On \((\in, \in \lor q)\)-intuitionistic fuzzy \(h\)-ideals of hemirings

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Received 28 June 2011; Revised 2 May 2012; Accepted 4 May 2012

Abstract. The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. Using the notion of “belongingness \((\in)\)” and “quasi-coincidence \((q)\)” of fuzzy points in fuzzy sets, we introduce the concepts of \((\in, \in \lor q)\)-intuitionistic fuzzy ideal, \((\in, \in \lor q)\)-intuitionistic fuzzy \(k\)-ideal and \((\in, \in \lor q)\)-intuitionistic fuzzy \(h\)-ideal of hemirings, and some interesting properties are investigated.

2010 AMS Classification: 16Y60, 13E05, 03G25

Keywords: Fuzzy set, Intuitionistic fuzzy set, \((\in, \in \lor q)\)-Intuitionistic fuzzy ideals, \((\in, \in \lor q)\)-Intuitionistic fuzzy \(k\)-ideals, \((\in, \in \lor q)\)-Intuitionistic fuzzy \(h\)-ideals.

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1. Introduction

Given a set \(H\), a fuzzy subset of \(H\) (or a fuzzy set in \(H\)) is, by definition, an arbitrary mapping \(\mu : H \rightarrow [0,1]\) where \([0,1]\) is the closed interval in reals whose endpoints are 0 and 1. This important concept of a fuzzy set has been introduced by Zadeh in [19]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications (see, for example, [2, 6]).

After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [3, 6] is one among them. An intuitionistic fuzzy set gives both a membership degree and a non-membership degree. The membership and non-membership values induce an indeterminacy index, which models the hesitancy of deciding the degree to which an object satisfies a particular property. As the basis for the study of intuitionistic fuzzy set theory, many operations and relations over intuitionistic fuzzy sets were introduced [4, 5]. Many concepts in fuzzy set theory were also extended to intuitionistic fuzzy set theory, such as intuitionistic fuzzy
relations, intuitionistic L-fuzzy sets, intuitionistic fuzzy implications, intuitionistic fuzzy grade of hypergroups, intuitionistic fuzzy logics, and the degree of similarity between intuitionistic fuzzy sets, etc., [10].

In [7] Biswas applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group.

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [9], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [8] gave the concepts of \((\alpha, \beta)\)-fuzzy subgroups by using the notion of \((\in, \notin)\) and \((q)\) between a fuzzy point and a fuzzy subgroup, where \(\alpha, \beta\) are any two of \(\{\in, q, \notin \} \) with \(\alpha \neq q \) and introduced the concept of an \((\in, q)\)-fuzzy subgroup. In [9] \((\in, q)\)-fuzzy subgroups and ideals defined. In [13] Jun and Song initiated the study of \((\alpha, \beta)\)-fuzzy ideal of a hemiring. Finally, in Section 4, we produce some relations between \((\in, q)\)-intuitionistic fuzzy ideals in hemirings. In [12] Jun studied \((\alpha, \beta)\)-fuzzy ideals of hemirings. This paper continues this line of research.

The paper is organized as follows: in Section 2 some fundamental definitions on fuzzy sets and intuitionistic fuzzy sets are explored; in Section 3, we define \((\in, \notin)\)-intuitionistic fuzzy ideals of hemirings, \((\in, \notin)\)-intuitionistic fuzzy \(k\)-ideal, and \((\in, q)\)-intuitionistic fuzzy \(h\)-ideal of a hemiring. Finally, in Section 4, we produce some relations between \((\in, q)\)-intuitionistic fuzzy ideals with \((\in, q)\)-fuzzy ideals and with anti \((\in, q)\)-fuzzy ideals, and then establish some useful theorems.

2. Preliminaries

A semiring is an algebraic system \((R, +, \cdot)\) consisting of a non-empty set \(R\) together with two binary operations called addition \(\{+\}\) and multiplication \(\{\cdot\}\), here \(x \cdot y\) will be denoted by juxtaposition for all \(x, y \in R\), such that \((R, +)\) and \((R, \cdot)\) are semigroups connected by the following distributive laws: \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\) for all \(a, b, c \in R\). An element \(0 \in R\) is called a zero of \(R\) if \(a + 0 = 0 + a = a\) and \(a0 = 0a = a\) for all \(a \in R\). A semiring with zero and a commutative addition is called a hemiring. A nonempty subset \(S\) of \(R\) is called a subsemiring of \(R\) if \(X \cdot X \subseteq X\) and \(X + X \subseteq X\). A non-empty subset \(I\) of a semiring \(R\) is said to be a left (resp. right) ideal of \(R\) if it is closed under the addition and \(RI \subseteq I\) (resp. \(IR \subseteq I\)). A left ideal which is also a right ideal is called an ideal. A left (resp. right) ideal \(I\) of a hemiring \(R\) is called a left (resp. right) \(h\)-ideal of \(R\) if for any \(a, b \in I\) and \(x \in R\) whenever \(x + a = b\) then \(x \in I\). A left (resp. right) ideal \(I\) of a hemiring \(R\) is called a left (resp. right) \(h\)-ideal of \(R\) if for any \(a, b \in I\) and all \(x, y \in R\) whenever \(x + a = b + y\) then \(x \in I\).

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [19] in 1965. Let \(X\) be a non-empty set. A mapping \(\mu : X \rightarrow [0; 1]\) is called a fuzzy set in \(X\). The complement of \(\mu\), denoted by \(\mu^c\), is the fuzzy set in \(X\) given by \(\mu^c(x) = 1 - \mu(x)\) for all \(x \in X\).
For any $t \in [0, 1]$ and fuzzy set $\mu$ of $X$, the set

$$U(\mu, t) = \{x \in X | \mu(x) \geq t\} \text{ (respectively, } L(\mu, t) = \{x \in X | \mu(x) \leq t\},$$
is called an upper (respectively, lower) $t$-level cut of $\mu$.

**Definition 2.1.** An intuitionistic fuzzy set (IFS for short) $A$ in a non-empty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\},$$

where the functions $\mu_A : X \rightarrow [0; 1]$ and $\lambda_A : X \rightarrow [0; 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ with respect to the set $A$, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$ (see [3] [4]). For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the IFS $A = \{x, \mu_A(x), \lambda_A(x)) | x \in X\}. Denote by IFS(X) the set of all intuitionistic fuzzy sets in $X$.

**Definition 2.2** ([3]). Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in $X$. Then

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(3) $A^c = \{(x, \lambda_A(x), \mu_A(x)) | x \in X\}$,
(4) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) | x \in X\}$,
(5) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) | x \in X\}$,
(6) $\Delta A = \{(x, \mu_A(x), \mu_A(x)) | x \in X\}$,
(7) $\Delta A = \{(x, \lambda_A(x), \lambda_A(x)) | x \in X\}$.

**Definition 2.3** ([16]). Let $Y \subseteq X$ and $t \in [0; 1]$. We define $t_Y \in F(X)$ as follows:

$$t_Y(x) = \begin{cases} 
  t & \text{if } x \in Y \\
  0 & \text{if } x \in X \setminus Y.
\end{cases}$$

In particular, if $Y$ is a singleton, say $x$, then $t_{\{x\}}$ is called a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

**Definition 2.4** ([16]). Let $\mu$ be a fuzzy subset of $X$ and $x_t$ be a fuzzy point.

(1) If $\mu(x) \geq t$, then we say $x_t$ belongs to $\mu$, and write $x_t \in \mu$.
(2) If $\mu(x) + t > 1$, then we say $x_t$ is quasi-coincident with $\mu$, and write $x_t \notin \mu$.
(3) $x_t \in \forall \mu \iff x_t \in \mu$ or $x_t \notin \mu$.
(4) $x_t \in \forall \mu \iff x_t \in \mu$ and $x_t \notin \mu$.

In what follows, unless otherwise specified, $\alpha$ and $\beta$ will denote any one of $\in, \notin \forall q$ or $\in \land q$ with $\alpha \neq \notin \forall q$. To say that $x_t \alpha \mu$ means that $x_t \alpha \mu$ does not hold. We defined

$$U(\alpha \mu, t) = \{x \in X | x_t \alpha \mu\},$$

where $\alpha \in \{\in, \notin \forall q \}$. 

**Definition 2.5** ([11]). A fuzzy subset $\mu$ of $R$ is said to be an ($\in, \notin \forall q$)-fuzzy left (resp. right) ideal of a hemiring $R$ if

$$x \in U(\in \mu, t), \ y \in U(\in \mu, r) \Rightarrow x + y \in U(\in \forall q, t \land r),$$
$$x \in U(\in \mu, t) \Rightarrow xy \in U(\in \forall q, t)(\text{resp. } xy \in U(\in \forall q, t)), \ 	ext{if } x \in U(\in \mu, t).$$
for all \( x, y \in R \) and \( t, r \in (0, 1] \). A fuzzy subset which is an \((\varepsilon, \in \mathcal{V}q)\)-fuzzy left and right ideal is called an \((\varepsilon, \in \mathcal{V}q)\)-fuzzy ideal.

An \((\varepsilon, \in \mathcal{V}q)\)-fuzzy ideal \( \mu \) of a hemiring \( R \) satisfying the following condition:

\[
x + a = b, \ a \in U(\in \mu, t), \ b \in U(\in \mu, r) \implies x \in U(\in \mathcal{V}q \mu, t \wedge r),
\]

for all \( a, b, x \in R \) and \( t, r \in (0, 1] \) is called an \((\varepsilon, \in \mathcal{V}q)\)-fuzzy \( k \)-ideal.

An \((\varepsilon, \in \mathcal{V}q)\)-fuzzy ideal \( \mu \) of a hemiring \( R \) satisfying the following condition:

\[
x + a + y = b + y, \ a \in U(\in \mu, t), \ b \in U(\in \mu, r) \implies x \in U(\in \mathcal{V}q \mu, t \wedge r),
\]

for all \( a, b, x, y \in R \) and \( t, r \in (0, 1] \) is called an \((\varepsilon, \in \mathcal{V}q)\)-fuzzy \( h \)-ideal.

**Lemma 2.6 (\[\Pi\])**. A fuzzy subset \( \mu \) of a hemiring \( R \) is an \((\varepsilon, \in \mathcal{V}q)\)-fuzzy \( h \)-ideal (resp. \( k \)-ideal) of \( R \) if and only if it satisfies:

\[
\begin{align*}
(a) & \ \mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\}, \\
(b) & \ \mu(xy) \geq \min\{\mu(x), 0.5\}, \\
(c) & \ \mu(xy) \geq \min\{\mu(x), 0.5\}, \\
(d) & \ x + a + y = b + y \implies \mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}, \\
& \ (\text{resp.} \ (e) \ x + a = b \implies \mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}),
\end{align*}
\]

for all \( a, b, x, y \in R \).

3. \((\varepsilon, \in \mathcal{V}q)\)-Intuitionistic Fuzzy Ideals of Hemirings

In what follows, let \( R \) denote a hemiring and \( t \in (0, 1] \).

**Definition 3.1.** Let \( \mu \) be a fuzzy set in \( X \). We define

\[
L(\in \mu, t) = \{x \in X | \mu(x) \leq t\},
\]

\[
L(\mathcal{V}q \mu, t) = \{x \in X | \mu(x) + t \leq 1\},
\]

\[
L(\in \mathcal{V}q \mu, t) = \{x \in X | \mu(x) + t \leq 1 \text{ or } \mu(x) \leq t\}.
\]

Then the set \( L(\alpha \mu, t) \) is called a lower \( t \)-level cut of \( \alpha \mu \), where \( \alpha \in \{\varepsilon, q, \in \mathcal{V}q\} \).

It is clear that \( L(\in \mu, t) = L(\mu, t) \).

**Corollary 3.2 \( (I) \).** Let \( \mu \) be a fuzzy set in \( X \). Then for all \( t \in (0, 1] \) we have

\[
\begin{align*}
(1) & \ U(\in \mathcal{V}q \mu, t) = U(\in \mu, t) \bigcup U(q \mu, t), \\
(2) & \ L(\in \mathcal{V}q \mu, t) = L(\in \mu, t) \bigcup L(q \mu, t).
\end{align*}
\]

**Corollary 3.3 \( (\Pi) \).** For any fuzzy subset \( \lambda \) of \( X \) and \( t \in (0, 1] \), we consider two subsets:

\[
Q(\lambda, t) = \{x \in X | x \in \mathcal{V}q \lambda\} \text{ and } |\lambda|_t = \{x \in X | x \in \mathcal{V}q \lambda\}.
\]

Then \( |\lambda|_t = U(\lambda, t) \bigcup Q(\lambda, t) \).

**Theorem 3.4 \( (I) \).** Let \( \mu \) be a fuzzy set in \( X \). Then we have

\[
\begin{align*}
(1) & \ \text{If } t \in (0, 0.5], \text{ then } U(\in \mathcal{V}q \mu, t) = U(\in \mu, t), \\
(2) & \ \text{If } t \in (0.5, 1], \text{ then } U(\in \mathcal{V}q \mu, t) = U(q \mu, t).
\end{align*}
\]

**Proof.** (1) If \( t \in (0, 0.5] \), then \( 1 - t \in [0.5, 1) \). Thus \( t \leq 1 - t \). By Corollary 3.2, it is clear that \( U(\in \mu, t) \subseteq U(\in \mathcal{V}q \mu, t) \). Let \( x \notin U(\in \mu, t) \). Then \( \mu(x) < t \) and so \( \mu(x) < 1 - t \). This shows that \( x \notin U(q \mu, t) \), and hence \( x \notin (U(\in \mu, t) \bigcup U(q \mu, t)) \).

Thus \( U(\in \mu, t) \supseteq U(\in \mathcal{V}q \mu, t) \). Therefore \( U(\in \mu, t) = U(\in \mathcal{V}q \mu, t) \).

(2) If \( t \in (0.5, 1] \), then \( 1 - t \in [0, 0.5) \). Thus \( 1 - t < t \). By Theorem 3.2, we have \( U(q \mu, t) \subseteq U(\in \mathcal{V}q \mu, t) \). Let \( x \notin U(q \mu, t) \), then \( \mu(x) + t \leq 1 \) and so...
Every fuzzy subset $x \subseteq \mathcal{A}$ is called an $(\alpha, \beta)$-intuitionistic fuzzy left (resp. right) ideal of hemiring $R$ if

$$\forall x \in U(\alpha_{x}, t), \; y \in U(\beta_{y}, t) \implies x + y \in U(\alpha_{x}, t \wedge r),$$

$$\forall x \in U(\alpha_{x}, t) \implies xy \in U(\beta_{x}, t) (\text{resp. } xy \in U(\beta_{x}, t)),$$

$$\forall x \in L(\alpha_{x}, t), \; y \in L(\beta_{y}, t) \implies x + y \in L(\alpha_{x}, t \vee r),$$

$$\forall x \in L(\alpha_{x}, t) \implies xy \in L(\beta_{x}, t) (\text{resp. } xy \in L(\beta_{x}, t)).$$

for all $x, y \in R$ and $t, r \in (0, 1]$. A fuzzy subset which is an $(\alpha, \beta)$-intuitionistic fuzzy left and right ideal is called an $(\alpha, \beta)$-intuitionistic fuzzy ideal.

A fuzzy subset $\mu$ (resp. $\lambda$) of $R$ is said to be an (resp. anti) $(\alpha, \beta)$-fuzzy ideal of hemiring $R$ if it satisfies the conditions (1) and (2) (resp. (3) and (4)) of Definition 3.7.

Definition 3.7. Let $A = (\mu_{A}, \lambda_{A}) \in IFS(R)$. Then $A = (\mu_{A}, \lambda_{A})$ is called an $(\alpha, \beta)$-intuitionistic fuzzy left (resp. right) ideal of hemiring $R$ if

$$\forall x \in U(\alpha_{x}, t), \; y \in U(\alpha_{y}, r) \implies x + y \in U(\beta_{x}, t \wedge r),$$

$$\forall x \in U(\alpha_{x}, t) \implies xy \in U(\beta_{x}, t) (\text{resp. } xy \in U(\beta_{x}, t)),$$

$$\forall x \in L(\alpha_{x}, t), \; y \in L(\beta_{y}, t) \implies x + y \in L(\beta_{x}, t \vee r),$$

$$\forall x \in L(\alpha_{x}, t) \implies xy \in L(\beta_{x}, t) (\text{resp. } xy \in L(\beta_{x}, t)).$$

A fuzzy subset $\mu$ (resp. $\lambda$) of $R$ is said to be an (resp. anti) $(\alpha, \beta)$-fuzzy ideal of hemiring $R$ if it satisfies the condition (1) (resp. (2)) of Definition 3.8.

Definition 3.8. An $(\alpha, \beta)$-intuitionistic fuzzy ideal $A = (\mu_{A}, \lambda_{A})$ of a hemiring $R$ satisfying the following condition:

$$\forall x = a, \; b \in U(\alpha_{x}, t), \; b \in U(\alpha_{y}, r) \implies x \in U(\beta_{x}, t \wedge r),$$

$$\forall x = a, \; b \in L(\alpha_{x}, t), \; b \in L(\alpha_{y}, r) \implies x \in L(\beta_{x}, t \vee r),$$

for all $a, b, x \in R$ and $t, r \in (0, 1]$ is called an $(\alpha, \beta)$-intuitionistic fuzzy $k$-ideal.

A fuzzy subset $\mu$ (resp. $\lambda$) of $R$ is said to be an (resp. anti) $(\alpha, \beta)$-fuzzy $k$-ideal of hemiring $R$ if it satisfies the condition (1) (resp. (2)) of Definition 3.8.

Definition 3.9. An $(\alpha, \beta)$-intuitionistic fuzzy ideal $A = (\mu_{A}, \lambda_{A})$ of a hemiring $R$ satisfying the following condition:

$$\forall x = a + y = b + y, \; a \in U(\alpha_{x}, t), \; b \in U(\alpha_{x}, r) \implies x \in U(\beta_{x}, t \wedge r),$$

$$\forall x = a + y = b + y, \; a \in L(\alpha_{x}, t), \; b \in L(\alpha_{x}, r) \implies x \in L(\beta_{x}, t \vee r),$$

for all $a, b, x, y \in R$ and $t, r \in (0, 1]$ is called an $(\alpha, \beta)$-intuitionistic fuzzy $h$-ideal.

A fuzzy subset $\mu$ (resp. $\lambda$) of $R$ is said to be an (resp. anti) $(\alpha, \beta)$-fuzzy $h$-ideal of hemiring $R$ if it satisfies the condition (1) (resp. (2)) of Definition 3.9.

Theorem 3.10. Let $\lambda$ be a fuzzy subset of a hemiring $R$ and $t, r \in (0, 1]$. Then:

1. $(a1) \; x \in L(\mathcal{E}, t), \; y \in L(\mathcal{E}, r) \implies x + y \in L(\mathcal{E}, t \wedge r)$ and
   $(a2) \; \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in R$ are equivalent.

2. $(b1) \; x \in L(\mathcal{E}, t) \implies xy \in L(\mathcal{E}, t)$ and
   $(b2) \; \lambda(xy) \leq \max\{\lambda(x), 0.5\}$ for all $x, y \in R$ are equivalent.

3. $(c1) \; x \in L(\mathcal{E}, t) \implies xy \in L(\mathcal{E}, t)$ and
   $(c2) \; \lambda(xy) \leq \max\{\lambda(x), 0.5\}$ for all $x, y \in R$ are equivalent.
(4) (d1) \( x + a + y = b + y, \ a \in L(\in \lambda, t), \ b \in L(\in \lambda, r) \implies x \in L(\in \vee \lambda, t \lor r) \) and
(d2) \( x + a + y = b + y \implies \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} \) for all \( a, b, x, y \in R \) are equivalent.

(5) \( (e1) \ x + a = b, \ a \in L(\in \lambda, t), \ b \in L(\in \lambda, r) \implies x \in L(\in \vee \lambda, t \lor r) \) and
(e2) \( x + a = b \implies \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} \) for all \( a, b, x \in R \) are equivalent.

Proof. (a1) \( \implies \) (a2). Assume that there exist \( x, y \in R \) such that
\[ \lambda(x + y) > \max\{\lambda(x), \lambda(y), 0.5\}. \]
Choose \( t \in (0, 1] \) such that \( \lambda(x + y) > t \geq \max\{\lambda(x), \lambda(y), 0.5\} \). Then \( x \in L(\in \lambda, t) \) and \( y \in L(\in \lambda, t) \). But \( \lambda(x + y) > t \), so \( x + y \in L(\in \lambda, t) \) and \( \lambda(x + y) + t > 2t \geq 1 \). Then we have
\[ x + y \in L(\in \vee \lambda, t) = L(\in \vee \lambda, t \lor r), \]
which is a contradiction. Thus \( \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\} \). Hence (a2) holds.

(a2) \( \implies \) (a1). Let
\[ \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\} \]
Assume that \( t, r \in (0, 1] \) such that \( x \in L(\in \lambda, t) \) and \( y \in L(\in \lambda, r) \). Then \( \lambda(x) \leq t \) and \( \lambda(y) \leq r \). Hence
\[ \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\} \leq \max\{t, r, 0.5\}. \]
If \( \max\{t, r\} \leq 0.5 \), then \( \lambda(x + y) \leq 0.5 \), and so \( \lambda(x + y) + \max\{t, r\} \leq 0.5 + 0.5 = 1 \), which implies \( x + y \in L(\lambda, t \lor r) \). If \( \max\{t, r\} > 0.5 \), then \( \lambda(x + y) \leq \max\{t, r\} \), which implies that \( x + y \in L(\in \lambda, t \lor r) \). Hence (a1) holds.

(b1) \( \implies \) (b2). Assume that there exist \( x, y \in R \) such that \( \lambda(yx) > \max\{\lambda(x), 0.5\} \).
Choose \( t \in (0, 1] \) such that \( \lambda(yx) > t \geq \max\{\lambda(x), 0.5\} \). Then \( x \in L(\in \lambda, t) \) but \( \lambda(yx) > t \), so \( yx \in L(\in \lambda, t) \) and \( \lambda(yx) + t > 2t \geq 1 \). Then we obtain \( yx \in L(\in \vee \lambda, t) \), which is a contradiction. Thus \( \lambda(yx) \leq \max\{\lambda(x), 0.5\} \). Hence (b2) holds.

(b2) \( \implies \) (b1). Let \( \lambda(yx) \leq \max\{\lambda(x), 0.5\} \). Assume that \( t \in (0, 1] \) such that \( x \in L(\in \lambda, t) \). Then \( \lambda(x) \leq t \). Hence \( \lambda(yx) \leq \max\{\lambda(x), 0.5\} \leq \max\{t, 0.5\} \). If \( t \leq 0.5 \), then \( \lambda(yx) \leq 0.5 \), and so \( \lambda(yx) + t \leq 0.5 + 0.5 = 1 \), which implies that \( yx \in L(\lambda, t) \). If \( t > 0.5 \), then \( \lambda(yx) \leq t \), which implies that \( yx \in L(\in \lambda, t) \). Hence (b1) holds.

(d1) \( \implies \) (d2). Suppose that there exist \( a, b, x, y \in R \) such that \( x + a + y = b + y \).
Assume that \( \lambda(x) > \max\{\lambda(a), \lambda(b), 0.5\} \). Choose \( t \in (0, 1] \) such that \( \lambda(x) > t \geq \max\{\lambda(a), \lambda(b), 0.5\} \). Then \( a, b \in L(\in \lambda, t) \). But \( x \in L(\in \lambda, t) \) and \( \lambda(x) + t > 2t \geq 1 \), so \( x \in L(\lambda, t) \). Then we obtain \( x \in L(\in \vee \lambda, t) \), which is a contradiction. Thus \( \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} \). Hence (d2) holds.

(d2) \( \implies \) (d1). Let \( a, b, x, y \in R \), \( t, r \in (0, 1] \), \( x + a + y = b + y \) and \( a \in L(\in \lambda, t) \), \( b \in L(\in \lambda, r) \). If \( \max\{\lambda(a), \lambda(b), 0.5\} = \lambda(a) \), then
\[ \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} = \lambda(a) \leq t \leq \max\{t, r\} \].
Thus \(x \in L(\in \lambda, t \lor r)\), implying that \(x \in L(\in \lor q\lambda, t \lor r)\).
Similarly, if \(\max\{\lambda(a), \lambda(b), 0.5\} = \lambda(b)\), then \(x \in L(\in \lor q\lambda, t \lor r)\).
Let \(\max\{\lambda(a), \lambda(b), 0.5\} = 0.5\). If \(\max\{t, r\} \geq 0.5\), then
\[
\lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} = 0.5 \leq \max\{t, r\},
\]
which implies \(x \in L(\in \lambda, t \lor r)\) and so \(x \in L(\in \lor q\lambda, t \lor r)\). If \(\max\{t, r\} < 0.5\), then \(0.5 < 1 - \max\{t, r\} < 1\). Thus \(\lambda(x) \leq 0.5 \leq 1 - \max\{t, r\}\), which implies that \(x \in L(q\lambda, t \lor r)\) and so \(x \in L(\in \lor q\lambda, t \lor r)\). Hence (d1) holds. □

**Corollary 3.11**. A fuzzy subset \(\lambda\) of a hemiring \(R\) is an anti \((\in, \in \lor q)\)-fuzzy \(h\)-ideal of \(R\) if and only if it satisfies:

1. \(\forall x, y \in R, \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\}\),
2. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\),
3. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\),
4. \(\forall a, b, x, y \in R, x + a + y = b + y \implies \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\}\).

**Corollary 3.12**. A fuzzy subset \(\lambda\) of a hemiring \(R\) is an anti \((\in, \in \lor q)\)-fuzzy \(k\)-ideal of \(R\) if and only if it satisfies:

1. \(\forall x, y \in R, \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\}\),
2. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\),
3. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\),
4. \(\forall a, b, x \in R, x + a = b \implies \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\}\).

**Corollary 3.13**. A fuzzy subset \(\lambda\) of a hemiring \(R\) is an anti \((\in, \in \lor q)\)-fuzzy ideal of \(R\) if and only if it satisfies:

1. \(\forall x, y \in R, \lambda(x + y) \leq \max\{\lambda(x), \lambda(y), 0.5\}\),
2. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\),
3. \(\forall x, y \in R, \lambda(xy) \leq \max\{\lambda(x), 0.5\}\).

**Example 3.14**. Let \(R = \{0, 1, 2, 3, 4\}\) and let the operations be given by the following tables:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
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<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Let \(\mu\) and \(\lambda\) be two fuzzy subset of \(R\) defined by

\[
\mu(x) = \begin{cases} 
1 & \text{if } x \in \{0, 1\}, \\
\frac{x-1}{2} & \text{if } x \in \{2, 3, 4\}
\end{cases}, \quad \lambda(x) = \begin{cases} 
0 & \text{if } x \in \{0, 1\}, \\
\frac{1}{2} & \text{if } x \in \{2, 3, 4\}
\end{cases}
\]

Then \((R, +, .)\) is a hemiring and \(A = (\mu, \lambda)\) is an \((\in, \in \lor q)\)-intuitionistic fuzzy \(h\)-ideal (resp. \(k\)-ideal) of \(R\).

**Theorem 3.15**. Let \(\lambda\) be an anti \((\in, \in \lor q)\)-fuzzy \(h\)-ideal of \(R\). Then we have

1. If \(t \in [0.5, 1]\), then \(L(\in \lambda, t) \neq \emptyset\) is a \(h\)-ideal of \(R\).
2. If \(t \in (0, 0.5]\), then \(L(q\lambda, t) \neq \emptyset\) is a \(h\)-ideal of \(R\).
Proof. (1) Let \( \lambda \) be an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \), and let \( t \in [0.5, 1] \) be such that \( L(\varepsilon, \lambda, t) \neq \emptyset \). Let \( x, y \in L(\varepsilon, \lambda, t) \) be such that \( x + y \not\in L(\varepsilon, \lambda, t) \). Then \( \lambda(x) \leq t \) and \( \lambda(y) \leq t \), but \( \lambda(x + y) > t \). Since \( \lambda \) is an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(1), we get

\[
t < \lambda(x + y) \leq \max \{\lambda(x), \lambda(y), 0.5\}.
\]

If \( \max \{\lambda(x), \lambda(y), 0.5\} = \lambda(x) \), then \( x \not\in L(\varepsilon, \lambda, t) \), which is a contradiction. Similarly, if \( \max \{\lambda(x), \lambda(y), 0.5\} = \lambda(y) \), then \( y \not\in L(\varepsilon, \lambda, t) \), which is a contradiction. If \( \max \{\lambda(x), \lambda(y), 0.5\} = 0.5 \), then

\[
0.5 \leq t < \lambda(x + y) \leq \max \{\lambda(x), \lambda(y), 0.5\} = 0.5,
\]

which is a contradiction. Thus \( x + y \not\in L(\varepsilon, \lambda, t) \).

If \( x \in L(\varepsilon, \lambda, t) \) and \( y \in R \) be such that \( yx \not\in L(\varepsilon, \lambda, t) \), then \( \lambda(x) \leq t \), but \( \lambda(yx) > t \). Since \( \lambda \) is an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(2), we get

\[
t < \lambda(yx) \leq \max \{\lambda(x), 0.5\},
\]

If \( \max \{\lambda(x), 0.5\} = \lambda(x) \), then \( x \not\in L(\varepsilon, \lambda, t) \), which is a contradiction. If \( \max \{\lambda(x), 0.5\} = 0.5 \), then

\[
0.5 \leq t < \lambda(yx) \leq \max \{\lambda(x), 0.5\} = 0.5,
\]

which is a contradiction. Thus \( yx \not\in L(\varepsilon, \lambda, t) \).

Now, let \( a, b \in L(\varepsilon, \lambda, t) \), \( x, y \in R \) and \( x + a + y = b + y \) be such that \( x \not\in L(\varepsilon, \lambda, t) \). Then \( \lambda(a) \leq t \) and \( \lambda(b) \leq t \), but \( \lambda(x) > t \). Since \( \lambda \) is an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(4), we get

\[
t < \lambda(x) \leq \max \{\lambda(x), 0.5\},
\]

If \( \max \{\lambda(a), \lambda(b), 0.5\} = \lambda(a) \), then \( a \not\in L(\varepsilon, \lambda, t) \), which is a contradiction. Similarly, if \( \max \{\lambda(a), \lambda(b), 0.5\} = \lambda(b) \), then \( b \not\in L(\varepsilon, \lambda, t) \), which is a contradiction. If \( \max \{\lambda(a), \lambda(b), 0.5\} = 0.5 \), then \( 0.5 \leq t < \lambda(x) \leq \max \{\lambda(a), \lambda(b), 0.5\} = 0.5 \), which is a contradiction. Thus \( x \not\in L(\varepsilon, \lambda, t) \).

(2) Let \( \lambda \) be an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \), and let \( t \in (0, 0.5] \) such that \( L(q\lambda, t) \neq \emptyset \). Let \( x, y \in L(q\lambda, t) \) be such that \( x + y \not\in L(q\lambda, t) \). Then \( \lambda(x) + t \leq 1 \) and \( \lambda(y) + t \leq 1 \) but \( \lambda(x + y) + t > 1 \). Since \( \lambda \) is an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(1), we get

\[
1 - t < \lambda(x + y) \leq \max \{\lambda(x), \lambda(y), 0.5\}.
\]

If \( \max \{\lambda(x), \lambda(y), 0.5\} = \lambda(x) \), then \( x \not\in L(q\lambda, t) \), which is a contradiction. Similarly, if \( \max \{\lambda(x), \lambda(y), 0.5\} = \lambda(y) \), then \( y \not\in L(q\lambda, t) \), which is a contradiction. Let \( \max \{\lambda(x), \lambda(y), 0.5\} = 0.5 \). Since \( t \in (0, 0.5] \), then \( 1 - t \in [0.5, 1] \) and so

\[
0.5 \leq 1 - t < \lambda(x + y) \leq \max \{\lambda(x), \lambda(y), 0.5\} = 0.5,
\]

which is a contradiction. Thus \( x + y \not\in L(q\lambda, t) \).

Let \( x \in L(q\lambda, t) \) and \( y \in R \) be such that \( yx \not\in L(q\lambda, t) \). Then \( \lambda(x) + t \leq 1 \), but \( \lambda(yx) + t > 1 \). Since \( \lambda \) is an anti \((\varepsilon, \in \lor \mathcal{Q})\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(2), we get

\[
1 - t < \lambda(yx) \leq \max \{\lambda(x), 0.5\},
\]
If \( \max\{\lambda(x), 0.5\} = \lambda(x) \), then \( \lambda(x) > 1 - t \) and so \( x \bar{\in} L(q\lambda, t) \), which is a contradiction. If \( \max\{\lambda(x), 0.5\} = 0.5 \), then

\[
0.5 \leq 1 - t < \lambda(yx) \leq \max\{\lambda(x), 0.5\} = 0.5,
\]

which is a contradiction. Thus \( yx \in L(q\lambda, t) \). Similarly, let \( x \in L(q\lambda, t) \) and \( y \in R \). Then \( xy \in L(q\lambda, t) \).

Now, let \( a, b \in L(q\lambda, t), x, y \in R \) and \( x + a + y = b + y \) be such that \( x \bar{\in} L(q\lambda, t) \). Then \( \lambda(a) + t \leq 1 \) and \( \lambda(b) + t \leq 1 \), but \( \lambda(x) + t > 1 \). Since \( \lambda \) is an anti \((\in, \notin)\)-fuzzy \( h \)-ideal of \( R \). By Corollary 3.11(4), we get

\[
1 - t < \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\},
\]

If \( \max\{\lambda(a), \lambda(b), 0.5\} = \lambda(a) \), then \( a \notin L(q\lambda, t) \), which is a contradiction.

Similarly, if \( \max\{\lambda(a), \lambda(b), 0.5\} = \lambda(b) \), then \( b \notin L(q\lambda, t) \), which is a contradiction.

If \( \max\{\lambda(a), \lambda(b), 0.5\} = 0.5 \), then \( 0.5 \leq 1 - t < \lambda(x) \leq \max\{\lambda(a), \lambda(b), 0.5\} = 0.5 \), which is a contradiction. Thus \( x \in L(q\lambda, t) \).

\[\square\]

**Corollary 3.16.** Let \( \lambda \) be an anti \((\in, \notin)\)-fuzzy \( k \)-ideal of \( R \). Then we have

1. If \( t \in [0, 0.5] \), then \( L(\in, \lambda, t) \neq \emptyset \) is a \( K \)-ideal of \( R \).
2. If \( t \in [0, 0.5) \), then \( L(\in, \lambda, t) \neq \emptyset \) is a \( K \)-ideal of \( R \).

**Corollary 3.17.** Let \( \lambda \) be an anti \((\in, \notin)\)-fuzzy \( k \)-ideal of \( R \). Then we have

1. If \( t \in [0, 0.5] \), then \( L(\in, \lambda, t) \neq \emptyset \) is an ideal of \( R \).
2. If \( t \in (0, 0.5] \), then \( L(\in, \lambda, t) \neq \emptyset \) is an ideal of \( R \).

**Theorem 3.18.** Let \( A \) be a \( h \)-ideal of \( R \), and let \( \lambda \) and \( \mu \) be fuzzy subset of \( R \) defined by

\[
\mu_A(x) = \begin{cases} 
\geq 0.5 & \text{if } x \in A, \\
0 & \text{a.w.}
\end{cases}
\quad \lambda_A(x) = \begin{cases} 
\leq 0.5 & \text{if } x \in A, \\
1 & \text{a.w.}
\end{cases}
\]

Then

1. \( A = (\mu_A, \lambda_A) \) is an \((\in, \notin)\)-intuitionistic fuzzy \( h \)-ideal of \( R \).
2. \( A = (\mu_A, \lambda_A) \) is an \((q, \in)\)-intuitionistic fuzzy \( h \)-ideal of \( R \).

**Proof.**

1. If \( t, r \in (0, 1] \), then \( A = (\mu_A, \lambda_A) \) must satisfies the following conditions,
   
   - \( a1 \) \( x \in L(\in, \lambda, t), y \in L(\in, \lambda, r) \Rightarrow x + y \in L(\in, \lambda, t \lor r) \),
   - \( a2 \) \( x \in U(\in, \mu, t), y \in U(\in, \mu, r) \Rightarrow x + y \in U(\in, \mu, t \lor r) \),
   - \( b1 \) \( x \in L(\in, \lambda, t) \Rightarrow xy \in L(\in, \lambda, t) \),
   - \( b2 \) \( x \in U(\in, \mu, t) \Rightarrow xy \in U(\in, \mu, t) \),
   - \( c1 \) \( x \in L(\in, \lambda, t) \Rightarrow xy \in L(\in, \lambda, t) \),
   - \( c2 \) \( x \in U(\in, \mu, t) \Rightarrow xy \in U(\in, \mu, t) \),
   - \( d1 \) \( x + a + y = b + y, a \in L(\in, \lambda, t), b \in L(\in, \lambda, r) \Rightarrow x \in L(\in, \lambda, t \lor r) \),
   - \( d2 \) \( x + a + y = b + y, a \in U(\in, \mu, t), b \in U(\in, \mu, r) \Rightarrow x \in U(\in, \mu, t \lor r) \)

   for all \( a, b, x, y \in R \).

   - \( a1 \) Let \( x, y \in R \) and \( t, r \in [0, 1] \) be such that \( x \in L(\in, \lambda, t), y \in L(\in, \lambda, r) \). Then \( \lambda_A(x) \leq t \) and \( \lambda_A(y) \leq r \). Let \( \max\{t, r\} = 1 \). Hence \( \lambda_A(x) = 1 \) or \( \lambda_A(y) = 1 \). Then \( \lambda(x + y) \leq 1 = \max\{\lambda(x), \lambda(y), 0.5\} \). By Theorem 3.10(1), we have \( x + y \in L(\in, \lambda, t \lor r) \). If \( \max\{t, r\} \neq 1 \), then \( \lambda_A(x) \leq 0.5 \) and...
\[ \lambda_a(x) \leq 0.5. \] Thus \( x, y \in A. \) Since \( A \) is a \( h \)-ideal of \( R, \) we have \( x + y \in A. \) This implies
\[ \lambda_a(x + y) \leq 0.5 = \max \{ \lambda(x), \lambda(y), 0.5 \}. \]
Therefore \( x + y \in L(\in \forall q \lambda_a, t \lor r). \)

(a2) Let \( x, y \in R \) and \( t, r \in [0, 1] \) be such that \( x \in U(\in \mu_a, t), \ y \in U(\in \mu_a, r). \) Then \( \mu_a(x) \geq 1 > 0 \) and \( \mu_a(y) \geq r > 0. \) Thus \( \mu_a(x) \geq 0.5 \) and \( \mu_a(y) \geq 0.5, \) and so \( x, y \in A. \) Since \( A \) is a \( h \)-ideal of \( R, \) we have \( x + y \in A. \) Thus \( \mu_a(x + y) \geq 0.5. \) If \( \max \{ t, r \} \leq 0.5, \) then \( \mu_a(x + y) = \max \{ t, r \}, \) and so \( x + y \in U(\in \mu_a, t \lor r). \) If \( \max \{ t, r \} > 0.5, \) then \( \mu_a(x + y) + \max \{ t, r \} > 0.5 + 0.5 = 1, \) and so \( x + y \in U(\in \forall q \mu_a, t \lor r). \) Therefore \( x + y \in L(\in \forall q \lambda_a, t \lor r). \)

(b1) Let \( x, y \in R \) and \( t \in (0, 1] \) be such that \( x \in L(\in \lambda_a, t). \) Then \( \lambda_a(x) \leq t. \) If \( \lambda_a(x) = 1. \) Since \( \lambda(y) \leq 1 = \max \{ \lambda(x), 0.5 \}. \) By Theorem 3.10, we have \( yx \in L(\in \forall q \lambda_a, t). \) If \( \lambda_a(x) \neq 1, \) then \( \lambda_a(x) \leq 0.5, \) Thus \( x, y \in R. \) Since \( A \) is a \( h \)-ideal of \( R, \) we have \( yx \in A. \) Thus \( \lambda_a(y) \leq 0.5 = \max \{ \lambda(x), 0.5 \}. \) Therefore \( yx \in L(\in \forall q \lambda_a, t). \)

Similarly we can prove (c1) and (c2).

(d1) Let \( a, b, x, y \in R, \ x + a + y = b + y \) and \( t, r \in (0, 1] \) be such that \( a \in L(\in \lambda_a, t), \ b \in L(\in \lambda_a, r). \) Then \( \lambda_a(a) \leq t \) and \( \lambda_a(b) \leq r. \) Let \( \max \{ t, r \} = 1. \) Then \( \lambda_a(a) = 1 \) or \( \lambda_a(b) = 1. \) Hence \( \lambda(x) = 1 = \max \{ \lambda(a), \lambda(b), 0.5 \}. \) By Theorem 3.10, we have \( yx \in L(\in \forall q \lambda_a, t \lor r). \) Let \( \max \{ t, r \} \neq 1. \) Then \( \lambda_a(a) \leq 0.5 \) and \( \lambda_a(b) \leq 0.5. \) Thus \( a, b \in A. \) Since \( A \) is a \( h \)-ideal of \( R, \) we have \( x, y \in A. \) Hence \( \lambda_a(x) \leq 0.5. \) This implies
\[ \lambda_a(x) \leq 0.5 = \max \{ \lambda(a), \lambda(b), 0.5 \}. \]
Therefore \( x \in L(\in \forall q \lambda_a, t \lor r). \)

(d2) Let \( a, b, x, y \in R, \ x + a + y = b + y \) and \( t, r \in (0, 1] \) be such that \( a \in U(\in \mu_a, t), b \in U(\in \mu_a, r). \) Then \( \mu_a(a) \geq t > 0 \) and \( \mu_a(b) \geq r > 0. \) Thus \( \mu_a(a) \geq 0.5 \) and \( \mu_a(b) \geq 0.5, \) and so \( a, b \in A. \) Since \( A \) is a \( h \)-ideal of \( R, \) we have \( x \in A. \) Thus \( \mu_a(x) \geq 0.5. \) If \( \max \{ t, r \} \leq 0.5, \) then \( \mu_a(x) = \max \{ t, r \}, \) and so \( x \in U(\in \mu_a, t \lor r). \) If \( \max \{ t, r \} > 0.5, \) then \( \mu_a(x) + \max \{ t, r \} > 0.5 + 0.5 = 1, \) and so \( x \in U(\in \forall q \mu_a, t \lor r). \) Therefore \( x \in U(\in \forall q \lambda_a, t \lor r). \)

\[ \square \]

**Theorem 3.19.** Let \( A \) be a \( k \)-ideal of \( R, \) and let \( \lambda \) and \( \mu \) be fuzzy subset of \( R \) defined by
\[ \mu_A(x) = \begin{cases} \geq 0.5 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}, \quad \lambda_A(x) = \begin{cases} \leq 0.5 & \text{if } x \in A \\ 1 & \text{o.w.} \end{cases} \]

Then
1. \( A = (\mu_A, \lambda_A) \) is an \((\in, \in \forall q)\)-intuitionistic fuzzy \( k \)-ideal of \( R. \)
2. \( A = (\mu_A, \lambda_A) \) is an \((\in, \in \forall q)\)-intuitionistic fuzzy \( k \)-ideal of \( R. \)

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Proof. The proof is similar to that of Theorem 3.18.

Corollary 3.20. Let $A$ be an ideal of $R$, and let $\lambda$ and $\mu$ be fuzzy subset of $R$ defined by

$$
\mu_A(x) = \begin{cases} 
0 & \text{if } x \in A \\
\lambda_A(x) = \begin{cases} 
0.5 & \text{if } x \in A \\
1 & \text{o.w.}
\end{cases}
\end{cases}
$$

Then

1. $A = (\mu_A, \lambda_A)$ is an $(\epsilon, \in \vee q)$-intuitionistic fuzzy ideal of $R$.
2. $A = (\mu_A, \lambda_A)$ is an $a (q, \in \vee q)$-intuitionistic fuzzy ideal of $R$.

4. $(\epsilon, \in \vee q)$-Intuitionistic fuzzy ideals with (anti) $(\epsilon, \in \vee q)$-Fuzzy ideals

In this section, let $R$ be a hemiring. It is clear that, $A = (\mu_A, \lambda_A)$ is an $(\epsilon, \in \vee q)$-intuitionistic fuzzy ideal of $R$ if and only if $\mu_A$ is an $(\epsilon, \in \vee q)$-fuzzy ideal and $\lambda_A$ is an anti $(\epsilon, \in \vee q)$-fuzzy ideal of $R$. But, we introduce some relations between $(\epsilon, \in \vee q)$-intuitionistic fuzzy ideals with $(\epsilon, \in \vee q)$-fuzzy ideals and with anti $(\epsilon, \in \vee q)$-fuzzy ideals.

Theorem 4.1. Let $R$ be a hemiring. Then, $\Box A = (\mu_A, \mu_A^c)$ is an $(\epsilon, \in \vee q)$-intuitionistic fuzzy h-ideal of $R$ if and only if $\mu_A$ is an $(\epsilon, \in \vee q)$-fuzzy h-ideal of $R$.

Proof. Let $\mu_A$ be an $(\epsilon, \in \vee q)$-fuzzy h-ideal of $R$. By Corollary 3.11, it is sufficient to show that $\mu_A^c$ satisfies the conditions:

1. $\forall x, y \in R, \mu_A^c(x + y) \leq \max\{\mu_A(x), \mu_A(y), 0.5\}$,
2. $\forall x, y \in R, \mu_A^c(xy) \leq \max\{\mu_A^c(x), 0.5\}$,
3. $\forall x, y \in R, \mu_A^c(x) \leq \max\{\mu_A^c(x), 0.5\}$,
4. $\forall a, b, x, y \in R, x + a + y = b + y \implies \mu_A^c(x) \leq \max\{\mu_A^c(a), \mu_A^c(b), 0.5\}$.

Since $\mu_A$ is an $(\epsilon, \in \vee q)$-fuzzy h-ideal of $R$. Then

Case (1) For $x, y \in R$, we have $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$, and so

$$1 - \mu_A^c(x + y) \geq \min\{1 - \mu_A^c(x), 1 - \mu_A^c(y), 0.5\}.$$ 

Which implies

$$\mu_A^c(x + y) \leq 1 - \min\{1 - \mu_A^c(x), 1 - \mu_A^c(y), 0.5\}.$$ 

Therefore

$$\mu_A^c(x + y) \leq \max\{\mu_A^c(x), \mu_A^c(y), 0.5\}.$$ 

Case (2) For $x, y \in R$, we have $\mu_A(xy) \geq \min\{\mu_A(x), 0.5\}$, and so

$$1 - \mu_A^c(xy) \geq \min\{1 - \mu_A^c(x), 0.5\}.$$ 

Which implies

$$\mu_A^c(xy) \leq 1 - \min\{1 - \mu_A^c(x), 0.5\}.$$ 

Therefore

$$\mu_A^c(xy) \leq \max\{\mu_A^c(x), 0.5\}.$$ 

Case (3) Similarly, for $x, y \in R$, we have $\mu_A(xy) \geq \min\{\mu_A(x), 0.5\}$, and so $1 - \mu_A^c(xy) \geq \min\{1 - \mu_A^c(x), 0.5\}$. Which implies $\mu_A^c(xy) \leq 1 - \min\{1 - \mu_A^c(x), 0.5\}$. Therefore $\mu_A^c(xy) \leq \max\{\mu_A^c(x), 0.5\}$.
Let the proof is similar to that of Theorem 4.5.

Corollary 4.2. Let R be a hemiring. Then, $\Diamond A = (\lambda^c_A, \lambda_A)$ is an $(\epsilon, 1 \lor q)$-intuitionistic fuzzy $h$-ideal of $R$ if and only if $\lambda_A$ is an anti $(\epsilon, 1 \lor q)$-fuzzy $h$-ideal of $R$.

Theorem 4.3. Let R be a hemiring. Then, $\Box A = (\mu_A, \mu^c_A)$ is an $(\epsilon, 1 \lor q)$-intuitionistic fuzzy $k$-ideal of $R$ if and only if $\mu_A$ is an $(\epsilon, 1 \lor q)$-fuzzy $k$-ideal of $R$.

Proof. The proof is similar to that of Theorem 4.1.

Corollary 4.4. Let R be a hemiring. Then, $\Diamond A = (\lambda^c_A, \lambda_A)$ is an $(\epsilon, 1 \lor q)$-intuitionistic fuzzy $k$-ideal of $R$ if and only if $\lambda_A$ is an anti $(\epsilon, 1 \lor q)$-fuzzy $k$-ideal of $R$.

Theorem 4.5. Let R be a hemiring. Then, $\Box A = (\mu_A, \mu^c_A)$ is an $(\epsilon, 1 \lor q)$-intuitionistic fuzzy ideal of $R$ if and only if $\mu_A$ is an $(\epsilon, 1 \lor q)$-fuzzy ideal of $R$.

Proof. The proof is similar to that of Theorem 4.1.

Corollary 4.6. Let R be a hemiring. Then, $\Diamond A = (\lambda^c_A, \lambda_A)$ is an $(\epsilon, 1 \lor q)$-intuitionistic fuzzy $h$-ideal of $R$ if and only if $\lambda_A$ is an anti $(\epsilon, 1 \lor q)$-fuzzy $h$-ideal of $R$.

Acknowledgements. The authors are grateful to the referee(s) for reading the paper carefully and for making helpful comments.

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