Antti fuzzy ideals in BE-algebra

S. Abdullah, T. Anwar, N. Amin, M. Taimur

Received 16 February 2013; Revised 14 March 2013; Accepted 18 March 2013

Abstract. In this paper, we apply the Biswas idea to BE-algebras and introduce the notion of an anti fuzzy ideal in BE-algebras. Furthermore, these sets are considered in the context of transitive and self distributive BE-algebras and their ideals, providing characterizations of one type, the generalized lower sets, in other type, ideals.

2010 AMS Classification: 06F35, 03G25, 03E72

Keywords: BE-algebra, Ideal, Anti fuzzy ideal, (Generalized) lower set, Self distributive.

Corresponding Author: Saleem Abdullah (saleemabdullah81@yahoo.com)

1. Introduction

The concept of fuzzy sets was first initiated by Zadeh [11] 1965. Since then these ideas have been applied to other algebraic structures such as semigroup, group, ring, etc. Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [6, 7]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Jun et al., [8] introduced the notion of BH-algebra, which is a generalization of BCK/BCI/BCH-algebras. In [10], Kim and Kim introduced the notion of a BE-algebra as a dualization of generalization of a BCK-algebra. In 1990, S. Biswas introduced the concept of anti fuzzy subgroup of group [4]. Recently, Hong and Jun, modifying Biswas idea, apply the concept to BCK-algebras. So, they defined the notion of anti fuzzy ideal of BCK algebras and obtain some useful results on it. In [9] Jun and Song introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. Further more see [11, 2].

In this paper, we apply the Biswas idea to BE-algebras, and introduce the concept of anti fuzzy ideal in BE-algebras and investigate some related properties. Also we characterize anti fuzzy ideals in BE-algebras.
2. Preliminaries

We recall some definitions and results [3, 9, 10].

**Definition 2.1.** An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE-algebra [10] if

\begin{align*}
(2.1) & \quad x \ast x = 1 \text{ for all } x \in X, \\
(2.2) & \quad x \ast 1 = 1 \text{ for all } x \in X, \\
(2.3) & \quad 1 \ast x = x \text{ for all } x \in X, \\
(2.4) & \quad x \ast (y \ast z) = y \ast (x \ast z) \text{ for all } x, y, z \in X.
\end{align*}

A relation "\(\leq\)" on a BE-algebra \(X\) is defined by

\begin{equation}
(\forall x, y \in X) \ (x \leq y \iff x \ast y = 1).
\end{equation}

A BE-algebra \((X; *, 1)\) is said to be transitive [3] if it satisfies:

\begin{equation}
(\forall x, y, z \in X) \ (y \ast z \leq (x \ast y) \ast (x \ast z)).
\end{equation}

A BE-algebra \((X; *, 1)\) is said to be self distributive [10] if it satisfies:

\begin{equation}
(\forall x, y, z \in X) \ (x \ast (y \ast z) = (x \ast y) \ast (x \ast z)).
\end{equation}

Note that every self distributive BE-algebra is transitive, but the converse is not true in general [3]

**Example 2.2 ([10]).** Let \(X := \{1, a, b, c, d, 0\}\) be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>a</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Then \((X; *, 1)\) is a BE-algebra.

**Definition 2.3 ([10]).** A BE-algebra \((X; *, 1)\) is said to be self distributive if

\[ x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \text{ for all } x, y, z \in X. \]
Example 2.4 ([10]). Let $X := \{1, a, b, c, d\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>1</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

It is easy to see that $X$ is a BE-algebra satisfying self distributivity. Note that the BE-algebra in Example 2.2 is not self distributive, since $d \ast (a \ast 0) = d \ast d = 1$, while $(d \ast a) \ast (d \ast 0) = 1 \ast a = a$.

Definition 2.5 ([9]). A non-empty subset $I$ of $X$ is called an ideal of $X$ if

(2.8) $\forall x \in X$ and $\forall a \in I \implies x \ast a \in I$, i.e., $X \ast I \subseteq I$,

(2.9) $\forall x \in X, \forall a, b \in I$ imply $(a \ast (b \ast x)) \ast x \in I$.

In Example 2.2, $\{1, a, b\}$ is an ideal of $X$, but $\{1, a\}$ is not an ideal of $X$, since $(a \ast (a \ast b)) \ast b = (a \ast a) \ast b = 1 \ast b = b \notin \{1; a\}$.

It was proved that every ideal $I$ of a BE-algebra $X$ contains 1, and if $a \in I$ and $x \in X$, then $(a \ast x) \ast x \in I$. Moreover, if $I$ is an ideal of $X$ and if $a \in I$ and $a \leq x$, then $x \in I$ [9].

3. Major section

In this section we introduce anti fuzzy ideals in BE-algebras and discuss some fundamental results.

Definition 3.1. A fuzzy subset $f$ of a BE-algebra $X$ is called an anti fuzzy ideal of $X$ if it satisfies:

(3.1) $(\forall x, y \in X)f(xy) \leq f(y)$

(3.2) $(\forall x, y, z \in X)(f((x \ast (y \ast z)) \ast z) \leq \max\{f(x), f(y)\})$.

Example 3.2. Consider the BE-algebra $X$ described in Example 2.2. Now we define a fuzzy set $f$ on $X$ as:

$$f(x) = \begin{cases} 0.4 & \text{if } x \in \{1, a, b\} \\ 0.7 & \text{if } x \in \{c, d, 0\} \end{cases}$$

Then, by routine calculation $f$ is an anti fuzzy ideal of $X$. Now we define a fuzzy set on $X$ as:

$$f(x) = \begin{cases} 0.4 & \text{if } x \in \{1, a\} \\ 0.7 & \text{if } x \in \{b, c, d, 0\} \end{cases}$$

Then, $f$ is a not an anti fuzzy ideal of $X$, i.e.,

$$f((a \ast (a \ast b)) \ast b) = f(b) = 0.7 > 0.4 = \max\{f(a), f(a)\}.$$
\begin{theorem} \normalfont Let \( f \) be a fuzzy set in \( X \). Then \( f \) is an anti fuzzy ideal of \( X \) if and only if it satisfies:
\[
(\forall \alpha \in [0,1])(L(f;\alpha) \neq \emptyset \implies L(f;\alpha) \text{ is an ideal of } X),
\]
where \( L(f;\alpha) := \{x \in X \mid f(x) \leq \alpha \} \).
\end{theorem}

\begin{proof} \normalfont Let \( f \) be an anti fuzzy ideal in \( X \). Let \( \alpha \in [0,1] \) be such that \( L(f;\alpha) \neq \emptyset \). Let \( x,y \in X \) be such that \( y \in L(f;\alpha) \). Then \( f(y) \leq \alpha \), and so \( f(x \ast y) \leq f(y) \leq \alpha \). Thus \( x \ast y \in L(f;\alpha) \). Let \( x \in X \) and \( a,b \in L(f;\alpha) \). Then \( f(a) \leq \alpha \), \( f(b) \leq \alpha \) and we have
\[
(f((a \ast (b \ast x)) \ast x) \leq \max\{f(a),f(b)\} \leq \alpha
\]
so that \( (a \ast (b \ast x)) \ast x \in L(f;\alpha) \). Hence \( L(f;\alpha) \) is an ideal of \( X \).

Conversely, suppose that \( f \) satisfies (3). If \( f(a \ast b) > f(b) \) for some \( a,b \in X \), then \( f(a \ast b) > \alpha_0 > f(b) \) by taking \( \alpha_0 := (f(a \ast b) + f(b))/2 \). Hence \( a \ast b \notin U(f;\alpha_0) \) and \( b \in U(f;\alpha_0) \), which is a contradiction. Let \( a,b,c \in X \) be such that
\[
f((a \ast (b \ast x)) \ast x) > \max\{f(a),f(b)\}.
\]
Taking \( \beta_0 = (f((a \ast (b \ast x)) \ast x) + \max\{f(a),f(b)\}) \), we have \( \beta_0 \in [0,1] \) and
\[
f((a \ast (b \ast x)) \ast x) > \beta_0 > \max\{f(a),f(b)\}.
\]
it follows that \( a,b \in U(f;\beta_0) \) and \( (a \ast (b \ast x)) \notin U(f;\beta_0) \). This is a contradiction and therefore \( f \) is an anti fuzzy ideal of \( X \).
\end{proof}

\begin{lemma} \normalfont Every anti fuzzy ideal of \( X \) satisfies the following inequality:
\[
(\forall x \in X)(\mu(1) \leq \mu(x)).
\]
\end{lemma}

\begin{proof} \normalfont Since in BE-algebra we have \( x \ast x = 1 \), thus we have
\[
\mu(1) = \mu(x \ast x) \leq \mu(x)
\]
for all \( x \in X \).
\end{proof}

\begin{proposition} \normalfont If \( f \) is an anti fuzzy ideal of \( X \), then
\[
(\forall x,y \in X)(f((x \ast y) \ast y) \leq f(x)).
\]
\end{proposition}

\begin{proof} \normalfont Taking \( y = 1 \) and \( z = y \) in (2), we get
\[
f((x \ast y) \ast y) = f((x \ast (1 \ast y)) \ast y) \leq \max\{f(x),f(1)\} = f(x)
\]
for all \( x,y \in X \).
\end{proof}

\begin{corollary} \normalfont Every anti fuzzy ideal \( f \) of \( X \) is reverse order preserving, that is, \( f \) satisfies:
\[
(\forall x,y \in X)(x \leq y \implies f(x) \geq f(y)).
\]
\end{corollary}

\begin{proof} \normalfont Let \( x,y \in X \) be such that \( x \leq y \). Then \( x \ast y = 1 \), and so
\[
f(y) = f(1 \ast y) = f((x \ast y) \ast y) \leq f(x)
\]
by (2.3) and (3.5).
\end{proof}
Proposition 3.7. Let \( f \) be a fuzzy set in \( X \) which satisfies (3.4) and 
(3.6) \( (\forall x, y, z \in X)(f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y)\}). \)
Then, \( f \) is reverse order preserving.

Proof. Let \( x, y \in X \) be such that \( x \leq y \). Then \( x \ast y = 1 \), and so 
\( f(y) = f(1 \ast y) \leq \max\{f(1 \ast (x \ast y)), f(x)\} = \max\{f(1 \ast 1), f(x)\} \)
by (2.1), (2.3), (3.7) and (3.4).

Theorem 3.8. Let \( X \) be a transitive BE-algebra. A fuzzy set \( f \) in \( X \) is an anti fuzzy ideal of \( X \) if and only if it satisfies conditions (3.4) and (3.7).

Proof. Let \( f \) be an anti fuzzy ideal of \( X \). By lemma (3.1), \( f \) satisfies (3.4). Since \( X \) is transitive, we have 
\( (y \ast z) \ast z \leq (x \ast (y \ast z)) \ast (x \ast z), \)
i.e., \((y \ast z)((x \ast (y \ast z)) \ast (x \ast z)) = 1\) for all \( x, y, z \in X \). It follows from (2.3), (3.2) and Proposition 3.1 that 
\[ f(x \ast z) = f(1 \ast (x \ast z)) = f((y \ast z) \ast z)((x \ast (y \ast z)) \ast (x \ast z)) \leq \max\{f((y \ast z) \ast z), f(x \ast (y \ast z))\} \leq \max\{f(x \ast (y \ast z)), f(y)\}. \]
Hence, \( f \) satisfies (3.7). Conversely suppose that \( f \) satisfies two conditions (3.4) and (3.7). Using (3.7), (2.1), (2.2) and (3.4), we have 
\[ f(x \ast y) \leq \max\{f(x \ast (y \ast y)), f(y)\} = \max\{f(x \ast 1), f(y)\} = \max\{f(1), f(y)\} = f(y) \]
and 
\[ f((x \ast y) \ast y) \leq \max\{f((x \ast y) \ast (x \ast y)), f(x)\} = \max\{f(1), f(x)\} = f(x) \]
for all \( x, y \in X \). Since \( f \) is reverse order preserving by Proposition 3.2, it follows from (3.8) that 
\[ f((y \ast z) \ast z) \geq f((x \ast (y \ast z)) \ast (x \ast z)) \]
and so from (3.7) and (3.10) that 
\[ f((y \ast z) \ast z) \leq \max\{f(((x \ast (y \ast z)) \ast (x \ast z)), f(x)\} \leq \max\{f((y \ast z) \ast z), f(x)\} \leq \max\{f(x), f(y)\} \]
for all \( x, y, z \in X \). Hence, \( f \) is a fuzzy ideal of \( X \).

Corollary 3.9. Let \( X \) be a self distributive BE-algebra. A fuzzy set \( f \) in \( X \) is an anti fuzzy ideal if and only if it satisfies condition (3.4) and (3.7).
Proof. Straightforward.

For every \( a, b \in X \), let \( f^b_a \) be a fuzzy set in \( X \) defined by

\[
f^b_a := \begin{cases} 
\alpha & \text{if } a \ast (b \ast c) = 1 \\
\beta & \text{otherwise}
\end{cases}
\]

for all \( x \in X \) and \( \alpha, \beta \in [0,1] \) with \( \alpha < \beta \).

The following example shows that there exist \( a, b \in X \) such that \( f^b_a \) is not an anti fuzzy ideal of \( X \).

Example 3.10. Let \( X = \{1, a, b, c\} \) with the following Cayley table:

\[
\begin{array}{cccc}
1 & a & b & c \\
\hline
1 & 1 & a & b & c \\
\hline
a & 1 & 1 & a & a \\
b & 1 & 1 & 1 & a \\
c & 1 & 1 & a & 1 \\
\end{array}
\]

Then, \( (X; \ast, 1) \) is a BE-algebra [3]. But \( f^1_a \) is not an anti fuzzy ideal of \( X \) since

\[
f^1_b((a \ast (a \ast c)) \ast c) = f^1_b((a \ast a) \ast c) = f^1_b(1 \ast c) = f^1_b(c) = \beta > \alpha = \max\{f^1_b(a), f^1_b(c)\}.
\]

Theorem 3.11. If \( X \) is self distributive, then the fuzzy set \( f^b_a \) in \( X \) is an anti fuzzy ideal of \( X \) for all \( a, b \in X \).

Proof. Let \( a, b \in X \). For every \( x, y \in X \), if \( a \ast (b \ast y) \neq 1 \), then \( f^b_a(y) = \beta \geq f^b_a(x \ast y) \).

Assume that \( a \ast (b \ast y) = 1 \). Then

\[
a \ast (b \ast (x \ast y)) = a \ast ((b \ast x) \ast (b \ast y)) = (a \ast (b \ast x)) \ast (a \ast (b \ast y)) = (a \ast (b \ast x)) \ast 1 = 1,
\]

and so \( f^b_a(x \ast y) = \alpha = f^b_a(y) \). Hence \( f^b_a(x \ast y) \leq f^b_a(y) \) for all \( x, y \in X \). Now, for every \( x, y, z \in X \), if \( a \ast (b \ast x) \neq 1 \) or \( a \ast (b \ast y) \neq 1 \), then \( f^b_a(x) = \beta \) or \( f^b_a(y) = \beta \). Thus

\[
f^b_a((x \ast (y \ast z)) \ast z) \leq \beta = \max\{f^b_a(x), f^b_a(y)\}.
\]

Suppose that \( a \ast (b \ast x) = 1 \) and \( a \ast (b \ast y) = 1 \). Then

\[
a \ast (b \ast ((x \ast (y \ast z)) \ast z)) = a \ast ((b \ast ((x \ast (y \ast z))) \ast (b \ast z)) = a \ast ((b \ast ((x \ast (y \ast z)))) \ast (a \ast (b \ast z)) = ((a \ast (b \ast x)) \ast (a \ast (b \ast (y \ast z)))) \ast (a \ast (b \ast z)) = (1 \ast (a \ast (b \ast (y \ast z)))) \ast (a \ast (b \ast z)) = (a \ast (b \ast (y \ast z))) \ast (a \ast (b \ast z)) = ((a \ast (b \ast y)) \ast (a \ast (b \ast z))) \ast (a \ast (b \ast z)) = (1 \ast (a \ast (b \ast z))) \ast (a \ast (b \ast z)) = (a \ast (b \ast z)) \ast (a \ast (b \ast z)) = 1
\]

which implies that \( f^b_a((x \ast (y \ast z)) \ast z) = \alpha < \beta = \max\{f^b_a(x), f^b_a(y)\} \).
A nonempty subset $I$ of $X$ is an ideal of $X$ if and only if it satisfies:

$$f(x, y) = f(y, x) \leq \max\{f(x, z), f(y, z)\}$$

for all $x, y, z \in X$. Consequently, $f_{\alpha}$ is an anti fuzzy ideal of $X$ for all $\alpha \in X$.

For any $a, b \in X$, the set $A(a, b) := \{x \in X \mid a * (b * x) = 1\}$ is called the upper set of $a$ and $b$ [4]. Clearly, $1, a, b \in A(a, b)$ for all $a, b \in X$ [5].

Theorem 3.1. A nonempty subset $I$ of $X$ is an ideal of $X$ if and only if it satisfies

$$1 \in I \quad (\forall x, z \in X)(\forall y \in X)(x * (y * z) \in I \implies x * z \in I).$$

Proof. Suppose that $f$ is an anti fuzzy ideal of $X$ and let $a, b \in L(f; \alpha)$. Then $f(a) \leq \alpha$ and $f(b) \leq \alpha$. Let $x \in A(a, b)$. Then, $a * (b * x) = 1$. Hence,

$$f(x) = f(1 * x) = f((a * (b * x)) * x) \leq \max\{f(a), f(b)\} \leq \alpha,$$

and so $x \in L(f; \alpha)$. Thus $A(a, b) \subseteq L(f; \alpha)$.

Conversely, since $1 \in A(a, b) \subseteq L(f; \alpha)$ thus for all $a, b \in X$. Let $x, y, z \in X$ be such that $x * (y * z) \in L(f; \alpha)$ and $y \in L(f; \alpha)$. Since

$$(x * (y * z)) * (y * (x * z)) = (x * (y * z))(y * (x * z)) = 1$$

by [2.4] and [2.1], we have $x * z \in A(x * (y * z), y) \subseteq L(f; \alpha)$. It follows from Lemma 3.2 that $L(f; \alpha)$ is an anti fuzzy ideal of $X$. Hence $f$ is an anti fuzzy ideal of $X$ by Theorem 3.1.

Corollary 3.14. If $f$ is an anti fuzzy ideal of $X$, then

$$(\forall \alpha \in [0, 1]) \quad (L(f; \alpha) \neq \emptyset \implies L(f; \alpha) = \bigcup_{a, b \in L(f; \alpha)} A(a, b)).$$

Proof. Let $\alpha \in [0, 1]$ be such that $L(f; \alpha) \neq \emptyset$. Since, we have

$$L(f; \alpha) \subseteq \bigcup_{a \in L(f; \alpha)} A(a, 1) \subseteq \bigcup_{a, b \in L(f; \alpha)} A(a, b).$$

Now let $x \in \bigcup_{a, b \in L(f; \alpha)} A(a, b)$. Then, there exist $u, v \in L(f; \alpha)$ such that $x \in A(u, v) \subseteq L(f; \alpha)$ by Theorem 4. Thus $\bigcup_{a, b \in L(f; \alpha)} A(a, b) \subseteq L(f; \alpha)$. This completes the proof.

4. Conclusions

Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [6, 7]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. Kim and Kim defined a new class of algebra called BE-algebra in [10]. In this article we studied ideal theory of BE-algebra in context of fuzzy set to introduced anti fuzzy ideals in BE-algebras. We discussed some characterizations of BE-algebras in terms of anti-fuzzy ideals. We also discussed
some basic properties of BE-algebras in terms of these notions which are necessary for further study of BE-algebras. We will be focus on further study in BE-algebras in terms of fuzzy sets as follows: We will defined further generalization of anti fuzzy ideals in BE-algebra. We will study BE-algebra in terms of rough set theory. We will define rough fuzzy ideals in BE-algebras.

References


S. Abdullah (saleemabdullah81@yahoo.com, saleem@math.qau.edu.pk)
Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan

T. Anwar (tariqanwar79@yahoo.co.in)
Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan

N. Amin (naminhu@gmail.com)
Department of Information Technology, Hazara University, Mansehra, KPK, Pakistan

M. Taimur (k.taimur@yahoo.com)
Department of Mathematics, Government Post Graduate College, Mansehra, KPK, Pakistan

494