : g_E \subseteq f_E \},
\]

\[
\text{cl}(f_E) = \bigcap \{ g_E \in \tau : f_E \subseteq g_E \}.
\]

Theorem 4.2. Let \( (U, \tau, E) \) be an IF soft topological space over \( U \). Then the following properties hold.

(1) \( U_E \) and \( \emptyset_E \) are IF soft closed sets over \( U \).

(2) The intersection of any number of IF soft closed sets is an IF soft closed set over \( U \).

(3) The union of any two IF soft closed sets is an IF soft closed set over \( U \).

Proof. This follows from Proposition 3.2. \( \Box \)
Theorem 4.3. Let \((U, \tau, E)\) be an IF soft topological space over \(U\) and let \(f_E \in \mathcal{IFS}(U)_E\). Then the following properties hold.

1. \(\text{int}(f_E) \subseteq f_E\).
2. \(f_E \subseteq g_E \Rightarrow \text{int}(f_E) \subseteq \text{int}(g_E)\).
3. \(\text{int}(f_E) \in \tau\).
4. \(f_E\) is an IF soft open set \(\iff \text{int}(f_E) = f_E\).
5. \(\text{int}(\text{int}(f_E)) = \text{int}(f_E)\).
6. \(\text{int}(\emptyset_E) = \emptyset_E, \text{int}(U_E) = U_E\).

**Proof.** (1) and (2) are obvious.
(3) Obviously, \(\bigcup \{g_E \in \tau : g_E \supseteq f_E\} \in \tau\). Note that \(\text{int}(f_E) = \bigcup \{g_E \in \tau : g_E \supseteq f_E\}\). Then \(\text{int}(f_E) \in \tau\).
(4) Necessity. By (1), \(\text{int}(f_E) \subsetneq f_E\).
Since \(f_E \in \tau\) and \(f_E \subsetneq f_E\), then \(f_E \subsetneq \bigcup \{g_E \in \tau : g_E \supseteq f_E\}\). By \(\text{int}(f_E) = \bigcup \{g_E \in \tau : g_E \supseteq f_E\}\). Thus \(\text{int}(f_E) = f_E\).
Sufficiency. This holds by (3).
(5) and (6) hold by (3) and (4).

Theorem 4.4. Let \((U, \tau, E)\) be an IF soft topological space over \(U\) and let \(f_E \in \mathcal{IFS}(U)_E\). Then the following properties hold.

1. \(f_E \subseteq \text{cl}(f_E)\).
2. \(f_E \subseteq g_E \Rightarrow \text{cl}(f_E) \subseteq \text{cl}(g_E)\).
3. \((\text{cl}(f_E))' \in \tau\).
4. \(f_E\) is an IF soft closed set \(\iff \text{cl}(f_E) = f_E\).
5. \(\text{cl}(\text{cl}(f_E)) = \text{cl}(f_E)\).
6. \(\text{cl}(\emptyset_E) = \emptyset_E, \text{cl}(U_E) = U_E\).

**Proof.** (1) and (2) are obvious.
(3) By Theorem 4.3 (3),
\[
\text{int}(\text{cl}(f_E)) \in \tau.
\]
Since \((\text{cl}(f_E))' = \bigcap \{g_E \in \tau' : f_E \supseteq g_E\} = \bigcup \{g_E \in \tau : g_E \supseteq f_E\} = \text{int}(f_E')\),
then \((\text{cl}(f_E))' \in \tau\).
(4) Necessity. By Theorem 4.4 (1),
\[
f_E \subseteq \text{cl}(f_E).
\]
Since \(f_E \in \tau'\) and \(f_E \subsetneq f_E\), \(\text{cl}(f_E) = \bigcap \{g_E \in \tau' : f_E \supseteq g_E\} \supsetneq \bigcap \{f_E \in \tau' : f_E \supseteq f_E\}\). Then \(\text{cl}(f_E) \subsetneq f_E\). Thus \(f_E = \text{cl}(f_E)\).
Sufficiency. This holds by (3).
(5) and (6) hold by (3) and (4).

Theorem 4.5. Let \((U, \tau, E)\) be an IF soft topological space and let \(f_E, g_E \in \mathcal{IFS}(U)_E\).

1. \(\text{int}(f_E) \cap \text{int}(g_E) = \text{int}(f_E \cap g_E)\).
2. \(\text{int}(f_E) \cup \text{int}(g_E) = \text{int}(f_E \cup g_E)\).
3. \(\text{cl}(f_E) \cup \text{cl}(g_E) = \text{cl}(f_E \cup g_E)\).
4. \(\text{cl}(f_E \cap g_E) = \text{cl}(f_E) \cap \text{cl}(g_E)\).
5. \((\text{int}(f_E))' = \text{cl}(f_E)\).
6. \((\text{cl}(f_E))' = \text{int}(f_E)\).
Proof. (1) Since \( f(e) \bar{\cap} g(e) \subset f(e) \) for any \( e \in E \), we have \( f_E \bar{\cap} g_E \subset f_E \). By Theorem 4.3(2),
\[
\text{int}(f_E \bar{\cap} g_E) \subset \text{int}(f_E).
\]
Similarly, \( \text{int}(f_E \bar{\cap} g_E) \subset \text{int}(g_E) \). Then \( \text{int}(f_E \bar{\cap} g_E) \subset \text{int}(f_E) \cap \text{int}(g_E) \).
By Theorem 4.3(1),
\[
\text{int}(f_E) \subset f_E \quad \text{and} \quad \text{int}(g_E) \subset g_E.
\]
Then \( \text{int}(f_E) \bar{\cap} \text{int}(g_E) \subset f_E \bar{\cap} g_E \). So \( \text{int}(f_E) \bar{\cap} \text{int}(g_E) \subset \text{int}(f_E \bar{\cap} g_E) \).
Hence, \( \text{int}(f_E) \bar{\cap} \text{int}(g_E) = \text{int}(f_E \bar{\cap} g_E) \).
Similarly, we can prove (2), (3) and (4).
(5) holds by Proposition 5.2
\[
\text{int}(f_E)' = \bigcap \{ h_E \in \tau : h_E \subset f_E \} = \bigcap \{ h_E \in \tau' : f_E \subset h_E \} = \text{cl}(f_E').
\]
(6) The proof is similar to (5). \( \square \)

Example 4.6. Let \( U = \{ x_1, x_2 \} \) and \( E = \{ e_1, e_2 \} \). Let
\[
f(e_1) = \frac{(0.2, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2}, f(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.9, 0.1)}{x_2}.
\]
Obviously, \( \tau = \{ f_E, \varnothing_E, U_E \} \) is an IF soft topology on \( U \).
\( h_E, l_E \) are defined as follows:
\[
g(e_1) = \frac{(0.1, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2}, g(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.9, 0.1)}{x_2};
\]
\[
h(e_1) = \frac{(0.2, 0.8)}{x_1} + \frac{(0.6, 0.3)}{x_2}, h(e_2) = \frac{(0.2, 0.5)}{x_1} + \frac{(0.7, 0.1)}{x_2}.
\]
(1) Obviously, \( \text{int}(g_E) = \varnothing_E = \text{int}(h_E) \) and \( g_E \bar{\cup} h_E = f_E \). Then
\[
\text{int}(g_E) \bar{\cup} \text{int}(h_E) = \varnothing_E \bar{\cup} \varnothing_E = \varnothing_E
\]
and
\[
\text{int}(g_E \bar{\cup} h_E) = \text{int}(f_E) = f_E.
\]
Thus
\[
\text{int}(g_E) \bar{\cup} \text{int}(h_E) \neq \text{int}(g_E \bar{\cup} h_E).
\]
(2) By Theorem 4.5(5),
\[
\text{cl}(g_E') = (\text{int}(g_E))' = \varnothing_E' = U_E.
\]
Similarly, \( \text{cl}(h_E') \neq \text{cl}(h_E \bar{\cup} \text{int}(h_E)) \).

Thus
\[
\text{cl}(g_E') \bar{\cap} \text{cl}(h_E') = U_E \bar{\cap} U_E = U_E.
\]
Similarly,
\[
\text{cl}(g_E' \bar{\cap} h_E') = \text{cl}((g_E \bar{\cup} h_E)') = (\text{int}(g_E \bar{\cup} h_E))' = f_E'.
\]

Thus
\[
\text{cl}(g_E' \bar{\cap} h_E') \neq \text{cl}(g_E') \bar{\cap} \text{cl}(h_E').
\]
5. Conclusions

In this paper, we introduced intuitionistic fuzzy soft topologies. The interior and closure of an intuitionistic fuzzy soft set, and bases for an intuitionistic fuzzy soft topology are introduced. Some basic properties of intuitionistic fuzzy soft topologies are given. In future work, we will research intuitionistic fuzzy soft subspace, intuitionistic fuzzy soft points and neighborhoods of a intuitionistic fuzzy soft point and bring them into the structure of intuitionistic fuzzy soft topological spaces.

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